Bose-Einstein condensation of ideal photons in a one-dimensional barrel cavity

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Our experimental scheme is based on a barrel optical microresonator filled with a dye solution. The barrel mirror provides a confining potential, a chemical potential, and an effective mass for a photon, making the system formally equivalent to a one-dimensional gas of harmonically trapped, number-conserving, and massive bosons. Within the framework of quantum statistical mechanics, we propose an exact analytical solution to the problem of Bose-Einstein condensation in harmonically trapped, one-dimensional, and ideal photons. It is found that the photon number of vapor is characterized by an analytical function, which involves a q-digamma function in mathematics. The numerical calculation of the analytical solution gives many interesting results. In the thermodynamic limit, the analytical expressions of the critical temperature and the condensate fraction are derived. We find that the spectral radiance of a one-dimensional barrel cavity has a sharp peak at the frequency of the cavity cutoff when the photon number exceeds the critical value determined by a temperature.

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I. INTRODUCTION

It is currently recognized that Bose-Einstein condensation (BEC) is a common quantum property of many-particle systems in which the number of particles is conserved. In 1995, the three research groups in the United States observed the BEC of ultracold Bose atomic gases in a trapping potential [1–3]. The BEC has been observed also in several systems of solid-state quasiparticles, which include excitons [4,5], exciton-polaritons [6,7], and magnons [8,9]. However, the most ubiquitous blackbody radiation does not reveal such condensation. The reason for this is that the photon number of a blackbody is not conserved. Thermalization of laser photons has been considered in the processes that involve Compton scattering with a gas of thermal electrons [10]. The superfluid behavior of photons in a nonlinear resonator configuration has been investigated theoretically [11–13]. Number-conserving thermalization has been observed experimentally [14] for a two-dimensional photon gas in a dye-filled optical microcavity, which acts as a "white-wall" box. In the presence of thermalization processes that conserve photon number, Weitz and colleagues have observed the BEC of two-dimensional photons in a dye-filled optical microcavity [15,16]. Kirton and Keeling have established a nonequilibrium model of photonic BEC in the dye-filled microcavity [17]. Recent progress in research into BEC of photons has been made both theoretically [18–24] and experimentally [25–27].

One-dimensional systems of many particles have fascinated physicists for more than 60 years now [28]. The physics in one-dimensional systems is drastically different from that in higher-dimensional systems. The one-dimensional character makes the problem simple enough that some exact analytical solutions can be obtained using specific methods, and these solutions can lead to incredibly rich physics. Originally, the BEC of atomic polaritons in a biconical waveguide cavity was investigated under the quasiclassical approximation [29]. The biconical waveguide cavity provides a linear confining potential for an atom polariton. In this paper, we study the BEC of ideal photons in a one-dimensional barrel cavity. Our experimental scheme is based on a barrel optical microresonator filled with a dye solution. The barrel mirror provides a confining potential, a chemical potential, and an effective mass for a photon, making the system formally equivalent to a one-dimensional gas of harmonically trapped, number-conserving, and massive bosons. The photons of the incident laser thermalize to the temperature of the dye solution (room temperature) by repeated absorption and reemission processes in the dye solution. In the one-dimensional barrel microcavity, the levels of a one-dimensional harmonic oscillator are nondegenerate, and so the Bose-Einstein condensate of a one-dimensional photon gas can be investigated analytically.

For simplicity, we neglect interactions between the photons. In fact, interactions between the photons are very small in a one-dimensional barrel microcavity. Therefore, the object of study is an ideal photon gas moving in a one-dimensional harmonic potential. The topic of BEC of a noninteracting boson gas in a one-dimensional harmonic potential is treated in all the references in the following manners: (1) the whole study is carried out numerically by a computer, (2) the thermodynamic limit is used, and (3) the discrete level structure is approximated by a continuous density of states under the assumption that the level spacing is negligible compared to the temperature (the quasiclassical approximation). For example, the thermodynamic properties of harmonically trapped low-dimensional boson systems have been studied under the quasiclassical approximation [30]. In the present paper, we derive an exact analytical solution of the BEC of an ideal photon gas in a one-dimensional barrel cavity. This solution is valid for arbitrary temperature and photon number. The solution is of great significance owing to the following reasons. First, the noninteracting Bose-Einstein condensate in a one-dimensional barrel cavity is a starting point for other various studies in the presence of interactions and disorders. Second, the form of this solution is so beautiful that the solution can be used for purposes of tuition. Third, the method and concept used in this study can be applied in the whole field of statistical physics. The exact analytical solution of the BEC of an ideal photon gas in a one-dimensional barrel cavity can be verified in present-day physics laboratories.

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The remainder of this paper is organized as follows. Section II describes the characters of a one-dimensional barrel microcavity. Section III depicts the microscopic theory of an ideal photon gas in a one-dimensional barrel microcavity. In Sec. IV, we derive an exact analytical solution to the BEC of a finite number of ideal photons in a one-dimensional barrel microcavity. In Sec. V, we derive an exact analytical solution to the BEC of ideal photons in the thermodynamic limit. Section VI describes the radiation properties of a one-dimensional barrel cavity. A comprehensive discussion is given in Sec. VII.

II. CHARACTERS OF A ONE-DIMENSIONAL BARREL MICROCAVITY

In an open cavity there are many losses, such as the loss from coupling to optical modes not confined in the cavity, the nonradiative decay, and the mirror loss. To compensate for the losses, we use an external laser beam to optically pump the dye. Our experimental setup shown in Fig. 1 is based on an optical resonator with a surface of revolution. The surface of revolution is formed by the rotation of an arc of circle in the y-z plane about the z axis. The resonator surface is a high-reflecting barrel optical mirror that has a radius R of spherical curvature ($R = 0.47$ m). The barrel cavity has the maximum inner radius r_0 ($r_0 = 1.60\,\mu$m). The resonator is filled with a drop of dye (rhodamine 6G) dissolved in an organic solvent (methanol) and is pumped with a laser beam that is near the transversal plane. It may be difficult for the present technology to create the barrel cavity. With the advance of science and technology, the future technology must create such a barrel cavity.

As shown in Fig. 1, the wave vector $\mathbf{k}$ of photons in a barrel cavity can be resolved into the sum of transversal wave vector $\mathbf{k}_\perp$ and longitudinal wave vector $\mathbf{k}_z$: $\mathbf{k} = \mathbf{k}_\perp + \mathbf{k}_z$. The barrel cavity has the maximum inner radius r_0 and a height d. For d large enough ($d > 2.03r_0$), the resonance frequency of the lowest TE mode is smaller than that for the lowest TM mode. Then the lowest TE mode is the fundamental oscillation of the cavity [31]. At distance z from the x-y plane, the barrel cavity has an inner radius

$$r(z) = r_0 - (R - \sqrt{R^2 - z^2}). \tag{1}$$

In cylindrical coordinates, for a TE mode the transverse wave equation for the magnetic field $B_z(\rho,\phi)$, subject to the boundary condition $\partial B_z/\partial\rho = 0$ at $\rho = r(z)$, has the solution

$$B_z(\rho,\phi) = J_m(k_\perp\rho)e^{\pm im\phi}, \tag{2}$$

where $k_\perp = x_{mn}/r(z)$. $J_m(x)$ is the Bessel function of the first kind of order m. x_{mn} is the nth root of the equation, $J'_m(x) = 0$. The integers m and n take on the values $m = 0,1,2,\ldots$ and $n = 1,2,3,\ldots$.

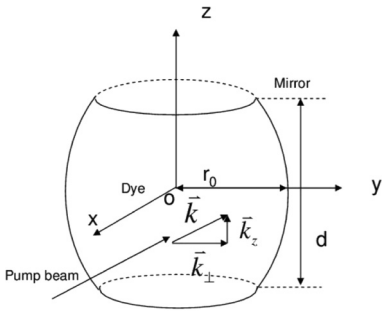

FIG. 1. Experimental scheme for thermalization of a one-dimensional photon gas.

At this point, we need to derive the energy-momentum relation for the photon in a barrel cavity. In general, this relation is given by $E(\mathbf{k}) = \hbar v\sqrt{\mathbf{k}_\perp^2 + \mathbf{k}_z^2}$. This is a relativistic relation. Here $v = c/n_0$ is the speed of light in the medium, and the medium has a linear index n_0 of refraction. If the cavity height d is much larger than the cavity radius r_0 ($d \gg r_0$), the transverse wave number $k_\perp$ is much greater than the longitudinal wave number k_z ($k_\perp \gg k_z$). This is contrary to the paraxial approximation in a slablike cavity [14–16]. Under the near-planar approximation, from a Taylor expansion of the relativistic relation we obtain a nonrelativistic energy-momentum relation,

$$E(\mathbf{k}_z) = \hbar v\left(k_\perp + \frac{\mathbf{k}_z^2}{2k_\perp}\right). \tag{3}$$

Under the near-planar approximation of $|z| \ll R$, we substitute Eq. (1) into $k_\perp = x_{mn}/r(z)$, and so $k_\perp(z)$ can be expressed as

$$k_\perp(z) = k_\perp(0) + k_\perp(0)\left(\frac{z}{\sqrt{2r_0R}}\right)^2, \tag{4}$$

where $k_\perp(0) = x_{mn}/r_0$. The substitution of Eq. (4) into Eq. (3) results in

$$E(\mathbf{k}_z) = m_{ph}v^2 + \frac{(\hbar\mathbf{k}_z)^2}{2m_{ph}} + \frac{1}{2}m_{ph}\Omega^2z^2. \tag{5}$$

It is evident that the barrel cavity maps the three-dimensional photons onto the one-dimensional photons. For near-planar propagation, the one-dimensional photons are the nonrelativistic massive particles that possess an effective mass $m_{ph} = \hbar k_\perp(0)/v = \hbar\omega_c/v^2$, where $\omega_c = k_\perp(0)v$ is the lower cutoff frequency of one-dimensional photons. The motion of nonrelativistic massive photons is restricted to the one-dimensional longitudinal resonator axis under harmonic confinement with trapping frequency $\Omega = v/\sqrt{r_0R}$.

III. MICROSCOPIC THEORY OF A TRAPPED IDEAL PHOTON GAS

A dilute-photon condensate of density n in an axially symmetric microcavity is characterized by four length scales: its transverse radius $R_\perp$, its axial radius R_z, the scattering length a which represents the strength of the two-body interaction, and the healing length $\xi = (4\pi na)^{-1/2}$. In the experiment on Bose-Einstein condensates of photons [14–16], both the radii and lengths are determined by the interaction between the photons, and thus, $R_\perp > R_z > \xi > a$. In this regime, a Bose-Einstein condensate is two-dimensional and is well described by the so-called Thomas-Fermi approximation. A qualitatively different behavior of a Bose-Einstein condensate is expected when the healing length is larger than $R_\perp$ since then the condensate becomes restricted to one dimension. New phenomena in this regime are, for example, quasicondensates with a fluctuating

phase and a Tonks-Girardeau gas of impenetrable bosons. In this paper, we investigate a barrel-shaped one-dimensional condensate with $R_z > \xi > R_\perp > a$.

We consider a one-dimensional gas of ideal photons with spin one. m_{ph} is the mass of photons. The system under study consists of N noninteracting photons moving in a one-dimensional harmonic potential. The one-dimensional photons possess a longitudinal momentum $\hat{p}_z = \hbar k_z$. In quantum mechanics, the canonical momentum of a photon at position z is given by the operator $\hat{p}_z = -i\hbar\frac{d}{dz}$, where $\hbar$ is Planck's constant reduced. In nonrelativistic quantum mechanics, the Hamiltonian of a photon in a one-dimensional harmonic potential is given by

$$\hat{H} = \frac{\hat{p}_z^2}{2m_{ph}} + \frac{1}{2}m_{ph}\Omega^2 z^2, \tag{6}$$

where Ω is the axial angular frequency of the microcavity. The stationary state of a photon at position z is described by the wave function $\Psi_l(z)$, where l is a quantum number. The wave function $\Psi_l(z)$ satisfies the Schrödinger equation,

$$\left(-\frac{\hbar^2}{2m_{ph}}\frac{d^2}{dz^2} + \frac{1}{2}m_{ph}\Omega^2 z^2\right)\Psi_l(z) = \mathcal{E}_l\Psi_l(z), \tag{7}$$

where $\mathcal{E}_l$ is the energy eigenvalue of a photon.

Equation (7) is the one-dimensional harmonic oscillator equation and has the oscillator levels

$$\mathcal{E}_l = \left(l + \frac{1}{2}\right)\hbar\Omega, \quad l = 0,1,2,\ldots. \tag{8}$$

The eigenfunctions $\Psi_l(z)$ corresponding to the oscillator levels are given by

$$\Psi_l(z) = \frac{1}{\pi^{1/4}a^{1/2}\sqrt{2^l l!}}\exp\left(-\frac{z^2}{2a^2}\right)H_l\left(\frac{z}{a}\right), \tag{9}$$

where $a = \sqrt{\hbar/m_{ph}\Omega}$ is a length characterizing the spread of the wave function in the z direction. $H_l(\xi)$ is the Hermite polynomial of the lth degree in $\xi = z/a$. The states described by the wave function Ψ_l are called oscillator states.

The energy of a longitudinal eigenmode belonging to the manifold of transverse electric mode mn is $E_l = \hbar\omega_c + \mathcal{E}_l$, where $\omega_c = x_{mn}v/r_0$. In order to undertake a numerical calculation, we notice that the linear refractive index for methanol is $n_0 = 1.33$. For simplicity, we assume that the low-frequency cutoff ω_c corresponds to the $n = 8$ zero of the derivative $J_0'(x)$ of the Bessel function of order $m = 0$, which is given by $x_{08} = 22.76008$. For the geometric parameters of the barrel cavity, we take $R = 0.47$ m, $r_0 = 1.60$ μm, and $d = 4.0$ mm. With the relation $\omega_c = x_{08}v/r_0$, the lower cutoff frequency of the barrel cavity is calculated as $\omega_c = 3.21 \times 10^{15}$ s^{-1}, or $\omega_c/2\pi = 5.10 \times 10^{14}$ Hz. According to Refs. [14,15], the lower cutoff frequency ω_c is in the low-lying tail of the dye emission (fluorescence) spectrum. At room temperature ($T = 300$ K), the energy of thermal excitation corresponds to the frequency $k_BT/h = 6.25 \times 10^{12}$ Hz. We see that the lower cutoff energy of photons is far larger than the thermal excitation energy. The number of laser photons in the barrel resonator is not altered by the temperature of the dye solution because purely thermal excitation is suppressed by a factor of the order of $\exp(-\hbar\omega_c/k_BT) = \exp(-81.64)$. The $n = 9$ zero of the derivative $J_0'(x)$ of the Bessel function of order $m = 0$ is given by $x_{09} = 25.90367$. As a result, the frequency spacing between adjacent transverse modes (the free spectral range) is $\Delta\omega/2\pi = 7.05 \times 10^{13}$ Hz, which is comparable to the spectral width of the dye emission (fluorescence). Furthermore, the free spectral range $\Delta\omega$ is much larger than the thermal excitation energy k_BT in frequency units. The photon frequencies will accumulate within a range $k_BT/h = 6.25 \times 10^{12}$ Hz above the low-frequency cutoff.

From the formula $\Omega = v/\sqrt{r_0 R}$, the trapping frequency of the harmonic potential is obtained as $\Omega = 2.599 \times 10^{11}$ s^{-1}, or $\Omega/2\pi = 4.137 \times 10^{10}$ Hz. The mean longitudinal excitation number (per plane) of the one-dimensional photon gas is $k_BT/\hbar\Omega = 151.10$. The ratio is quite considerable, so that the longitudinal motion is quasicontinuous. From the formula $m_{ph} = \hbar\omega_c/v^2$, the effective mass of one-dimensional photons is computed as $m_{ph} = 6.655 \times 10^{-36}$ kg. The de Broglie wavelength associated with the thermal motion in the resonator axis is defined by $\lambda_{th} = h/\sqrt{2\pi m_{ph}k_BT}$. The computation gives $\lambda_{th} = 1.592$ μm. Now we introduce the quantity ρ to denote the number density in photons per length. The mean interphoton spacing is of order ρ^{-1}. The BEC in a photon gas sets in when λ_{th} is comparable to ρ^{-1}. According to Refs. [14,15], the dye absorption spectrum overlaps the upper-lying tail of the dye emission (fluorescence) spectrum. This overlapping leads to the nonradiative decay of photons. To compensate for the loss rate, the dye is pumped with an external laser beam. The electronic excitations in the dye form a thermal bath that exchanges particles with the photon gas. The photon gas is thermalized to a thermal bath near room temperature. Thus, the photon gas is seen as an open system in the sense of a grand-canonical ensemble. The pumping maintains a steady state in which the average photon number N_{ph} will be proportional to the number of electronic excitations $N_{\rm exc}$ and is determined by $N_{ph}/N_{\rm exc} = \tau_{ph}/\tau_{\rm exc}$, where τ_{ph} and $\tau_{\rm exc}$ denote the lifetimes of photons and electronic excitations, respectively. $\tau_{ph} \cong 20$ ps and $\tau_{\rm exc}$ is of the order of a nanosecond.

IV. QUANTUM STATISTICAL PROPERTIES OF A FINITE NUMBER OF IDEAL PHOTONS

Because the reflectivity of the cavity mirror is near unity in the relevant wavelength regime and the dye molecules are pumped with a laser beam, the number of one-dimensional photons is conserved in the experiment under study. As a result, the one-dimensional photon gas possesses a nonzero chemical potential μ. The projection of the spin operator along the z direction is called the helicity σ, and spin-1 photons have $\sigma = \pm 1$. At temperature T, $N_{l\sigma}$ denotes the thermal population of the photon number occupying level l and helicity σ. In the usual method, one finds that

$$N_{l\sigma} = \frac{1}{e^{(\mathcal{E}_l-\mu)/k_BT} - 1}, \tag{10}$$

where k_B is Boltzmann's constant. Equation (10) is the well-known Bose-Einstein distribution. The chemical potential μ is determined by the constraint that the total number of massive

photons in the system is N:

$$\sum_{l\sigma} N_{l\sigma} = N. \tag{11}$$

The phenomenon of BEC for noninteracting massive photons is fully described by Eqs. (10) and (11). The remaining thing is to determine the chemical potential as a function of N and T.

To this end, we first rewrite Eq. (8) as $\mathcal{E}_l = l\hbar\Omega + \frac{1}{2}\hbar\Omega$. On putting the last equation into Eq. (10), Eq. (10) can be rewritten as follows:

$$N_{l\sigma} = \frac{ze^{-\beta l\hbar\Omega}}{1 - ze^{-\beta l\hbar\Omega}}, \tag{12}$$

where $\beta = 1/k_B T$. The fugacity z can be expressed as $z = \exp(\beta\mu^*)$, and we have introduced an effective chemical potential $\mu^* = \mu - \frac{1}{2}\hbar\Omega$. The ground state of the system is the state with $l = 0$. Since we have moved the zero-point energy into the effective chemical potential, the energy of the ground state has been taken to be zero. From Eq. (12), the number of massive photons in the ground state is $N_0 = 2z/(1 - z)$. The ground-state population diverges as $z \to 1$, and hence, BEC occurs at $z = 1$. For this reason N_0 is called the photon number of the condensate. When we separate the ground-state population in Eq. (11), Eq. (11) becomes

$$\frac{2z}{1 - z} + \sum_{l\neq 0,\sigma} N_{l\sigma} = N. \tag{13}$$

Putting Eq. (12) into Eq. (13) and completing the summation over l and σ, we then obtain

$$\frac{2z}{1 - z} + 2\sum_{j=1}^{\infty} \frac{z^j q^j}{(1 - q^j)} = N, \tag{14}$$

where $q = \exp(-\beta\hbar\Omega)$. In the above summation, we use the degeneracy factor 2 to account for the two possible polarizations. The infinite series converges, and its analytical expression can be obtained as

$$F_q(x) = \sum_{j=1}^{\infty} \frac{z^j q^j}{(1 - q^j)} = \frac{\ln(1 - q) + \psi_q(x)}{\ln q}, \tag{15}$$

where $x = 1 - \mu^*/\hbar\Omega$. $\psi_q(x)$ is the q-digamma function defined by $\psi_q(x) = d[\ln\Gamma_q(x)]/dx$, where $\Gamma_q(x)$ is the q-gamma function defined by

$$\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}}, \tag{16}$$

when $|q| < 1$ and $x \neq 0, -1, -2, \ldots$. The q-gamma function was introduced by Jackson [32], and the q-digamma function was introduced by Krattenthaler and Srivastava [33]. In recent decades the q-gamma function and the q-polygamma function have gained extensive applications in science and technology [34].

To acquire a simple expression of $F_q(x)$, we consider the high-temperature limit of $k_B T \gg \hbar\Omega$. In this case, we find that $q \to 1$ and

$$|\ln(1 - q)| \gg |\psi_q(x)|. \tag{17}$$

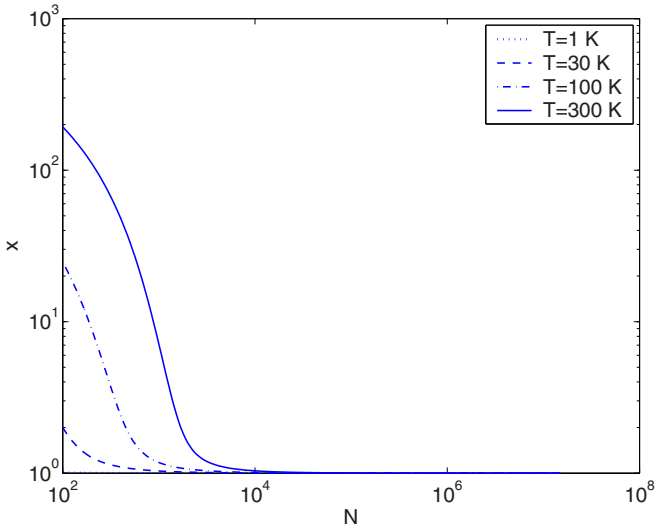

FIG. 2. The variation of the reduced chemical potential x with the photon number N in a gas of one-dimensional photons.

Equation (15) is then reduced to the form:

$$F_q(x) = \frac{\ln(1 - q)}{\ln q}, \quad k_B T \gg \hbar\Omega. \tag{18}$$

One must notice that $F_q(x) = 0$ at $T = 0$ K.

Because the temperature T appears in the fugacity z, the fugacity z does not parametrize the chemical potential μ^* by much, and so z is not a good physical quantity. The quantity x parameterizes the chemical potential μ^* a lot, and so x is a good physical quantity. As a result, the quantity x is called the reduced chemical potential, and the reduced chemical potential x is dimensionless. In the same way, the quantity q parameterizes the temperature T a lot, and so q is a good physical quantity. In terms of the good physical quantities x and q, Eq. (14) is cast into a simple form:

$$\frac{2q^{x-1}}{1 - q^{x-1}} + 2F_q(x) = N, \tag{19}$$

where $F_q(x)$ is given by Eq. (15) and $2F_q(x)$ represents the photon number in vapor. The reduced chemical potential x can be determined numerically from Eq. (19). x is a function of temperature T and photon number N. Once x is known, the number of massive photons in the ground state can be obtained from the relation $N_0 = 2q^{x-1}/(1 - q^{x-1})$. To satisfy Eq. (19), it is necessary that $x \geqslant 1$. When $x = 1$, a one-dimensional photon gas is in the state of BEC. All the information about the BEC of a one-dimensional photon gas is embraced in the analytical function $F_q(x)$.

According to the definition $k_B T = \hbar\Omega$, the axial frequency of the harmonic potential corresponds to the temperature $T = 1.985$ K. In the above statistical mechanics, we introduce the concept of temperature of a one-dimensional photon gas. The temperature is meaningless when the photon number is smaller than 40, and it is meaningful when $N > 40$. According to Eq. (19), the variation of the reduced chemical potential x with the photon number N is given in Fig. 2 for various T. It is interesting to note that when $T \leqslant 1$ K, $x = 1$ for all N. Namely, when $\Omega/2\pi = 4.137 \times 10^{10}$ Hz and $T \leqslant 1$ K, a

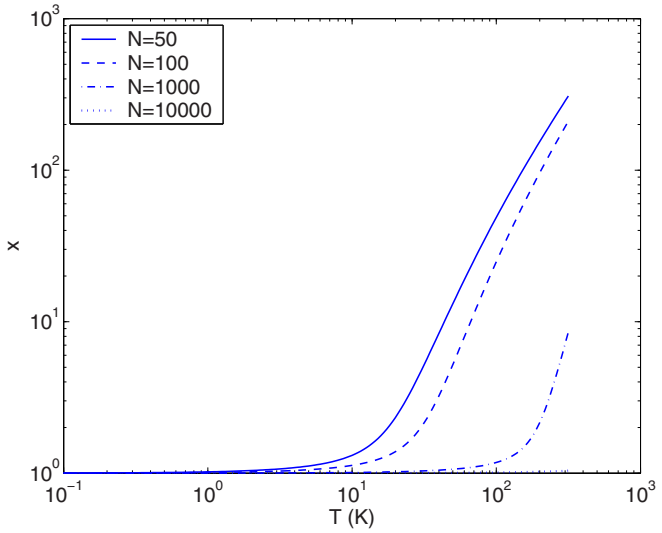

FIG. 3. The variation of the reduced chemical potential x with the temperature T in a gas of one-dimensional photons.

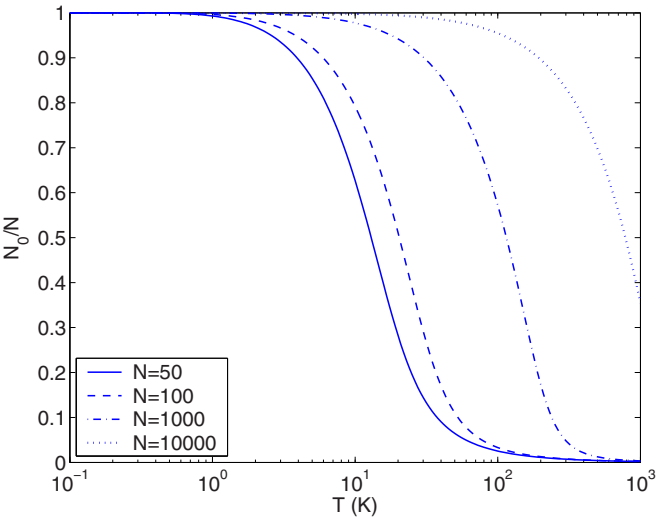

FIG. 4. The variation of the condensate fraction N_0/N with the temperature T in a gas of one-dimensional photons.

one-dimensional photon gas is always in the state of BEC no matter how large the photon number N is. Note that when $T \leqslant 300$ K and $N \geqslant 10^4$, $x = 1$. Namely, when $T \leqslant 300$ K and $N \geqslant 10^4$, a one-dimensional photon gas is always in the state of BEC. Furthermore, Fig. 2 shows that when $N \to \infty$, $x = 1$ for all T. According to Eq. (19), the variation of the reduced chemical potential x with the temperature T is given in Fig. 3 for various N. It is interesting to note that at $N = 10^4$, $x = 1$ for $T \leqslant 300$ K. For a fixed N, x is a monotonically increasing function of temperature T.

At this point, we need to calculate the condensate fraction N_0/N, which is given by

$$N_0/N = \frac{2q^{x-1}}{N(1-q^{x-1})}. \tag{20}$$

The reduced chemical potential can then be used in Eq. (20) to obtain the condensate fraction. N_0/N is a function of temperature T and photon number N. When calculating N_0/N, we must combine Eq. (19) with Eq. (20). According to Eq. (20), the variation of the condensate fraction N_0/N with the temperature T is given in Fig. 4 for various N. Figure 4 shows that for a finite number of one-dimensional photons, there is no exact transition temperature T_c. For a fixed N, N_0/N decreases smoothly to zero when T is near T_c. The phenomenon of phase transitions becomes clearer and clearer as the photon number becomes very large. Namely, when $N \geqslant 10^4$, there is an approximate transition temperature T_c. As $T \geqslant T_c$, $N_0/N = 0$. According to Eq. (20), the variation of the condensate fraction N_0/N with the photon number N is given in Fig. 5 for various T. It is interesting to note that at $T = 1$ K, $N_0/N = 1$ for all N. Figure 5 shows that for a fixed T, there is a critical photon number N_c, above which a one-dimensional photon gas is in the state of BEC and below which $N_0/N = 0$. A nice feature of the exact results in Eqs. (19) and (20) is that they are valid for arbitrary T and N.

V. QUANTUM STATISTICAL PROPERTIES OF IDEAL PHOTONS IN THE THERMODYNAMIC LIMIT

We are now going to investigate the thermodynamic limit when $N \to \infty$. To this end, let us rewrite Eq. (19) in the form

$$N_0 + 2F_q(x) = N. \tag{21}$$

It is known that when $N \to \infty, x = 1$. The critical temperature T_c can now be found by setting $N_0 = 0$ and $x = 1$ in Eq. (21). This results in the following equation for the critical temperature:

$$2F_{q_c}(1) = N, \tag{22}$$

where $q_c = \exp(-\hbar\Omega/k_B T_c)$. The function $F_{q_c}(1)$ can be rewritten as

$$L(q_c) = F_{q_c}(1) = \frac{\ln(1-q_c) + \psi_{q_c}(1)}{\ln q_c}, \tag{23}$$

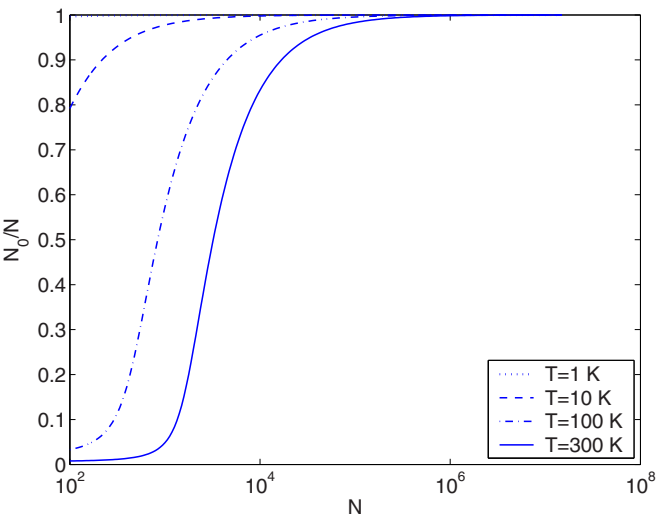

FIG. 5. The variation of the condensate fraction N_0/N with the photon number N in a gas of one-dimensional photons.

where $L(q)$ is a particular Lambert series. The critical temperature T_c given by Eq. (22) is a monotonically increasing function of the number of photons N.

In the limit as $N \to \infty$, we obtain the solution of Eq. (21) as

$$x = \begin{cases} 1, & T \leqslant T_c, \\ \text{the root of } 2F_q(x) = N, & T > T_c. \end{cases} \tag{24}$$

By virtue of Eq. (22), from Eq. (21) we find that the condensate fraction of massive photons is given by

$$\frac{N_0}{N} = \begin{cases} 1 - \frac{L(q)}{L(q_c)}, & T \leqslant T_c, \\ 0, & T > T_c, \end{cases} \tag{25}$$

where $L(q)$ and $L(q_c)$ are given by Eq. (23) and $L(q)/L(q_c)$ represents the vapor fraction. The condensate fraction N_0/N given by Eq. (25) is a monotonically decreasing function of temperature T. $N_0/N = 1$ at $T = 0$ K and $N_0/N = 0$ at $T = T_c$.

We consider the high-temperature limit of $k_B T \gg \hbar\Omega$. In this case, one can find that

$$L(q) = -\frac{k_B T}{\hbar\Omega} \ln[1 - \exp(-\hbar\Omega/k_B T)]. \tag{26}$$

As the first result of Eq. (26), we reduce Eq. (22) into the form

$$T_c = \frac{N/2}{\ln(N/2)} \frac{\hbar\Omega}{k_B}, \tag{27}$$

which is close to the corresponding expression obtained in Ref. [30]. As the second result of Eq. (26), we reduce Eq. (25) into the form

$$\frac{N_0}{N} = \begin{cases} 1 - \frac{T \ln[1-\exp(-\hbar\Omega/k_B T)]}{T_c \ln[1-\exp(-\hbar\Omega/k_B T_c)]}, & T \leqslant T_c, \\ 0, & T > T_c, \end{cases} \tag{28}$$

which is close to the corresponding expression obtained in Ref. [30].

Although Eqs. (22) and (25) are derived in the case of $N \to \infty$, they are valid for arbitrary T and large N ($N \geqslant 10^4$). When $10^3 \leqslant N < 10^4$, Eqs. (22) and (25) are tenable too. Practically, we first employ Eq. (22) to determine the transition temperature T_c and then utilize Eq. (25) to determine the condensate fraction N_0/N. Qualitatively, the phase transition of one-dimensional ideal photons can happen at an exact transition temperature T_c, clearly indicating that the BEC in one dimension exists in a one-dimensional harmonic potential. According to Eq. (22), the variation with N of transition temperature T_c is shown in Fig. 6. The transition temperature T_c is not directly proportional to the total number N of ideal photons, and hence T_c is a complicated function of N. Note that at $N = 1679$, $T_c = 298.11$ K, which is the room temperature. According to Eq. (25), Fig. 7 shows the condensate fraction N_0/N versus the temperature T for $N = 10^4$, 10^5, 10^6, and 10^7. From Eq. (22), we find that at $N = 10^4$, 10^5, 10^6, and 10^7, $T_c = 1.39 \times 10^3$, 1.08×10^4, 8.80×10^4, and 7.40×10^5 K, respectively. Figure 7 demonstrates the clear behavior of phase transitions of BEC in one dimension. From Fig. 7, one sees that for a fixed $T < T_c$, the condensate fraction N_0/N is a monotonically increasing function of the number of photons N. According to Eq. (24), a graph of x is given in Fig. 8. For a fixed N and at $T > T_c$, the reduced chemical potential

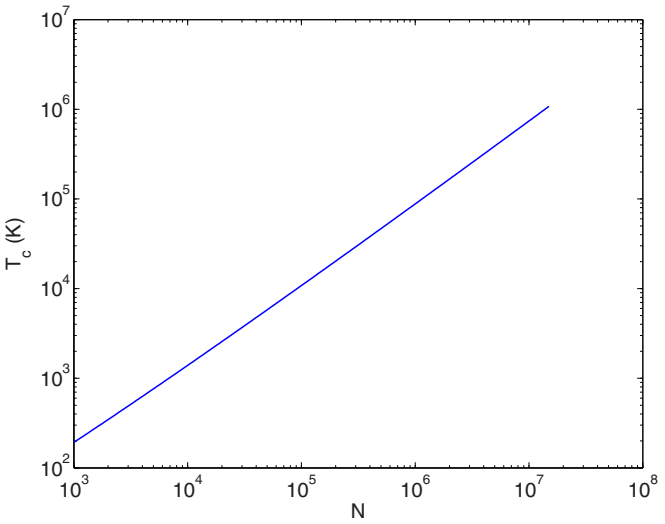

FIG. 6. The variation of transition temperature T_c with photon number N according to Eq. (22).

x increases steeply with temperature T. Many properties of BEC in harmonically trapped one-dimensional photons are exhibited in the present paper.

Because the required experiment for this theory will be implemented at room temperature (say $T = 300$ K), the transition temperature T_c must be larger than the room temperature. Here we take $T_c = 1.392 \times 10^3$ K, which corresponds to the total photon number $N = 10^4$. At $T = 300$ K and $N = 10^4$, from Eq. (20) we find that the condensation fraction is $N_0/N = 0.831$. The temperature is meaningful only when $N > 40$. This requires that the intensity of pumping light should exceed a threshold. On the other hand, if the cavity losses are too high, we will not be able to assign a temperature. The photon system will be out of thermal equilibrium when the intensity of pumping light exceeds a saturation. In order for a photonic Bose-Einstein condensate to be observed,

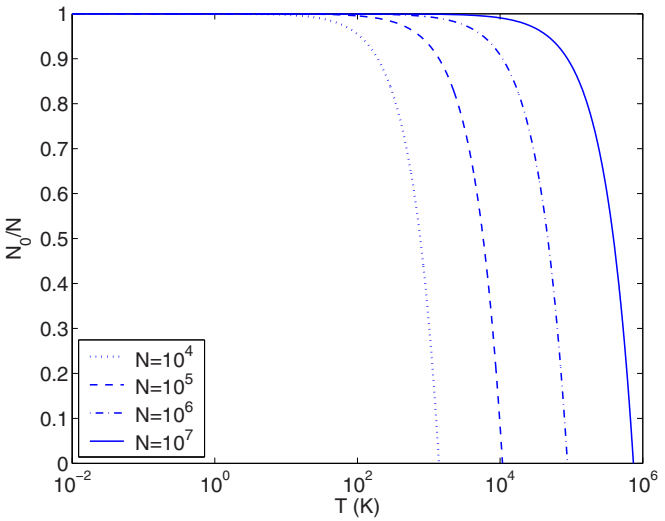

FIG. 7. The condensate fraction of N photons in a one-dimensional harmonic potential versus the temperature. Plots are shown for $N = 10^4$, 10^5, 10^6, and 10^7.

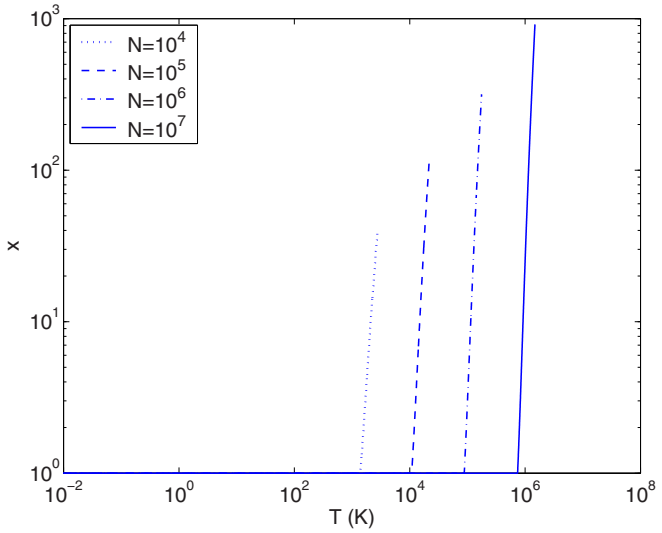

FIG. 8. The reduced chemical potential of N photons in a one-dimensional harmonic potential versus the temperature. Plots are shown for $N = 10^4$, 10^5, 10^6, and 10^7.

the intensity I of pumping light must be in the interval [14,15] 10^2 W cm$^{-2} \lesssim I \lesssim 10^3$ W cm^{-2}. It is plausible that the characteristic intensities of a one-dimensional photon gas are smaller than those of a two-dimensional photon gas by two orders of magnitude. At a fixed temperature T, there is a critical photon number N_c, which is the precise onset of BEC in this one-dimensional and harmonically trapped system. According to the above statistical mechanics, the critical photon number N_c is determined by

$$N_c = 2L(q), \tag{29}$$

where $L(q)$ is given by Eq. (23). At room temperature ($T = 300$ K), the calculation of Eq. (29) gives $N_c = 1691$. It is interesting to note that the critical number N_c of a one-dimensional photon gas is smaller than that of a two-dimensional photon gas by two orders of magnitude. There is a critical optical power P_c, which is directly proportional to the critical photon number N_c. The experiment in a curved-mirror microresonator has demonstrated that the critical optical power P_c grows linearly with the mirror radius of curvature R and possesses the form

$$P_c = N_c \hbar \omega_c \Omega^2 R/4v. \tag{30}$$

We assume that Eq. (30) holds also in a one-dimensional barrel cavity. The calculation of Eq. (30) gives $P_c = 0.02$ W. The statement that the critical power P_c of a one-dimensional photon gas is smaller than that of a two-dimensional photon gas by two orders of magnitude is to be proven. When the intensity I of pumping light is of the order of 10 W cm^{-2}, a one-dimensional photon gas will be in a normal state, in which the one-dimensional photon gas possesses a nonzero chemical potential μ^* ($x > 1$). The chemical potential μ^* is determined by measuring the power P_{out} of light transmitted by an area of the cavity mirror, which corresponds to a photon number N in the cavity, and solving Eq. (19) for $x = 1 - \mu^*/\hbar\Omega$ and $q = \exp(-\hbar\Omega/k_B T)$.

VI. RADIATION PROPERTIES OF A ONE-DIMENSIONAL BARREL CAVITY

In this section we describe the electromagnetic radiation emitted by a one-dimensional barrel cavity in thermal equilibrium at a definite temperature. At first, we must point out that strict one-dimensional systems are very unstable and hence do not exist in nature. The one-dimensional barrel cavity is a quasi-one-dimensional system. The one-dimensional barrel cavity has a specific spectral radiance that depends on the wavelength, the temperature, and the photon number. In order to derive this spectral radiance, we rewrite the oscillator levels of photons as $\mathcal{E}_l = \hbar\omega_{\mathbf{k}} - \hbar\omega_c$, where $\omega_{\mathbf{k}} = c|\mathbf{k}|$ is the frequency of a photon with three-dimensional wave vector $\mathbf{k}$. Now we let $N_{\mathbf{k}\sigma}$ denote the thermal population of the photon number occupying wave vector $\mathbf{k}$ and helicity σ, and from Eq. (10) one has the expression

$$N_{\mathbf{k}\sigma} = \frac{1}{e^{(\hbar\omega_{\mathbf{k}} - \hbar\omega_c - \mu^*)/k_B T} - 1}. \tag{31}$$

The main thermodynamic quantity in a one-dimensional barrel cavity is the total energy E of photons, as given by

$$E = \sum_{\mathbf{k}\sigma} \hbar\omega_{\mathbf{k}} N_{\mathbf{k}\sigma}\ . \tag{32}$$

Because the temperature concerned is much larger than the level spacing, the wave vector $\mathbf{k}$ is quasicontinuous. In the usual way we can alter the summation to an integration in three-dimensional momentum space. Since the trapping potential or the shape of the cavity is incorporated into the chemical potential μ^* in Eq. (31), the one-dimensional barrel cavity can be regarded as uniform in three-dimensional momentum space. Consequently, in the integral we can use the density of states $V/(2\pi)^3$ for a three-dimensional box with volume V. Putting Eq. (31) into Eq. (32), we immediately obtain

$$E = \frac{\hbar c V}{\pi^2} \int_0^\infty \frac{k^3}{e^{(\hbar c k - \hbar\omega_c - \mu^*)/k_B T} - 1} dk\ . \tag{33}$$

Using the relation $k = 2\pi/\lambda$, where λ is the wavelength of light, we can simplify Eq. (33) as

$$E = V \int_0^\infty \rho(\lambda, T, N) d\lambda, \tag{34}$$

where $\rho(\lambda, T, N)$ stands for the spectral energy density of a one-dimensional barrel cavity and is given by

$$\rho(\lambda, T, N) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{(hc/\lambda - hc/\lambda_c - \mu^*)/k_B T} - 1}, \tag{35}$$

where λ_c is the wavelength of the cavity cutoff defined by the relation $\omega_c = 2\pi c/\lambda_c$ and the computation gives $\lambda_c = 587.46$ nm. Equation (35) is called Planck's law.

Since the radiation is the same in all directions and propagates at the speed of light c, the spectral radiance of light transmitted by an area of the cavity mirror is defined by

$$I(\lambda, T, N) = \frac{\rho(\lambda, T, N) c}{4\pi} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{(hc/\lambda - hc/\lambda_c - \mu^*)/k_B T} - 1}, \tag{36}$$

where Eq. (35) is used. Because the chemical potential μ^* is a function of temperature T and photon number N, the

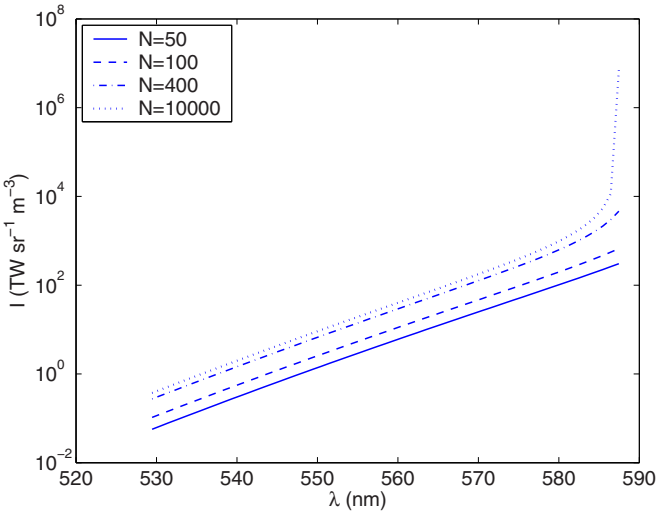

FIG. 9. The spectral radiance of N photons in a one-dimensional harmonic potential versus the wavelength. Plots are shown for $N = 50$, 100, 400, 10 000 and $T = 300$ K.

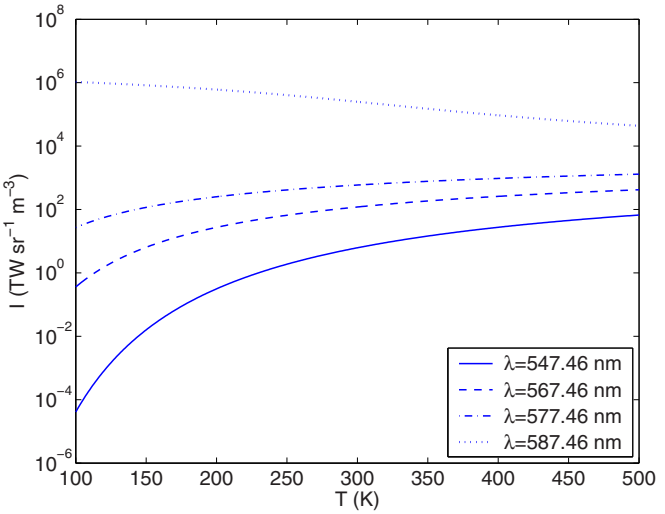

FIG. 10. The spectral radiance of a one-dimensional barrel cavity versus the temperature. Plots are shown for $\lambda = 547.46$, 567.46, 577.46, 587.46 nm and $N = 1679$.

spectral radiance I is a function of wavelength λ, temperature T, and photon number N. The spectral radiance of the cavity is measured in terms of the power emitted per unit area of the cavity mirror, per unit solid angle, per unit wavelength. The SI units of the spectral radiance I are W sr^{-1} m^{-3}.

At this point, we need to calculate the spectral radiance I, which is given by Eq. (36). The chemical potential μ^* is related to the reduced chemical potential x by the relation $\mu^* = \hbar\Omega(1 - x)$. I is a function of wavelength λ, temperature T, and photon number N. When calculating I, we must combine Eq. (19) with Eq. (36). According to Eq. (36), the variation of the spectral radiance I with the wavelength λ is given in Fig. 9 for various N at $T = 300$ K. From Eq. (29), one finds that at $T = 300$ K, $N_c = 1691$. Figure 9 shows that for a fixed T and N, the spectral radiance I is a monotonically increasing function of the wavelength λ. From Fig. 9, we see that the spectral radiance of a one-dimensional barrel cavity has a sharp peak at the wavelength of the cavity cutoff when $N > N_c$. This is a signature of BEC of photons when $N > N_c$. According to Eq. (36), the variation of the spectral radiance I with the temperature T is given in Fig. 10 for various λ at $N = 1679$ K. From Eq. (22), one finds that at $N = 1679$ K, $T_c = 298.11$ K. For a fixed N, the spectral radiance I is a monotonically increasing function of the temperature T when $\lambda < \lambda_c$. From Fig. 10, we see that the spectral radiance of a one-dimensional barrel cavity has a rise at the wavelength of the cavity cutoff when $T < T_c$. This is also a signature of BEC of photons when $T < T_c$. According to Eq. (36), the variation of the spectral radiance I with the photon number N is given in Fig. 11 for various λ at $T = 300$ K. Figure 11 reveals that for a fixed T, the spectral radiance I is a slowly increasing function of the photon number N when $\lambda < \lambda_c$. Further, Fig. 11 shows that for a fixed T, the spectral radiance I is a fast increasing function of the photon number N when $\lambda = \lambda_c$. A nice feature of the exact results in Eq. (36) is that they are valid for arbitrary λ, T, and N. The present theory can address the anomaly of the spectral radiance at the cutoff frequency of a one-dimensional barrel cavity. In fact, the anomaly of the spectral radiance at the cutoff frequency of a two-dimensional cavity has been observed and analyzed in Refs. [15,16]. As is well known, this anomaly is associated with the properties of resonance fluorescence near the cutoff (or a band edge) [35].

VII. DISCUSSION

In this paper we investigate the BEC properties of ideal photons in a one-dimensional harmonic potential. One-dimensional photons in a barrel cavity have an effective mass $m_{ph} = 6.655 \times 10^{-36}$ kg. This agrees with the value $m_{ph} = 6.7 \times 10^{-36}$ kg reported for the two-dimensional photon gas experiment [14]. One-dimensional barrel cavity has a trapping frequency $\Omega = 2.599 \times 10^{11}$ s^{-1}. This also agrees with the value $\Omega = 2.576 \times 10^{11}$ s^{-1} of the two-dimensional

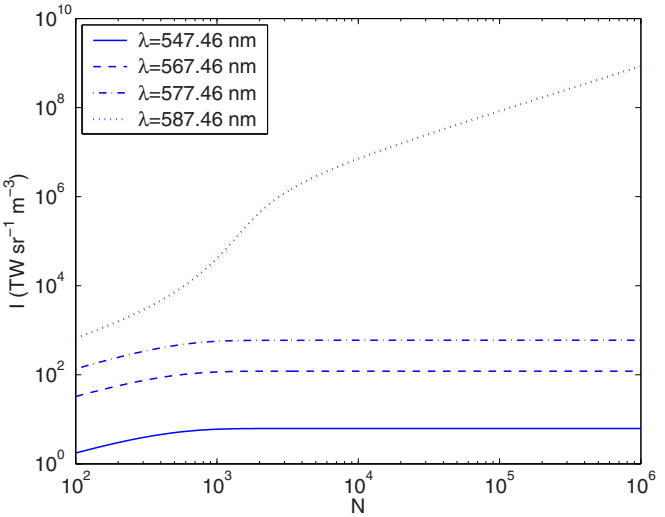

FIG. 11. The spectral radiance of a one-dimensional barrel cavity versus the photon number. Plots are shown for $\lambda = 547.46$, 567.46, 577.46, 587.46 nm and $T = 300$ K.

curved-mirror cavity. Further, a one-dimensional barrel cavity possesses a lower cutoff frequency $\omega_c/2\pi = 5.10 \times 10^{14}$ Hz, which is the same as the value of the two-dimensional curved-mirror cavity. As a result of these three reasons, the BEC properties of a one-dimensional photon gas are very similar to those of a two-dimensional photon gas. With the geometric parameters of the barrel cavity, $R = 0.47$ m, $r_0 = 1.60$ μm, and $d = 4.0$ mm, this type of microcavity can be experimentally designed by future experimental technology.

The principal point in this study is that the problem of BEC in harmonically trapped one-dimensional photons can be solved analytically. We next point out that the analytical solution is associated with an analytical function, which involves a q-digamma function. The q-digamma function was introduced in mathematics 20 years ago and now finds a variety of applications in science and technology. In this study we introduce a new physical quantity called the "reduced chemical potential" to replace the fugacity. In the problem of BEC in harmonically trapped one-dimensional photons, the fugacity z is not a good physical quantity, but the reduced chemical potential x is. The concept of fugacity was introduced in investigating the quantum statistical properties of three-dimensional, number-conserving, uniform, and ideal fermion or boson gases [36]. In these problems the fugacity z is a good physical quantity. As is well known, these problems can be solved analytically, and the analytical solutions are associated with some special functions in mathematics. For example, the analytical solution of the problem of BEC in three-dimensional, uniform, and ideal boson gases is associated with the polylogarithm function $\mathrm{Li}_\nu(z)$ in mathematics, which is also called the Bose function $g_\nu(z)$ in quantum statistical mechanics and is defined by

$$\mathrm{Li}_\nu(z) = g_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}, \quad |z| < 1, \tag{37}$$

where ν is complex and z is the fugacity. Since the discovery of BEC in ultracold dilute atomic gases in 1995, it has been believed that the problem of BEC in harmonically trapped one-dimensional bosons may be solved analytically. In the present paper this is what we succeed in doing.

In this paper we only deal with a gas of harmonically trapped, one-dimensional, and ideal photons. For a gas of harmonically trapped, one-dimensional, and interacting photons, we need to adopt the Bogoliubov theory of BEC. In the Bogoliubov theory of BEC, the expectation value $\langle\hat{\Psi}\rangle$ of the field operator $\hat{\Psi}(\mathbf{r})$ of the photon system is a classical field. Thus, one can write $\Psi_0(\mathbf{r}) = \langle\hat{\Psi}(\mathbf{r})\rangle$. The function $\Psi_0(\mathbf{r})$ is called the wave function of the condensate and plays the role of an order parameter. It is a complex quantity. The order parameter characterizes the BEC phase and vanishes above the critical temperature. At zero temperature, the condensate wave function can be described by a Gross-Pitaevskii equation, in which the interphotonic interaction potential is replaced by a mean-field pseudopotential [37,38]. For a spatially homogeneous gas with repulsive interaction, there are stable solutions to this equation. The off-diagonal long-range order in the one-particle density matrix in BEC theory is associated with the macroscopic condensation of massive photons.

To sum up, we have proposed the BEC theory of a one-dimensional photon gas in a barrel optical microresonator filled with a dye solution. The BEC of one-dimensional photons in a barrel optical microresonator possesses some peculiar properties. We have proposed an analytical solution to the problem of BEC in harmonically trapped, one-dimensional, and ideal photons. It is found that the photon number of vapor is characterized by an analytical function, which involves a q-digamma function in mathematics. The numerical calculation of the analytical solution gives many interesting results. In the thermodynamic limit, the analytical expressions of the critical temperature and the condensate fraction are derived. We address the anomaly of the spectral radiance at the cutoff frequency. In this study we introduce a new physical quantity called the reduced chemical potential to replace the fugacity. Our investigation in one dimension provides an example of the BEC of photons in an optical microcavity.

ACKNOWLEDGMENTS

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