

What happens in a measurement?

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It is assumed that in a measurement the system under study interacts with a macroscopic measuring apparatus, in such a way that the density matrix of the measured system evolves according to the Lindblad equation. Under an assumption of nondecreasing von Neumann entropy, conditions on the operators appearing in this equation are given that are necessary and sufficient for the late-time limit of the density matrix to take the form appropriate for a measurement. Where these conditions are satisfied, the Lindblad equation can be solved explicitly. The probabilities appearing in the late-time limit of this general solution are found to agree with the Born rule, and are independent of the details of the operators in the Lindblad equation.

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I. INTRODUCTION

According to the Copenhagen interpretation of quantum mechanics, during a complete measurement the initial density matrix ρ_{initial} undergoes a collapse

$$\rho_{\text{initial}} \mapsto \rho_{\text{final}} = \sum_{\alpha} p_{\alpha} \Lambda_{\alpha}, \quad (1)$$

where $\Lambda_{\alpha} = |\alpha\rangle\langle\alpha|$ are projection operators onto the complete set of orthonormal eigenvectors $|\alpha\rangle$ of whatever is being measured, satisfying the usual conditions

$$\Lambda_{\alpha} \Lambda_{\beta} = \delta_{\alpha\beta} \Lambda_{\alpha}, \quad \sum_{\alpha} \Lambda_{\alpha} = \mathbf{1}, \quad \text{Tr} \Lambda_{\alpha} = 1, \quad \Lambda_{\alpha}^{\dagger} = \Lambda_{\alpha}, \quad (2)$$

and p_{α} are probabilities, given by the Born rule

$$p_{\alpha} = \langle\alpha|\rho_{\text{initial}}|\alpha\rangle = \text{Tr}(\Lambda_{\alpha}\rho_{\text{initial}}). \quad (3)$$

In a closed system in ordinary quantum mechanics the state vector evolves unitarily and deterministically, so as is well known the collapse (1) cannot occur if the initial density matrix ρ_{initial} describes a pure state (or an ensemble of fewer pure states than the number of terms in ρ_{final}). In the original formulation [1] of the Copenhagen interpretation it was simply accepted that the change in a system during measurement in principle departs from quantum mechanics. We will instead adopt the popular modern view that the Copenhagen interpretation refers to open systems in which the transition (1) is driven by the interaction of the microscopic system under study with a suitable environment, a macroscopic external measuring apparatus (which may include an observer) chosen to bring this transition about.

Of course, this view of the Copenhagen interpretation just pushes the hard problems of interpreting quantum mechanics to a larger scale. We make no attempt to address these problems in the present paper, beyond noting the conjecture [2] that the unitary evolution of microscopic systems is merely a very good approximation, while the density matrix of combined systems with macroscopic parts in general evolves rapidly and nonunitarily, and in particular undergoes the collapse (1) during a measurement. Although in this paper we are focusing

on ordinary quantum mechanics in an open system, most of our analysis applies equally to closed systems in modified versions of quantum mechanics.

Whether in open systems in ordinary quantum mechanics or in closed systems in some modified version of quantum mechanics, in order to avoid instantaneous communication at a distance in entangled states, it is important to require that the density matrix at one time depends on the density matrix at any earlier time, but not otherwise on the state vector at the earlier time. [3]. This evolution can be linear but nonunitary in ordinary quantum mechanics if the system under study interacts with an environment that fluctuates randomly more rapidly than the rate at which the density matrix evolves (set by the interaction strength) if we average over these fluctuations. We do not need to go into details regarding this interaction with the environment, because it is known that in the most general linear evolution that preserves the unit trace and Hermiticity of the density matrix and satisfies the condition of complete positivity [4], the density matrix satisfies the Lindblad equation [5]:

$$\frac{d\rho(t)}{dt} = -i[\mathcal{H}, \rho(t)] + \sum_n \left(L_n \rho(t) L_n^{\dagger} - \frac{1}{2} L_n^{\dagger} L_n \rho(t) - \frac{1}{2} \rho(t) L_n^{\dagger} L_n \right), \quad (4)$$

with constant matrices¹ L_n and \mathcal{H} . We then face three questions:

(1) What are the necessary conditions on the operators L_n and \mathcal{H} for the density matrix to approach a time-independent linear combination such as (1) of specific projection operators Λ_{α} at late times?

(2) Are these conditions sufficient?

(3) For such L_n and \mathcal{H} , are the coefficients p_{α} of the Λ_{α} in this linear combination given by the Born rule (3)?

The answer to the first question is given in Sec. II, under the assumption that the L_n satisfy the necessary and sufficient

¹We limit the considerations of this paper to a Hilbert space of a finite dimensionality d . Presumably they can be extended to infinite-dimensional spaces, on which \mathcal{H} and the L_n act as suitably defined operators.

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condition [6] that the von Neumann entropy $-\text{Tr}(\rho \ln \rho)$ should never decrease:

$$\sum_n L_n^\dagger L_n = \sum_n L_n L_n^\dagger. \quad (5)$$

The second and third questions are answered in Sec. III, where we give a general solution of the Lindblad equation, under the conditions found in Sec. II.

II. NECESSARY CONDITIONS FOR A MEASUREMENT

First, let us consider some general aspects of the late-time behavior of the solutions of the Lindblad equation (4), without yet specializing to L_n satisfying Eq. (5). Because Eq. (4) is linear with time-independent coefficients, it has solutions that are generically of the form

$$\rho(t) = \sum_k v_k \exp(\lambda_k t), \quad (6)$$

where v_k and λ_k are the eigenmatrices and eigenvalues of the operator \mathcal{L} in Eq. (4):

$$\mathcal{L} v_k = \lambda_k v_k, \quad (7)$$

$$\mathcal{L} v \equiv -i[\mathcal{H}, v] + \sum_n \left(L_n v L_n^\dagger - \frac{1}{2} L_n^\dagger L_n v - \frac{1}{2} v L_n^\dagger L_n \right), \quad (8)$$

with the normalization of each v_k in Eq. (6) of course depending on initial conditions. [It is only for the nondegenerate case that the solution of Eq. (4) necessarily takes the form (6); if an eigenvalue λ_k has an N -fold degeneracy, then $\exp(\lambda_k t)$ may be accompanied with a polynomial in t of order up to $N - 1$.] Because \mathcal{L} is in general not Hermitian the eigenvalues may be complex, and the individual v_k need not be Hermitian or positive, although the sum (6) must be both Hermitian and positive,

Even if there are eigenvalues λ_k with positive-definite real parts, such terms cannot contribute to the sum (6). If they did contribute then the sum of such terms would dominate $\rho(t)$ at late times. But $\text{Tr}\rho(t)$ must remain constant, so the sum of terms with $\text{Re}\lambda_k > 0$ would have to be traceless. Also, $\rho(t)$ must remain Hermitian and positive, so the sum of terms with $\text{Re}\lambda_k > 0$ would have to be Hermitian and positive. But then the eigenvalues of this sum would have to be real and positive and add up to zero, which is impossible unless all the eigenvalues vanish, in which case the sum vanishes. The same argument rules out any contribution of powers of time for any eigenvalues with $\text{Re}\lambda_k = 0$. So we conclude that the asymptotic behavior of $\rho(t)$ is dominated by the sum of $v_k \exp(\lambda_k t)$ over all eigenmatrices with $\text{Re}\lambda_k = 0$, if there are any.

In fact, as required by the constancy of the trace, there always is at least one eigenmatrix with $\lambda_k = 0$. We can think of \mathcal{L} as a $d^2 \times d^2$ matrix, acting on the space of $d \times d$ matrices. Because Eq. (4) preserves the trace of ρ , the unit $d \times d$ matrix is a left eigenvector of \mathcal{L} with eigenvalue zero, so $\text{Det}\mathcal{L} = 0$, and therefore \mathcal{L} also has a right eigenvector (not necessarily the unit matrix) with eigenvalue zero. But in general there may be several v_k with $\lambda_k = 0$.

In order to separate the real and imaginary parts of general eigenvalues, let us consider the quantity

$$\text{Tr}(v_k^\dagger v_k) \lambda_k = \text{Tr}(v_k^\dagger \mathcal{L} v_k). \quad (9)$$

A straightforward calculation gives

$$\begin{aligned} \text{Tr}(v_k^\dagger v_k) \text{Re}\lambda_k &= -\frac{1}{2} \text{Tr} \left(\sum_n [v_k, L_n^\dagger]^\dagger [v_k, L_n^\dagger] \right) \\ &\quad - \frac{1}{2} \text{Tr} \left(v_k v_k^\dagger \sum_n (L_n^\dagger L_n - L_n L_n^\dagger) \right), \end{aligned} \quad (10)$$

$$\text{Tr}(v_k^\dagger v_k) \text{Im}\lambda_k = -\text{Tr}(v_k^\dagger [\mathcal{H}, v_k]) + \text{Im} \text{Tr} \sum_n L_n v_k^\dagger [v_k, L_n^\dagger]. \quad (11)$$

(See Appendix A.) It is difficult to make further progress without invoking some assumption that limits the nature of the L_n . As mentioned in Sec. I, we shall assume that the L_n satisfy the necessary and sufficient condition (5) for non-decreasing entropy. In this case, Eq. (10) simplifies to

$$\text{Tr}(v_k^\dagger v_k) \text{Re}\lambda_k = -\frac{1}{2} \text{Tr} \left(\sum_n [v_k, L_n^\dagger]^\dagger [v_k, L_n^\dagger] \right). \quad (12)$$

We see immediately that the real parts of all λ_k are negative or zero. The behavior of $\rho(t)$ for $t \rightarrow \infty$ is then dominated by the modes v_k for which $\text{Re}\lambda_k = 0$, for which according to Eq. (12) v_k must commute with all L_n^\dagger , and hence with all L_n . [By taking the adjoint of Eq. (7) we see that if v_k is an eigenmatrix of \mathcal{L} then so is v_k^\dagger , which must appear in (6) along with v_k to keep ρ Hermitian. The adjoint of the condition that v_k^\dagger commutes with L_n^\dagger tells us that v_k must commute with L_n .]

Also, for such modes Eq. (11) gives

$$\text{Tr}(v_k^\dagger v_k) \text{Im}\lambda_k = -\text{Tr}(v_k^\dagger [\mathcal{H}, v_k]). \quad (13)$$

But it must not be thought that a v_k that commutes with all L_n is necessarily an eigenmatrix of \mathcal{L} with the real part of the eigenvalue zero and its imaginary part of first order in \mathcal{H} . With L_n subject to Eq. (5) and v_k commuting with all L_n , the eigenvalue equation (7) becomes

$$\lambda_k v_k = -i[\mathcal{H}, v_k],$$

but this is impossible if the space of matrices that commute with all L_n is not invariant under commutation with \mathcal{H} . Otherwise the commutator with \mathcal{H} in Eq. (7) will mix the eigenmatrices v_k that commute with all L_n with other matrices that do not commute with some L_n , giving an eigenvalue with negative-definite real part, whose contribution vanishes for $t \rightarrow \infty$.

Here is an example. Take $d = 2$, with a single L_n given by $L = \ell \sigma_3$ [which trivially satisfies Eq. (5)], and $\mathcal{H} = h \sigma_1$, with h real. This L commutes with the projection operators $(1 \pm \sigma_3)/2$, as required in a measurement of σ_3 , but for $h \neq 0$ the commutator of \mathcal{H} with these projection operators does not commute with them, so the measurement does not work. We can see this in the late-time behavior of the solutions of the Lindblad equation. In general the eigenmatrices of \mathcal{L} are $v_0 \propto 1$, with eigenvalue zero; $v_1 \propto \sigma_1$, with eigenvalue $-2|\ell|^2 < 0$; and two mixtures of σ_2 and σ_3 , with eigenvalues $-|\ell|^2 \pm$

$(|\ell|^4 - 4h^2)^{1/2}$. For $h = 0$ there are two eigenmatrices with eigenvalue zero, which can be taken as 1 and σ_3 , or equivalently as the projection matrices $(1 + \sigma_3)/2$ and $(1 - \sigma_3)/2$ as needed in a measurement of σ_3 ; the other eigenmatrices both have eigenvalues $-2|\ell|^2$, corresponding to modes that disappear for $t \rightarrow \infty$. On the other hand the late-time behavior of the density matrix is entirely different if h is nonzero, although arbitrarily small. In this case all eigenvalues have negative-definite real part, except the eigenvalue $\lambda_0 = 0$ associated with $v_0 \propto 1$, and for $t \rightarrow \infty$ the density matrix approaches the maximum entropy matrix $1/2$, for which all probabilities are the same.

With this background, let us now consider what happens in a measurement. We suppose that the microscopic system under study interacts with a macroscopic measuring apparatus, in such a way that the density matrix of the microscopic system evolves according to the Lindblad equation (4), with the measuring apparatus chosen so that the matrices L_n and \mathcal{H} have whatever properties are needed so that $\rho(t)$ at late times approaches a linear combination of projection operators Λ_α on the eigenstates $|\alpha\rangle$ of whatever is being measured. As we have seen in Eq. (12), in order for this to be the case without putting any constraints on the initial conditions that determine the coefficients in this linear combination, it is necessary that the matrices L_n should commute with any linear combination of the Λ_α , and hence with each Λ_α :

$$[L_n, \Lambda_\alpha] = 0, \quad (14)$$

from which it follows immediately that each L_n must itself be a linear combination of the Λ_α :

$$L_n = \sum_{\alpha} \ell_{n\alpha} \Lambda_{\alpha}, \quad (15)$$

with coefficients $\ell_{n\alpha}$ that are in general complex numbers. (From Eq. (14) it follows that the eigenstates $|\alpha\rangle$ satisfying $\Lambda_\beta|\alpha\rangle = \delta_{\alpha\beta}|\alpha\rangle$ must be eigenstates of the L_n :

$$L_n|\alpha\rangle = L_n\Lambda_\alpha|\alpha\rangle = \Lambda_\alpha L_n|\alpha\rangle = |\alpha\rangle\langle\alpha|L_n|\alpha\rangle.$$

Then L_n has the same action on any $|\alpha\rangle$ as does the sum (15) with $\ell_{n\alpha} = \langle\alpha|L_n|\alpha\rangle$, and since the $|\alpha\rangle$ form a complete set, L_n must equal the sum (15).) From Eq. (15) the condition (5) for nondecreasing entropy follows trivially.

This leaves us with the matrix \mathcal{H} . As remarked earlier, in order that the limiting behavior of $\rho(t)$ for general initial conditions should be a linear combination of the Λ_α , it is necessary that the space of such linear combinations should be invariant under commutation with \mathcal{H} :

$$[\mathcal{H}, \Lambda_\alpha] = \sum_{\beta} h_{\alpha\beta} \Lambda_{\beta}.$$

By multiplying this commutator on both the left and right with any Λ_β , we see that $0 = h_{\alpha\beta}$, and therefore \mathcal{H} must commute with all Λ_α . By the same argument used above for the L_n , we see then that \mathcal{H} must be a linear combination of the Λ_α :

$$\mathcal{H} = \sum_{\alpha} h_{\alpha} \Lambda_{\alpha}, \quad (16)$$

with real coefficients h_{α} .

This is a good place to bring up a complication. The late-time behavior (1) is expected only for a *complete* measurement. It is more common for measurements to be

incomplete, in the sense that they do not lead to definite states $|\alpha\rangle$ with definite probabilities, but to equivalence classes of states that are not distinguished by the measurement. For instance, in a system consisting of two spins $1/2$, we might measure only the first spin, leaving the other undisturbed. The states then fall into two classes, labeled by the z -components of the two spins: one class consists of $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$, and the other consists of $|-1/2, 1/2\rangle$ and $|-1/2, -1/2\rangle$. In incomplete measurements, instead of (1), the expected late-time limit of the density matrix is

$$\rho_{\text{initial}} \mapsto \rho_{\text{final}} = \sum_C \Lambda_C \rho_{\text{initial}} \Lambda_C, \quad (17)$$

where

$$\Lambda_C = \sum_{\alpha \in C} \Lambda_{\alpha}, \quad (18)$$

As far as the states within a single class are concerned, Λ_C acts just like a unit matrix, so Eq. (17) says that the measurement does nothing to what is not being measured. For a complete measurement, where each state belongs to a different class, Eq. (17) reduces to Eqs. (1) and (3).

Equation (12) shows that in order for $\rho(t)$ to have some given asymptotic limit ρ_{final} , it is necessary for all L_n to commute with this limit, and since this must be true for all ρ_{initial} , the L_n here must in particular commute with $\sum_C \Lambda_C \Lambda_{\alpha} \Lambda_C = \Lambda_{\alpha}$. The same argument as given above for complete measurements then shows that, here too, each L_n must be a linear combination (15) of the Λ_{α} . Only now there is a constraint on the coefficients. The commutator of the sum (15) with the limit (17) is

$$\begin{aligned} & \left[\sum_{\alpha} \ell_{n\alpha} \Lambda_{\alpha}, \sum_C \Lambda_C \rho_{\text{initial}} \Lambda_C \right] \\ &= \sum_C \sum_{\beta, \gamma \in C} [\ell_{n\beta} - \ell_{n\gamma}] \Lambda_{\beta} \rho_{\text{initial}} \Lambda_{\gamma} \end{aligned}$$

which vanishes for all initial density matrices if $\ell_{n\beta} = \ell_{n\gamma}$ for all β and γ in the same class. The same argument shows that $h_{\beta} = h_{\gamma}$ if β and γ are in the same class. This is reasonable, because for an incomplete measurement the Lindblad equation must not distinguish between different states in the same class. We will see in the next section that in this case the late-time limit of the density matrix does have the form (17).

III. COLLAPSE OF THE DENSITY MATRIX

First let us give the solution of the Lindblad equation under the condition that the matrices L_n and \mathcal{H} in this equation are linear combinations (15) and (16) of projection operators Λ_{α} satisfying Eq. (2):

$$L_n = \sum_{\alpha} \ell_{n\alpha} \Lambda_{\alpha}, \quad \mathcal{H} = \sum_{\alpha} h_{\alpha} \Lambda_{\alpha}.$$

It is straightforward to check that Eq. (4) is then satisfied by

$$\rho(t) = \sum_{\alpha\beta} \Lambda_{\alpha} M \Lambda_{\beta} \exp(\lambda_{\alpha\beta} t), \quad (19)$$

where

$$\lambda_{\alpha\beta} = -\frac{1}{2} \sum_n |\ell_{n\alpha} - \ell_{n\beta}|^2 + i \operatorname{Im} \sum_n \ell_{n\alpha} \ell_{n\beta}^* - i(h_\alpha - h_\beta). \quad (20)$$

and M is an arbitrary matrix, independent of α , β , and time. (See Appendix B.) To relate M to the initial value of $\rho(t)$ at $t = 0$, set $t = 0$ in Eq. (19) and use the completeness condition $\sum_\alpha \Lambda_\alpha = 1$. We see that $\rho(0) = M$, and so

$$\rho(t) = \sum_{\alpha\beta} \Lambda_\alpha \rho(0) \Lambda_\beta \exp(\lambda_{\alpha\beta} t), \quad (21)$$

This is our general solution [7].

Now consider the behavior of this solution at late times. The only terms in the sum (21) that do not decay exponentially are those with $\ell_{n\alpha} = \ell_{n\beta}$ for all n . If for the moment we rule out degeneracy, so that $\ell_{n\alpha}$ can equal $\ell_{n\beta}$ for all n only for $\alpha = \beta$, then all $\lambda_{\alpha\beta}$ have negative-definite real part except those with $\alpha = \beta$, for which the imaginary as well as the real parts of $\lambda_{\alpha\alpha}$ vanish. These terms then dominate the asymptotic behavior [8] of the density matrix for $t \rightarrow \infty$:

$$\rho(t) \rightarrow \sum_\alpha \Lambda_\alpha \rho(0) \Lambda_\alpha = \sum_\alpha \Lambda_\alpha \langle \alpha | \rho(0) | \alpha \rangle. \quad (22)$$

This is just the behavior (1) called for by the Copenhagen interpretation, with probabilities p_α given by the Born rule (3).

The case of degeneracy arises in an incomplete measurement, in which we measure only whether the system is in some state or other in a class of states that are not distinguished by the measurement. As indicated at the end of the previous section, in this case we expect $\ell_{n\alpha}$ to equal $\ell_{n\beta}$ for all n and $h_\alpha = h_\beta$ if (and only if) $|\alpha\rangle$ and $|\beta\rangle$ are in the same class. Then Eq. (21) has the expected late-time behavior (17).

It is striking that although the detailed time-dependence of the density matrix depends on the coefficients $\ell_{n\alpha}$ and h_α appearing in the matrices in the Lindblad equation, the asymptotic limit for $t \rightarrow \infty$ for both complete and incomplete measurements does not depend on these details, depending only on the initial condition $\rho(0)$ and on what it is that is being measured. This, of course, is just what we require of a measurement.

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APPENDIX A: DERIVATION OF EQS. (10) AND (11)

We start with the desired result, and work back to the problem it solves. For a general matrix v , consider

the quantity

$$\begin{aligned} R \equiv & -\frac{1}{2} \operatorname{Tr} \left(\sum_n [v, L_n^\dagger]^\dagger [v, L_n^\dagger] \right) \\ & - \frac{1}{2} \operatorname{Tr} \left(v v^\dagger \sum_n (L_n^\dagger L_n - L_n L_n^\dagger) \right) \\ & + i \operatorname{Im} \operatorname{Tr} \sum_n L_n v^\dagger [v, L_n^\dagger] - i \operatorname{Tr} (v^\dagger [\mathcal{H}, v]). \end{aligned} \quad (A1)$$

Expanding each term, this is

$$\begin{aligned} R = & -\frac{1}{2} \operatorname{Tr} \sum_n L_n v^\dagger v L_n^\dagger + \frac{1}{2} \operatorname{Tr} \sum_n v^\dagger L_n v L_n^\dagger \\ & + \frac{1}{2} \operatorname{Tr} \sum_n L_n v^\dagger L_n^\dagger v - \frac{1}{2} \operatorname{Tr} \sum_n v^\dagger L_n L_n^\dagger v \\ & - \frac{1}{2} \operatorname{Tr} \sum_n v v^\dagger L_n^\dagger L_n + \frac{1}{2} \operatorname{Tr} \sum_n v v^\dagger L_n L_n^\dagger \\ & + \frac{1}{2} \operatorname{Tr} \sum_n v^\dagger L_n v L_n^\dagger - \frac{1}{2} \operatorname{Tr} \sum_n L_n v^\dagger L_n^\dagger v \\ & - i \operatorname{Tr} (v^\dagger [\mathcal{H}, v]). \end{aligned} \quad (A2)$$

The third and eighth terms cancel; the fourth and sixth terms cancel; the second and seventh terms add to give the term $\operatorname{Tr} v^\dagger \sum_n L_n v L_n^\dagger$ in $\operatorname{Tr} v^\dagger \mathcal{L} v$; the first and fifth terms give the terms $-\operatorname{Tr} v^\dagger v \sum_n L_n^\dagger L_n / 2$ and $-\operatorname{Tr} v^\dagger \sum_n L_n^\dagger L_n v / 2$ in $\operatorname{Tr} v^\dagger \mathcal{L} v$; and the last term gives the Hamiltonian term in $\operatorname{Tr} v^\dagger \mathcal{L} v$. We conclude that

$$R = \operatorname{Tr} (v^\dagger \mathcal{L} v). \quad (A3)$$

The first two terms in (A1) are real, while the last two are imaginary, so

$$\begin{aligned} \operatorname{Re} \operatorname{Tr} v^\dagger \mathcal{L} v = & -\frac{1}{2} \operatorname{Tr} \left(\sum_n [v, L_n^\dagger]^\dagger [v, L_n^\dagger] \right) \\ & - \frac{1}{2} \operatorname{Tr} \left(v v^\dagger \sum_n (L_n^\dagger L_n - L_n L_n^\dagger) \right), \end{aligned} \quad (A4)$$

$$\operatorname{Im} \operatorname{Tr} v^\dagger \mathcal{L} v = \operatorname{Im} \operatorname{Tr} \sum_n L_n v^\dagger [v, L_n^\dagger] - \operatorname{Tr} (v^\dagger [\mathcal{H}, v]). \quad (A5)$$

Taking v to be one of the eigenmatrices v_k of \mathcal{L} , with $\mathcal{L} v_k = \lambda_k v_k$ then gives Eqs. (10) and (11).

APPENDIX B: DERIVATION OF EQS. (19) AND (20)

We try a solution of the Lindblad equation

$$\rho(t) = \sum_{\alpha\beta} \Lambda_\alpha M \Lambda_\beta f_{\alpha\beta}(t). \quad (B1)$$

With Λ_α and \mathcal{H} given by Eqs. (15) and (16), the Lindblad equation (4) becomes

$$\sum_{\alpha\beta} \Lambda_\alpha M \Lambda_\beta \frac{d}{dt} f_{\alpha\beta}(t) = \sum_{\alpha\beta} \lambda_{\alpha\beta} \Lambda_\alpha M \Lambda_\beta f_{\alpha\beta}(t), \quad (B2)$$

‘ where

$$\lambda_{\alpha\beta} = C_{\alpha\beta} - \frac{1}{2}C_{\alpha\alpha} - \frac{1}{2}C_{\beta\beta} - i(h_\alpha - h_\beta) \quad (\text{B3})$$

and

$$C_{\alpha\beta} = \sum_n \ell_{n\alpha} \ell_{n\beta}^*. \quad (\text{B4})$$

This has an obvious solution of the same form as (19):

$$f_{\alpha\beta}(t) = \exp(\lambda_{\alpha\beta}t) f_{\alpha\beta}(0). \quad (\text{B5})$$

To get a more useful expression for $\lambda_{\alpha\beta}$, we note that

$$\begin{aligned} -\frac{1}{2}|\ell_\alpha - \ell_\beta|^2 &= -\frac{1}{2}C_{\alpha\alpha} - \frac{1}{2}C_{\beta\beta} + \text{Re} \sum_n \ell_{n\alpha} \ell_{n\beta}^* \\ &= C_{\alpha\beta} - \frac{1}{2}C_{\alpha\alpha} - \frac{1}{2}C_{\beta\beta} - i \text{Im} \sum_n \ell_{n\alpha} \ell_{n\beta}^* \end{aligned}$$

so Eq. (B3) is the same as Eq. (20).

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