PHYSICAL REVIEW A **93**, 032120 (2016)

Ground-state cooling of quantum systems via a one-shot measurement

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(Received 13 March 2015; published 15 March 2016)

We prove that there exists a family of quantum systems that can be cooled to their ground states by a one-shot projective measurement on the ancillas coupled to these systems. Consequently, this proof gives rise to the conditions for achieving the one-shot measurement ground-state cooling (OSMGSC). We also propose a general procedure for finding unitary propagators and corresponding Hamiltonians to realize such cooling by means of inverse engineering techniques.

DOI: 10.1103/PhysRevA.93.032120

I. INTRODUCTION

Quantum ground-state cooling of small objects exemplified by nanosystems has long been a challenge and one of the most desirable quantum technologies. Physically, the cooling process can be formulated as a transformation from an initial thermal state of a small object into its ground state. The transformation is irreversible and cannot be realized when the object is isolated. It is an indispensable part in the initialization of quantum devices such as an adiabatic quantum computer [1–5], and it also plays a crucial role in the ultrahigh-precision measurements using mechanical resonators [6–8]. Over the years, scientists have made great efforts to develop ground-state cooling techniques [8–19], in particular, sideband cooling [9–11].

Recently, the ground-state cooling of small objects via quantum measurements has been proposed theoretically [8] and verified experimentally [12]. In this approach, the target system A is coupled to an ancilla B. The composite system $A+B$ undergoes a unitary evolution for a *random* interval of time before a projective measurement is taken on the ancilla. If the outcome of this projective measurement on B is found to be the ground state, the evolution-measurement procedure is repeated. It was reported that efficient ground-state cooling can be achieved by repeating such random-time-interval evolutions and measurements, and the cooling efficiency hardly depends on time intervals between any two consecutive measurements [8] but increases with the frequency of measurements. The major disadvantage of this cooling approach is that it requires many measurements to achieve ground cooling, and consequently the survival probability becomes so small that a very large ensemble of identical systems is required. Here we prove an existence theorem that, for a family of physical systems, guarantees ground-state cooling by making a *one-shot projective measurement* at a specified time, and we derive explicit conditions for this one-shot measurement cooling method to be valid. Furthermore, we show a general approach to engineering Hamiltonians that are able to realize the one-shot measurement cooling by means of inverse engineering techniques. For existing Hamiltonians, our approach can be used to find the optimal times when the projective measurement is taken, and it is interesting to note that the probability to realize the one-shot measurement cooling remains high even if the abovementioned conditions are not strictly satisfied.

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II. EXISTENCE PROOF OF ONE-SHOT MEASUREMENT GROUND-STATE COOLING (OSMGSC)

Consider an n-level quantum system A coupled with an ancilla B with m levels. The system A may not be experimentally accessible so that ground-state cooling cannot be processed, exemplified by a mechanical resonator [8]. To remedy this, we choose a fully controllable ancilla B to manipulate the system A through the A-B interaction and measurements on B, for instance, a flux qubit as the ancilla [8]. Without loss of generality, we prepare the ancilla in its ground state $|g\rangle$. The preparation for a qubit ancilla has recently been done experimentally [12]. In general, the ancilla could be initialized by a measurement before it interacts with the system A. The joint unitary propagator for the composite system $A+B$ is expressed by $U(t)=\sum_{i,\alpha;j,\beta}U_{i,\alpha;j,\beta}(t)|i\rangle_A\langle j|\otimes|\alpha\rangle_B\langle\beta|$ with $nm\times nm$ independent real parameters. It has been shown that it is possible to achieve efficient ground-state cooling with repeated projective measurements.

Theorem. For the composite system $A+B$, there exist joint unitary propagators U which allow the ground-state cooling of subsystem A through a one-shot selective projective measurement on the ancilla B.

Proof. Consider a general initial state of system A,

$$\rho_A^i=\sum_{l=0}^{n-1}p_l\,|l\rangle\,\langle l|\,, \tag{1}$$

where p_l are probabilities arising from the thermal bath, $|l\rangle$ are energy eigenstates of A, and $n>1$ is the dimension of the system Hilbert space. After the joint unitary evolution, the state becomes $\rho_{A+B}^f(t)=U\rho_A^i\otimes|g\rangle\langle g|U^\dagger$. We then make a projective measurement on B. Given that the outcome of the measurement is $|g\rangle$, the output of the composite $A+B$ is $\rho_A^f(t)\otimes|g\rangle\langle g|$, where $\rho_A^f(t)$ is the final state of the target A. A schematic diagram of the cooling procedure is depicted in Fig. 1.

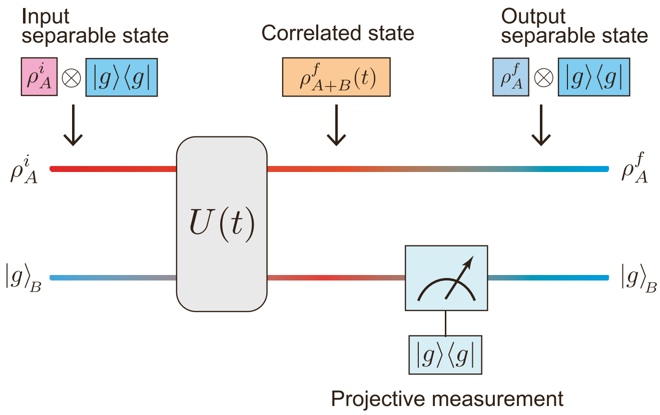

FIG. 1. One-shot measurement ground-state cooling scheme. An ancilla B initially in its ground state is coupled to the target A that is expected to be cooled to the ground state. The composite system undergoes a unitary evolution for a duration of t and the final state $\rho_{A+B}^f(t)$ is correlated. A projective measurement is then applied on the ancilla. We discard the result if the output of the ancilla is not the ground state.

After the projective measurement, the probability $P_{l,g}$ of finding the composite system $\rho_{A+B}^f(t)$ in state $|l,g\rangle$ is

$$P_{l,g} = \sum_k p_k |U_{l,g;k,g}|^2. \tag{2}$$

It manifests that the conditions to achieve ground-state cooling are all $P_{l,g} = 0$ for $l \geqslant 1$ such that there is no population in the excited states. Seeing that all $p_k \geqslant 0$, if we require stronger constraints,

$$U_{l,g;k,g} \equiv 0,\ l \geqslant 1, \forall k, \tag{3}$$

the conditions $P_{l,g} = 0$ will always hold true. The equation provides $2n^2 - 2n$ constraints upon the $nm \times nm$ real parameters of the propagator U, leaving $n[n(m^2-2)+2]$ free parameters. This clearly shows that there always exists a propagator U such that the system can be cooled to its ground states by a one-shot measurement.

Here we would like to point out that for any given *finite* system A with Hamiltonian H_A, one can always have H_B and H_{AB} such that the total Hamiltonian $H = H_A + H_B + H_{AB}$ will produce the desired propagator U for ground-state cooling. As shown above, the number of free parameters for the total H is $p = n(m-1)[2n(m+1)-1]$. When we have a given H_A it means that the number of free parameters is reduced to $p' = p - n^2$, we still have $p' > 1$ for any $n > 1$ and $m \geqslant 2$. Therefore, our theorem remains valid for *any finite system*.

Corollary. One can always construct a Hamiltonian H for the composite system $A + B$, whose propagator $U(t)$ satisfies the OSMGSC condition, Eq. (3).

Proof. Generally, a Hamiltonian can be expressed in terms of the propagator:

$$H = i\dot{U}U^\dagger. \tag{4}$$

We take the Planck constant $\hbar = 1$ throughout the paper. By imposing the Hermiticity of the Hamiltonian, it can be shown that there are $p = n(m-1)[2n(m+1)-1]$ free parameters to choose in the construction of the Hamiltonian H. Since the number of free parameters $p > 1$ for any $n \geqslant 1$ and $m \geqslant 2$, it is self-evident that there always exists a family of Hamiltonians H which can be used to realize the ground-state cooling by a one-shot measurement.

The Hamiltonians of these *good for cooling* systems can be found by using Eqs. (3) and (4). Note that p is even greater than $nm \times nm$, the number of real parameters of a Hermitian matrix H. As such, one can always construct a Hamiltonian from the unitary propagator by using inverse engineering control techniques [20].

III. INVERSE ENGINEERING

As an illustrative example, we now consider a two-level system as the target A, coupled to an ancillary qubit B. Our purpose is to construct a propagator U that satisfies the condition (3) and then construct the corresponding Hamiltonian by inverse engineering [20].

The unitary propagator of a two-level system can be generally written as

$$U(t) = \cos\theta(t) + i\sin\theta(t)\vec{\sigma}\cdot\vec{n}(t), \tag{5}$$

where $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ represents a vector of Pauli operators and $\vec{n}$ denotes a unit vector. The corresponding Hamiltonian can be written as [21,22]

$$\begin{aligned} H &= i\dot{U}U^\dagger \\ &= -\vec{\sigma}\cdot[\dot{\theta}\vec{n} + \sin\theta\cos\theta\dot{\vec{n}} + \sin^2\theta(\dot{\vec{n}}\times\vec{n})]. \end{aligned} \tag{6}$$

Assume that the two-qubit unitary propagator is a direct sum of two 2×2 block-diagonal matrices,

$$U = \begin{pmatrix} U_{0,g;0,g} & U_{0,g;0,e} & 0 & 0 \\ U_{0,e;0,g} & U_{0,e;0,e} & 0 & 0 \\ 0 & 0 & U_{1,g;1,g} & U_{1,g;1,e} \\ 0 & 0 & U_{1,e;1,g} & U_{1,e;1,e} \end{pmatrix}; \tag{7}$$

we can directly use Eq. (5) to inversely engineer the blocks of U and then the corresponding blocks of H by Eq. (6). The OSMGSC conditions $U_{1,g;0,g} = U_{1,g;1,g} = 0$ in Eq. (3) are accordingly satisfied. Another OSMGSC condition $U_{1,g;1,g} = 0$ in the bottom block U_2,

$$U_2 = \begin{pmatrix} U_{1,g;1,g} & U_{1,g;1,e} \\ U_{1,e;1,g} & U_{1,e;1,e} \end{pmatrix}, \tag{8}$$

is inversely engineered by using Eq. (5). Consequently the vector $\vec{n}$ is not allowed to have the z component because otherwise the component will lead to $U_{1,g;1,g} = \cos\theta + ia\sin\theta \neq 0$ ($a \neq 0$). For simplicity, assume that $\vec{n}(t) = (-1,0,0)$ and $\theta(t) = t$, then the U_2 block reads

$$U_2 = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}, \tag{9}$$

with $U_{1,g;1,g} = 0$ at time instants $t = \pi/2 + \pi n$ ($n = 0,1,2,\ldots$). The corresponding H block is $H_2 = \sigma_x$. Equation (5) does not impose constraints on U_1. For simplicity, we set $\theta(t) = t$ and $\vec{n}(t) = (0,0,-1)$, such that Eq. (6) becomes

$$U_1 = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

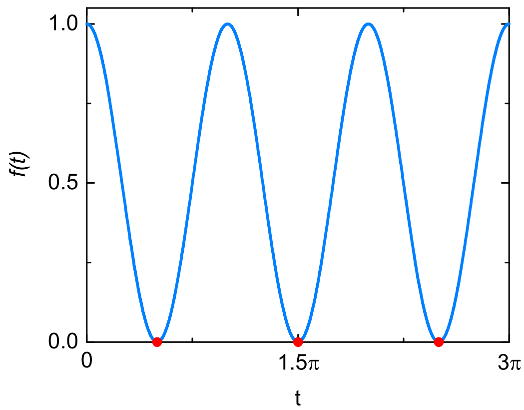

FIG. 2. The measure $f(t)$ as a function of time. The optimal times for our one-shot projective measurements show up periodically and are highlighted by solid red circles.

We now combine H_1 and H_2 and write the total Hamiltonian of system $A+B$:

$$H = |0,g\rangle\langle 0,g| - |0,e\rangle\langle 0,e| + |1,e\rangle\langle 1,g| + |1,g\rangle\langle 1,e| = |0\rangle\langle 0| \otimes \sigma_z + |1\rangle\langle 1| \otimes \sigma_x. \tag{11}$$

Driven by this Hamiltonian, it is easy to verify that the system A reaches its ground state by a one-shot measurement on B, when the measurement is made at $t = \pi/2 + \pi n$ (n's are integers). Specifically, let us consider the general initial state $(p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|) \otimes |g\rangle\langle g|$, where $0 < p \leqslant 1$. At the moment of *good for cooling*, the system evolves to $p\,|0\rangle\langle 0| \otimes |g\rangle\langle g| + (1-p)\,|1\rangle\langle 1| \otimes |e\rangle\langle e|$. Therefore, as long as $p \neq 0$, we always have a nonzero probability of making the measurement on the ancilla and find it to be in its ground state. Since other measurement results are discarded, we deterministically have the system cooled to its ground state after this one-shot selective measurement on the ancilla. Notice that the good-for-cooling joint density matrix above is a classically correlated state [23–26].

To quantify the ability of achieving OSMGSC, we define the measure f,

$$f = \frac{1}{\sum_m |U_{0,g;m,g}|^2} \sum_{n \geqslant 1, m} |U_{n,g;m,g}|^2. \tag{12}$$

This measure entails that if we make measurements at times when $f = 0$, the OSMGSC scheme will be fulfilled. Figure 2 shows $f(t)$ for the system driven by the Hamiltonian (11), where the optimal times for OSMGSC indicated by $f = 0$ show up periodically over the course of time.

We can also engineer $\theta(t)$ in U_2 to create a *steady good-for-cooling* state to allow our OSMGSC scheme to be preformed in a wide time domain. This relaxes the experimental constraints of making the measurement precisely at the optimal time instants. For example, if $\theta(t) = \pi(1 - e^{-\omega t})/2$, the corresponding Hamiltonian is

$$H(t) = |0\rangle\langle 0| \otimes \sigma_z + h(t)|1\rangle\langle 1| \otimes \sigma_x, \tag{13}$$

where $h(t) = \dot{\theta}(t) = \pi\omega e^{-t}/2$. Here a part of the interaction decays with time and becomes effectively switched off for a longer time. In Fig. 3 we plot the measure $f(t)$ for the system under the Hamiltonian (13) with $\omega = 1$. This Hamiltonian drives the system into a steady state $p|0,g\rangle\langle 0,g| + (1-p)|1,e\rangle\langle 1,e|$, which satisfies the OSMGSC condition (3).

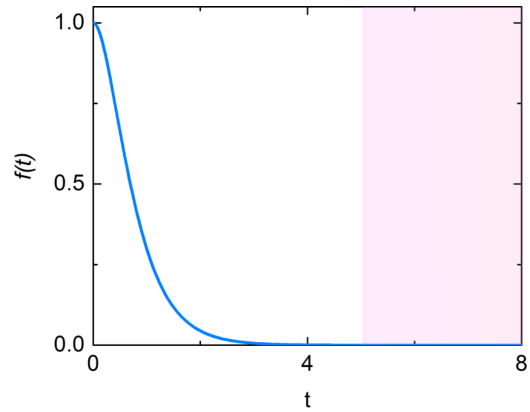

FIG. 3. The measure $f(t)$ as a function of time. The light pink (shaded) area marks the time domain where $f(t) < 10^{-4}$.

Under the constraints (3), we are free to choose H and U in different manners, for example, by including the transition term $|0\rangle\langle 1| + \mathrm{H.c.}$ in the constructed Hamiltonian. Consider the propagator U,

$$U = \mathrm{diag}(u_1, U_2, u_3), \tag{14}$$

where u_1 and u_3 are complex numbers, with $|u_1| = |u_3| = 1$, and U_2 is a 2×2 unitary matrix different from the previous case. By using the above method, we can obtain the corresponding H with transition terms $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$:

$$H = \omega_1\,|0,g\rangle\langle 0,g| + \omega_2\,|1,e\rangle\langle 1,e| + |1,g\rangle\langle 0,e| + |0,e\rangle\langle 1,g|, \tag{15}$$

where ω_1 and ω_2 are free parameters.

The last approach can be applied to engineering U and H matrices for an arbitrary $A+B$ system. Suppose that A is an n-level system and the ancilla B is a qubit, U is engineered in the same manner as Eq. (14), i.e.,

$$U = \mathrm{diag}(u_1, U_2, U_3, \ldots, U_n, u_2). \tag{16}$$

We then use Eqs. (5) and (6) to obtain all U_i ($i = 2,3,\ldots,n$), where we select $\theta(t)$ and time t carefully such that, for each U_i block, all $U_{i,g;i,g} = 0$ simultaneously. We underline that the inverse engineering scheme is only *one of possible ways* to find Hamiltonians which satisfy the OSMGSC theorem.

IV. EXISTING HAMILTONIANS AND ONE-SHOT MEASUREMENT COOLING

The ground-state cooling of nanomechanical resonators (NAMR) has become increasingly important in ultrahigh-precision measurements, classical to quantum transitions, preparations of nonclassical states, and quantum information processing. Efficient ground-state cooling of the NAMR with repeated measurements on an ancillary qubit has been proposed [8]. Below we analyze the possibility of achieving OSMGSC for such a system.

When the coupling strength between the NAMR and the qubit is much smaller than the qubit frequency, the rotating wave approximation becomes valid and the total Hamiltonian is reduced to the standard Jaynes-Cummings model [8,27]:

$$H = \omega a^\dagger a + \frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + g(a \otimes |e\rangle\langle g| + a^\dagger \otimes |g\rangle\langle e|), \quad (17)$$

where $a^\dagger$ (a) are creation (annihilation) operators of phonons, ω is the fundamental mode frequency of the NAMR, Δ is the tunneling amplitude between the two qubit states, and g is the coupling strength. For this existing model, we study the possibility of OSMGSC by using Eq. (3). Since the propagator U has a block structure [8], we can analytically give

$$\begin{aligned} U_{0,g;0,g}(t) &= e^{i\Delta t/2}, \\ U_{n-1,e;n-1,e}(t) &= e^{-i\varepsilon_n^+ t}\cos^2\theta_n + e^{-i\varepsilon_n^- t}\sin^2\theta_n, \\ U_{n,g;n,g}(t) &= e^{-i\varepsilon_n^+ t}\sin^2\theta_n + e^{-i\varepsilon_n^- t}\cos^2\theta_n, \end{aligned} \quad (18)$$

where $\varepsilon_n^\pm = \omega(n-1/2) \pm \sqrt{(\Delta-\omega)^2 + 4g^2 n}/2$ and $\tan(2\theta_n) = 2g\sqrt{n}/(\Delta-\omega)$.

It is seen that the constraints $U_{n,g;n,g} = 0$ cannot be fulfilled for all $n \geqslant 1$ no matter how we adjust the parameters. Although the *exact* OSMGSC is impossible for this model, we may still numerically explore optimal times when the measurement should be taken. This amounts to numerical estimation of the time instants when the measure f,

$$f = \frac{1}{|U_{0,g;0,g}|^2}\sum_{n=1}^{\infty}|U_{n,g;n,g}|^2, \quad (19)$$

approximates to zero, though does not completely vanish.

If we consider a thermal input state as in the conventional cooling process,

$$\rho_A^i = \frac{1}{Z}\sum_{n=0} e^{-n\omega/T}|n\rangle\langle n|, \quad Z = \sum_{n=0} e^{-n\omega/T}, \quad (20)$$

our numerical analysis shows that only the first few blocks of U determine the possibility of OSMGSC when T is not too large. Therefore, we can truncate the sum to the kth level:

$$f_k = \frac{1}{|U_{0,g;0,g}|^2}\sum_{n=1}^{k}|U_{n,g;n,g}|^2. \quad (21)$$

Using Eqs. (2), (20), and (21) we can derive an inequality for the cooling success probability P_c:

$$P_c = \frac{P_{0,g}}{P_g} > 1 - \frac{\langle n\rangle}{1+\langle n\rangle}[f_k + \langle n\rangle^k(1+\langle n\rangle)^{2-k}], \quad (22)$$

where $P_g = \sum_{n=0}^{\infty} P_{n,g}$ is the probability of obtaining the qubit ground state $|g\rangle$. We illustrate f_k in a resonate case with $\omega = \Delta = 1$ and $g = 0.2$. Figure 4 shows $f_3(t)$ for the parameters given in the figure caption. Interestingly, we indeed find $f_3(150) = 0.04 \approx 0$, which is the optimal time to implement a projective measurement to achieve the ground-state cooling of the NAMR. The inequality (22) also concludes that the probability of achieving the ground-state cooling is greater than 93% at $T = 1$. The example suggests the possibility that the propagator U generated by certain existing Hamiltonians may directly result in OSMGSC at specific time instants or domains.

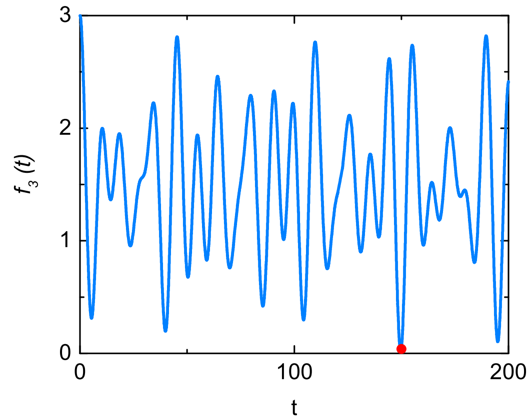

FIG. 4. The measure $f_3(t)$ as a function of time for the NAMR with $\omega = \Delta = 1$ and $g = 0.2$. The optimal times for our one-shot projective measurements is highlighted by a solid red circle at $t \approx 150$.

As another example, Ref. [12] uses a specific joint unitary matrix U, an ordered product of a Hadamard gate, a phase γ shifter on ancillary qubit B, a controlled-$\mathcal{U}(s)$, and another Hadamard gate on B. Here $\mathcal{U}(s) = \exp(-i\mathcal{H}s)$, where $\mathcal{H}$ is the Hamiltonian of the target system A. A direct calculation shows that $U_{1,g;0,g} = 0$ and $U_{1,g;1,g}(s) = 1 - i\exp(i\gamma)[\cos s - i\sin s]$. Interestingly, our requirement (3) can be met when $s = \pi/2$ and $\gamma = 0$. Our OSMGSC scheme helps to find the optimal measurement instant of the experiment and provides clear guidance for the experiment.

V. CONCLUSION

In conclusion, we have proven the existence of a family of systems that can be efficiently cooled to their ground states by making one-shot projective measurement on a coupled ancilla. The explicit condition for achieving this one-measurement cooling has been given, and we have also shown an example of a general procedure for finding the corresponding Hamiltonian to realize this technique by means of inverse engineering. For existing Hamiltonians, our method can be used to find an optimal time when the projective measurement should be taken, which provides clear guidance for the existing experiments and for designing experiments in rapid-cooling techniques.

ACKNOWLEDGMENTS

We acknowledge grant support from the Basque Government Grant No. IT472-10 and the Spanish MICINN Grants No. FIS2012-36673-C03-03 and No. FIS2015-67161-P.

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