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Degeneracy and non-Abelian statistics

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A non-Abelian anyon can only occur in the presence of ground-state degeneracy in the plane. It is conceivable that for some strange anyon with quantum dimension > 1 that the resulting representations of all n-strand braid groups B_n are overall phases, even though the ground-state manifolds for n such anyons in the plane are in general Hilbert spaces of dimensions > 1. We observe that degeneracy is all that is needed: For an anyon with quantum dimension > 1 the non-Abelian statistics cannot all be overall phases on the degeneracy ground-state manifold. Therefore, degeneracy implies non-Abelian statistics, which justifies defining a non-Abelian anyon as one with quantum dimension > 1. Since non-Abelian statistics presumes degeneracy, degeneracy is more fundamental than non-Abelian statistics.

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I. INTRODUCTION

In 1991, the potential realization of non-Abelian statistics in fractional quantum Hall states was proposed [1,2]. Recently, the Majorana zero-mode version of non-Abelian statistics has been intensively pursued in experiments using nanowires (see Ref. [3]). More generally, non-Abelian statistics occurs in topological phases of matter—quantum phases of matter that exhibit topological orders [4]. A direct observation of non-Abelian statistics will be to braid the non-Abelian objects. But an easier experiment than braiding non-Abelian objects is to observe the Ising fusion rule $\sigma \otimes \sigma = 1 \oplus \psi$. This Ising fusion rule implies degeneracy, and more generally any anyon with quantum dimension $d > 1$ has degeneracy. Does non-Abelian statistics follow from degeneracy? In this Rapid Communication, we point out that indeed degeneracy implies non-Abelian statistics. Since degeneracy is a prerequisite for non-Abelian statistics, degeneracy is more fundamental than non-Abelian statistics in a sense. Without our observation, replacing non-Abelian statistics by degeneracy is unjustified.

Non-Abelian statistics is a fundamentally new form of particle interaction. This "spooky action" is a manifestation of entanglement in the degenerate ground states—*the* characteristic attribute of quantum mechanics according to Schödinger. The central role that braiding plays in recent work on ultracold atoms can be seen in, e.g., Refs. [5–7]. Besides its general interest as a new form of particle interaction, non-Abelian statistics underlies the idea of topological quantum computation—the braiding matrices are inherently fault-tolerant quantum circuits [8–10]. Therefore, it is crucial to confirm non-Abelian statistics by experiments. The Ising fusion rule $\sigma \otimes \sigma = 1 \oplus \psi$ is now amenable to experimental test in nanowire technology [11]. Our result means that if we can verify a non-Abelian fusion rule by experiments, then on one hand, it is also a verification of non-Abelian particle interactions, and on the other hand, it establishes the feasibility of the construction of a topological quantum computer.

II. ANYON MODELS

Anyons are topological quantum fields materialized as finite energy particlelike excitations in topological phases of matter. As particles, they can be moved, but cannot be created or destroyed by local operators alone. Two anyons have the same anyon type or topological charge if they differ by local operators. There are two equivalent ways to model anyon systems. We can focus on the ground-state manifold $V(Y)$ of an anyonic system on any possible space Y, and then the anyon system is modeled in low energy by a $(2+1)$-dimensional topological quantum field theory (TQFT) $\{V(Y)\}$. An alternative is to consider the fusion and braiding structures of all elementary excitations in the plane. The anyon system is then equivalently modeled by a unitary modular (tensor) category $\mathcal{C}$. The two notions $(2+1)$-TQFT and modular category are essentially the same [12]. Therefore, anyon systems can be modeled either by TQFTs or unitary modular categories. In this Rapid Communication, we will use unitary modular categories to model anyon systems (see Ref. [13]).

In the modular category model, an anyon X is a simple object that abstracts an irreducible representation of some symmetry algebra. The topological charge or anyon type x of an anyon X is an equivalent class of anyons [14]. All possible topological charges in an anyon system form a finite label set $L = \{a,b,\ldots\}$ with fusion rules $\{N_{ab}^c\}$, where N_{ab}^c are non-negative integers [13]. The fusion rule N_{ab}^c encodes the possible topological charges c that will appear when two anyons of types a,b are fused: If $N_{ab}^c = 0$, then anyons of type c will not appear; otherwise $N_{ab}^c > 0$ and there are N_{ab}^c different fusion channels for anyons A,B to fuse to anyon C. There is always a label 1 in L that corresponds to the ground state or vacuum. In the famous Ising theory, the label set is $L = \{1,\sigma,\psi\}$. Usually, we write the fusion rules as a tensor-sum $a \otimes b = \oplus_{c\in L} N_{ab}^c c$. There are always the trivial fusion rules $1 \otimes x = x \otimes 1 = x$. In this tensor-sum notation, the nontrivial fusion rules for the Ising theory are $\sigma \otimes \sigma = 1 \oplus \psi$, $\sigma \otimes \psi = \psi \otimes \sigma = \sigma$, $\psi \otimes \psi = 1$. The anyon σ is called the Ising anyon. The anyon ψ is a fermion. Generally, an anyon is self-dual if it has the same anyon type as its antiparticle. Both σ and ψ are self-dual, hence ψ is Majorana—a real fermion.

III. QUANTUM DIMENSION AND DEGENERACY

An important quantum number of an anyon X is its quantum dimension d_X—a positive real number $\geqslant 1$. The quantum dimension of an anyon can be easily computed from its fusion rules: Regard all anyon types as unknown variables and the fusion rules as polynomial equations, then the maximal real solutions of these polynomial equations are the quantum dimensions. For the Ising fusion rules, the quantum dimension of the Ising anyon σ is $d_\sigma = \sqrt{2}$ and the Majorana fermion $d_\psi = 1$. The quantum dimension d_X of an anyon X determines the asymptotic growth rate when n identical anyons X are confined to the sphere: The dimension of the degeneracy ground-state manifold $V_{X,n,1}$ grows as d_X^n as $n \to \infty$. Therefore, an anyon X leads to degeneracy in the plane if and only if its quantum dimension $d_X > 1$.

IV. BRAID GROUPS AND NON-ABELIAN STATISTICS

In two spatial dimensions, statistics of quasiparticles can be more general than bosons and fermions (see Ref. [15]). An exotic form of statistics is not an overall phase, but a unitary matrix: The overall change when two anyons X in n identical anyons X are exchanged is a unitary matrix on the degeneracy ground-state manifolds $V_{x,n,a}$ for some total topological charge a. It follows that non-Abelian statistics presumes degeneracy.

The ground-state manifold $V_{x,n,a}$ is a representation of the n-stand braid group B_n. It is well known that the braid group B_n is generated by n elementary braids $\{\sigma_i\}$, $i = 1,2,\ldots,n-1$. *Non-Abelian statistics* means that the image of the representation for some B_n is a non-Abelian subgroup of the unitary group $U(V_{x,n,a})$. It is conceivable that for some particular anyon X with quantum dimension > 1 that all representations of B_n are overall phases, even though $V_{X,n,a}$ are Hilbert spaces of dimensions > 1 for general n. We will see below that this cannot occur. At the end of Sec. II B in Ref. [16], a weaker version of degeneracy implies non-non-Abelian statistics is proved [17].

V. DEGENERACY IMPLIES NON-NON-ABELIAN STATISTICS

When n anyons X are pinned in the plane, the ground-state manifold $V_{X,n,a}$ for some total charge a consists of exponentially closed degenerate ground states. An orthonormal basis of $V_{X,n,a}$ is usually represented by labeled fusion trees (see

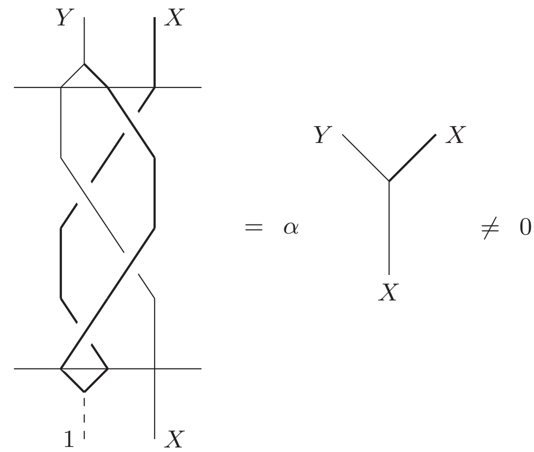

FIG. 2. Self-dual case: A nonzero state.

Ref. [13]). The statistics of the anyon X is computed by stacking braids on top of any state in $V_{X,n,a}$. Physical states have to satisfy fusion rules at each trivalent vertex.

An anyon X with $d_X > 1$ is self-dual if and only if $X \otimes X = 1 \oplus Y \oplus \cdots$ for some nontrivial anyon Y, which could have multiplicity. Otherwise, there is a different anyon X^* such that $X \otimes X^* = 1 \oplus Y \oplus \cdots$ for some nontrivial anyon Y.

Theorem. Suppose X is an anyon with quantum dimension $d_X > 1$. Then:

(1) If X is self-dual, the image of the afforded representation of the 3-strand braid group B_3 is non-Abelian.

(2) If X is non-self-dual, the image of the afforded representation of the 4-strand pure braid group P_4 is not trivial up to scalars.

If an anyon X with $d_X > 1$ is self-dual, to show that the 3-strand braid group B_3 has an image which is non-Abelian, we consider the following braid $b = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1$ in the 3-strand braid group B_3. Note the braid is the commutator of the two elementary braids σ_1 and σ_2. We choose the representation $V_{X,3,X}$ [18]. Note that dim $V_{X,3,X} \geqslant 2$. Starting with the state of three anyons X in $V_{X,3,X}$ represented by the fusion tree below the bottom horizontal line, we braid the three anyons X through the braid b. We want to compute the resulting state

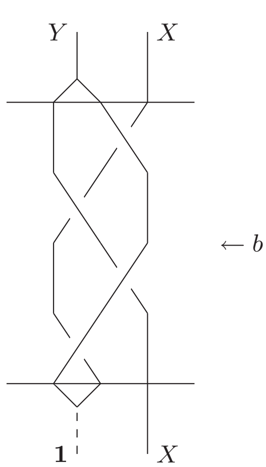

FIG. 1. The self-dual case.

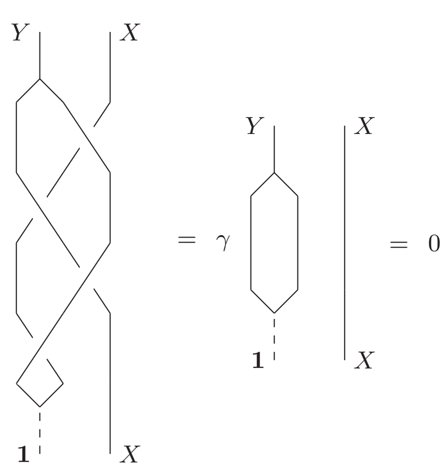

FIG. 3. Self-dual case: A vanishing state.

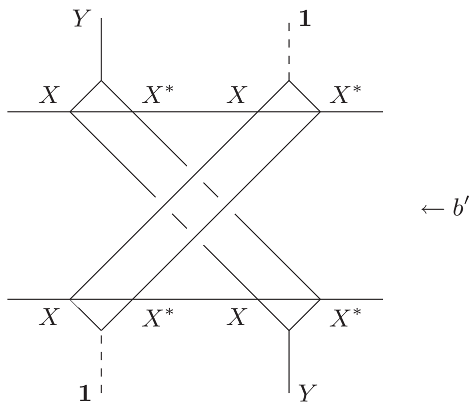

FIG. 4. The non-self-dual case.

after braiding b with the constraint that the total charge of the first two anyons X is Y (see Fig. 1).

By sliding and twisting, we can deform the thick braided arc in Fig. 2 to the thick interval in the trivalent vertex without deforming the other arc, hence deforming the braided fusion tree to a trivalent vertex. Therefore, up to an overall nonzero scalar $\alpha = \theta_X^2 R_{XX}^Y$, the resulting state is a trivalent vertex state (see Fig. 2). If Y has multiplicity, we choose a trivalent vertex state where the braiding R_{XX}^Y acts as a scalar.

The trivalent vertex state is nonzero because the fusion rule is admissible. On the other hand, if the braid b has the same image as the identity up to some scalar γ, we can replace the braid b between the two horizontal lines in Fig 1 by the identity braid. Then the resulting state will be 0 due to the no-tadpole rule (see Fig. 3).

This contradiction implies that the image matrix of b is not a scalar. It follows that the images of the elementary braids σ_1 and σ_2 do not commute.

If X is non-self-dual, then we will show that there is a braid in B_4 whose image is not the identity up to an overall scalar.

Consider the following braid b' in the 4-strand braid group B_4 (see Fig. 4).

Similarly to the argument above, on one hand we have the identity in Fig. 5, which shows the resulting state is nonzero.

On the other hand, if the image of the middle braid b' is the same as the identity up to an overall phase, then Fig. 6 implies that the resulting state would be 0.

The contradiction implies that the image of the braid b' is not a scalar.

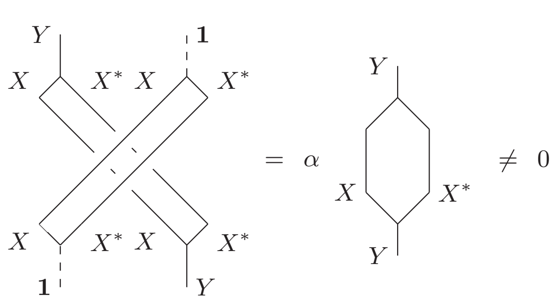

FIG. 5. Non-self-dual: The overstrands result in a nonzero scalar. The right-hand side is a nonzero state.

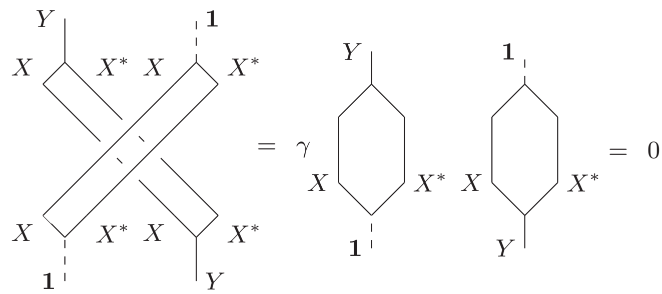

FIG. 6. Non-self-dual: Replacing b' by the identity braid results in a zero state.

VI. CONCLUSIONS

In this Rapid Communication, we find that degeneracy is more fundamental than non-Abelian statistics. One consequence is that experimental confirmation of non-Abelian fusion rules implies non-Abelian braiding statistics if anyon systems are modeled by TQFTs or unitary modular categories up to overall phases.

ACKNOWLEDGMENTS

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APPENDIX

Our theorem is proved using graphical calculus–geometric manipulation of operators in the algebraic theory of anyons, i.e., ribbon categories. One may also perform these calculations directly. We illustrate this by verifying that the state in Fig. 2 (self-dual case) is nonzero, making free use of the standard notation and axioms of ribbon categories (see, e.g., Ref. [19]).

First, we choose a basis for $\mathrm{Hom}(X \otimes X, Y)$ with respect to which the braiding $c_{X,X}$ acts diagonally (on the right), and fix a nonzero element $h \in \mathrm{Hom}(X \otimes X, Y)$. The map in Fig. 2 is, with right to left composition read from bottom to top,

$$(h \otimes I_X)(c_{X\otimes X,X}^{-1})(I_X \otimes c_{X,X})(c_{X,X} \otimes I_X)(b_X \otimes I_X),$$

where $b_X \in \mathrm{Hom}(\mathbf{1}, X \otimes X)$ is the creation operator and I_X is the identity map. We wish to show that this is a nonzero vector. Two key axioms will be used: (1) the functoriality of the braiding, $c_{X,Y}(f \otimes g) = (g \otimes f)c_{X',Y'}$, where $f \in \mathrm{Hom}(X', X)$ and $g \in \mathrm{Hom}(Y', Y)$, and (2) rigidity, $(d_X \otimes I_X)(I_X \otimes b_X) = I_X$, where $d_X \in \mathrm{Hom}(X \otimes X, \mathbf{1})$ is the annihilation operator. Using the fact that $c_{X,X} b_X = \alpha_1 b_X$ for some $\alpha_1 \neq 0$ and $(h \otimes I_X)(c_{X\otimes X,X}^{-1}) = c_{Y,X}^{-1}(I_X \otimes h)$ we simplify to

$$\alpha_1 c_{Y,X}^{-1}(I_X \otimes h)(I_X \otimes c_{X,X})(b_X \otimes I_X).$$

Now since $(I_X \otimes h)(I_X \otimes c_{X,X}) = \alpha_2(I_X \otimes h)$ for some $\alpha_2 \neq 0$ and $c_{Y,X}^{-1}$ is invertible, it is enough to see that

$$(I_X \otimes h)(b_X \otimes I_X) \neq 0.$$

For this we observe that

$$0 \neq h = (h)(d_X \otimes I_X \otimes I_X)(I_X \otimes b_X \otimes I_X),$$

which then becomes

$$(d_X \otimes I_Y)(I_X \otimes I_X \otimes h)(I_X \otimes b_X \otimes I_X).$$

Factoring this we obtain

$$(d_X \otimes I_Y)(I_X \otimes [(I_X \otimes h)(b_X \otimes I_X)]) \neq 0.$$

Thus the factor $(I_X \otimes h)(b_X \otimes I_X) \neq 0$, as desired.

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[4] Our terminologies are as follows: A topological phase of matter is an equivalence class of Hamiltonians or states without any phase transitions among them, while a topological order is the topological quantum field theory or unitary modular tensor category encoded in the Hamiltonian or state class at low energy.

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[14] In general, it is important to distinguish anyon types with anyon objects. We will use uppercase letters to denote anyon objects and the corresponding lowercase letters to denote their types.

[15] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian Anyons and Topological Quantum Computation, Rev. Mod. Phys. **80**, 1083 (2008).

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[17] There, it is proved that for any anyon X with $d_X > 1$, there is another anyon Y, possible $Y = X$, that the double braiding between X and Y is not a phase. In particular, if an anyon system has only one type of non-Abelian anyon, then its double braiding is not a phase.

[18] For convenience, we choose the total charge to be X. The argument applies to other admissible total charges, too.

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[20] Answer to how non-Abelian anyons arise in solid-state systems at http://physics.stackexchange.com/.