Quantum-coherence quantifiers based on the Tsallis relative α entropies

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The concept of coherence is one of cornerstones in physics. The development of quantum information science has lead to renewed interest in properly approaching the coherence at the quantum level. Various measures could be proposed to quantify coherence of a quantum state with respect to the prescribed orthonormal basis. To be a proper measure of coherence, each candidate should enjoy certain properties. It seems that the monotonicity property plays a crucial role here. Indeed, there is known an intuitive measure of coherence that does not share this condition. We study coherence measures induced by quantum divergences of the Tsallis type. Basic properties of the considered coherence quantifiers are derived. Tradeoff relations between coherence and mixedness are examined. The property of monotonicity under incoherent selective measurements has to be reformulated. The proposed formulation can naturally be treated as a parametric extension of its standard form. Finally, two coherence measures quadratic in moduli of matrix elements are compared from the monotonicity viewpoint.

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I. INTRODUCTION

Interest in the nature of coherent objects and processes in physics has a very long history. Probably, the notion of coherence is most known due to the role of phase coherence in optical phenomena [1]. It is now clear that quantum coherence is very essential in studying thermodynamic properties of small systems at low temperatures [2–6]. Understanding quantum phenomena such as multipartite entanglement is also connected with a description of coherence. Entangled states play a central role in quantum information science [7]. Due to interaction with the environment, coherent superpositions of states will be altered. Physical processes describing decoherence are also the subject of active research. Recently, many efforts have been made in studies of quantum coherence as a physical resource. Quantum resource theories are speciated by a restriction on the quantum operations that can be implemented [8,9]. To reveal this question with respect to coherence, a unified framework for its quantification is desired. The authors of [10] considered properties that should be satisfied by any proper measure of coherence. They also proposed some ways to construct easily computable measures of coherence. Further development of this approach was established in [11].

In principle, any measure of distinguishability of quantum states leads to a candidate for a coherence quantifier [10]. The following negative result should be emphasized here. It turns out that the measure induced by the squared Hilbert-Schmidt norm does not enjoy a valid coherence monotonicity [10]. In this regard, monotonicity properties play the crucial role in development of proper coherence measures. The coherence quantifiers of [10] were used for obtaining complementarity relations for quantum coherence with respect to mutually unbiased bases [12,13]. The authors of [12] also claimed a conjecture related to the negative result mentioned above. Due to a simple structure, the conjectured quantifier of coherence deserves further investigation. Relations between coherence and multipath interference phenomena were considered in [14,15]. The role of coherence in the Deutsch-Jozsa and related algorithms is considered in [16]. The authors of [17] examined under which conditions the coherence of an open quantum system is unaffected by noise. The paper of [18] is devoted to quantum processes that can neither create nor use coherence. Quantification of coherence in infinite-dimensional systems is studied in [19,20].

In this work, we study coherence quantifiers based on the Tsallis relative entropies. Quantum relative entropies of the Tsallis type are expressed in terms of powers of density matrices. Hence, we may expect a relatively simple character of induced coherence measures. The paper is organized as follows. In Sec. II, we recall the approach developed in [10] and list some preliminaries. In Sec. III, we consider relative entropies of the Tsallis type and prove the two results required. In Sec. IV, we study properties of coherence measures based on the Tsallis relative entropies. In particular, tradeoff relations between coherence and mixedness are obtained. The case of a single qubit is separately discussed in Sec. V. The monotonicity property is satisfied with an interesting form found in Sec. VI. The obtained family of coherence quantifiers includes a homogeneous quadratic function of moduli of matrix elements. Another quadratic function of such a kind is induced by the squared ℓ_2-norm. In Sec. VII, the two quadratic measures of coherence are compared within a concrete example. In Sec. VIII, we conclude the paper with a summary of results.

II. PRELIMINARIES

In this section, we briefly recall basic points of the approach of [10] to quantifying quantum coherence. In principle, measures of coherence may be introduced with using operator norms. Some genuine properties of coherence measures are related to their behavior with respect to quantum operations. Thus, main results of quantum operation formalism should be used. Let $\mathcal{L}(\mathcal{H})$ be the space of linear operators on finite-dimensional Hilbert space $\mathcal{H}$. By $\mathcal{L}_+(\mathcal{H})$, we denote the set of positive semidefinite operators on $\mathcal{H}$. By $\mathrm{ran}(\hat{X})$, we denote the range of operator $\hat{X}$. In the following, we use the convention that powers of a positive operator are taken only on its support. For any $\hat{Z} \in \mathcal{L}_+(\mathcal{H})$, we treat $\hat{Z}^0$ as the orthogonal projector onto $\mathrm{ran}(\hat{Z})$. Let $\hat{P}$ and $\hat{Q}$ be operators of the orthogonal projection. In the finite-dimensional case, we define $\hat{P} \vee \hat{Q}$

as the projector onto the sum of subspaces $\mathrm{ran}(\hat{P})+\mathrm{ran}(\hat{Q})$. In the infinite-dimensional case, this definition should be modified. In the following, we will deal with finite dimensions only.

A distance between operators of interest can be characterized by norms. With respect to the given orthonormal basis, each operator $\hat{X}:\mathcal{H}\to\mathcal{H}$ is represented by the square matrix with elements x_{ij}. The ℓ_1-norm is then defined as [21]

$$\|\hat{X}\|_{\ell_1}:=\sum_{ij}|x_{ij}|. \tag{1}$$

Further, the ℓ_2-norm is defined as

$$\|\hat{X}\|_{\ell_2}:=\left(\sum_{ij}|x_{ij}|^2\right)^{1/2}. \tag{2}$$

This norm is also known as the Frobenius or Hilbert-Schmidt norm [21]. There are other frequently used norms such as the Schatten norms and the Ky Fan norms. These norms, defined in terms of singular values, are unitarily invariant [21].

A state of the quantum system of interest is represented by positive operator $\hat{\rho}$ normalized as $\mathrm{Tr}(\hat{\rho})=1$. To formulate the desired properties of measures of coherence, we will use some basic facts about quantum operations. Let us consider a linear map,

$$\Phi:\ \mathcal{L}(\mathcal{H}_A)\to\mathcal{L}(\mathcal{H}_B), \tag{3}$$

where the input space $\mathcal{H}_A$ and the output space $\mathcal{H}_B$ can be different. This map is positive, when $\Phi(\hat{X})\in\mathcal{L}_+(\mathcal{H}_B)$ for each $\hat{X}\in\mathcal{L}_+(\mathcal{H}_A)$ [7]. Physical processes are described by completely positive maps [7]. Let id_R be the identity map on $\mathcal{L}(\mathcal{H}_R)$, where the Hilbert space $\mathcal{H}_R$ is related to an imagined reference system. The complete positivity implies that the map $\Phi\otimes\mathrm{id}_R$ is positive for arbitrary dimensionality of $\mathcal{H}_R$. Each completely positive map can be represented in the form [7]

$$\Phi(\hat{X})=\sum_n \hat{K}_n\hat{X}\hat{K}_n^{\dagger}, \tag{4}$$

with the Kraus operators $\hat{K}_n:\ \mathcal{H}_A\to\mathcal{H}_B$. The map preserves the trace, when these operators satisfy

$$\sum_n \hat{K}_n^{\dagger}\hat{K}_n=\mathbb{1}_A. \tag{5}$$

Trace-preserving completely positive (TPCP) maps are usually referred to as quantum channels.

The authors of [10] developed an approach to quantum coherence with the use of a fixed preferred basis for a physical situation of interest. They also collected desirable properties a proper measure of coherence should satisfy. Some applications of these ideas were further developed in [12,15,19]. Let $\mathcal{E}=\{|e_i\rangle\}$ be a prescribed orthonormal basis in $\mathcal{H}_A$. The set of incoherent states contains all states that are diagonal with respect to $\mathcal{E}$, namely,

$$\hat{\delta}=\sum_i \delta_i\,|e_i\rangle\langle e_i|. \tag{6}$$

By $\mathcal{I}(\mathcal{E})\subset\mathcal{L}_+(\mathcal{H}_A)$, we mean the set of all such states. Quantifiers of coherence should map from the set of states to the set of non-negative real numbers. The following two quantifiers of coherence are intuitively natural [10]. Using the ℓ_1-norm finally gives

$$C_{\ell_1}(\mathcal{E}|\hat{\rho}):=\min_{\hat{\delta}\in\mathcal{I}(\mathcal{E})}\|\hat{\rho}-\hat{\delta}\|_{\ell_1}=\sum_{i\neq j}|\langle e_i|\hat{\rho}|e_j\rangle|. \tag{7}$$

Another natural candidate is the one based on the squared ℓ_2-norm [10]. That is, we write

$$C_{\ell_2}(\mathcal{E}|\hat{\rho}):=\min_{\hat{\delta}\in\mathcal{I}(\mathcal{E})}\|\hat{\rho}-\hat{\delta}\|_{\ell_2}^2=\sum_{i\neq j}|\langle e_i|\hat{\rho}|e_j\rangle|^2. \tag{8}$$

Unfortunately, this seemingly natural measure does not obey the monotonicity requirement [10]. The trace norm also induces an interesting candidate for quantification of coherence [22]. The authors of [23] proposed a common frame to quantify quantumness in terms of coherence and entanglement. They also derived the geometric measure of coherence based on the notion of fidelity of quantum states [24,25].

III. QUANTUM DIVERGENCES OF THE TSALLIS TYPE

In this section, we recall the definition of relative entropies of the Tsallis type. Many fundamental results of quantum information theory are connected with the properties of the standard relative entropy [7]. There exist several extensions of the standard entropic functions [26]. Many quantum relative entropies can be unified within the concept of f divergences [27]. This approach is a quantum counterpart of the Csiszár f divergence [28]. For $0<\alpha\neq 1$, the Tsallis relative α entropy is defined as [29,30]

$$D_\alpha(p\|q):=\frac{1}{\alpha-1}\left(\sum_j p_j^{\alpha}q_j^{1-\alpha}-1\right). \tag{9}$$

If for some j we have $p_j\neq 0$ and $q_j=0$, and then the relative α entropy with $\alpha>1$ is set to be $+\infty$. In the limit $\alpha\to 1$, the above divergence gives the standard relative entropy $D_1(p\|q)=\sum_j p_j\,\ln(p_j/q_j)$. Here, one assumes $-0\ln 0\equiv 0$ and $-p_j\ln 0\equiv+\infty$ for $p_j>0$ [7]. Basic properties of quantity (9) were discussed in [29,30]. We mention only several of them. It was shown in [30] $D_\alpha(p\|q)\geqslant 0$. Necessary conditions for vanishing $D_\alpha(p\|q)$ are a more complicated question. For the class of Csiszár f divergences, this question was considered in [31]. The answer is connected with the notion of strict convexity of a certain function at 1. It follows from the results of [31] that $D_\alpha(p\|q)=0$ only when the distributions p and q coincide (see, e.g., example 2 of [31]).

We shall now recall the notion of quantum relative entropy. For density operators $\hat{\rho}$ and $\hat{\sigma}$, the quantum relative entropy is expressed as [7]

$$D_1(\hat{\rho}\|\hat{\sigma}):=\begin{cases}\mathrm{Tr}(\hat{\rho}\,\ln\hat{\rho}-\hat{\rho}\,\ln\hat{\sigma}), & \text{if } \mathrm{ran}(\hat{\rho})\subseteq\mathrm{ran}(\hat{\sigma}),\\ +\infty, & \text{otherwise.}\end{cases} \tag{10}$$

For a discussion of the role of (10) in quantum information theory, see [7,32] and references therein. The divergence (10) can be generalized in several ways. For $\alpha\in(1;+\infty)$, the Tsallis α divergence is defined as

$$D_\alpha(\hat{\rho}\|\hat{\sigma}):=\begin{cases}\frac{\mathrm{Tr}(\hat{\rho}^{\alpha}\hat{\sigma}^{1-\alpha})-1}{\alpha-1}, & \text{if } \mathrm{ran}(\hat{\rho})\subseteq\mathrm{ran}(\hat{\sigma}),\\ +\infty, & \text{otherwise.}\end{cases} \tag{11}$$

When $\text{ran}(\hat{\rho}) \subseteq \text{ran}(\hat{\sigma})$, the trace is assumed to be taken over $\text{ran}(\hat{\sigma})$. For $\alpha \in (0;1)$, the first entry can be used without such conditions. The formula (11) can be represented similarly to (10) with the use of the α logarithm. For $0 < \alpha \neq 1$ and real $\xi > 0$, the α logarithm is defined as [29]

$$\ln_\alpha(\xi) := \frac{\xi^{1-\alpha} - 1}{1-\alpha}. \tag{12}$$

For $\alpha \to 1$, the function (12) reduces to the usual logarithm. Up to a factor, the relative entropy (11) is a particular case of quasientropies proposed by Petz [33]. A more general family of quantum f divergences is studied in [27]. The following extension will also be useful. Let $\hat{A}$ and $\hat{B}$ be positive operators such that $\text{ran}(\hat{A}) \subseteq \text{ran}(\hat{B})$. The α divergence of $\hat{A}$ with respect to $\hat{B}$ is defined by

$$D_\alpha(\hat{A}\|\hat{B}) := \frac{1}{\alpha-1}[\text{Tr}(\hat{A}^\alpha \hat{B}^{1-\alpha}) - \text{Tr}(\hat{A})]. \tag{13}$$

Recall several properties of the quantum α divergence. They follow from the corresponding results on the quantum f divergences [27]. For all $\lambda \in [0;+\infty)$, we have

$$D_\alpha(\lambda\hat{A}\|\lambda\hat{B}) = \lambda\, D_\alpha(\hat{A}\|\hat{B}). \tag{14}$$

Let four positive semidefinite operators $\hat{A}_1, \hat{B}_1, \hat{A}_2, \hat{B}_2$ obey $\hat{A}_1^0 \vee \hat{B}_1^0 \perp \hat{A}_2^0 \vee \hat{B}_2^0$; then

$$D_\alpha(\hat{A}_1 + \hat{A}_2\|\hat{B}_1 + \hat{B}_2) = D_\alpha(\hat{A}_1\|\hat{B}_1) + D_\alpha(\hat{A}_2\|\hat{B}_2). \tag{15}$$

The latter can be proved for quantum f divergences under certain conditions [27]. We will use (15) in studies of the monotonicity of coherence quantifiers.

One of the fundamental properties of (10) is its monotonicity under TPCP maps [7]. That is, for any TPCP map (3) we have $D_1(\Phi(\hat{\rho})\|\Phi(\hat{\sigma})) \leqslant D_1(\hat{\rho}\|\hat{\sigma})$. In the classical regime, the relative Tsallis entropy (9) is monotone under stochastic maps for all $\alpha \geqslant 0$ [30]. This is not the case for the quantum regime. The quantum α divergence (11) is monotone under TPCP maps for $\alpha \in (0;2]$, namely,

$$D_\alpha(\Phi(\hat{\rho})\|\Phi(\hat{\sigma})) \leqslant D_\alpha(\hat{\rho}\|\hat{\sigma}). \tag{16}$$

This claim follows from the results of [27] (see Theorem 4.3 therein) and the two facts about functions on positive matrices. The function $\xi \mapsto \xi^\alpha$ is matrix concave on $[0;+\infty)$ for $0 \leqslant \alpha \leqslant 1$ and matrix convex on $[0;+\infty)$ for $1 \leqslant \alpha \leqslant 2$ (see, e.g., Theorems 4.2.3 and 1.5.8 in [34]). The monotonicity further yields the joint convexity of the f divergences (see, e.g., Corollary 4.7 of [27]). In particular, the quantum α divergences of the Tsallis type are jointly convex for $\alpha \in (0;2]$. Let $\{\hat{\rho}_n\}$ and $\{\hat{\sigma}_n\}$ be two collections of density matrices, and let p_n's be positive numbers that are summarized to 1. For $\alpha \in (0;2]$, we then have

$$D_\alpha\left(\sum_n p_n\hat{\rho}_n \middle\| \sum_n p_n\hat{\sigma}_n\right) \leqslant \sum_n p_n\, D_\alpha(\hat{\rho}_n\|\hat{\sigma}_n). \tag{17}$$

The properties (16) and (17) will be very important in the verification of corresponding properties of induced coherence measures. We shall also discuss other properties of the quantum α divergences. They are essential from the viewpoint of constructing measures of coherence. So, we present them as separate statements.

Theorem 1. For $\alpha > 0$, the quantum α divergence is non-negative,

$$D_\alpha(\hat{\rho}\|\hat{\sigma}) \geqslant 0, \tag{18}$$

with equality if and only if $\hat{\rho} = \hat{\sigma}$.

The proof of this statement is carried out similarly to the case of the standard relative entropy (see, e.g., theorem 11.7 in [7]). We refrain from presenting the details here. It should be noted that positivity of the Tsallis α divergence *per se* was considered in Proposition 2.4 of [30]. Although the authors of [30] focused on the range $0 < \alpha < 1$, their arguments are applicable for all positive α. We are also interested in conditions for equality. For this aim, we merely modify the proof of Theorem 11.7 of [7] with the α logarithm (12). Another property of the Tsallis α divergence is essential in studying the monotonicity of the induced coherence measures.

Theorem 2. Let $\{\hat{K}_n\}$ be a set of operators such that $\sum_n \hat{K}_n^\dagger\hat{K}_n = \mathbb{1}_A$. With the given normalized density matrices $\hat{\rho}$ and $\hat{\sigma}$ on $\mathcal{H}_A$, one associates two probability distributions with the corresponding probabilities

$$p_n = \text{Tr}(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger), \qquad q_n = \text{Tr}(\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger). \tag{19}$$

For $\alpha > 0$, the quantum α divergences obey

$$\sum_n D_\alpha(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\|\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger) \geqslant \sum_n p_n^\alpha q_n^{1-\alpha}\, D_\alpha(\hat{\rho}_n\|\hat{\sigma}_n), \tag{20}$$

where the states $\hat{\rho}_n = p_n^{-1}\hat{K}_n\hat{\rho}\hat{K}_n^\dagger$ and $\hat{\sigma}_n = q_n^{-1}\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger$ are normalized.

Proof. The right-hand side of (20) is focused on those values of n that $p_n \neq 0$ and $q_n \neq 0$ simultaneously. When $p_n \neq 0$ and $q_n = 0$ for some n, we have $\hat{K}_n\hat{\rho}\hat{K}_n^\dagger \neq \mathbf{0}$ and $\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger = \mathbf{0}$. Then the corresponding term in the left-hand side of (20) becomes $+\infty$ [see the second line of (11)], whence the statement holds. So, we will prove (20) for the case in which $p_n \neq 0$ implies $q_n \neq 0$. Due to the definition (11), we can write

$$D_\alpha(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\|\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger) = D_\alpha(p_n\hat{\rho}_n\|q_n\hat{\sigma}_n) = p_n^\alpha q_n^{1-\alpha}\, D_\alpha(\hat{\rho}_n\|\hat{\sigma}_n) + \frac{p_n^\alpha q_n^{1-\alpha} - p_n}{\alpha-1}. \tag{21}$$

Hence, the left-hand side of (20) minus the right-hand side is equal to $D_\alpha(p\|q) \geqslant 0$. ■

In the case $\alpha = 1$, the inequality (20) is reduced to

$$\sum_n D_1(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\|\hat{K}_n\hat{\sigma}\hat{K}_n^\dagger) \geqslant \sum_n p_n\, D_1(\hat{\rho}_n\|\hat{\sigma}_n). \tag{22}$$

This property of the standard relative entropy was formulated and proved in [35] [see item (F4) therein]. So, we obtained an extension of the formula (22) to quantum divergences of the Tsallis type. Such an extension does not seem to have been previously recognized in the literature. It should be noted, however, that the right-hand side of (20) is more complicated in character. We will use (20) in studying changes of coherence quantifiers under incoherent selective measurements.

IV. COHERENCE QUANTIFIERS BASED ON THE TSALLIS DIVERGENCES

The authors of [10] pointed out a general way to obtain candidates for quantification of coherence. To find more coherence measures, we can try to consider generalized relative entropies. This is formally posed as follows. Let us pick the Tsallis α divergence as a distinguishability measure. For $\alpha > 0$, we define the quantity

$$C_\alpha(\mathcal{E}|\hat{\rho}) := \min_{\hat{\delta}\in\mathcal{I}(\mathcal{E})} D_\alpha(\hat{\rho}\|\hat{\delta}). \tag{23}$$

In principle, this approach could be used with divergences of a more general type. The optimization problem is generally not easy. However, it is simply resolved in the case of Tsallis divergences. The following statement takes place.

Theorem 3. For all $0 < \alpha \neq 1$, the corresponding coherence measure is expressed as

$$C_\alpha(\mathcal{E}|\hat{\rho}) = \frac{1}{\alpha-1}\left\{\left(\sum_j \langle e_j|\hat{\rho}^\alpha|e_j\rangle^{1/\alpha}\right)^\alpha - 1\right\}. \tag{24}$$

Proof. Since the α divergence $D_\alpha(\hat{\rho}\|\hat{\delta})$ should be minimized, we will assume $\mathrm{ran}(\hat{\rho}) \subseteq \mathrm{ran}(\hat{\delta})$. If δ_j's are eigenvalues of $\hat{\delta}$, we set $\delta_j = 0$ whenever $\langle e_j|\hat{\rho}|e_j\rangle = 0$. As any $\hat{\delta} \in \mathcal{I}(\mathcal{E})$ is diagonal with respect to $\mathcal{E}$, we write

$$D_\alpha(\hat{\rho}\|\hat{\delta}) = \frac{1}{\alpha-1}\left\{\sum_j \langle e_j|\hat{\rho}^\alpha|e_j\rangle\, \delta_j^{1-\alpha} - 1\right\}, \tag{25}$$

where the sum is taken over nonzero matrix elements. Let us define the probabilities r_j such that $r_j^\alpha \propto \langle e_j|\hat{\rho}^\alpha|e_j\rangle$. Together with the normalization condition, one gets

$$r_j = \frac{1}{N}\, \langle e_j|\hat{\rho}^\alpha|e_j\rangle^{1/\alpha}, \tag{26}$$

$$N = \sum_i \langle e_i|\hat{\rho}^\alpha|e_i\rangle^{1/\alpha}. \tag{27}$$

Substituting $\langle e_j|\hat{\rho}^\alpha|e_j\rangle = N^\alpha r_j^\alpha$ into (25), we obtain

$$D_\alpha(\hat{\rho}\|\hat{\delta}) = N^\alpha D_\alpha(r\|\delta) + \frac{N^\alpha - 1}{\alpha - 1}. \tag{28}$$

Here, the probabilities r_j and the normalization denominator N depends only on the state $\hat{\rho}$. As was already mentioned, we always have $D_\alpha(r\|\delta) \geqslant 0$. To minimize the right-hand side of (28), we should therefore reach $D_\alpha(r\|\delta) = 0$ by setting $\delta_j = r_j$. Combining the result with the formula for N completes the proof. ∎

Thus, the result of minimizing is expressed in terms of matrix elements of the power $\hat{\rho}^\alpha$. For the given $\hat{\rho}$ and α, the minimum in (23) is reached with the state

$$\hat{\delta}_\rho = \frac{1}{N}\sum_j \langle e_j|\hat{\rho}^\alpha|e_j\rangle^{1/\alpha}\, |e_j\rangle\langle e_j|. \tag{29}$$

We avoid stating explicitly that $\hat{\delta}_\rho$ depends on the value of α. It the case $\alpha = 1$, we obtain the formulation with the standard relative entropy. Then the state (29) is obtained from $\hat{\rho}$ by deleting all off-diagonal elements. Then the coherence measure is merely equal to the von Neumann entropy of this diagonal state minus the von Neumann entropy of $\hat{\rho}$ [10]. Let us consider another interesting case $\alpha = 2$. As the density matrix is Hermitian, we obtain

$$C_2(\mathcal{E}|\hat{\rho}) = \left(\sum_j \sqrt{\sum_i |\rho_{ij}|^2}\right)^2 - 1, \tag{30}$$

where $\rho_{ij} = \langle e_i|\hat{\rho}|e_j\rangle$. This coherence measure is a function of squared moduli $|\rho_{ij}|^2$ but more complicated in comparison with (8). We now consider basic properties of the presented coherence quantifiers.

First of all, the quantity (23) is zero for all incoherent states. It follows from (24) by substituting $\langle e_j|\hat{\rho}^\alpha|e_j\rangle = \rho_{jj}^\alpha$ and $\sum_j \rho_{jj} = 1$. Further, we have $C_\alpha(\mathcal{E}|\hat{\rho}) = 0$ only for incoherent states. According to Theorem 1, $D_\alpha(\hat{\rho}\|\hat{\sigma})$ is zero only for $\hat{\rho} = \hat{\sigma}$. So, for $\hat{\rho} \notin \mathcal{I}(\mathcal{E})$ and any $\hat{\delta} \in \mathcal{I}(\mathcal{E})$ we have $D_\alpha(\hat{\rho}\|\hat{\delta}) > 0$. Thus, the coherence measure (23) satisfies one of the conditions listed in [10]. An upper bound on the α quantifiers of coherence can be expressed in terms of the purity.

Theorem 4. For $0 < \alpha \leqslant 2$, we have

$$C_\alpha(\mathcal{E}|\hat{\rho}) \leqslant -\ln_\alpha\left(\frac{1}{d\,\mathrm{Tr}(\hat{\rho}^2)}\right). \tag{31}$$

For $2 < \alpha < \infty$, we have

$$C_\alpha(\mathcal{E}|\hat{\rho}) \leqslant \frac{1}{\alpha-1}\{d\,\mathrm{Tr}(\hat{\rho}^2)(1+\sqrt{d-1}\sqrt{d\,\mathrm{Tr}(\hat{\rho}^2)-1})^{\alpha-2} - 1\}. \tag{32}$$

Proof. We first consider the case $\alpha \neq 1$. Due to (23), for $0 < \alpha \neq 1$ we immediately obtain

$$C_\alpha(\mathcal{E}|\hat{\rho}) \leqslant \frac{d^{\alpha-1}\mathrm{Tr}(\hat{\rho}^\alpha) - 1}{\alpha - 1}. \tag{33}$$

The right-hand side of (33) is the α divergence of $\hat{\rho}$ with respect to the completely mixed state. For $\alpha \leqslant 2$, the function $\xi \mapsto \xi^{\alpha-1}/(\alpha-1)$ is concave. Calculating traces in the eigenbasis of $\hat{\rho}$ and applying Jensen's inequality, we then have

$$\frac{\mathrm{Tr}(\hat{\rho}^\alpha)}{\alpha-1} = \sum_j \lambda_j\, \frac{\lambda_j^{\alpha-1}}{\alpha-1} \leqslant \frac{[\mathrm{Tr}(\hat{\rho}^2)]^{\alpha-1}}{\alpha-1}. \tag{34}$$

Here, the eigenvalues λ_j of $\hat{\rho}$ obey the normalization condition. Combining (34) with (33) and (12) finally gives (31) for $\alpha \neq 1$. To complete the proof of (31), we write

$$C_1(\mathcal{E}|\hat{\rho}) \leqslant \ln d + \mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) \tag{35}$$

and repeat the above reasons with the concave function $\xi \mapsto \ln\xi$.

Let us proceed to the case $\alpha > 2$. As follows from Lemma 3 of [36], the maximal eigenvalue of $\hat{\rho}$ satisfies

$$\lambda_{\max} \leqslant \frac{1}{d}(1 + \sqrt{d-1}\sqrt{d\,\mathrm{Tr}(\hat{\rho}^2) - 1}). \tag{36}$$

Combining this with $\mathrm{Tr}(\hat{\rho}^\alpha) \leqslant \lambda_{\max}^{\alpha-2}\,\mathrm{Tr}(\hat{\rho}^2)$ and (33) completes the proof. ∎

The results (31) and (32) provide an upper bound on the coherence quantifiers in terms of the purity $\mathrm{Tr}(\hat{\rho}^2)$. They are similar to the complementarity relation derived in [12] with the coherence measure (7) taken for $d+1$ mutually unbiased bases (MUBs). The distinction of the formulas (31) and (32) is that only a single quantifier is involved. The purity is closely related to the Brukner-Zeilinger concept of operationally invariant measure of information in quantum measurements [37]. The method of [37] is based on the use of a complete set of $d+1$ MUBs. Except for prime power d, the existence of such sets is an open problem [38]. Then three other schemes to approach the Brukner-Zeilinger information can be used [39]. Hence, we have a way to estimate the right-hand sides of both (31) and (32) in experiment.

The result (31) can be reformulated as a tradeoff relation between coherence and mixedness. For d-dimensional state $\hat{\rho}$, one of the natural quantifiers of the mixedness is given by [40]

$$\mathrm{M}(\hat{\rho}) := \frac{d}{d-1}[1-\mathrm{Tr}(\hat{\rho}^2)]. \tag{37}$$

This figure is zero for pure states and reaches 1 for the completely mixed state. The purity can be expressed via the mixedness and then substituted to (31) and (32). However, the resulting inequalities will be too complicated. A convenient method is to approach the right-hand side of (31) from above by a linear function of the variable $d\,\mathrm{Tr}(\hat{\rho}^2)=y\in[1;d]$. Here, we deal with the function

$$f_\alpha(y) := -\ln_\alpha\left(\frac{1}{y}\right) = \frac{y^{\alpha-1}-1}{\alpha-1}\,, \tag{38}$$

which is concave for $\alpha\leqslant 2$. By the Taylor formula with remainder written in Lagrange's form, with $1<c<d$, one gets

$$f_\alpha(y) = f_\alpha(1) + f'_\alpha(1)\,(y-1) + \tfrac{1}{2}\,f''_\alpha(c)\,(y-1)^2 \leqslant y-1. \tag{39}$$

Here, we used $f_\alpha(1)=0$, $f'_\alpha(1)=1$, and $f''_\alpha(c)\leqslant 0$. The claim (39) poses that the graph of concave $f_\alpha(y)$ goes under its tangent line drawn at the point $y=1$. Combining (39) with (31), for $0<\alpha\leqslant 2$ we have

$$C_\alpha(\mathcal{E}|\hat{\rho}) \leqslant d\,\mathrm{Tr}(\hat{\rho}^2)-1. \tag{40}$$

Due to (40), we obtain a tradeoff relation between coherence and mixedness in the form

$$\frac{1}{d-1}\,C_\alpha(\mathcal{E}|\hat{\rho}) + \mathrm{M}(\hat{\rho}) \leqslant 1, \tag{41}$$

where $0<\alpha\leqslant 2$. When a degree of mixedness increases, an upper bound on values of the coherence α quantifier decreases. It is instructive to compare (41) with Theorem 1 of the paper [41], where tradeoff between coherence and mixedness is expressed in terms of the quantities (7) and (37).

V. ON COHERENCE OF A SINGLE QUBIT

In this section, we examine coherence of a single qubit with the use of the α quantifiers. Here we can express results more explicitly. With respect to the prescribed basis, the density matrix is written as

$$\hat{\omega} = \begin{pmatrix} u & w^* \\ w & 1-u \end{pmatrix}. \tag{42}$$

For brevity, we will further omit the symbol of the reference basis in notation. Of course, the real parameter u lies between 0 and 1. The eigenvalues of (42) are expressed as

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{\left(u-\frac{1}{2}\right)^2 + |w|^2}. \tag{43}$$

They should be both positive and no greater than 1, whence $|w|\leqslant\sqrt{u(1-u)}$. For integer values of α, sufficiently simple expressions take place. For $\alpha=1$, we obtain

$$C_1(\hat{\omega}) = h(u) - h(\lambda_+), \tag{44}$$

where $h(u):=-u\ln u-(1-u)\ln(1-u)$ is the binary Shannon entropy. Furthermore, we have

$$C_2(\hat{\omega}) = \left(\sqrt{u^2+|w|^2}+\sqrt{(1-u)^2+|w|^2}\right)^2-1. \tag{45}$$

Note also that $C_{\ell_1}(\hat{\omega})=2\,|w|$ and $C_{\ell_2}(\hat{\omega})=2\,|w|^2$. For another integer α, the resulting expressions are obtained similarly to (45).

One way to study coherence of a single qubit is posed as follows. For the given u, we consider an interval of changes of the corresponding quantifier. The minimum is clearly 0, whereas the maximum is found as for the function of $|w|\leqslant\sqrt{u(1-u)}$. For example, we have

$$\max\{C_2(\hat{\omega}):\ |w|\leqslant\sqrt{u(1-u)}\} = 2\sqrt{u(1-u)}, \tag{46}$$

$$\max\{C_{\ell_1}(\hat{\omega}):\ |w|\leqslant\sqrt{u(1-u)}\} = 2\sqrt{u(1-u)}. \tag{47}$$

For the given u, the coherence quantifiers $C_2(\hat{\omega})$ and $C_{\ell_1}(\hat{\omega})$ cover the same interval of values. Their maximal values are reached for the same states. These states are pure, since $|w|^2=u(1-u)$ implies $\lambda_+=1$ and $\lambda_-=0$. The coherence quantifiers are zero for incoherent states when $w=0$.

Let us consider the maximum of $C_\alpha(\hat{\omega})$ for the given u similarly to (46) and (47). In Fig. 1, this maximum is shown as a function of u for several integer values of α. In particular, by

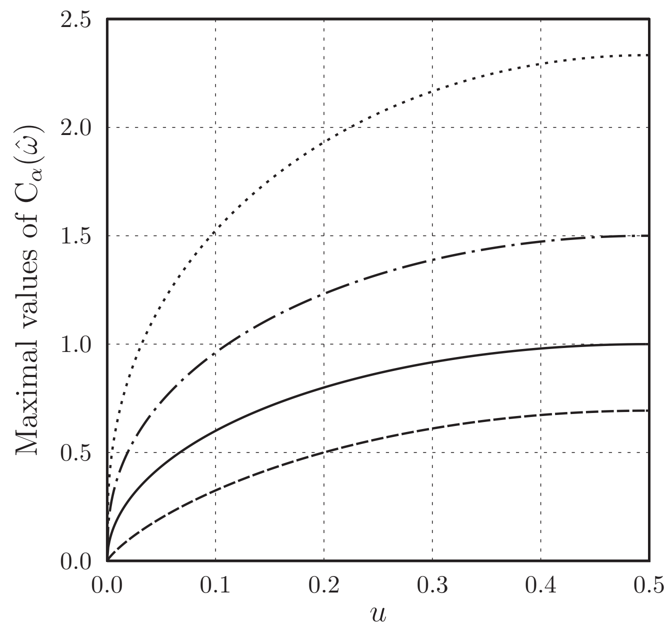

FIG. 1. Maximal values of $C_\alpha(\hat{\omega})$ versus u are shown by a dashed line for $\alpha=1$, by a solid line for $\alpha=2$, by a dash-dotted line for $\alpha=3$, and by a dotted line for $\alpha=4$.

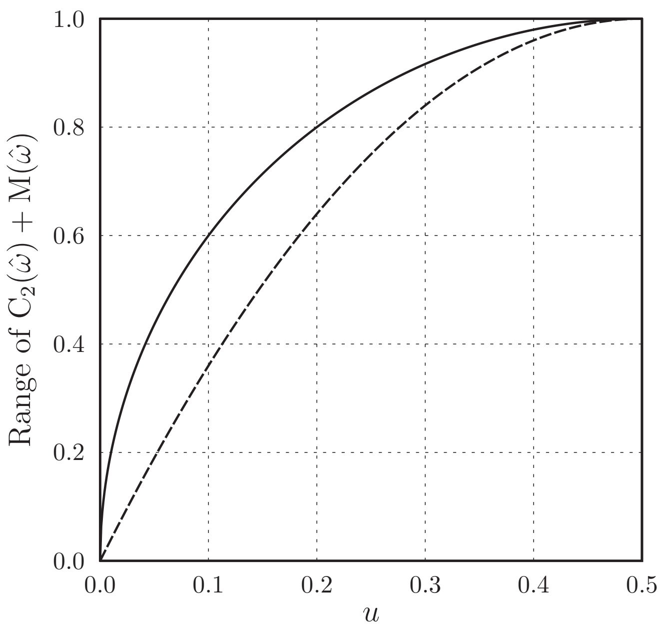

FIG. 2. The minimal and maximal values of the sum $C_2(\hat{\omega}) + \mathrm{M}(\hat{\omega})$ as functions of u.

the solid line we represent both (46) and (47). The curves are shown only for the half of the interval $u \in [0;1]$, since they are symmetric with respect to the line $u = 1/2$. The four curves all show a similar behavior. In each case, the range between the abscissa and the curve shows those values that are covered by the corresponding quantifier. Hence, the coherence α quantifier seems to be more sensitive for larger values of α. On the other hand, coherence measures should obey monotonicity properties. Without them, any candidate to quantify coherence cannot be accepted.

For $d = 2$, the formula (41) gives $C_\alpha(\hat{\omega}) + \mathrm{M}(\hat{\omega}) \leqslant 1$ with $0 < \alpha \leqslant 2$. For a single qubit, we can obtain more precise tradeoff bounds. In particular, we have

$$4u(1-u) \leqslant C_2(\hat{\omega}) + \mathrm{M}(\hat{\omega}) \leqslant 2\sqrt{u(1-u)}. \tag{48}$$

It is instructive to compare (48) with the exact equality

$$C_{\ell_1}(\hat{\omega})^2 + \mathrm{M}(\hat{\omega}) = 4u(1-u). \tag{49}$$

In Fig. 2, we plot the left-hand side of (48) by a dashed line and the right-hand side of (48) by a solid line. Here, the former is reached for incoherent states and the latter is reached for pure states. For each u, the range between these lines shows values that are covered by the sum $C_2(\hat{\omega}) + \mathrm{M}(\hat{\omega})$. So, this sum ranges in a narrow interval. It is similar to the sum (49), but the latter is quadratic in the coherence measure. Thus, a lack of coherence will rather be accompanied by some increasing of the mixedness. Figure 2 also illustrates that the relation $C_2(\hat{\omega}) + \mathrm{M}(\hat{\omega}) \leqslant 1$ is sufficiently tight when the diagonal elements of (42) do not differ essentially.

In the case of a single qubit, the considered coherence quantifiers enjoy a behavior similarly to the measure (7). There are also natural tradeoff relations between coherence and mixedness. They are brightly exposed with the quantifier (30). It seems that the quadratic measure (30) provides a useful alternate approach to quantify coherence. To support this claim, the question of monotonicity should be resolved. We address this in the next section.

VI. FORMULATION OF THE MONOTONICITY PROPERTY

Desired properties of coherence measures concern their behavior with respect to state transformations [10]. It is natural to demand that coherence quantifiers cannot increase under mixing [10]. Let $\{\hat{\rho}_n\}$ be a collection of density matrices, and let positive numbers p_n obey $\sum_n p_n = 1$. For all $\alpha \in (0;2]$, we have

$$C_\alpha\left(\mathcal{E}\,\middle|\,\sum_n p_n\hat{\rho}_n\right) \leqslant \sum_n p_n\, C_\alpha(\mathcal{E}|\hat{\rho}_n). \tag{50}$$

This result immediately follows from the the joint convexity (17) and the definition (23). We refrain from presenting the details here. For $\alpha \in (0;2]$, therefore, the quantity (23) fulfills one of the properties listed in [10]. Changes of coherence measures under some forms of quantum operations are of great importance [10]. Here, the following two classes of incoherent operations should be considered. The first form of monotonicity property is posed as follows. The notion of coherence is basis dependent. Let $\mathcal{E}'$ be the prescribed orthonormal basis with respect to which incoherent states are defined in $\mathcal{H}_B$. We define incoherent quantum operation as a TPCP map $\Phi_I : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ such that its Kraus operators all obey the property

$$\hat{\rho} \in \mathcal{I}(\mathcal{E}) \implies \frac{\hat{K}_n\hat{\rho}\hat{K}_n^\dagger}{\mathrm{Tr}(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger)} \in \mathcal{I}(\mathcal{E}'). \tag{51}$$

For $\alpha \in (0;2]$, the coherence quantifier (23) is monotone under incoherent quantum operations, namely,

$$C_\alpha(\mathcal{E}'|\Phi_I(\hat{\rho})) \leqslant C_\alpha(\mathcal{E}|\hat{\rho}). \tag{52}$$

This follows from the property (16) and the definition (23), which includes the minimization.

Monotonicity under incoherent selective measurements seems to be more sophisticated [10]. Formulating (52), we assume the loss of information about the measurement outcome. When measurement outcomes are retained, one further allows a subselection according to these outcomes. Such operations are also described by a set of Kraus operators $\{\hat{K}_n\}$, but now these operators may have different output spaces though the input space is the same. So, we consider a set of operators $\hat{K}_n : \mathcal{H}_A \to \mathcal{H}_{Bn}$ that satisfy (5). To each output space $\mathcal{H}_{Bn}$, we assign the orthonormal basis $\mathcal{E}_n'$ used for determining incoherent density matrices. The authors of [10] formulated the monotonicity under incoherent selective measurements as

$$\sum_n p_n\, C(\mathcal{E}_n'|\hat{\rho}_n) \leqslant C(\mathcal{E}|\hat{\rho}). \tag{53}$$

Here, $p_n = \mathrm{Tr}(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger)$ is the probability of nth outcome resulting in nth particular output

$$\hat{\rho}_n = p_n^{-1}\hat{K}_n\hat{\rho}\hat{K}_n^\dagger. \tag{54}$$

The authors of [22] called (53) the strong monotonicity under incoherent channels. In the context of transport phenomena, incoherent quantum channels are considered in [42]. It turns out that the coherence α quantifiers obey the monotonicity property in the following form:

Theorem 5. Let the incoherent state $\hat{\delta}_\rho \in \mathcal{I}(\mathcal{E})$ be such that $C_\alpha(\mathcal{E}|\hat{\rho}) = D_\alpha(\hat{\rho}\|\hat{\delta}_\rho)$. For all $\alpha \in (0;2]$, the coherence measures (23) are changed under any incoherent selective measurement in line with

$$\sum_n p_n^\alpha q_n^{1-\alpha}\, C_\alpha(\mathcal{E}'_n|\hat{\rho}_n) \leqslant C_\alpha(\mathcal{E}|\hat{\rho}), \tag{55}$$

where $p_n = \mathrm{Tr}(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger)$, $q_n = \mathrm{Tr}(\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger)$, and $\hat{\rho}_n$ is defined by (54).

Proof. We first note that we can consider the set of Kraus operators with the same output space. More precisely, we define the space

$$\widetilde{\mathcal{H}}_B := \bigoplus_n \mathcal{H}_{Bn}, \tag{56}$$

where $\mathcal{H}_{Bn}$ is the output space of the nth Kraus operator $\hat{K}_n$. To each $\hat{K}_n : \mathcal{H}_A \to \mathcal{H}_{Bn}$, we assign the operator

$$\hat{L}_n = \begin{pmatrix} \mathbf{0} \\ \dots \\ \hat{K}_n \\ \dots \\ \mathbf{0} \end{pmatrix}. \tag{57}$$

That is, this operator is represented as a block column, whose nth block is $\hat{K}_n$ and the others are all zero. The operator $\hat{L}_n$ maps vectors of $\mathcal{H}_A$ to vectors of $\widetilde{\mathcal{H}}_B$. For each n, the input states $\hat{\rho}$ and $\hat{\delta}_\rho$ are respectively mapped into subnormalized outputs

$$\hat{L}_n\hat{\rho}\hat{L}_n^\dagger = \mathrm{diag}(\mathbf{0}\dots\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\dots\mathbf{0}), \tag{58}$$

$$\hat{L}_n\hat{\delta}_\rho\hat{L}_n^\dagger = \mathrm{diag}(\mathbf{0}\dots\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger\dots\mathbf{0}). \tag{59}$$

Thus, the output (58) is the diagonal block matrix with the (n,n) block $\hat{K}_n\hat{\rho}\hat{K}_n^\dagger$ and other zero blocks. Similarly, the output (59) is the diagonal block matrix with the (n,n) block $\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger$. According to the definition (11), we then have

$$D_\alpha(\hat{L}_n\hat{\rho}\hat{L}_n^\dagger\|\hat{L}_n\hat{\delta}_\rho\hat{L}_n^\dagger) = D_\alpha(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\|\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger). \tag{60}$$

We define a TPCP map $\widetilde{\Phi}_I : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\widetilde{\mathcal{H}}_B)$ by the formula

$$\widetilde{\Phi}_I(\hat{X}) := \sum_n \hat{L}_n\hat{X}\hat{L}_n^\dagger. \tag{61}$$

It is trace preserving due to $\hat{L}_n^\dagger\hat{L}_n = \hat{K}_n^\dagger\hat{K}_n$ and the fact that Kraus operators of any incoherent selective measurement obey (5). With the input state $\hat{\rho}$, the output of the quantum operation (61) can be represented as

$$\widetilde{\Phi}_I(\hat{\rho}) = \sum_n p_n\hat{\varpi}_n, \tag{62}$$

where $\hat{\varpi}_n = p_n^{-1}\hat{L}_n\hat{\rho}\hat{L}_n^\dagger$. Denoting $\hat{\delta}_{\rho n} = q_n^{-1}\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger$, we write the following relations:

$$D_\alpha(\hat{\rho}\|\hat{\delta}_\rho) \geqslant D_\alpha(\widetilde{\Phi}_I(\hat{\rho})\|\widetilde{\Phi}_I(\hat{\delta}_\rho)) \tag{63}$$

$$= \sum_n D_\alpha(\hat{L}_n\hat{\rho}\hat{L}_n^\dagger\|\hat{L}_n\hat{\delta}_\rho\hat{L}_n^\dagger) \tag{64}$$

$$= \sum_n D_\alpha(\hat{K}_n\hat{\rho}\hat{K}_n^\dagger\|\hat{K}_n\hat{\delta}_\rho\hat{K}_n^\dagger) \tag{65}$$

$$\geqslant \sum_n p_n^\alpha q_n^{1-\alpha}\, D_\alpha(\hat{\rho}_n\|\hat{\delta}_{\rho n}). \tag{66}$$

Here, step (63) follows from (16), and step (64) follows from (15). Indeed, the construction of operators (57) implies orthogonality of subspaces $\mathrm{ran}(\hat{L}_n\hat{\rho}\hat{L}_n^\dagger)$ for different indices n. Further, step (65) follows from (60), and step (66) is based on Theorem 2. The inequality $D_\alpha(\hat{\rho}_n\|\hat{\delta}_{\rho n}) \geqslant C_\alpha(\mathcal{E}'_n|\hat{\rho}_n)$, clear from definition (23), completes the proof. ■

In the case $\alpha = 1$, the statement of Theorem 5 reduces to (53) written with the coherence measure $C_1(\mathcal{E}|\hat{\rho})$. This property was first proved in [10]. In a certain sense, the relation (55) is a natural extension of the formula (53). The coherence α measures can also be treated as monotone, but the inequality is posed formally in a more sophisticated manner. In particular, the formulation now involves the particular probabilities q_n calculated for the incoherent state (29). Thus, the probabilities q_n are also dependent on the considered state $\hat{\rho}$, but not so directly as p_n's. In view of the above results, constructing coherence measures that obey monotonicity just in the form (53) seems to be difficult. The only known examples are the measures $C_{\ell_1}(\mathcal{E}|\hat{\rho})$ and $C_1(\mathcal{E}|\hat{\rho})$. Of course, we do not consider here any linear combination of the mentioned two measures. As the properties imposed are linear in a coherence measure, a linear combination of two (or more) particular measures will obey these properties whenever each particular measure does.

VII. TWO QUADRATIC MEASURES COMPARED

In this section, we will compare two quantifiers of coherence obtained as homogeneous quadratic functions of matrix elements. These measures are respectively defined by the formulas (8) and (30). The authors of [10] exemplified that the coherence measure (8) is not monotone under incoherent selective measurements. It is instructive to examine the property (55) just with this example. Let us check monotonicity of the coherence measure (30). In the example considered, the input and output reference bases are the same. For brevity, we will omit the symbols $\mathcal{E}$ and $\mathcal{E}'$ in further calculations. The two Kraus operators are written as

$$\hat{K}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}, \qquad \hat{K}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \tag{67}$$

where the complex numbers a and b obey $|a|^2 + |b|^2 = 1$. Further, one considers the density matrix

$$\hat{\varrho} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{68}$$

The normalized particular outputs are expressed as

$$\hat{\varrho}_1 = \frac{1}{2+|a|^2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |a|^2 \end{pmatrix}, \tag{69}$$

$$\hat{\varrho}_2 = \frac{1}{1+|b|^2}\begin{pmatrix} 1 & b^* & 0 \\ b & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{70}$$

The corresponding probabilities are written as

$$p_1 = \frac{2+|a|^2}{4}, \qquad p_2 = \frac{1+|b|^2}{4}. \tag{71}$$

For $\alpha = 2$, the α divergence is minimized with the incoherent state, whose nonzero entries are proportional to the square roots of the diagonal elements of $\hat{\varrho}^2$:

$$\hat{\delta}_\varrho = \frac{1}{2+\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{72}$$

Calculations of the coherence measure (30) result in

$$C_2(\hat{\varrho}) = D_2(\hat{\varrho}\|\hat{\delta}_\varrho) = \frac{2\sqrt{2}-1}{4}. \tag{73}$$

Using $\hat{\delta}_\varrho$ as the input, we get the probabilities of particular outcomes,

$$q_1 = \text{Tr}(\hat{K}_1\hat{\delta}_\varrho\hat{K}_1^\dagger) = \frac{\sqrt{2}+|a|^2}{2+\sqrt{2}}, \tag{74}$$

$$q_2 = \text{Tr}(\hat{K}_2\hat{\delta}_\varrho\hat{K}_2^\dagger) = \frac{1+|b|^2}{2+\sqrt{2}}. \tag{75}$$

For the density matrices (70), we obtain $C_2(\hat{\varrho}_1) = 0$ and

$$C_2(\hat{\varrho}_2) = \frac{2\,|b|}{1+|b|^2}. \tag{76}$$

For all $|b| \in [0;1]$, we consider the quantity

$$p_1^2 q_1^{-1} C_2(\hat{\varrho}_1) + p_2^2 q_2^{-1} C_2(\hat{\varrho}_2) = \frac{2+\sqrt{2}}{8}\,|b| \leqslant \frac{2+\sqrt{2}}{8} \approx 0.426\,8, \tag{77}$$

which is strictly less than $C_2(\hat{\varrho}) = (2\sqrt{2}-1)/4 \approx 0.457\,1$. The latter point illustrates the result (55). The example also shows that the formulation (55) is actually necessary. Indeed, the quantifier (30) does not share the monotonicity formulation (53). To see this fact, we write

$$p_1\, C_2(\hat{\varrho}_1) + p_2\, C_2(\hat{\varrho}_2) = \frac{|b|}{2}. \tag{78}$$

The right-hand side of (78) increases up to 0.5 for $|b| = 1$ and can exceed $C_2(\hat{\varrho}) \approx 0.457\,1$. The above findings give an evidence that the α quantifiers do not generally obey monotonicity in the form of (53). On the other hand, these measures certainly satisfy monotonicity in the form of (55). Thus, the monotonicity of coherence under selective measurements is sophisticated in character. This property does not follow immediately from the monotonicity of quantum relative entropies.

The considered example allows us to resolve the following natural question. The coherence measure (8) does not share monotonicity in the form of (53). In principle, we may ask for monotonicity of (8) similarly to (55). In other words, we consider the quantity

$$\sum_n p_n^\alpha r_n^{1-\alpha}\, C_{\ell_2}(\hat{\rho}_n), \tag{79}$$

where $r_n = \text{Tr}(\hat{K}_n\hat{\delta}_*\hat{K}_n^\dagger)$ and $\hat{\delta}_*$ is obtained from $\hat{\rho}$ by vanishing all off-diagonal entries. The above example shows that (79) can exceed $C_{\ell_2}(\hat{\rho})$. Indeed, the input state (68) is such that $r_n = p_n$ for $n = 1,2$. Hence, the quantity (79) is equal to $\sum_n p_n\, C_{\ell_2}(\hat{\varrho}_n)$ and violates monotonicity in the form of (53), as already known. Thus, the coherence measure (8) is not monotone even in the sense of (55). This fact shows that the formulation (55) is not trivial. It is also reduced to (53) in the limit $\alpha \to 1$. Thus, we can treat (55) as a natural extension of the standard form (53).

VIII. CONCLUSIONS

We have examined quantum-coherence measures based on α divergences of the Tsallis type. Tradeoff relations between coherence and mixedness were obtained. Some properties were further exemplified with a single qubit. Most of the desired properties immediately follow from general properties of quantum relative entropies. The monotonicity of coherence under selective measurements is a more interesting and complicated question. This monotonicity has been shown for the two measures based on the ℓ_1-norm and on the standard relative entropy [10]. For the coherence measure based on the trace distance, only particular monotonicity results are known [22]. We have proved that coherence α measures enjoy desired monotonicity in the form of (55), where the parameter α is involved. For $\alpha \to 1$, this formulation is directly reduced to the standard formulation proposed in [10]. In this regard, the result (55) is a parametric extension of the standard form (53). It may be supposed that the two known examples satisfying just (53) are the only such.

The obtained family includes the quantity expressed in terms of the squared moduli of matrix elements. In several respects, this quantity differs from the coherence measure induced by the squared ℓ_2-norm. In both (7) and (8), the closest incoherent state is obtained by vanishing all off-diagonal entries of $\hat{\rho}$. Except for $\alpha = 1$, the incoherent state that minimizes the α divergence in (23) is reached by a more complicated procedure. Nevertheless, for quantum α divergences of the Tsallis type, the required minimization can be solved with an explicit answer. It seems to be difficult for quantum f divergences in general. Currently, so-called "sandwiched" relative entropies are the subject of active research [43]. Such quantities could be used for obtaining coherence measures, but the required minimization seems to be difficult. By comparing two quadratic measures of coherence, we also have shown that the measure induced by the squared ℓ_2-norm violates monotonicity even in a generalized form. It was conjectured in [12] that the square root of (8) may obey all the desired properties. Due to our results, this conjecture seems to be sufficiently difficult to resolve.

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