

## Adiabaticity in open quantum systems

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We provide a rigorous generalization of the quantum adiabatic theorem for open systems described by a Markovian master equation with time-dependent Liouvillian  $\mathcal{L}(t)$ . We focus on the finite system case relevant for adiabatic quantum computing and quantum annealing. Adiabaticity is defined in terms of closeness to the instantaneous steady state. While the general result is conceptually similar to the closed-system case, there are important differences. Namely, a system initialized in the zero-eigenvalue eigenspace of  $\mathcal{L}(t)$  will remain in this eigenspace with a deviation that is inversely proportional to the total evolution time  $T$ . In the case of a finite number of level crossings, the scaling becomes  $T^{-\eta}$  with an exponent  $\eta$  that we relate to the rate of the gap closing. For master equations that describe relaxation to thermal equilibrium, we show that the evolution time  $T$  should be long compared to the corresponding minimum inverse gap squared of  $\mathcal{L}(t)$ . Our results are illustrated with several examples.

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### I. INTRODUCTION

The origins of the celebrated quantum adiabatic theorem (QAT) date back to Einstein's "Adiabatenhypothese": "If a system be affected in a reversible adiabatic way, allowed motions are transformed into allowed motions" [1]. Ehrenfest was the first to appreciate the importance of adiabatic invariance, guessing (before the advent of a complete quantum theory) that quantum laws would only allow motions which are invariant under adiabatic perturbations [2]. The more familiar, modern version of the QAT was put forth by Born and Fock already in 1928 for the case of discrete spectra [3]. Since then, a series of increasingly sophisticated techniques have been developed in order to generalize the QAT to include degeneracy, unbounded models, continuous spectra, and exponential error estimates [4–11].

This long history of adiabatic theorems is almost exclusively concerned with closed systems undergoing unitary evolution. Previous approaches to formulating an adiabatic condition for open quantum systems [12] have focused on a Jordan block decomposition of the dissipative generator [13], the weak coupling limit [14,15], zero temperature [16], or on a noiseless subsystem decomposition [17,18]. Here, we prove that in analogy to the closed-system case, where the system follows the instantaneous (pure) eigenstates of the Hamiltonian in the adiabatic limit of arbitrarily large total evolution time  $T$ , the open system follows the instantaneous steady state (ISS) of the Liouvillian. In doing so, we extend the seminal closed-system result by Kato [4] to infinite order in  $1/T$ .

Rigorous extensions of the adiabatic theorem for generators of contractive semigroups, similar to ours, have also appeared in the mathematical literature [19–22]. Our focus is on estimating the adiabatic error in terms of the physical parameters of the theory, thus making the result more suitable for applications. We demonstrate that in the case of thermal baths satisfying the Kubo-Martin-Schwinger (KMS) condition [23], where the ISS is the instantaneous Gibbs state, a sufficient criterion

for adiabaticity is  $T \gg \Delta_{\min}^{-2}$ , where  $\Delta_{\min}$  is the smallest Liouvillian gap in absolute value. Our QAT also allows for a finite number of level crossings in the Liouvillian spectrum, for which we demonstrate that the error scales asymptotically as  $T^{-\eta}$  with a known exponent  $\eta \in (0, 1)$  related to the rate at which the gap closes. This setting is directly relevant to recent theoretical and experimental work on quantum annealing [15,24–26], where the Liouvillian gap may close at the end of the evolution, and we verify this prediction using numerical simulations.

### II. INSTANTANEOUS STEADY STATES

We assume that the evolution of a  $d$ -dimensional ( $d < \infty$ ) quantum system with state  $\rho(t)$  can be described by a linear, time-local master equation  $d\rho/dt = \mathcal{L}_T(t)\rho$ , where  $T$  is the total evolution time. We also assume that  $\mathcal{L}_T(sT) = \mathcal{L}(s)$  where  $s = t/T \in [0, 1]$  is a rescaled, dimensionless time coordinate, and  $\mathcal{L}(s)$  is  $T$  independent. Setting  $\rho_T(s) = \mathcal{E}_T(s, 0)\rho_T(0)$ , the evolution operator  $\mathcal{E}_T(s, s_0)$  satisfies

$$\mathcal{E}'_T(s, s_0) = T\mathcal{L}(s)\mathcal{E}_T(s, s_0), \quad (1)$$

with  $\mathcal{E}(s, s) = \mathbb{1}$  [we drop the subscript  $T$  from now on; we also write  $\mathcal{E}(s)$  for  $\mathcal{E}(s, 0)$  for simplicity], and where the prime denotes  $\partial_s$ . We are interested in the solutions of Eq. (1) for large  $T$ . We further assume that the Liouvillian  $\mathcal{L}(s)$  can be written in Lindblad form for all  $s$ , i.e.,  $\mathcal{L}(s)\bullet = -i[H(s), \bullet] + \sum_l [L_l(s)\bullet L_l^\dagger(s) - \frac{1}{2}\{L_l^\dagger(s)L_l(s), \bullet\}]$ , where  $H(s)$  is the system Hamiltonian and  $\{L_l(s)\}$  are the Lindblad operators. Equation (1) then describes a Markovian master equation with a time-dependent Lindblad generator, and the corresponding evolution operator  $\mathcal{E}(s_2, s_1)$  is a completely positive trace-preserving (CPTP) map for any  $s_2 \geq s_1$  [15,27,28]. We formulate our results in terms of Lindblad operators and CPTP maps, but in fact all our results are valid in the more general case where  $\mathcal{L}(s)$  generates a contractive semigroup (i.e.,  $\|e^{t\mathcal{L}(s)}\| \leq 1 \forall s$  and  $t > 0$ ). A special role is played by the ISSs, i.e.,

the states in the kernel of  $\mathcal{L}(s)$ . In the time-independent case [ $\mathcal{L}(s) = \mathcal{L}, \forall s$ ], it follows from the CPTP property that any initial state evolves to  $\text{Ker}\mathcal{L}$  in the long time limit [29]. Let us denote by  $P(s)$  the (instantaneous) spectral projection of  $\mathcal{L}(s)$  with eigenvalue zero. The Lindblad form guarantees that zero is a semisimple, possibly degenerate, eigenvalue of  $\mathcal{L}(s)$  (see Appendix A for a proof), and so there are no nilpotent terms in the zero sector, i.e.,  $\mathcal{L}(s)P(s) = P(s)\mathcal{L}(s) = 0$ .

We are now ready to informally state the QAT for open systems: *If a system is initialized at  $s = 0$  in  $\text{Ker}\mathcal{L}(0)$ , the final state at  $s = 1$  will be close to  $\text{Ker}\mathcal{L}(1)$ , provided the Lindbladian changes sufficiently slowly.*

In principle, one could formulate an open-system QAT considering other (nonzero) eigenvalues of  $\mathcal{L}(s)$ . However, their corresponding invariant subspaces contain no physical states, so that the physical interest in such a generalization is questionable.<sup>1</sup> We proceed to rigorously establish the QAT and identify the time scales it entails.

### III. GAPPED CASE

We start by assuming that the zero eigenvalue is separated by a finite gap  $\Delta_{\min}$  from the rest of the spectrum  $\sigma[\mathcal{L}(s)]$  for all  $s \in [0, 1]$ , i.e.,  $\text{dist}[\sigma[\mathcal{L}(s)] \setminus \{0\}, 0] = \Delta(s) \geq \Delta_{\min} > 0$ , a condition we relax later. The ideal adiabatic evolution is represented by an operator  $V(s)$  that satisfies the intertwining property:  $V(s)P(0) = P(s)V(s)$ . It is well known [4] (see Appendix B) that a possible choice for  $V(s)$  is given by the solution of the differential equation  $V'(s) = [P'(s), P(s)]V(s)$  with  $V(0) = \mathbb{1}$ . We are interested in quantifying the deviation of the actual evolution, governed by the CPTP map  $\mathcal{E}(s)$ , from the ideal adiabatic evolution. However,  $V(s)$  is not CPTP in general. Instead, we can prove that  $W(s) := V(s)P(0)$  is a CPTP map since it can be written as a product of projectors (CPTP maps):  $W(s) = \lim_{N \rightarrow \infty} P(s) \dots P(2s/N)P(s/N)P(0)$  (see Appendix C or Proposition 3 in [22]). Therefore, to state the QAT we wish to bound the deviation from the ideal adiabatic evolution projected to  $\text{Ker}\mathcal{L}$ ,  $\|\mathcal{E}(s)P(0) - V(s)P(0)\|$ , in the large- $T$  limit.<sup>2</sup>

To proceed, we introduce the reduced resolvent  $S(s) = \lim_{z \rightarrow 0} Q(s)[\mathcal{L}(s) - z]^{-1}Q(s)$ , where  $Q(s) = \mathbb{1} - P(s)$  [30]. We further assume that  $\mathcal{L}$  is  $m$  times differentiable, and let  $X_{n+1}(s) = S(s)X'_n(s)$ , with  $X_1(s) = S(s)$ . Under the additional simplifying assumption that the ISS is unique, we then prove the following in Appendix D using integration by

<sup>1</sup>To see this, let  $P_0$  be the projector onto the zero eigenvalue. Since it can be realized as the infinite time limit of a CPTP map,  $P_0$  is itself a CPTP map, and in particular it is trace preserving. Let  $P_j$  be the projector onto another invariant subspace, with  $j \neq 0$ . Assume there is a state in its range, i.e.,  $\exists x | \rho = P_j x$  is a state. But  $P_0 P_j = 0$  so  $\text{Tr}(\rho) = \text{Tr}(P_0 \rho) = 0$  (where the first equality holds since  $P_0$  is a CPTP map), a contradiction.

<sup>2</sup>Unless explicitly noted otherwise, from hereon the norm is the induced trace norm, i.e.,  $\|A\| = \sup_{x \neq 0} \|A(x)\|_1 / \|x\|_1$ , where  $\|x\|_1$  denotes the trace norm, i.e., the sum of the singular values [46]. For a discussion of the properties of the induced trace norm see, e.g., Ref. [49], where it is denoted  $\|\dots\|_{\infty,1}$ , or Ref. [50].

parts:

$$\begin{aligned} [\mathcal{E}(s) - V(s)]P(0) &= \sum_{n=1}^m \frac{\Omega_n}{T^n} - \frac{1}{T^m} \int_0^s d\sigma \mathcal{E}(s, \sigma) X'_m(\sigma) W'(\sigma), \\ \Omega_n &= \mathcal{E}(s, \sigma) X_n(\sigma) W'(\sigma) \Big|_0^s \\ &\quad - \int_0^s d\sigma \mathcal{E}(s, \sigma) X_n(\sigma) W''(\sigma). \end{aligned} \quad (2)$$

The general result, valid also for degenerate kernels, is given in Appendix F.

It turns out that Eq. (2) is valid in the  $m = 1$  case even without requiring that the ISS be unique, and as we show in Appendix D we can bound the deviation from the ideal adiabatic evolution in general as

$$\|[\mathcal{E}(s) - V(s)]P(0)\| \leq C/T. \quad (3)$$

A similar result has been derived in [22] where, however, the constant  $C$  is left undetermined. We show that the constant  $C$ , independent of  $T$ , can be taken to be

$$\begin{aligned} C &= \|S(s)\| \|P'(s)\| + \|S(0)\| \|P'(0)\| \\ &\quad + \sup_{\sigma \in [0, s]} \|[S'P' + SP''](\sigma)\|. \end{aligned} \quad (4)$$

Following, we discuss how  $C$  relates to the physical parameters of the model associated with  $\mathcal{L}$ . *Inequality (3) states the QAT for open systems and implies the QAT in the standard form for states.*

To see the latter, let us initialize the system in a state  $\tilde{\rho}(0)$  in  $\text{Ker}\mathcal{L}(0)$ , i.e.,  $\tilde{\rho}(0) = P(0)\tilde{\rho}(0)$ . Then  $\tilde{\rho}(s) := V(s)\tilde{\rho}(0)$  is an instantaneous steady state at time  $s$  since  $\mathcal{L}(s)\tilde{\rho}(s) = \mathcal{L}(s)V(s)P(0)\tilde{\rho}(0) = \mathcal{L}(s)P(s)V(s)\tilde{\rho}(0) = 0$ , i.e.,  $\tilde{\rho}(s) \in \text{Ker}\mathcal{L}(s)$ . Under the actual evolution, the state is mapped to  $\rho(s) = \mathcal{E}(s)\tilde{\rho}(0)$  and one has  $\|\rho(s) - \tilde{\rho}(s)\|_1 \leq \|[\mathcal{E}(s) - V(s)]P(0)\| \|\tilde{\rho}(0)\|_1 \leq C/T$ . Namely, if  $T \geq C/\epsilon$  ( $\epsilon > 0$ ), then the system is guaranteed to find itself  $\epsilon$ -close in norm to the instantaneous steady state, at the end of the evolution.

### IV. CLOSED-SYSTEM LIMIT

It is useful to comment on how our result relates to adiabatic theorems for closed systems, described by a system Hamiltonian  $H(s)$  with eigenvalues  $E_n$ . First, if one is interested in initial states belonging to the  $-iE_0$  level, one may simply set  $\mathcal{L}(s) = -i[H(s) - E_0]$ , as our formalism encompasses (with minor modifications) the case where  $\mathcal{L}$  is anti-Hermitian. In this way, one recovers the standard adiabatic theorem for closed system. The relevant gap is given by the eigenvalue closest to  $E_0$  in modulus, i.e.,  $|E_1 - E_0|$ . The bound we obtained for the constant  $C$  in this case is similar to that given in Ref. [10], at least for what concerns the dependence on the gap.

Another possibility is to write  $\mathcal{L}(s) = \mathcal{K}(s) = -i[H(s), \bullet]$ . The eigenvalues of  $\mathcal{K}$  are  $\{-i(E_n - E_m)\}$ . There is a  $\geq d$ -fold degenerate zero eigenvalue arising from  $E_n = E_m$  constituting  $\text{Ker}\mathcal{K}$ . The relevant energy scale is determined by the next eigenvalue which is closest to zero in modulus. This is given by the smallest difference  $|E_n - E_m|$  with  $E_n \neq E_m$  (nonzero since we assume  $d < \infty$  and hence discrete spectra). This is

consistent with the previous result because in  $\text{Ker}\mathcal{K}$  one has the freedom to pick any state  $|n\rangle\langle n|$  leading to a gap  $\min_m |E_n - E_m|$ . In this manner, one obtains an adiabatic theorem for closed systems in the Liouvillian (superoperator) formalism. We discuss the closed-system limit further in Appendix E.

## V. THERMAL BATH

It turns out that the open-system version of the QAT can have additional structure that is absent in the closed-system case. To demonstrate this, we consider the important class of Lindbladians generated by the interaction of a system with a thermal bath, for which we can make the bounds above more specific. As a result of the KMS condition, such Lindbladians satisfy the quantum detailed balance condition [31,32]. This fact has important consequences, namely, (i) the Gibbs state is an ISS, i.e.,  $\mathcal{L}(s)\rho_G(s) = 0$  with  $\rho_G(s) \equiv \exp[-\beta H(s)]/Z$ , where  $Z = \text{Tr} \exp[-\beta H(s)]$  is the partition function; (ii) the generator  $\mathcal{L}(s)$  is normal.

Let us now show how we can relate  $C$  to standard quantities such as the gap and  $H'(s)$  using the assumption of a thermal bath. Assume for simplicity that  $\rho_G(s)$  is the unique state in  $\text{Ker}\mathcal{L}(s)$ . For such thermal baths the projector onto the ISS manifold is  $P(s) = |\rho_G(s)\rangle\langle \mathbb{1}|$ . Then,  $P^{(n)}(s) = |\rho_G^{(n)}(s)\rangle\langle \mathbb{1}|$ , so that  $\|P^{(n)}(s)\| = \|\rho_G^{(n)}(s)\|_1$ . Thus, if  $H(s)$  is bounded with bounded derivatives,  $\|P^{(n)}(s)\|$  is bounded for all  $s$ , and hence  $P'$  and  $P''$  in the constant  $C$  [Eq. (4)] do not introduce any singularities. In addition, since  $\mathcal{L}$  is normal,  $\|S\| = c/\Delta$  where the constant  $c$  depends only on the norm used [33]. Moreover, the identity  $S' = S^2\mathcal{L}'P + P\mathcal{L}'S^2 - S\mathcal{L}'S$  (Appendix G) implies that  $\|S'\| \leq 3\|S\|^2\|\mathcal{L}'\|$ . Thus, from Eq. (4), for thermal baths the dependence on the Liouvillian gap is  $C = O(\Delta_{\min}^{-2})$ . Note that in the absence of the KMS condition, one only has  $C = O(\Delta_{\min}^{-3})$ , implying  $T = O(\Delta_{\min}^{-3})$  as a criterion for adiabaticity in accordance with the closed-system result of Ref. [10]. However, in particular cases the dependence can be even milder. For example, if  $\mathcal{L}(s)$  is a unitary family, i.e.,  $\mathcal{L}(s) = e^{s\mathcal{K}}\mathcal{L}(0)e^{-s\mathcal{K}}$ , with  $\mathcal{K}$  an anti-Hermitian superoperator, one has (Appendix H)  $P' = \mathcal{K}P - P\mathcal{K}$ , so neither  $P'$  (nor  $P''$ ) depends on  $\Delta_{\min}$ . Moreover,  $S' = \mathcal{K}S - S\mathcal{K}$  so that in this case  $C = O(\Delta_{\min}^{-1})$ , as also shown and extensively exploited in [34,35].

In addition, for thermal baths the constants appearing in Eq. (4) bear an explicit dependence on  $H(s)$ . For example, one can show that  $\|\rho_G'(s)\|_1 \leq 2\beta\sqrt{\langle [H'(s)]^2 \rangle_G}$ , where  $\langle \bullet \rangle_G = \text{Tr}[\rho_G \bullet]$  is the thermal average (see Appendix I). This fact has important consequences for adiabatic quantum computation where the complexity of a computation is encoded into  $H(s)$  and depends on the system size  $L$ . In general, we expect  $\|P^{(n)}(s)\|$  to display a stronger divergence with  $L$ , for some  $n$ , e.g., at (positive temperature) phase transition points of  $H(s)$ .<sup>3</sup>

When the gap  $\Delta_{\min}$  is very small and is attained inside the interval  $[0, s]$ , the constant  $C$  is dominated by the third term in Eq. (4), i.e.,  $C \simeq \sup_{\sigma} \|S'(\sigma)P'(\sigma)\|$ . Using the above estimates for  $S'$  and  $P'$  we obtain  $C \lesssim$

$6c^2\beta\|\mathcal{L}'(\sigma)\|_{\max}\sqrt{\langle [H'(\sigma)]^2 \rangle_{G,\max}}\Delta_{\min}^{-2}$ , where the subscript “max” means that the corresponding quantities must be maximized over  $\sigma \in [0, s]$ . In other words, taking  $T \gtrsim c^2\beta\|\mathcal{L}'(\sigma)\|_{\max}\sqrt{\langle [H'(\sigma)]^2 \rangle_{G,\max}}\Delta_{\min}^{-2}/\epsilon$  guarantees adiabaticity up to an error  $O(\epsilon)$  in trace norm.

## VI. CASE OF LEVEL CROSSINGS

The gapped case is typical since a random Lindbladian will have a gap above zero with probability one for all values of  $s$ . However, symmetries may give rise to degeneracies, and so we would like to extend our result and consider the case where a finite number of level crossings with the zero eigenvalue may take place along the path.<sup>4</sup>

Since singularities are only algebraic in the finite-dimensional case, it is reasonable to expect that, in the case of level crossing, one has  $\|[\mathcal{E}(s) - V(s)]P(0)\| \sim 1/T^\eta$  for large  $T$ , with a positive exponent  $\eta < 1$ . We are interested in estimating  $\eta$  for large  $T$ . For definiteness assume that at the level crossings the gap vanishes as  $\Delta_{\min}(s) \simeq v_i(s - s_i^*)^{\alpha_i}$  with some positive exponents  $\alpha_i$ . The analysis is detailed in Appendix J. The final result is

$$\|[\mathcal{E}(s) - V(s)]P(0)\| \leq \sum_{i=1}^N \frac{D_i}{T^{\eta_i}}, \quad \eta_i = \frac{1}{1 + \alpha_i}, \quad (5)$$

where  $D_i$  are positive constants. Clearly, the asymptotic behavior of the right-hand side is dictated by the smallest exponent  $\eta_i$ , i.e., by the largest  $\alpha_i$ , and hence the most divergent of the  $N$  gaps.

## VII. EXAMPLES

We now illustrate our results with a few examples.

### A. Example 1

Let us first consider a time-dependent generalization of the amplitude damping master equation. The Lindbladian is  $\mathcal{L}(s) = \mathcal{K}(s) + \mathcal{L}_0$  with  $\mathcal{K}(s) = -i[H(s), \bullet]$ ,  $H(s) = \mathbf{m}(s) \cdot \boldsymbol{\sigma}$ , and  $\mathcal{L}_0 = 2\gamma[\sigma^- \bullet \sigma^+ - (1/2)\{\sigma^+ \sigma^-, \bullet\}]$  ( $\sigma^\alpha$  Pauli matrices and  $\sigma^\pm = \sigma^x \pm i\sigma^y$ ). The steady-state manifold is one dimensional. The ISS is given by the solution of  $\mathcal{L}(s)\tilde{\rho}(s) = 0$  and is (in the  $\sigma^z$  basis)

$$\tilde{\rho} = \frac{1}{c} \begin{pmatrix} \mathbf{m}^2 - m_z^2 & -m_-(2m_z + i\gamma) \\ -m_+(2m_z - i\gamma) & \mathbf{m}^2 + 3m_z^2 + \gamma^2 \end{pmatrix}, \quad (6)$$

where  $c = 2(\mathbf{m}^2 + m_z^2) + \gamma^2$  and  $m_\pm = m_x \pm im_y$ . A plot of the absolute values of the Lindbladian's eigenvalues is shown in Fig. 1(a). Parameters are chosen to illustrate a phenomenon which is not possible in the unitary case. Namely, the eigenvalues can have algebraic singularities of the form  $(s - s^*)^r$  with  $r$  a noninteger rational number. Also shown, in Fig. 1(b), is the expected scaling of  $\|\rho(1) - \tilde{\rho}(1)\|_1$  as  $C/T$ .

<sup>3</sup>In general, one can directly relate  $\|P^{(n)}(s)\|$  to expectation values of powers of the Hamiltonian and its derivatives.

<sup>4</sup>A generalization where the spectrum becomes continuous at some point (e.g., at a second-order quantum phase transition) is possible in the unitary case [8]. Such an extension is not considered here.

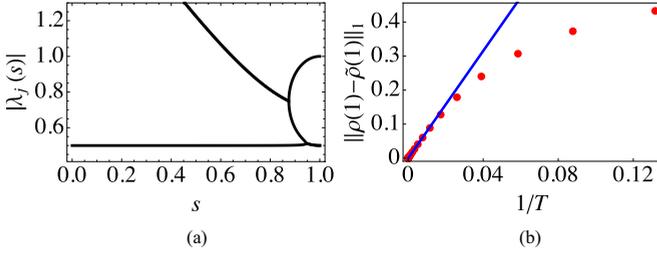


FIG. 1. QAT for the gapped case using the model of Example 1. (a) Absolute values of the eigenvalues of the Lindbladian (zero is not shown). The eigenvalues show square-root singularities at  $s \simeq 0.88$  and  $0.94$ . (b) Trace norm distance between the actual state and the ISS for increasing  $T$ . The blue line is given by  $\|\rho(1) - \tilde{\rho}(1)\|_1 \sim 7.88/T$ . Parameters:  $m_x(s) = 1 - s$ ,  $m_y(s) = 0$ ,  $m_z(s) = s/150$ ,  $\gamma = \frac{1}{2}$  arb. units Initial condition is  $\tilde{\rho}(0)$  as given in Eq. (6).

### B. Example 2

Consider a qubit with system Hamiltonian  $H(s) = \mathbf{m}(s) \cdot \boldsymbol{\sigma}$  interacting with a heat bath at inverse temperature  $\beta$ , so that the total Hamiltonian is  $H_{\text{tot}}(t) = H(t) + H_{\text{int}} + H_B$ , with  $H_{\text{int}} = A \otimes B$ , where  $A$  ( $B$ ) is a system (bath) operator, and  $H_B$  is the bath Hamiltonian. For  $H(s)$  of the form of Landau-Zener driving, exact expressions for the transition probabilities for a particular zero-temperature bosonic bath have been obtained in Ref. [36]. To treat the more general case we use the time-dependent Lindblad master-equation approximation for a slowly varying system Hamiltonian [15]:  $\mathcal{L}_{\text{tot}}(t) = \mathcal{K}(t) + \mathcal{L}(t)$ , where  $\mathcal{K}(t) = -i[H(t) + H_{\text{LS}}(t), \bullet]$ .

The dissipative part, specialized to the single-qubit case, reads as

$$\mathcal{L}(t) = \sum_{\omega} \gamma(\omega) \left[ A_{\omega}(t) \bullet A_{\omega}(t)^{\dagger} - \frac{1}{2} \{A_{\omega}(t)^{\dagger} A_{\omega}(t), \bullet\} \right], \quad (7)$$

with rates  $\gamma(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\tau\omega} \langle e^{i\tau H_B} B e^{-i\tau H_B} B \rangle$ . The Lindblad operators are given by the Fourier resolution  $e^{i\tau H(t)} A e^{-i\tau H(t)} = \sum_{\omega} e^{i\omega\tau} A_{\omega}(t)$ , where  $\omega$  are the Bohr frequencies of  $H(t)$ .  $H_{\text{LS}}(t) = \sum_{\omega} S(\omega) A_{\omega}(t)^{\dagger} A_{\omega}(t)$  is the Lamb shift Hamiltonian, with  $S(\omega) = \int_{-\infty}^{\infty} d\omega' \gamma(\omega') \mathcal{P}(\frac{1}{\omega - \omega'})$ , where  $\mathcal{P}$  denotes the principal value. Moreover,  $\gamma_{\alpha}(-\omega) = e^{-\beta\omega} \gamma_{\alpha}(\omega)$  as a consequence of the KMS condition. This implies that the generator  $\mathcal{L}(t)$  together with  $\mathcal{K}(t)$  and  $\mathcal{K}_{\text{LS}}(t) = -i[H_{\text{LS}}(t), \bullet]$  all commute at the same time.

The corresponding Lindbladian has the following instantaneous eigenvalues (derived in Appendix K):

$$\lambda = \{0, -\gamma(\delta)|A_{01}|^2(1 + e^{-\beta\delta}), -\Gamma \pm i\mu\}, \quad (8)$$

with  $2\Gamma = \gamma(0)(|A_{00}|^2 + |A_{11}|^2) + \gamma(\delta)|A_{01}|^2(1 + e^{-\beta\delta})$  and  $\mu = \delta - S(0)(|A_{00}|^2 - |A_{11}|^2) + |A_{01}|^2[S(\delta) - S(-\delta)]$ , where  $\delta = 2\|\mathbf{m}\|$  is the instantaneous Hamiltonian gap,  $A_{ab} = \langle \epsilon_a | A | \epsilon_b \rangle$ , and  $|\epsilon_a\rangle$ ,  $a = \{0, 1\}$ , denotes the instantaneous eigenvectors of  $H(s)$ .

We now illustrate the QAT for  $H(s) = \omega_x(1-s)\sigma^x + \omega_z s\sigma^z$  and system operator  $A = g\sigma^{y,z}$ , where  $g$  is a coupling constant. For  $A = g\sigma^y$ , there is always a gap in the Liouvillian spectrum above the zero eigenvalue. Correspondingly, the decay is  $T^{-1}$  as shown in Fig. 2(a). For  $A = g\sigma^z$ , the system Hamiltonian commutes with  $A$  when  $s = s^* = 1$ , at which point the spectrum becomes degenerate. Indeed, one can check

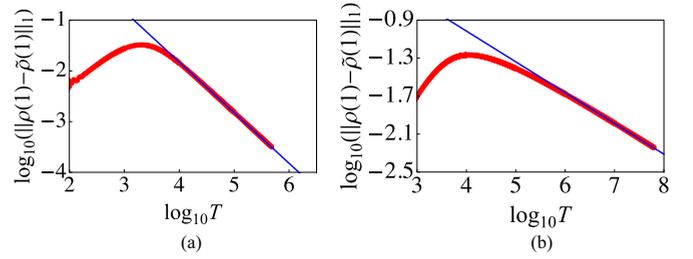


FIG. 2. Illustration of the QAT for a single qubit coupled to a thermal bath (Example 2). (a) Gapped case with  $A = g\sigma^y$ . The fit (blue line) gives  $\|\rho(1) - \tilde{\rho}(1)\|_1 = 148.5/T^{0.9990}$ . (b) Gapless case with  $A = g\sigma^z$ . Now the fit gives  $\|\rho(1) - \tilde{\rho}(1)\|_1 = 1.910/T^{0.324}$ . Fits are for  $T \geq 10^{5.4}$ . Parameters:  $g = 10^{-2}$ ,  $\omega_x = \omega_z = -\frac{1}{2}$  arb. units,  $\beta = 1$  arb. units,  $\gamma(\omega) = \frac{2\pi\omega e^{-|\omega|/8\pi}}{1 - e^{-\beta\omega}}$ .

that  $|A_{01}|^2 = \omega_x^2(1-s)^2/\|\mathbf{m}(s)\|^2$  and, correspondingly, from Eq. (8), the second eigenvalue goes to zero quadratically in  $s - s^*$ :  $\lambda_2(s) \simeq -\gamma(\delta)(\omega_x/\omega_z)^2(1-s)^2(1 + e^{-\beta\delta})$ . This corresponds to  $\alpha = 2$  and hence [recall Eq. (5)] an exponent  $\eta = \frac{1}{3}$ . Our numerical simulations are in agreement with this prediction, as seen in Fig. 2(b). Note that the Hamiltonian gap  $\delta$  enters only indirectly via  $\lambda$ .

## VIII. CONCLUSIONS

Using an adiabatic expansion we have extended the adiabatic theorem of quantum mechanics to open systems described by a time-dependent master equation with generator  $\mathcal{L}(t)$  in Lindblad form. The theorem, first proven using different methods in [20–22], states that if one initializes the system in  $\text{Ker}\mathcal{L}(0)$  and a gap condition is satisfied, the evolution brings the system close to  $\text{Ker}\mathcal{L}(T)$  up to an error  $C/T$ , where  $C$  is a constant and  $T$  the total time. Our approach allowed us to extend the results of [20–22] in two directions particularly relevant for quantum state preparation and quantum annealing in open systems. On the one hand, we related the constant  $C$  to the smallest (in absolute value) gap  $\Delta_{\text{min}}$  of the Liouvillian. For general Liouvillians, we obtained  $C = O(\Delta_{\text{min}}^{-3})$ , whereas for thermal baths satisfying the KMS condition we found an improved scaling  $C = O(\Delta_{\text{min}}^{-2})$ . More precisely, we showed that taking  $T \gtrsim \beta\|\mathcal{L}'(\sigma)\|_{\text{max}}\sqrt{([H'(\sigma)]^2)_{G,\text{max}}}\Delta_{\text{min}}^{-2}/\epsilon$  guarantees adiabaticity up to an error  $O(\epsilon)$  in trace norm. On the other hand, we extended previous results to the case of level crossing, for which the error becomes  $O(T^{-\eta})$  with an exponent  $\eta$  that depends on the rate of the gap closing. Thus, level crossings with the instantaneous steady state can slow convergence down. We provided several examples to illustrate our findings, which confirm the predicted scaling with  $T$ . An interesting open question is whether the growing body of techniques developed for bath engineering [37–39] can be used to enact boundary cancellation methods and reduce the error to  $O(T^{-n})$  with controllable  $n > 1$ , as in the closed-system case [11,40–42]. Our results have implications for adiabatic quantum computation and quantum annealing in the presence of dissipation, where the closed-system adiabatic theorem cannot be directly applied.

**ACKNOWLEDGMENTS**

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**APPENDIX A: PROOF THAT ZERO IS A SEMISIMPLE EIGENVALUE OF  $\mathcal{L}(s)$** 

We assume in this section that  $\mathcal{L}(s)$  is finite dimensional. We show that zero is a semisimple eigenvalue of  $\mathcal{L}(s)$ , i.e., that there are no idempotents in the Jordan block decomposition of  $\mathcal{L}$  corresponding to its zero eigenvalue. Assume the contrary, i.e., that one can write  $\mathcal{L}(s) = 0 \times P(s) + D(s) + R(s)$  where  $D(s)$  is nilpotent, i.e.,  $D(s)^m = 0$ ,  $D(s)^{m-1} \neq 0$ , and  $R(s)P(s) = P(s)R(s) = 0$ . Then,

$$e^{t\mathcal{L}(s)}P(s) = P(s) + \sum_{n=1}^{m-1} \frac{t^n D(s)^n}{n!}, \quad (\text{A1})$$

where we also used that  $D(s)P(s) = P(s)D(s) = D(s)$ . By assumption,  $\mathcal{L}(s)$  is Markovian, which means that for each  $s$  and  $t > 0$ ,  $e^{t\mathcal{L}(s)}$  is CPTP and so  $\|e^{t\mathcal{L}(s)}\| = 1$ . Taking the norm of both sides of Eq. (A1) we obtain that the left-hand side is bounded while the right-hand side grows unboundedly with  $t$ , which is a contradiction. Hence,  $D(s) = 0$  and  $\mathcal{L}(s)P(s) = P(s)\mathcal{L}(s) = 0$ . For an alternative proof see, e.g., Ref. [43] or Proposition 5 of Ref. [22].

Note that when the generator  $\mathcal{L}(s)$  is normal (i.e.,  $[\mathcal{L}(s), \mathcal{L}^*(s)] = 0$ , where  $*$  denotes the adjoint with respect to the scalar product  $\langle X, Y \rangle_G := \text{Tr}[\rho_G(s)X^\dagger Y]$ ), as is the case for a thermal bath, this has the pleasant consequence that all the eigenprojectors of  $\mathcal{L}(s)$  are bounded (in fact have norm one) and there are no nilpotent terms in its Jordan block decomposition.

**APPENDIX B: PROOF THAT  $P(s)V(s) = V(s)P(0)$** 

Recall that we defined the intertwiner to be the solution of the following differential equation:  $V'(s) = [P'(s), P(s)]V(s)$ . To check that  $V(s)$  satisfies the intertwining relation, define  $W(s) = P(s)V(s)$ . From now on, we drop the  $s$  dependence when not explicitly needed. Differentiating  $P^2 = P$  gives  $P' = P'P + PP'$ , and after right multiplying by  $P$  gives  $PP'P = 0$  and likewise  $QQ'Q = 0$  [which also implies  $QP'Q = 0$  since  $0 = Q\mathbb{1}'Q = Q(P' + Q')Q$ ]. Thus,  $[[P', P], P] = P'P - 2PP'P + PP' = P'$ . Using this, note that  $W' = P'V + PV' = (P' + P[P', P])V = [P', P]PV = [P', P]W$ , which is the same differential equation as the one satisfied by  $V$ . Since  $W$  and  $V$  differ in their initial condition, i.e.,  $V(0) = \mathbb{1}$  and  $W(0) = P(0)$ , it follows that  $W(s) = V(s)P(0)$ , i.e.,

$$W(s) = P(s)V(s) = V(s)P(0). \quad (\text{B1})$$

**APPENDIX C: COMPLETE POSITIVITY OF  $V(s)P(0)$ , AND LACK OF POSITIVITY OF  $V(s)$** 

Here, we show that  $W(s) = V(s)P(0)$  is a CPTP map even though the intertwiner  $V(s)$  itself is not in general. From

Appendix B, we obtain

$$\begin{aligned} W' &= [P', P]W \\ &= P'PW - PP'W = P'PW - PP'PW \\ &= P'PW = P'W. \end{aligned} \quad (\text{C1})$$

Now, we write the solution of the differential equation using the method of Euler lines [44,45], i.e.,  $W(s) = \lim_{N \rightarrow \infty} [\mathbb{1} + \epsilon P'(s - \epsilon)] \dots [\mathbb{1} + \epsilon P'(0)]P(0)$ , with  $\epsilon = s/N$ . Using  $P(s)^2 = P(s)$  we get, in the  $\epsilon \rightarrow 0$  limit,

$$[\mathbb{1} + \epsilon P'(0)]P(0) = [P(0) + \epsilon P'(0)]P(0) = P(\epsilon)P(0),$$

where the second equality is up to first order in  $\epsilon$ . Then (with the same notation),

$$[\mathbb{1} + \epsilon P'(\epsilon)]P(\epsilon) = [P(\epsilon) + \epsilon P'(\epsilon)]P(\epsilon) = P(2\epsilon)P(\epsilon),$$

etc., until  $W(s) = \lim_{N \rightarrow \infty} P(s) \dots P(2\epsilon)P(\epsilon)P(0)$ . In this form,  $W(s)$  is an infinite product of CPTP maps, so it is a CPTP map itself. It follows from submultiplicativity of the norm [46] that  $\|W(s)\| = \|V(s)P(0)\| \leq 1$ .

Next, let us demonstrate that  $V(s)$ , in general, is not even positive. Consider, e.g., the case where  $P(s)$  is one dimensional, so that  $P(s) = |\rho(s)\rangle\langle\mathbb{1}|$ , where  $\langle x|$  is the adjoint of  $|x\rangle$  with respect to the Hilbert-Schmidt scalar product  $\langle A|B\rangle = \text{Tr}[A^\dagger B]$ . We assume that  $\rho(s)$  is differentiable. Then,  $P'(s) = |\rho'(s)\rangle\langle\mathbb{1}|$  and so  $P'P = P'$  while  $PP' = 0$  since  $\rho'(s)$  is traceless. Thus,  $V' = [P', P]V$  becomes  $V' = P'V$  [with  $V(0) = \mathbb{1}$ ], with the solution

$$V(s) = \text{Texp}\left[\int_0^s P'(\sigma)d\sigma\right], \quad (\text{C2})$$

where  $\text{Texp}$  denotes the time-ordered exponential. However,  $P'(s_1)P'(s_2) = P'(s_2)P'(s_1) = 0$  [again, since  $\rho'(s)$  is traceless] so the time-ordered exponential in the above equation reduces to a standard exponential. The solution is then

$$\begin{aligned} V(s) &= \exp\{[|\rho(s)\rangle - |\rho(0)\rangle]\langle\mathbb{1}|\} \\ &= \mathbb{1} + [|\rho(s)\rangle - |\rho(0)\rangle]\langle\mathbb{1}|, \end{aligned} \quad (\text{C3})$$

where in the second line we used the fact that  $\{[|\rho(s)\rangle - |\rho(0)\rangle]\langle\mathbb{1}|\}^2 = 0$ . Now, it is easy to see that, unless  $\rho(s) = \rho(0)$  such that  $V(s) = \mathbb{1}$ ,  $V(s)$  does not even preserve positivity. Let us define, for clarity,  $\delta\rho := \rho(s) - \rho(0)$ . This is a Hermitian and traceless operator, so it must have a negative eigenvalue, i.e., there exist a vector  $|\alpha\rangle$  and a real number  $\alpha > 0$ , such that  $\delta\rho|\alpha\rangle = -\alpha|\alpha\rangle$ . Let us consider the following state:

$$x_0 := \lambda \frac{\mathbb{1}}{d} + (1 - \lambda)|\alpha^\perp\rangle\langle\alpha^\perp|, \quad (\text{C4})$$

where  $d$  is the dimension of the ISS,  $|\alpha^\perp\rangle$  is a vector satisfying  $\langle\alpha|\alpha^\perp\rangle = 0$  and  $\lambda \in [0, 1]$ . It is clear that  $x_0$  is a positive operator. However,

$$V(s)x_0|\alpha\rangle = \left(\frac{\lambda}{d} - \alpha\right)|\alpha\rangle, \quad (\text{C5})$$

such that, for  $\lambda < \min(\alpha d, 1)$ ,  $V(s)x_0$  has a negative eigenvalue, i.e.,  $V(s)$  does not preserve positivity.

**APPENDIX D: PROOF OF EQ. (2)**

We assume that  $\mathcal{L}$  is  $m$ -fold differentiable.

### 1. The $m = 1$ case

The proof of Eq. (2) of the main text for  $m = 1$  closely follows the classic reference [4]. As above, let  $P(s)$  be the projector onto the instantaneous, zero eigenvalue of  $\mathcal{L}(s)$ . In this section, we do not require the ISS to be unique. Since we established in Appendix A that  $\mathcal{L}P = P\mathcal{L} = 0$  (from now on we omit the explicit dependence on  $s$  if not strictly needed), it follows that  $\mathcal{L} = Q\mathcal{L} = \mathcal{L}Q$ . Therefore, the reduced resolvent  $S = \lim_{z \rightarrow 0} Q(\mathcal{L} - z)^{-1}Q$  satisfies  $S\mathcal{L} = \lim_{z \rightarrow 0} Q(\mathcal{L} - z)^{-1}Q\mathcal{L} = Q\lim_{z \rightarrow 0}(\mathcal{L} - z)^{-1}\mathcal{L} = Q$ . To summarize,

$$\mathcal{L}S = S\mathcal{L} = Q, \quad (\text{D1a})$$

$$SP = PS = 0. \quad (\text{D1b})$$

It follows from Eq. (C1) and  $PP'P = 0$  that  $PW' = 0$  and so

$$W' = QW'. \quad (\text{D2})$$

Recall [Eq. (1) of the main text] that the evolution operator  $\mathcal{E}$  satisfies  $\mathcal{E}' = T\mathcal{L}\mathcal{E}$ . Now,  $0 = (\mathcal{E}\mathcal{E}^{-1})' = \mathcal{E}'\mathcal{E}^{-1} + \mathcal{E}(\mathcal{E}^{-1})'$ , so that

$$(\mathcal{E}^{-1})' = -T\mathcal{E}^{-1}\mathcal{L}. \quad (\text{D3})$$

Therefore,  $(\mathcal{E}^{-1})'W = -T\mathcal{E}^{-1}\mathcal{L}PV = 0$  since  $\mathcal{L}P = 0$ . Hence,  $(\mathcal{E}^{-1}W)' = \mathcal{E}^{-1}W'$ , which integrated over  $[0, s]$  gives

$$\mathcal{E}^{-1}(s)W(s)P(0) - P(0) = \int_0^s \mathcal{E}^{-1}(\sigma)W'(\sigma)d\sigma. \quad (\text{D4})$$

Now, we use

$$\begin{aligned} \mathcal{E}^{-1}W' &= \mathcal{E}^{-1}QW' \\ &= \mathcal{E}^{-1}\mathcal{L}SW' \\ &= -T^{-1}(\mathcal{E}^{-1})'SW', \end{aligned} \quad (\text{D5})$$

where we used Eqs. (D2), (D1a), and (D3). We plug this into Eq. (D4) and integrate by parts, to obtain

$$\begin{aligned} \mathcal{E}^{-1}(s)W(s)P(0) &= \frac{1}{T} \left\{ \int_0^s \mathcal{E}^{-1}(\sigma)[S(\sigma)W'(\sigma)]'d\sigma - (\mathcal{E}^{-1}SW')|_0^s \right\}. \end{aligned} \quad (\text{D6})$$

Now, we act with  $\mathcal{E}(s)$  from the left and use the property  $\mathcal{E}(a,b)\mathcal{E}(b,0) = \mathcal{E}(a,0)$  to obtain

$$\begin{aligned} [\mathcal{E}(s) - V(s)]P(0) &= \frac{1}{T} \mathcal{E}(s,\sigma)S(\sigma)W'(\sigma)|_0^s \\ &\quad - \frac{1}{T} \int_0^s d\sigma \mathcal{E}(s,\sigma)(S'W' + SW'')(\sigma). \end{aligned} \quad (\text{D7})$$

At this point recall that  $W' = P'W$  [Eq. (C1)] and hence [using Eq. (B1)]  $W'(s) = P'(s)V(s)P(0)$ , and also  $W'' = [P'' + (P')^2]W$ . It follows from the latter, together with  $S = SQ$  [from Eq. (D1b)] and  $W = PW$  [from Eq. (B1)], that  $SW'' = SQ[P'' + (P')^2]PW$ . But,  $Q(P')^2P = QP'(P + Q)P'P = 0$  because  $PP'P = 0$  and  $QP'Q = 0$  (see Appendix B), so

that  $SW'' = SP'W$ . Collecting all these results we finally obtain

$$\begin{aligned} &[\mathcal{E}(s) - V(s)]P(0) \\ &= \frac{1}{T} \left\{ [S(s)P'(s)V(s)P(0) - \mathcal{E}(s)S(0)P'(0)P(0)] \right. \\ &\quad \left. - \int_0^s d\sigma \mathcal{E}(s,\sigma)[S'P' + SP''](\sigma)V(\sigma)P(0) \right\}. \end{aligned} \quad (\text{D8})$$

Using submultiplicativity along with  $\|P(s)\| = 1$  and  $\|V(s)P(0)\| \leq 1$  (as shown in Appendix C) and  $\|\mathcal{E}(s)\| \leq 1$  (see immediately below) we obtain

$$\|[\mathcal{E}(s) - V(s)]P(0)\| \leq C/T \quad (\text{D9})$$

with

$$\begin{aligned} C &= \|S(s)\| \|P'(s)\| + \|S(0)\| \|P'(0)\| \\ &\quad + \sup_{\sigma \in [0,s]} \|[S'P' + SP''](\sigma)\|, \end{aligned} \quad (\text{D10})$$

as stated in Eqs. (3) and (4) of the main text.

Note that when restricted to acting on normalized states, clearly  $\|\mathcal{E}\| = 1$  by trace preservation. More generally, since  $\mathcal{L}(s)$  is a Lindbladian we have  $\|\exp[t\mathcal{L}(s)]\| \leq 1$  for each fixed  $s$  and  $t > 0$  [i.e.,  $\mathcal{L}(s)$  generates a contraction for any fixed  $s$ ]. Using the method of Euler lines (see Appendix C) one can show that this implies that  $\|\mathcal{E}(s,t)\| \leq 1$  because  $\mathcal{E}(s,t)$  can be written as an infinite product of evolutions of the form  $\exp[t\mathcal{L}(s_i)]$ .

### 2. Extension to arbitrary order

Next, we show how to extend the integration by parts technique of the previous subsection to arbitrary order  $m$ . A particularly simple series will result under the assumption that the ISS is unique, i.e.,  $P(s) = |\rho(s)\rangle\langle\mathbb{1}|$ , as in Appendix C (this assumption is not essential, but it significantly simplifies our calculations below). As before, we also assume that  $\mathcal{L}$  generates a trace-preserving map. Both of the latter two assumptions are usually satisfied for thermal (i.e., Davies) generators.

We begin by considering the first term of Eq. (D6), i.e.,  $(SW')' = S'W' + SW''$ . Our strategy will be to repeat the integration by parts on the term arising from  $W'$ , while keeping (i.e., not integrating by parts) the term arising from  $W''$ . We will show how this can be done repeatedly and thus extend the result to arbitrary order.

Now, thanks to the assumption of uniqueness  $P(s) = |\rho(s)\rangle\langle\mathbb{1}|$ . Hence, from  $PS = 0$  we obtain  $\langle\mathbb{1}|S = 0$  and so  $P'S = |\rho'(s)\rangle\langle\mathbb{1}|S = 0$ . Using  $P'S + PS' = 0$  we get  $PS' = 0$  so that writing  $S'Q = (P + Q)S'Q = P'S'Q + QS'Q$  it thus follows that

$$S'Q = QS'Q. \quad (\text{D11})$$

This means that, using Eq. (D2), we can write

$$S'W' = S'QW' = QS'W'. \quad (\text{D12})$$

Returning to the term of interest in Eq. (D6), we have

$$\mathcal{E}^{-1}S'W' = \mathcal{E}^{-1}QS'W' \quad (\text{D13a})$$

$$= \mathcal{E}^{-1}\mathcal{L}SS'W' \quad (\text{D13b})$$

$$= -T^{-1}(\mathcal{E}^{-1})'SS'W', \quad (\text{D13c})$$

where in the last equality we used Eq. (D3). This term can now be integrated by parts again, in analogy to the term in Eq. (D5). Differentiating  $SS'W'$ , we obtain two terms, one of which is  $SS'W''$ , which we keep. The other is  $(SS')'W'$ , which we would like to integrate by parts again (because it diverges more strongly when  $\Delta_{\min} \rightarrow 0$ ). We now show that this procedure can be iterated to any order provided  $\mathcal{L}$  is differentiable sufficiently many times.

Let us assume that at order  $n$  we obtained the term

$$\frac{1}{T^n} \int_0^s d\sigma \mathcal{E}^{-1}X'_n W'. \quad (\text{D14})$$

Assume for the moment that  $X'_n W' = QX'_n W'$ ; we will show shortly that this is legitimate. In this case, we can repeat the calculation of Eq. (D13) and write  $\mathcal{E}^{-1}X'_n W' = -T^{-1}(\mathcal{E}^{-1})'SX'_n W'$ . We can thus use the integration by parts trick:

$$\begin{aligned} & \frac{1}{T^n} \int_0^s d\sigma \mathcal{E}^{-1}X'_n W' \\ &= -\frac{1}{T^{n+1}} \int_0^s d\sigma (\mathcal{E}^{-1})'SX'_n W' \\ &= -\frac{1}{T^{n+1}} \mathcal{E}^{-1}SX'_n W'|_0^s + \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1}(SX'_n W')' \\ &= -\frac{1}{T^{n+1}} \mathcal{E}^{-1}SX'_n W'|_0^s + \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1}SX'_n W'' \\ & \quad + \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1}(SX'_n)'W'. \end{aligned} \quad (\text{D15})$$

The first integral (with  $W''$ ) is the one we keep, while the second (with  $W'$ ) is the one we continue to integrate by parts. This shows that the process can be iterated with  $X_{n+1} = SX'_n$ .

We now verify that we can plug in a  $Q$  term at each order as claimed above. We prove it by induction. Assume that  $X'_n W' = QX'_n W'$ . We wish to show that this implies that  $X'_{n+1} W' = QX'_{n+1} W'$ . But, this is clear since

$$\begin{aligned} X'_{n+1} W' &= (S'X'_n + SX''_n)W' \\ &= (S'QX'_n + QSX''_n)QW' \\ &= Q(S'QX'_n + SX''_n)QW' \\ &= Q(S'X'_n + SX''_n)W' = QX'_{n+1} W', \end{aligned} \quad (\text{D16})$$

where we used the induction hypothesis, Eqs. (D2) and (D11). We have already shown that the claim holds for  $n = 1$  [Eq. (D12) for  $X_1 = S$ ] so we are done.

We thus obtain

$$\mathcal{E}^{-1}(s)W(s) - P(0) = \sum_{n=1}^m \frac{\Gamma_n}{T^n} + \frac{1}{T^m} \int_0^s d\sigma \mathcal{E}^{-1}X'_m W', \quad (\text{D17})$$

$$\Gamma_n = -\mathcal{E}^{-1}X_n W'|_0^s + \int_0^s d\sigma \mathcal{E}^{-1}X_n W'', \quad (\text{D18})$$

$$X_{n+1} = SX'_n, \quad X_1 = S. \quad (\text{D19})$$

Multiplying from the left by  $\mathcal{E}(s)$  we obtain

$$\begin{aligned} & [\mathcal{E}(s) - V(s)]P(0) \\ &= \sum_{n=1}^m \frac{\Omega_n}{T^n} - \frac{1}{T^m} \int_0^s d\sigma \mathcal{E}(s, \sigma)X'_m(\sigma)W''(\sigma), \quad (\text{D20}) \\ \Omega_n &= \mathcal{E}(s, \sigma)X_n(\sigma)W'(\sigma)|_0^s - \int_0^s d\sigma \mathcal{E}(s, \sigma)X_n(\sigma)W''(\sigma). \end{aligned} \quad (\text{D21})$$

If  $\mathcal{L}$  is differentiable infinitely many times, we can write

$$[\mathcal{E}(s) - V(s)]P(0) = \sum_{n=1}^{\infty} \frac{\Omega_n}{T^n} \quad (\text{D22})$$

under the assumption of convergence. We expect this assumption to hold in the finite-dimensional case if  $\mathcal{L}$  depends analytically on  $s$ .

#### APPENDIX E: ADDITIONAL REMARKS ON THE CLOSED-SYSTEM LIMIT

The closed-system limit may also be achieved by considering  $\mathcal{L}_x(s) = \mathcal{K}(s) + x\mathcal{D}(s)$ , where  $\mathcal{D}(s)$  is a dissipative generator, and taking the (weak coupling) limit  $x \rightarrow 0$ . This is a singular limit: for  $x \neq 0$  there is typically only one steady state, whereas for  $x = 0$  there is a large degeneracy ( $\geq d$  Hilbert's space dimension). Calling  $P(x)$  the spectral projection onto the zero eigenvalue for fixed  $s$ , one has  $\lim_{x \rightarrow 0} P(x) \neq P(0)$  since the left-hand side has rank one, whereas  $P(0)$  has rank  $\geq d$ . Moreover, for sufficiently small  $x$  the smallest gap will be  $\propto x$  and hence also  $(1/\text{gap})^\eta$  ( $\eta > 0$ ) will diverge. Thus, the constant  $C$  in Eq. (4), and even the optimal constant  $C_{\text{opt}}$  [obtained taking the infimum of all the constants  $C$  satisfying the bound Eq. (3) for all  $T$ ] will, in general, likely be large. In other words, in general the limit  $x \rightarrow 0$  is singular and the adiabatic theorem for closed system cannot be recovered in this form. A possible way to recover it is to abandon projectors and move to states, at least in the case of nondegenerate ISS for  $x \neq 0$ . In other words, let us call  $\rho_x(s)$  the state evolved with generator  $\mathcal{L}_x(s)$  from  $\sigma_x(0)$  [ $\mathcal{L}_x(s)\sigma_x(s) = 0$  is assumed to be unique for  $x \neq 0$ ]. Now, it is possible that  $\rho_x(s)$  and  $\sigma_x(s)$  have a well-defined limit as  $x \rightarrow 0$  such that the following bound

$$\|\rho_x(s) - \sigma_x(s)\| \leq \tilde{C}_x/T \quad (\text{E1})$$

may admit a nontrivial limit for  $x \rightarrow 0$  with  $\lim_{x \rightarrow 0} \tilde{C}_x < \infty$ . This may be achieved by applying Eq. (D8) to  $\sigma_x(0)$  and bounding the resulting expression.

#### APPENDIX F: EXTENSION TO THE CASE OF A DEGENERATE KERNEL

If the kernel of  $\mathcal{L}(s)$  is not one dimensional, in general we will also have a  $P$ - $Q$  block in  $X_n$ . The procedure, however,

can still be iterated. Assume that at order  $n$  we obtained

$$\begin{aligned} & \frac{1}{T^n} \int_0^s d\sigma \mathcal{E}^{-1} X_n W' \\ &= \frac{1}{T^n} \int_0^s d\sigma [\mathcal{E}^{-1} P X_n Q W' + \mathcal{E}^{-1} Q X_n Q W']. \end{aligned} \quad (\text{F1})$$

We keep the first term (we will have to show later that it scales nicely), and integrate the second by parts, for which we use  $\mathcal{E}^{-1} Q X_n Q W' = \mathcal{E}^{-1} \mathcal{L} S X_n Q W' = -T^{-1} (\mathcal{E}^{-1})' S X_n W'$ . The above equation becomes

$$\begin{aligned} & \frac{1}{T^n} \int_0^s d\sigma \mathcal{E}^{-1} P X_n Q W' - \frac{1}{T^{n+1}} \mathcal{E}^{-1} S X_n W' \Big|_0^s \\ &+ \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1} (S X_n W')' \end{aligned} \quad (\text{F2})$$

$$\begin{aligned} &= \frac{1}{T^n} \int_0^s d\sigma \mathcal{E}^{-1} P X_n Q W' - \frac{1}{T^{n+1}} \mathcal{E}^{-1} S X_n W' \Big|_0^s \\ &+ \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1} (S X_n) W'' \\ &+ \frac{1}{T^{n+1}} \int_0^s d\sigma \mathcal{E}^{-1} (S X_n)' W', \end{aligned} \quad (\text{F3})$$

from which we see that the recurrence is  $X_{n+1} = (S X_n)'$  with  $X_0 = \mathbb{1}$  as previously. However, now the series reads as

$$\mathcal{E}^{-1}(s) W(s) - P(0) = \sum_{n=1}^{\infty} \frac{\Gamma_n}{T^n}, \quad (\text{F4})$$

$$\begin{aligned} \Gamma_n &= -\mathcal{E}^{-1} X_n W' \Big|_0^s + \int_0^s d\sigma \mathcal{E}^{-1} X_n W'' \\ &+ \int_0^s d\sigma \mathcal{E}^{-1} P X_n' Q W', \end{aligned} \quad (\text{F5})$$

with  $X_{n+1} = S(X_n)'$ ,  $X_1 = S$ . Multiplying from the left by  $\mathcal{E}(s)$  we obtain

$$[\mathcal{E}(s) - V(s)] P(0) = \sum_{n=1}^{\infty} \frac{\Omega_n}{T^n}, \quad (\text{F6})$$

$$\begin{aligned} \Omega_n &= \mathcal{E}(s, \sigma) X_n(\sigma) W'(\sigma) \Big|_0^s - \int_0^s d\sigma \mathcal{E}(s, \sigma) X_n(\sigma) W''(\sigma) \\ &- \int_0^s d\sigma \mathcal{E}(s, \sigma) [P X_n' Q W'](\sigma). \end{aligned} \quad (\text{F7})$$

In order to repeat the arguments for the case of level crossings, we now have to assess the scaling of  $P X_n' Q$  as one eigenvalue goes to zero. We have separately shown that  $X_n \sim 1/\delta^{\beta_n}$  with  $\beta_n = n\alpha + n - 1$ . Now,  $X_n = S(X_{n-1})'$ , so

$$P X_n' Q = P [S(X_{n-1})]' Q \quad (\text{F8})$$

$$= P S' X_{n-1} Q \quad (\text{F9})$$

$$= P S' Q X_{n-1}' Q, \quad (\text{F10})$$

where the first equation holds because  $PS = 0$  and the second because  $PS'P = 0$ . Now, as we know,  $PS'Q = -P'S$  introduces only a power  $1/\delta^\alpha$ . On the other hand, we know

that  $X_{n-1}' \sim 1/\delta^{\beta_{n-1}+1}$ . Hence,

$$P X_n' Q \sim \frac{1}{\delta^{\beta_{n-1}+1+\alpha}} = \frac{1}{\delta^{\beta_n}}. \quad (\text{F11})$$

We see hence that this potentially more dangerous term has the same scaling as  $X_n$ . In other words, all our earlier conclusions also hold for the case with a degenerate kernel.

### APPENDIX G: PROOF OF AN IDENTITY FOR $S'$

Since  $S = SQ$  [Eq. (D1b)] we have  $S' = S'Q + SQ'$ . Note that

$$SQ' = -SP' = -SQP' = -S^2\mathcal{L}P' = S^2\mathcal{L}'P, \quad (\text{G1})$$

where we used that  $S\mathcal{L} = Q$  and  $\mathcal{L}P' = -\mathcal{L}'P$ . Differentiating  $S\mathcal{L} = Q$  yields  $S'\mathcal{L} + S\mathcal{L}' = -P'$ . Multiplying from the right by  $S$  we obtain

$$S'Q = -P'S - S\mathcal{L}'S. \quad (\text{G2})$$

Therefore, we have

$$\begin{aligned} S'Q &= -P'QS - S\mathcal{L}'S \\ &= -P'\mathcal{L}S^2 - S\mathcal{L}'S = P\mathcal{L}'S^2 - S\mathcal{L}'S. \end{aligned} \quad (\text{G3})$$

Combining the last three equations, we have

$$S' = S^2\mathcal{L}'P + P\mathcal{L}'S^2 - S\mathcal{L}'S. \quad (\text{G4})$$

### APPENDIX H: GAP DEPENDENCE FOR UNITARY FAMILIES

Assume here that  $\mathcal{L}(s)$  is a unitary family, i.e.,  $\mathcal{L}(s) = e^{s\mathcal{K}}\mathcal{L}(0)e^{-s\mathcal{K}}$  with  $\mathcal{K}$  an anti-Hermitian superoperator. In this case, one has (dropping the  $s$  dependence)  $\mathcal{L}' = \mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K}$  and so, using first-order perturbation theory ([33], p. 77),

$$P' = -P\mathcal{L}'S - S\mathcal{L}'P \quad (\text{H1})$$

$$= -P\mathcal{K}Q + Q\mathcal{K}P \quad (\text{H2})$$

$$= \mathcal{K}P - P\mathcal{K}. \quad (\text{H3})$$

Differentiating the above equation we get that both  $P'$  and  $P''$  are sums of products of  $P$  and  $\mathcal{K}$  and so are everywhere bounded under the assumption that  $P$  and  $\mathcal{K}$  are bounded. From Eq. (G4) and repeatedly using  $S\mathcal{L} = \mathcal{L}S = Q$ ,  $\mathcal{L}P = P\mathcal{L} = 0$ , and  $S = SQ = SQ$  [from Eq. (D1b)], we obtain

$$\begin{aligned} S' &= S^2(\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K})P + P(\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K})S^2 - S(\mathcal{K}\mathcal{L} - \mathcal{L}\mathcal{K})S \\ &= -SQ\mathcal{K}P + P\mathcal{K}QS - S\mathcal{K}Q + Q\mathcal{K}S \\ &= -S\mathcal{K}P + P\mathcal{K}S - S\mathcal{K}Q + Q\mathcal{K}S \\ &= -S\mathcal{K} + \mathcal{K}S. \end{aligned} \quad (\text{H4})$$

Hence, in this case  $S'$  carries only one inverse Liouvillian gap (hidden in  $S$ ). Combining these results we see that the constant  $C$  appearing in Eq. (D10) is in this case at most  $O(\Delta_{\min}^{-1})$ .

### APPENDIX I: BOUND ON $\|P'\|$ FOR THE KMS CASE

In the KMS case with a unique steady state we have already seen (Appendix C) that  $P(s)x = \rho_G(s)\text{Tr}(x)$  with

$\rho_G = e^{-\beta H}/Z$  and the partition function  $Z = \text{Tr}e^{-\beta H}$ . Then, obviously  $P'(s)x = \rho'_G(s)\text{Tr}(x)$  and  $\|P'\| = \|\rho'\|_1$ . We first recall the Bogoliubov-Duhammel scalar product [47,48]

$$(A, B) = \frac{1}{Z} \int_0^1 dx \text{Tr}[e^{-\beta(1-x)H} A^\dagger e^{-\beta x H} B], \quad (I1)$$

and define  $\langle A \rangle_G := \text{Tr}[Ae^{-\beta H}]/Z$ . Moreover, one has the inequality [48]

$$(A, A) \leq \frac{1}{2} \langle A^\dagger A + A A^\dagger \rangle. \quad (I2)$$

Differentiating the Gibbs state we obtain

$$\rho'_G = -\beta \frac{1}{Z} \int_0^1 dx e^{-\beta(1-x)H} H' e^{-\beta x H} + \beta \rho_G \langle H' \rangle, \quad (I3)$$

where the second term arises from the differentiation of the partition function. Now, let  $Y$  denote the first term of the above equation. We wish to bound its trace norm. We use [46]

$$\|Y\|_1 = \sup_U |\text{Tr}(YU)|, \quad (I4)$$

where the supremum is taken over the set of unitary matrices  $U$ . Note that  $\text{Tr}(YU) = -\beta \langle H', U \rangle$  since  $H'$  is Hermitian. Using the Schwartz inequality we have  $|\text{Tr}(YU)| \leq \beta \sqrt{\langle H', H' \rangle \langle U, U \rangle}$ . We now use Eq. (I2) to obtain

$$\|Y\|_1 \leq \beta \sqrt{\langle (H')^2 \rangle_G}. \quad (I5)$$

On the other  $\|\rho_G\| = 1$ . Combining the bounds we get, all in all,

$$\|\rho'\|_1 \leq \beta [\sqrt{\langle (H')^2 \rangle_G} + |\langle H' \rangle_G|]. \quad (I6)$$

Since in general  $\langle X^2 \rangle > \langle X \rangle^2$ , we can write an even more compact bound

$$\|P'\| = \|\rho'\|_1 \leq 2\beta \sqrt{\langle (H')^2 \rangle_G}, \quad (I7)$$

as reported in the main text.

## APPENDIX J: EXTENSION TO THE CASE OF LEVEL CROSSING

The results obtained so far assume that there is a finite gap from the zero eigenvalue to the rest of the spectrum. However, this is not necessary and all of the above can be generalized to include the case in which there are level crossings along the adiabatic path  $[0, s]$ . Suppose that there are  $N$  level crossings at positions  $s_i^*$ ,  $i = 1, \dots, N$ . We isolate each level crossing by a short segment of length  $\delta_i$ , so that no two segments overlap. Then, define two sets  $A, B$  such that  $[0, s] = A \cup B$ , with  $A = \cup_i [s_i^* + \delta_i/2, s_{i+1}^* - \delta_{i+1}/2]$ ,  $i = 0, \dots, N$ , with  $s_0^* = 0$ ,  $s_{N+1}^* = s$ ,  $\delta_0 = \delta_{N+1} = 0$ , and  $B = \cup_i [s_i^* - \delta_i/2, s_i^* + \delta_i/2]$ ,  $i = 1, \dots, N$ . By construction,  $A$  avoids the singularities while  $B$  comprises short segments around each level crossing. We start with Eq. (D4), which we separate into integrals over  $A$  and  $B$ :

$$\begin{aligned} & \mathcal{E}^{-1}(s)V(s)P(0) - P(0) \\ &= \int_{A \cup B} \mathcal{E}^{-1} W' d\sigma \\ &= -\frac{1}{T} \int_A (\mathcal{E}^{-1})' S W' d\sigma + \int_B \mathcal{E}^{-1} W' d\sigma, \end{aligned} \quad (J1)$$

where in the integral over  $A$  we used Eq. (D5). From the above equation, using the classic argument in Ref. [4] one can then show that

$$\lim_{T \rightarrow \infty} [\mathcal{E}(s) - V(s)]P(0) = 0. \quad (J2)$$

However, we are interested in assessing the way in which the limit is approached. In order to obtain a better estimate of the asymptotic behavior, it is necessary to repeat the integration by parts trick at all orders. Since in region  $A$  there are no exceptional points we can immediately repeat all the steps from Appendix D. Assuming convergence of the series, we then obtain

$$\begin{aligned} [\mathcal{E}(s) - V(s)]P(0) &= \sum_{n=1}^{\infty} \frac{\Omega_n^A}{T^n} - \int_B d\sigma \mathcal{E}(s, \sigma) W'(\sigma), \quad (J3) \\ \Omega_n^A &= \sum_{i=0}^N \mathcal{E}(s, \sigma) X_n(\sigma) W'(\sigma) \Big|_{s_i^* + \delta_i/2}^{s_{i+1}^* - \delta_{i+1}/2} \\ &\quad - \int_A d\sigma \mathcal{E}(s, \sigma) X_n(\sigma) W''(\sigma). \end{aligned} \quad (J4)$$

Each term in the above equations is bounded for finite  $\delta_i$ , however, when  $\delta_i \rightarrow 0$  some terms display singular behavior. We are interested in estimating the size of the most singular terms. Recall that we are assuming a Davies generator. In this case,  $W' = P'W$  [Eq. (C1)] is bounded (because  $P'$  is bounded [30] and  $\|W\| = 1$  because  $W$  is a CPTP map), and similarly  $W'' = [P'' + (P')^2]W$  is bounded because  $P''$  is bounded [30]. Hence, when  $\delta_i \rightarrow 0$ , the only singularities in Eqs. (J3) and (J4) arise from  $X_n(\sigma)$ . Now, assume that at each singular point  $s_i^*$ , there is an eigenvalue going to zero as  $v_i(s - s_i^*)^{\alpha_i}$  (where the coefficients  $v_i$  can be complex). This algebraic vanishing of the gap is the only possibility in the finite-dimensional case on which we focus. Moreover, for normal operators and analytic dependence on  $s$ , the exponents  $\alpha_i$  must be positive integers (see Ref. [33]), but we will not be needing this. Consider then the finite-dimensional case. The normality of  $\mathcal{L}$  guarantees that there are no nilpotent terms in the spectral resolution of  $\mathcal{L}$  and so the reduced resolvent can be written as

$$S(s) = \sum_{j>0} \frac{P_j(s)}{\lambda_j(s)}, \quad (J5)$$

where  $\lambda_j(s)$  [ $P_j(s)$ ] are the eigenvalues (spectral projections) of  $\mathcal{L}(s)$ , with  $P_0(s) = P(s)$ . Hence,

$$S' = - \sum_{j>0} \frac{P_j}{(\lambda_j)^2} \lambda_j' + \sum_{j>0} \frac{P_j'}{\lambda_j}. \quad (J6)$$

Now, it is natural to assume that  $P_j'$  are piecewise differentiable also for  $j > 0$ . For normal operators and analytic dependence on  $s$  this follows from a theorem of Kato (see Ref. [33], Theorem 1.10, p. 71). In this case, the largest divergence of  $S'$ , when  $s - s_i^* = \delta_i \rightarrow 0$ , comes from the vanishing of the gap in the first term of Eq. (J6) and is of the form

$$S'(s) \sim \frac{1}{\delta_i^{\alpha_i+1}}. \quad (J7)$$

Note that the prefactors may be different depending on whether  $\delta_i \rightarrow 0^\pm$ . This reasoning can be extended to  $X_n(s)$ . Recall that

$X_n = S X'_{n-1}$ , with  $X_1 = S$ , and note that  $X_n$  contains  $n$  powers of  $S$  and a total of  $n - 1$  derivatives with respect to  $s$ . So, we conclude that  $X_n(s) \sim 1/\delta_i^{n\alpha_i+n-1}$  as  $\delta_i \rightarrow 0$ . More precisely, we have

$$\lim_{\delta_i \rightarrow 0^\pm} \delta_i^{\beta_n^i} X_n(s_i^* + \delta_i) = Y_n^\pm, \quad (\text{J8})$$

where the  $Y_n^\pm$  exist and where

$$\beta_n^i = n\alpha_i + n - 1. \quad (\text{J9})$$

Thanks to the above, there exist positive constants  $A_n^i$  such that, for sufficiently small  $\delta_i$ ,

$$\|\Omega_n^A\| \leq \sum_{i=1}^N \frac{A_n^i}{\delta_i^{\beta_n^i}}. \quad (\text{J10})$$

The  $A_n^i$  can be made independent of  $\delta_i$ , for sufficiently small  $\delta_i$ . In fact, they can be taken to be the max of the norm of the most diverging terms plus a small constant. On the other hand, to bound the second term of Eq. (J3) we just need to notice that  $W'$  is piecewise continuous:

$$\begin{aligned} & \left\| - \int_B d\sigma \mathcal{E}(s, \sigma) W'(\sigma) \right\| \\ &= \left\| - \sum_{i=1}^N \int_{s_i^* - \delta_i/2}^{s_i^* + \delta_i/2} d\sigma \mathcal{E}(s, \sigma) W'(\sigma) \right\| \leq \sum_{i=1}^N \delta_i B_i, \end{aligned} \quad (\text{J11})$$

with positive  $B_i$ , because the integrand is piecewise continuous  $\{B_i \sim (1/2)[\|P'(s_i^* + 0^+)\| + \|P'(s_i^* - 0^+)\|]\}$ . All in all we obtained

$$\|[\mathcal{E}(s) - V(s)]P(0)\| \leq \sum_{i=1}^N f_i(T, \delta_i), \quad (\text{J12})$$

where

$$f_i(T, \delta_i) = \sum_{n=1}^{\infty} \frac{A_n^i}{T^n \delta_i^{\beta_n^i}} + B_i \delta_i. \quad (\text{J13})$$

Now, for each fixed  $T$ , we are free to minimize each  $f_i$  over  $\delta_i$ , to which we turn next.

Differentiating, we obtain

$$\partial_{\delta_i} f_i = - \sum_{n=1}^{\infty} \frac{\beta_n^i A_n^i}{T^n \delta_i^{\beta_n^i + 1}} + B_i = 0, \quad (\text{J14})$$

where  $\beta_n^i + 1 = n(\alpha_i + 1)$ . Recall that all the  $A_n^i \geq 0$  and  $B_i > 0$ , so the only way Eq. (J14) can have a solution is that, asymptotically as  $T \rightarrow \infty$ , some of the terms in the series in

Eq. (J14) scale to constant, while some others may scale to zero. Plugging in the ansatz  $\delta_i = (c_i T)^{-\eta_i}$  one can check that  $\eta_i = 1/(\alpha_i + 1)$  is the only possible solution. In fact, all the terms scale to a constant. The constants  $c_i$  must satisfy

$$\sum_{n=1}^{\infty} \beta_n^i A_n^i c_i^n = B_i. \quad (\text{J15})$$

The extremum is indeed a minimum as it is easy to check that the second derivative is always positive for positive  $T, \delta$ . Substituting the solution back into Eq. (J13) we obtain

$$f_i(T, \delta_{\min}) = \frac{D_i}{(c_i T)^{\eta_i}}, \quad (\text{J16})$$

$$D_i = \left[ \sum_{n=1}^{\infty} A_n^i (c_i)^n + B_i \right]. \quad (\text{J17})$$

Returning to Eq. (J12), we finally obtain

$$\|[\mathcal{E}(s) - V(s)]P(0)\| \leq \sum_{i=1}^N \frac{D_i}{(c_i T)^{\eta_i}}, \quad (\text{J18})$$

as stated in Eq. (5) of the main text.

#### APPENDIX K: DERIVATION OF THE EIGENVALUES OF THE LINDBLADIAN OF EXAMPLE 2

Denote the instantaneous Hamiltonian gap by  $\delta(s)$  and the instantaneous ground state and first excited state by  $|\epsilon_0(s)\rangle$  and  $|\epsilon_1(s)\rangle$ , respectively. The action of the Lindbladian on  $\rho$  can be written in this basis as (dropping the  $s$  dependence for clarity)

$$\langle \epsilon_0 | \mathcal{L} \rho | \epsilon_0 \rangle = \gamma(\delta) [|A_{01}|^2 \rho_{11} - e^{-\beta\delta} |A_{10}|^2 \rho_{00}], \quad (\text{K1a})$$

$$\langle \epsilon_0 | \mathcal{L} \rho | \epsilon_1 \rangle = (i\mu - \Gamma) \rho_{01}, \quad (\text{K1b})$$

$$\langle \epsilon_1 | \mathcal{L} \rho | \epsilon_0 \rangle = (-i\mu - \Gamma) \rho_{10}, \quad (\text{K1c})$$

$$\langle \epsilon_1 | \mathcal{L} \rho | \epsilon_1 \rangle = \gamma(\delta) [e^{-\beta\delta} |A_{10}|^2 \rho_{00} - |A_{01}|^2 \rho_{11}], \quad (\text{K1d})$$

where  $\rho_{ab} \equiv \langle \epsilon_a | \rho | \epsilon_b \rangle$ ,  $\mu = \delta - S(0)(A_{00}^2 - A_{11}^2) + [S(\delta) - S(-\delta)]|A_{01}|^2$ , and  $2\Gamma = \gamma(0)(|A_{00}|^2 + |A_{11}|^2) + \gamma(\delta)|A_{01}|^2(1 + e^{-\beta\delta})$ . This form immediately allows us to read off two of the eigenvalues of  $\mathcal{L}$  as  $\pm i\mu - \Gamma$ . The remaining two are the eigenvalues of the following  $2 \times 2$  matrix:

$$\gamma(\delta) |A_{10}|^2 \begin{pmatrix} -e^{-\beta\delta} & 1 \\ e^{-\beta\delta} & -1 \end{pmatrix}, \quad (\text{K2})$$

whose eigenvalues are 0 and  $-|A_{01}|^2 \gamma(\delta)(1 + e^{-\beta\delta})$ .

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