# Trade-off relation between information and disturbance in quantum measurement

Tomohiro Shitara,[1,*] Yui Kuramochi,[2] and Masahito Ueda[1,3]
[1]*Department of Physics, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-8654, Japan*
[2]*Department of Nuclear Engineering, Kyoto University, Kyoto 615-8540, Japan*
[3]*RIKEN Center for Emergent Matter Science, Wako 351-0198, Japan*
(Received 19 January 2016; published 29 March 2016)

We formulate a trade-off relation between information and disturbance in quantum measurement from an estimation-theoretic point of view. The information and disturbance are characterized in terms of the classical Fisher information and the average loss of the quantum Fisher information, respectively. We identify the necessary condition for various divergences between two quantum states to satisfy similar relations.

DOI: 10.1103/PhysRevA.93.032134

## I. INTRODUCTION

When we perform a quantum measurement and extract information from a system, the system is disturbed due to the backaction of the measurement. The more information we extract by quantum measurement, the more strongly the system is disturbed. Such a trade-off relation has been recognized since Heisenberg pointed out the uncertainty relation between measurement error and disturbance [1]. With tremendous advances in quantum information theory, various methods of quantifying information and disturbance and their trade-off relations have been proposed [2–18].

These studies are classified into two types: an information-theoretic approach [2–8] and an estimation-theoretic one [9–18]. The former is based on entropic quantities such as the mutual information [2–5], the conditional entropy [6], and the Groenewold-Ozawa information [7,8], each of which has its own operational meaning in the information-theoretic setting, i.e., decoding the message encoded in quantum states by a measurement. The latter is mainly based on either the fidelity [9–18] between the true state and the estimated state for information or the fidelity between the premeasurement state and the postmeasurement state for disturbance. In the formulation of the trade-off relations, a uniform distribution, i.e., the Haar measure over input pure states, is often assumed, and the average information and disturbance are calculated for this measure. Since there is no unique uniform distribution over all quantum states including mixed states, such a formulation cannot naturally be extended to the case in which input states are not necessarily pure. Moreover, information and disturbance depend, in general, on the state to be measured. Therefore, a general trade-off relation should be calculated over each individual input state rather than over the uniform measure.

In this paper, we apply quantum estimation theory [19,20] to characterize estimation theoretical quantities for each quantum state. We formulate information and disturbance in the setting of estimating an unknown quantum state by quantum measurements, with an emphasis on the estimation accuracy. We derive an inequality that shows the trade-off relation between information and disturbance and give sufficient conditions to achieve the equality. We also discuss the condition for divergences, which measure the distinguishability between two quantum states, to satisfy a similar inequality.

This paper is organized as follows. In Sec. II, after briefly reviewing the quantum estimation theory, we define information and disturbance in terms of the Fisher information and derive the trade-off relation between them. To make its physical implication clear, we investigate conditions for the equality in Sec. III. By interpreting the quantum Fisher information as a metric on the state space, we discuss the generalization of the trade-off relation for divergences between probability distributions in Sec. IV. Finally, we conclude the paper in Sec. V.

## II. INFORMATION-DISTURBANCE RELATION BASED ON THE FISHER INFORMATION

Suppose that we perform a quantum measurement to estimate an unknown quantum state described by a density operator $\hat{\rho}_{\boldsymbol{\theta}}$. Here, $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^m$ represents $m$ real parameters that characterize the unknown state, so that estimating the state is equivalent to estimating the parameters. Such a parameterized family of states $\{\hat{\rho}_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}\in\Theta}$ is called a quantum statistical model. If we do not have any knowledge about the system, we can use the full model, which includes all possible density operators on the $d$-dimensional Hilbert space with $m = d^2 - 1$. A quantum measurement is characterized by a mapping from quantum states to both the probability distribution of outcomes and the postmeasurement state corresponding to each outcome. If the measurement outcome is discrete, the probability $p_{\boldsymbol{\theta},i}$ of obtaining an outcome $i \in I$ ($I$ represents the entire set of outcomes) and the postmeasurement state $\hat{\rho}_{\boldsymbol{\theta},i}$ are respectively given by

$$p_{\boldsymbol{\theta},i} = \sum_j \mathrm{tr}[\hat{K}_{ij}\hat{\rho}_{\boldsymbol{\theta}}\hat{K}_{ij}^\dagger] \tag{1}$$

and

$$\hat{\rho}_{\boldsymbol{\theta},i} = \frac{1}{p_{\boldsymbol{\theta},i}} \sum_j \hat{K}_{ij}\hat{\rho}_{\boldsymbol{\theta}}\hat{K}_{ij}^\dagger, \tag{2}$$

where the measurement operators $\{\hat{K}_{ij}\}$ satisfy the completeness relation $\sum_{i,j}\hat{K}_{ij}^\dagger\hat{K}_{ij} = \hat{I}$ with $\hat{I}$ being the identity operator.

What is the natural quantification of the information in the setting of estimating an unknown state from the outcome

*shitara@cat.phys.s.u-tokyo.ac.jp

 

of the quantum measurement? The estimation process is characterized by a mapping $\boldsymbol{\theta}^{\mathrm{est}} : I \to \Theta$, which is called an estimator. Since the outcome $i \in I$ is a random variable, the estimator, which is calculated from the outcome, is also a random variable and should be distributed around the true parameter $\boldsymbol{\theta}$. According to the classical Cramér-Rao inequality, the variance-covariance matrix of the locally unbiased estimator has a lower bound which is determined only by a family of the probability distributions $\{p_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}\in\Theta}$:

$$\mathrm{Var}_{\boldsymbol{\theta}}[\boldsymbol{\theta}_{\mathrm{est}}] \geqslant \left(J_{\boldsymbol{\theta}}^{C}\right)^{-1}. \tag{3}$$

Here, $J_{\boldsymbol{\theta}}^{C}$ is an $m \times m$ matrix called the classical Fisher information whose matrix elements are defined by

$$\left[J_{\boldsymbol{\theta}}^{C}\right]_{ab} := \sum_{i} p_{\boldsymbol{\theta},i} \frac{\partial \ln p_{\boldsymbol{\theta},i}}{\partial \theta_a} \frac{\partial \ln p_{\boldsymbol{\theta},i}}{\partial \theta_b}. \tag{4}$$

Therefore, a measurement which gives a larger classical Fisher information allows us to estimate the state more accurately. We will quantify the information obtained by quantum measurement in terms of the classical Fisher information.

The disturbance can be evaluated as the loss of information about the parameter $\boldsymbol{\theta}$ that can be extracted from a quantum state. The information on the parameter $\boldsymbol{\theta}$ that the quantum state potentially possesses can be quantified by the quantum Fisher information, which is defined as

$$\left[J_{\boldsymbol{\theta}}^{Q}\right]_{ab} := \mathrm{tr}\left[\frac{\partial \hat{\rho}_{\boldsymbol{\theta}}}{\partial \theta_a} \boldsymbol{K}_{\hat{\rho}_{\boldsymbol{\theta}}}^{-1} \frac{\partial \hat{\rho}_{\boldsymbol{\theta}}}{\partial \theta_b}\right]. \tag{5}$$

Here, the superoperator $\boldsymbol{K}_{\hat{\rho}}$ is defined as

$$\boldsymbol{K}_{\hat{\rho}} := f\left(\boldsymbol{L}_{\hat{\rho}} \boldsymbol{R}_{\hat{\rho}}^{-1}\right) \boldsymbol{R}_{\hat{\rho}}, \tag{6}$$

where $f : (0,\infty) \to (0,\infty)$ is an operator monotone function, i.e., $0 < \hat{A} \leqslant \hat{B}$ implies $f(\hat{A}) \leqslant f(\hat{B})$, satisfying $f(1) = 1$ and $\boldsymbol{R}_{\hat{\rho}}$ ($\boldsymbol{L}_{\hat{\rho}}$) is the right (left) multiplication of $\hat{\rho}$:

$$\boldsymbol{R}_{\hat{\rho}}(\hat{A}) := \hat{A}\hat{\rho}, \quad \boldsymbol{L}_{\hat{\rho}}(\hat{A}) := \hat{\rho}\hat{A}. \tag{7}$$

The quantum Fisher information is the unique class of metrics on the space of quantum states that monotonically decrease under an arbitrary completely positive and trace-preserving (CPTP) mapping $\boldsymbol{\mathcal{E}}$ [21]:

$$J_{\boldsymbol{\theta}}^{Q}(\{\hat{\rho}_{\boldsymbol{\theta}}\}) \geqslant J_{\boldsymbol{\theta}}^{Q}(\{\boldsymbol{\mathcal{E}}(\hat{\rho}_{\boldsymbol{\theta}})\}). \tag{8}$$

From the monotonicity, the quantum Fisher information gives an upper bound of the classical Fisher information which is obtained by all the possible measurements; in particular, the symmetric logarithmic derivative (SLD) Fisher information [22], which corresponds to $f(x) = \frac{1+x}{2}$, is known to be the least upper bound. Therefore, the quantum Fisher information, especially the SLD Fisher information, can be interpreted as the information on the parameter $\boldsymbol{\theta}$ that can be extracted from the quantum state. We define the disturbance of the measurement as

$$\Delta J_{\boldsymbol{\theta}}^{Q} := J_{\boldsymbol{\theta}}^{Q} - \sum_{i} p_{\boldsymbol{\theta},i} J_{i,\boldsymbol{\theta}}^{\prime Q}, \tag{9}$$

where $J_{\boldsymbol{\theta}}^{Q}$ ($J_{i,\boldsymbol{\theta}}^{\prime Q}$) is the quantum Fisher information for the quantum statistical model $\{\hat{\rho}_{\boldsymbol{\theta}}\}$ ($\{\hat{\rho}_{\boldsymbol{\theta},i}\}$). For the sake of generality, we consider the disturbance using the general quantum Fisher information.

The information (4) and the disturbance (9) satisfy the following inequality that shows a trade-off relation between them:

$$J_{\boldsymbol{\theta}}^{C} \leqslant \Delta J_{\boldsymbol{\theta}}^{Q}. \tag{10}$$

This inequality means that if we perform a quantum measurement on an unknown state and extract information on the state, the state is disturbed and hence loses some intrinsic information. Since the inequality (10) is valid independently of the choice of the quantum Fisher information to define the disturbance, we obtain

$$J_{\boldsymbol{\theta}}^{C} \leqslant \inf_{Q} \Delta J_{\boldsymbol{\theta}}^{Q}, \tag{11}$$

where the infimum is taken over all types of quantum Fisher information with which the disturbance is defined. We note that it is nontrivial which quantum Fisher information gives the minimum disturbance, although the minimum and the maximum of the quantum Fisher information are known to be the SLD Fisher information and the real right logarithmic derivative Fisher information (real RLD), respectively. The monotone function of the latter is given by $f(x) = \frac{2x}{x+1}$.

The proof of the inequality (10) is based on the monotonicity and the chain rule of the quantum Fisher information. For a given measurement $\{\hat{K}_{ij}\}$, we define a CPTP mapping $\boldsymbol{\mathcal{E}}^{\mathrm{meas}}$ as

$$\boldsymbol{\mathcal{E}}^{\mathrm{meas}}(\hat{\rho}) := \bigoplus_{i} \left(\sum_{j} \hat{K}_{ij} \hat{\rho} \hat{K}_{ij}^{\dagger}\right), \tag{12}$$

which represents the direct sum of the unnormalized postmeasurement states. Then the quantum Fisher information of the quantum statistical model $\{\boldsymbol{\mathcal{E}}^{\mathrm{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\}$ is given by the sum of the classical Fisher information about the measurement outcome and the average quantum Fisher information of the postmeasurement state:

$$J_{\boldsymbol{\theta}}^{Q}(\{\boldsymbol{\mathcal{E}}^{\mathrm{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\}) = J_{\boldsymbol{\theta}}^{C}(\{p_{\boldsymbol{\theta}}\}) + \sum_{i} p_{\boldsymbol{\theta},i} J_{i,\boldsymbol{\theta}}^{\prime Q}, \tag{13}$$

which is the chain rule (see Appendix A for the proof). By applying the monotonicity under the CPTP mapping $\boldsymbol{\mathcal{E}}^{\mathrm{meas}}$, we obtain

$$\begin{aligned} J_{\boldsymbol{\theta}}^{Q}(\{\hat{\rho}_{\boldsymbol{\theta}}\}) &\geqslant J_{\boldsymbol{\theta}}^{Q}(\{\boldsymbol{\mathcal{E}}^{\mathrm{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\}) \\ &= J_{\boldsymbol{\theta}}^{C}(\{p_{\boldsymbol{\theta}}\}) + \sum_{i} p_{\boldsymbol{\theta},i} J_{i,\boldsymbol{\theta}}^{\prime Q}, \end{aligned} \tag{14}$$

which proves the inequality (10).

## III. CONDITIONS FOR THE EQUALITY

To clarify the physical meaning of the inequality (10), we investigate conditions for the equality. Measurements that achieve the equality of the inequality (10) are also important since they are efficient in the sense that they cause the minimum disturbance among those that extract a given amount of information. Since the equality conditions depend on both the quantum statistical model and the measurement, it is difficult to find the general necessary and sufficient condition to achieve the equality. Here, we provide three sufficient conditions.

The first rather trivial condition is the case in which the CPTP mapping $\mathcal{E}^{\text{meas}}$ is reversible by a CPTP mapping $\tilde{\mathcal{E}}$, i.e., $\tilde{\mathcal{E}} \circ \mathcal{E}^{\text{meas}} = \mathcal{I}$, where $\mathcal{I}$ is the identity mapping. In this case, by using the monotonicity twice, we can show that the equality in the inequality (10) holds. However, if $\mathcal{E}^{\text{meas}}$ is reversible by a CPTP mapping, every measurement operator must be proportional to some unitary operator, and the probability distribution $p_{\boldsymbol{\theta},i}$ does not depend on the measured state. Therefore, no information is extracted by this measurement process.

The second sufficient condition is classical measurements. Suppose that the quantum statistical model $\{\hat{\rho}_{\boldsymbol{\theta}}\}$ is given by

$$\hat{\rho}_{\boldsymbol{\theta}} = \sum_j q_{\boldsymbol{\theta},j} \, |j\rangle \langle j| , \tag{15}$$

where $\{q_{\boldsymbol{\theta},j}\}$ is a classical statistical model and $\{|j\rangle\}$ is an orthonormal basis set which is independent of the parameter $\boldsymbol{\theta}$. When we perform a measurement corresponding to measurement operators

$$\hat{K}_i = \sum_j \sqrt{\kappa_{ij}} \, |j\rangle \langle j| , \tag{16}$$

where the coefficient $\kappa_{ij}$ represents the probability of obtaining outcome $i$ on the condition that the state is $|j\rangle$, the obtained information $J^C_{\boldsymbol{\theta}}$ is equal to $\Delta J^Q_{\boldsymbol{\theta}}$. This is because, in classical theory, all the observables are simultaneously diagonalizable, and therefore, all the components in the parameter $\boldsymbol{\theta}$ are simultaneously measurable.

The third sufficient condition is obtained when we adopt the right logarithmic derivative (RLD) Fisher information [23], which corresponds to $f(x) = x$, to define the disturbance. Then a class of measurements called pure and logically reversible measurements achieves the equality in the inequality (10) (see Appendix B for the proof):

$$J^C_{\boldsymbol{\theta}} = \Delta J^{\text{RLD}}_{\boldsymbol{\theta}}. \tag{17}$$

Here, a measurement is called pure (also referred to as efficient [24,25] or ideal [26]) if the number of measurement operators is one for each measurement outcome, so that the measurement operators can be written as $\{\hat{K}_i\}$. A measurement is called logically reversible if each $\hat{K}_i$ has a left inverse $\hat{K}_i^{-1}$ [27]. Note that the physical reversibility [27,28] is not necessarily needed. As an example, a measurement on a spin-1/2 system proposed by Royer [29] is pure, with measurement operators

$$\hat{K}_1 = \begin{pmatrix} \cos(\theta/2 - \sigma/4) & 0 \\ 0 & \cos(\theta/2 + \sigma/4) \end{pmatrix}, \tag{18}$$

$$\hat{K}_2 = \begin{pmatrix} \sin(\theta/2 - \sigma/4) & 0 \\ 0 & \sin(\theta/2 + \sigma/4) \end{pmatrix}. \tag{19}$$

This measurement is logically reversible if $\theta/2 \pm \sigma/4 \neq n\pi/2$.

In fact, pure measurements are the least disturbing measurements in the following sense. Suppose that two measurements $\{\hat{K}'_{ij}\}$ and $\{\hat{K}_i\}$ give the same positive operator-valued measure (POVM)

$$\sum_j \hat{K}'^{\dagger}_{ij} \hat{K}'_{ij} = \hat{K}^{\dagger}_i \hat{K}_i, \quad \forall i \in I, \tag{20}$$

and hence give the same probability distribution. Then, the pure measurement causes less disturbance, while it gives the same amount of information:

$$\Delta J^Q_{\boldsymbol{\theta}}(\{\hat{K}_i\}) \leqslant \Delta J^Q_{\boldsymbol{\theta}}(\{\hat{K}'_{ij}\}), \tag{21}$$

$$J^C_{\boldsymbol{\theta}}(\{\hat{K}_i\}) = J^C_{\boldsymbol{\theta}}(\{\hat{K}'_{ij}\}), \tag{22}$$

since the measurement process $\{\hat{K}'_{ij}\}$ can be expressed as the pure measurement $\{\hat{K}_i\}$ followed by a CPTP mapping $\mathcal{E}_i$, which depends on the measurement outcome $i$.

## IV. INFORMATION-DISTURBANCE RELATION BASED ON DIVERGENCES

In Ref. [30], by using the classical and quantum relative entropies

$$S^C(p\|q) = \sum_i p_i \ln\left(\frac{p_i}{q_i}\right), \tag{23}$$

$$S^Q(\hat{\rho}\|\hat{\sigma}) = \text{tr}[\hat{\rho}(\ln\hat{\rho} - \ln\hat{\sigma})], \tag{24}$$

an inequality similar to Eq. (10) was derived:

$$S^C(p\|q) \leqslant S^Q(\hat{\rho}\|\hat{\sigma}) - \sum_i p_i S^Q(\hat{\rho}_i\|\hat{\sigma}_i), \tag{25}$$

where $p,q$ and $\hat{\rho}_i,\hat{\sigma}_i$ are the probability distributions and the postmeasurement states of a quantum measurement performed on quantum states $\hat{\rho},\hat{\sigma}$. Since the quantum relative entropy is a measure of the distinguishability of two quantum states [31,32], Eq. (25) can also be interpreted as a trade-off relation between information and disturbance. In particular, if we choose two similar states $\hat{\rho}_{\boldsymbol{\theta}}$ and $\hat{\rho}_{\boldsymbol{\theta}+\mathrm{d}\boldsymbol{\theta}}$ as the arguments of the relative entropy, Eq. (25) reproduces the inequality (10) with the disturbance defined by the Bogoliubov-Kubo-Mori (BKM) Fisher information, the monotone function of which is given by $f(x) = \frac{x-1}{\ln x}$.

In the following, we discuss an extension of the inequality (25) to general divergences. Let $D^C(\cdot\|\cdot)$ be a divergence between two probability distributions [33] and $D^Q(\cdot\|\cdot)$ be its quantum extension; that is, if two quantum states $\hat{\rho}$ and $\hat{\sigma}$ commute and therefore are simultaneously diagonalizable as $\hat{\rho} = \sum_j r_j \, |j\rangle \langle j|$ and $\hat{\sigma} = \sum_j s_j \, |j\rangle \langle j|$, we obtain

$$D^Q(\hat{\rho}\|\hat{\sigma}) = D^C(r\|s). \tag{26}$$

We note that the quantum generalization of a divergence is not unique in general.

Let us consider a condition for divergences $D^C(\cdot\|\cdot)$ and $D^Q(\cdot\|\cdot)$ to satisfy the information-disturbance trade-off relation

$$D^C(p\|q) \leqslant D^Q(\hat{\rho}\|\hat{\sigma}) - \sum_i p_i D^Q(\hat{\rho}_i\|\hat{\sigma}_i). \tag{27}$$

The essential properties needed for the proof of the inequality (10) are the monotonicity and the chain rule of the quantum Fisher information. We require these two properties for divergences:

$$D^Q(\hat{\rho}\|\hat{\sigma}) \geqslant D^Q(\mathcal{E}(\hat{\rho})\|\mathcal{E}(\hat{\sigma})), \tag{28}$$

$$D^Q(\mathcal{E}^{\text{meas}}(\hat{\rho})\|\mathcal{E}^{\text{meas}}(\hat{\sigma})) = D^C(p\|q) + \sum_i p_i D^Q(\hat{\rho}_i\|\hat{\sigma}_i). \tag{29}$$

These two relations give sufficient conditions for Eq. (27). If we also require a continuity of $D^C(p\|q)$ with respect to $p$ and $q$, then the classical divergence $D^C(p\|q)$ satisfies Hobson's five conditions that axiomatically characterize the classical relative entropy [34]. Therefore, the divergence from the monotonicity, the chain rule, and the continuity must be consistent with the relative entropy at least for classical probability distributions:

$$D^C(p\|q) = S^C(p\|q). \tag{30}$$

As shown in [30], the standard quantum relative entropy satisfies the trade-off relation (27) because it satisfies the monotonicity and the chain rule. Another quantum extension of the relative entropy proposed by Belavkin and Staszewski [35] is given by

$$S^{\text{BS}}(\hat{\rho}\|\hat{\sigma}) = \text{tr}[\hat{\rho}\ln(\hat{\rho}^{1/2}\hat{\sigma}^{-1}\hat{\rho}^{1/2})], \tag{31}$$

which also satisfies the inequality (27). Here, $S^{\text{BS}}(\cdot\|\cdot)$ is known to be maximal among all the possible quantum extensions of the classical relative entropy [36]. By substituting $\hat{\rho}_{\boldsymbol{\theta}}$ and $\hat{\rho}_{\boldsymbol{\theta}+d\boldsymbol{\theta}}$ into the inequality (27), we again obtain the inequality (10) with the disturbance defined by the real RLD Fisher information.

## V. CONCLUSION

In this paper, we have formulated the trade-off relation between information and disturbance in quantum measurement from the viewpoint of estimating parameters that characterize an unknown quantum state. The information is defined as the classical Fisher information of the probability distribution of measurement outcomes, and the disturbance is defined as the average loss of the quantum Fisher information due to the backaction of the measurement. We have shown the trade-off relation (10) between them. When we use the RLD Fisher information, the equality of the inequality (10) is achieved by pure and logically reversible measurements. In fact, pure measurements are the least disturbing among those that provide us with a given amount of information.

We have also discussed the necessary condition for divergences between two quantum states to satisfy a similar trade-off relation (27). It is necessary for divergences to coincide with the relative entropy at least for classical probability distributions. In addition to the well-known relative entropy, the maximum relative entropy also satisfies the trade-off relation (27), which reproduces the inequality (10) for the disturbance defined by the real RLD Fisher information. If there are other quantum extensions of the relative entropy that give an arbitrary quantum Fisher information, another systematic derivation of the inequality (10) should be possible.

## ACKNOWLEDGMENTS

T.S. thanks Y. Murashita and N. Kura for helpful discussions. This work was supported by KAKENHI Grant No. 26287088 from the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research on Innovation Areas "Topological Materials Science" (KAKENHI Grant No. 15H05855), the Photon Frontier Network Program from MEXT of Japan, and the Mitsubishi Foundation. T.S. was supported by the Japan Society for the Promotion of Science through the Program for Leading Graduate Schools (ALPS). Y.K. acknowledges support from the Japan Society for the Promotion of Science (KAKENHI Grant No. 269905).

## APPENDIX A: PROOF OF THE CHAIN RULE (13)

We introduce the logarithmic derivative operator of the quantum statistical model $\{\hat{\rho}_{\boldsymbol{\theta}}\}$ defined as

$$\hat{L}_a := \boldsymbol{K}^{-1}_{\hat{\rho}_{\boldsymbol{\theta}}}\left(\frac{\partial\hat{\rho}_{\boldsymbol{\theta}}}{\partial\theta_a}\right), \tag{A1}$$

so that the quantum Fisher information is rewritten as

$$[J^Q_{\boldsymbol{\theta}}(\{\hat{\rho}_{\boldsymbol{\theta}}\})]_{ab} := \text{tr}\left[\frac{\partial\hat{\rho}_{\boldsymbol{\theta}}}{\partial\theta_a}\hat{L}_b\right]. \tag{A2}$$

Let $\hat{L}'_a$ and $\hat{L}_{i,a}$ denote the logarithmic derivative of the quantum state models $\{\mathcal{E}^{\text{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\}$ and $\{\hat{\rho}_{\boldsymbol{\theta},i}\}$, respectively. The relation between $\hat{L}'_a$ and $\hat{L}_{i,a}$ is given by

$$\hat{L}'_a = \left(\bigoplus_i \hat{L}_{i,a}\right) + \left(\bigoplus_i \frac{\partial \ln p_{\boldsymbol{\theta},i}}{\partial\theta_a}\hat{I}_i\right). \tag{A3}$$

Therefore, the matrix elements of the quantum Fisher information can be calculated as follows:

$$\begin{aligned}
&[J^{\text{RLD}}_{\boldsymbol{\theta}}(\{\mathcal{E}^{\text{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\})]_{ab}\\
&= \sum_i\left\{p_{\boldsymbol{\theta},i}\text{tr}\left[\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_a}\hat{L}_{i,b}\right] + p_{\boldsymbol{\theta},i}\frac{\partial\ln p_{\boldsymbol{\theta},i}}{\partial\theta_b}\text{tr}\left[\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_a}\right]\right\}\\
&\quad + \sum_i\left\{\frac{\partial p_{\boldsymbol{\theta},i}}{\partial\theta_a}\text{tr}[\hat{\rho}_{\boldsymbol{\theta},i}\hat{L}_{i,b}] + \frac{\partial p_{\boldsymbol{\theta},i}}{\partial\theta_a}\frac{\partial\ln p_{\boldsymbol{\theta},i}}{\partial\theta_b}\text{tr}[\hat{\rho}_{\boldsymbol{\theta},i}]\right\}\\
&= \sum_i p_{\boldsymbol{\theta},i}J'^Q_{i,\boldsymbol{\theta}} + 0 + 0 + J^C_{\boldsymbol{\theta}}.
\end{aligned} \tag{A4}$$

Here, in obtaining the last equality, we use the fact that the second and third terms vanish because

$$\text{tr}\left[\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_a}\right] = \frac{\partial}{\partial\theta_a}\text{tr}[\hat{\rho}_{\boldsymbol{\theta},i}] = 0, \tag{A5}$$

$$\begin{aligned}
\text{tr}[\hat{\rho}_{\boldsymbol{\theta},i}\hat{L}_{i,b}] &= \text{tr}\left[\hat{\rho}_{\boldsymbol{\theta},i}\boldsymbol{K}^{-1}_{\hat{\rho}_{\boldsymbol{\theta},i}}\left(\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_b}\right)\right]\\
&= \text{tr}\left[\boldsymbol{K}^{\dagger-1}_{\hat{\rho}_{\boldsymbol{\theta},i}}(\hat{\rho}_{\boldsymbol{\theta},i})\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_b}\right] = \text{tr}\left[\hat{I}_i\frac{\partial\hat{\rho}_{\boldsymbol{\theta},i}}{\partial\theta_b}\right] = 0.
\end{aligned} \tag{A6}$$

## APPENDIX B: PROOF OF EQUATION (17)

It is sufficient to prove

$$J^{\text{RLD}}_{\boldsymbol{\theta}}(\{\hat{\rho}_{\boldsymbol{\theta}}\}) = J^{\text{RLD}}_{\boldsymbol{\theta}}(\{\mathcal{E}^{\text{meas}}(\hat{\rho}_{\boldsymbol{\theta}})\}) \tag{B1}$$

because the disturbance can be rewritten as

$$\Delta J_{\theta}^{\mathrm{RLD}} = J_{\theta}^{\mathrm{RLD}}(\{\hat{\rho}_{\theta}\}) - J_{\theta}^{\mathrm{RLD}}(\{\mathcal{E}^{\mathrm{meas}}(\hat{\rho}_{\theta})\}) + J_{\theta}^{C}. \tag{B2}$$

Let $\hat{L}_a$ and $\hat{L}'_a$ denote the right logarithmic derivatives of the quantum state models $\{\hat{\rho}_{\theta}\}$ and $\{\mathcal{E}^{\mathrm{meas}}(\hat{\rho}_{\theta})\}$. Then, we obtain the following relation between $\hat{L}_a$ and $\hat{L}'_a$:

$$\begin{aligned}\hat{L}'_a &= \bigoplus_i (K_i \hat{\rho}_{\theta} \hat{K}_i^{\dagger})^{-1} \frac{\partial}{\partial \theta_a} K_i \hat{\rho}_{\theta} \hat{K}_i^{\dagger} \\ &= \bigoplus_i (\hat{K}_i^{\dagger})^{-1} \hat{L}_a \hat{K}_i^{\dagger}.\end{aligned} \tag{B3}$$

Therefore, we obtain

$$\begin{aligned}&\left[J_{\theta}^{\mathrm{RLD}}(\{\mathcal{E}^{\mathrm{meas}}(\hat{\rho}_{\theta})\})\right]_{ab} \\ &= \sum_i \mathrm{tr}[\hat{K}_i \hat{\rho}_{\theta} \hat{K}_i^{\dagger} (\hat{K}_i^{\dagger})^{-1} \hat{L}_a \hat{K}_i^{\dagger} (\hat{K}_i^{\dagger})^{-1} \hat{L}_b \hat{K}_i^{\dagger}] \\ &= \mathrm{tr}\left[\sum_i \hat{K}_i^{\dagger} \hat{K}_i \hat{\rho}_{\theta} \hat{L}_a \hat{L}_b\right] \\ &= \mathrm{tr}[\hat{\rho}_{\theta} \hat{L}_a \hat{L}_b] \\ &= \left[J_{\theta}^{\mathrm{RLD}}(\{\hat{\rho}_{\theta}\})\right]_{ab},\end{aligned} \tag{B4}$$

which completes the proof.


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