Universality of finite-time disentanglement

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In this paper we investigate how common the phenomenon of finite time disentanglement (FTD) is with respect to the set of quantum dynamics of bipartite quantum states with finite-dimensional Hilbert spaces. Considering a quantum dynamics from a general sense as just a continuous family of completely positive trace preserving maps (CPTP) (parametrized by the time variable) acting on the space of the bipartite systems, we conjecture that FTD happens for all dynamics but those when all maps of the family are induced by local unitary operations. We prove that this conjecture is valid for two important cases: (i) when all maps are induced by unitaries and (ii) for pairs of qubits, when all maps are unital. Moreover, we prove some general results about unitaries that preserve product states and about CPTP maps preserving pure states.

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I. INTRODUCTION

Following the definition of entanglement as a resource for nonlocal tasks, as a consequence being quantified [1], the time evolution of this quantity was the subject of intense interest. Typically a composite system will lose its entanglement whenever its parts interact with an environment. It is of great interest then for practical implementations of quantum information protocols, which require entanglement, to understand how the amount of entanglement behaves in time [2].

One characteristic of entanglement dynamics that has drawn a great deal of attention was the possibility of an initially entangled state to lose all its entanglement in a finite time, instead of asymptotically. The phenomenon was initially called entanglement sudden death [3], or finite time disentanglement (FTD). The simplest explanation for this fact is essentially topological: For finite-dimensional Hilbert spaces, the set $\mathcal{S}$ of separable states, where entanglement is null, has nonempty interior, i.e., there are balls consisting entirely of separable states. Therefore, whenever an initially entangled state approaches a separable state in the interior of $\mathcal{S}$ and given that the dynamics of the state is continuous, it must spend at least a finite amount of time inside the set, so entanglement will be null during this time interval [4].

In Refs. [5,6] we explored how typical the phenomenon is (for several paradigmatic dynamics of two qubits and two harmonic oscillators) when one varies the initial states for a fixed dynamics. Here we explore how typical it is with respect to the dynamics themselves. More explicitly, given a dynamics for a composite system, should one expect to find some initially entangled state exhibiting FTD? Here we argue that the answer is generally positive.

The paper is organized as follows. In Sec. II we discuss the generic existence of FTD and illustrate this discussion with a well-known example of a family of maps. In Sec. III we state and prove the technical lemmas and theorems already used in Sec. II. We conclude this work in Sec. IV, discussing further questions and open problems.

II. FINITE-TIME DISENTANGLEMENT

In a very broad sense, we can think of a (continuous-time) quantum dynamical system as given by a family of completely positive trace preserving (CPTP) maps Λ_t, parametrized by the real time variable t for, say, $t \geqslant 0$. If a quantum system is in some state given by a density operator ρ_0 at $t = 0$, for any $t \geqslant 0$ we have the system at the quantum state $\rho(t) = \Lambda_t(\rho_0)$. Of course, one must have $\Lambda_0 = I$, where I is the identity map. Although in some cases a discontinuous family of maps can be a good approximation to describe a process (for example, when a very fast operation is performed on a system or when the system will not be accessed during some time interval), strictly speaking, the family of maps should be at least continuous.

Generally speaking, for a fixed dynamics Λ_t, we say that it shows FTD if there exist an entangled state ρ_{ent} and a time interval (a,b), with $0 < a < b \leqslant \infty$, such that $\Lambda_t(\rho_{\text{ent}})$ is a separable state for all $t \in (a,b)$. In Refs. [4,5] we pointed out that the occurrence of such an effect is a natural consequence of the set of separable states $\mathcal{S}$ having a nonempty interior. Indeed, if an initially entangled state is mapped at some time $\bar{t}$ to a state in the interior of $\mathcal{S}$, given the dynamics continuity, it must spend some finite time inside S to reach that state. During that time interval entanglement is null, although initially the system had some entanglement. We formally state this fact for future reference as follows.

Proposition 1. If a bipartite quantum dynamical system is such that, for some $\bar{t} > 0$, there exists an initially entangled state ρ_{ent} where its evolved state at time $\bar{t}$ is in the interior of the separable states, there is FTD.

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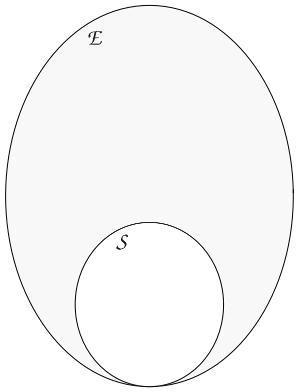

FIG. 1. Pictorial representation of the set of quantum states when $\dim(\mathcal{H}) < \infty$.

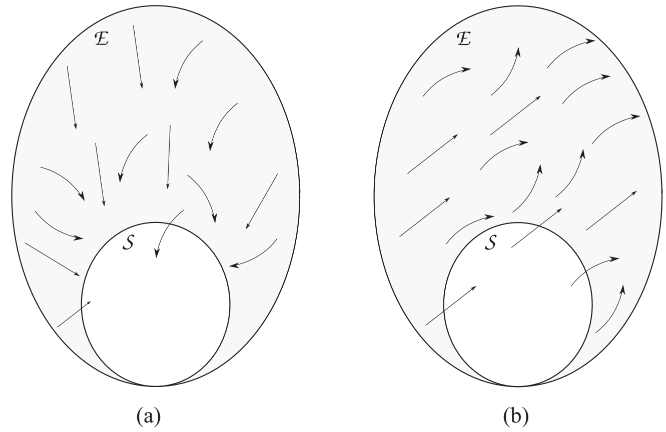

FIG. 2. Arrows represent how initial states are mapped to time-evolved ones: (a) flow directed towards a separable state and (b) flow directed towards an entangled one. In fact, we should stress that it is not always true that the whole family keeps fixed some ρ, i.e., $\Lambda_t(\rho) = \rho\ \forall t \geqslant 0$ for some state ρ.

This proposition is one of the main reasons why we believe the following general conjecture is valid.

Conjecture 1. Given a bipartite quantum dynamical system with finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and a continuous family of CPTP maps Λ_t, there is no finite time disentanglement if and only if for all $t > 0$ there exist unitary operations $U_{A,t}$ and $U_{B,t}$ acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $\Lambda_t(\cdot) = (U_{A,t} \otimes U_{B,t})(\cdot)(U_{A,t} \otimes U_{B,t})^*$.

In physical terms, this says that FTD does not take place only in the extremely special situation in which the pair of systems is closed (or at most interacting with a classical external field) *and* noninteracting. That is, whatever interaction they may have, with each other *or* with a third quantum system (such as a reservoir), FTD takes place for some entangled state. From now on, we denote the family of dynamics contained in Conjecture 1 by $\mathcal{F}_{\mathcal{H}_A,\mathcal{H}_B}$, that is,

$$\mathcal{F}_{\mathcal{H}_A,\mathcal{H}_B} = \{\{\Lambda_t(\cdot)\}_{t\geqslant 0}; \{\Lambda_t(\cdot)\}_{t\geqslant 0} \quad \text{is continuous and} \\ \Lambda_t(\cdot) = (U_{A,t} \otimes U_{B,t})(\cdot)(U_{A,t} \otimes U_{B,t})^*\}. \tag{1}$$

Once again, the intuition behind Conjecture 1 is geometric. Figure 1 shows a pictorial representation of the set of quantum states when the Hilbert space is finite dimensional, with the distinguishing property of the set of separable states having a nonempty interior. In Fig. 2 the arrows indicates the mapping of initial states to their corresponding evolved ones, at an instant of time $\bar{t} > 0$. Note that all CPTP maps must have at least one fixed point and all other states cannot increase their distance to that fixed one; therefore for each instant of time $t \geqslant 0$ we can identify a direction for the flow of states. It is expected that if the flow is directed towards a separable state, some entangled states will be mapped inside the separable set (2 a). However, even in the case where the flow is directed towards an entangled one, if the displacement is small enough, some entangled state located "behind" the set of separable states will be mapped inside it (2 b). Below we prove this statement under some special conditions.

A. Closed systems

We start with the additional assumption that the bipartite system dynamics is induced by unitary operations for all $t > 0$ [there is some U_t acting on $\mathcal{H}_{AB}$ such that $\Lambda_t(\cdot) = U_t(\cdot)U_t^*$]. That is, the pair of systems may have any interaction with each other and they can even interact with classical external sources (for instance, their Hamiltonian may vary in time due to an external control of some of its parameters). Under such conditions, FTD is a consequence of Proposition 1 above and Theorem 5 (discussed in Sec. III).

Theorem 1. If a bipartite system has dynamics given by $\Lambda_t(\cdot) = U_t(\cdot)U_t^*$ for all $t > 0$, there is no FTD if and only if $\{\Lambda_t\}_{t\geqslant 0} \in \mathcal{F}_{\mathcal{H}_A,\mathcal{H}_B}$.

Proof. Indeed, if the family Λ_t is such that, for some $\bar{t} > 0$, $U_{\bar{t}}$ is not a local unitary operation, there exists an entangled state $|\psi_E\rangle$ such that $|\psi_P\rangle = U_{\bar{t}}\,|\psi_E\rangle$ is a product state (see Corollary 2). Take small enough $0 < \lambda < 1$ such that $\rho_E = \lambda\frac{I}{d_A d_B} + (1-\lambda)|\psi_E\rangle\langle\psi_E|$ is still an entangled state. We then have that $\Lambda(\rho_E) = \lambda\frac{I}{d_A d_B} + (1-\lambda)|\psi_P\rangle\langle\psi_P|$ is a state in the interior of the set of separable states (a convex combination of an arbitrary point of a convex set with a point in the interior of it results in an element also in its interior [7]). By Proposition 1, FTD takes place. ■

B. Pair of qubits

Physically, although Theorem 1 allows for very general interactions between the systems, it is restrictive with respect to their interaction with their environment, since this environment must be effectively classic. Here we greatly relax this restriction, with the consequence of diminishing the range of quantum systems considered.

Theorem 2. If a bipartite system with Hilbert space $\mathcal{H}_{AB}$, where $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$, has a dynamics such that $\Lambda_t(\mathbb{1}) = \mathbb{1}$ for all $t \geqslant 0$ (i.e., each map is unital), there is no FTD if and only if $\{\Lambda_t\}_{t\geqslant 0} \in \mathcal{F}_{\mathcal{H}_A,\mathcal{H}_B}$.

Proof. For an arbitrary instant of time t, we have the following four possibilities for the corresponding CPTP map Λ_t: (i) It is induced by a local unitary operation, (ii) it is induced by a composition of a local unitary operation with the SWAP operator, (iii) it is induced by a unitary operation that is neither local nor the composition of a local unitary with the SWAP operator, and (iv) it is not induced by any unitary. Let us look at each situation.

(i) Of course, if this holds for all $t > 0$, we do not have FTD.

(iii) Here we can just apply Theorem 1 to show that there is FTD.

(iv) We can find a maximally entangled state ρ_E such that $\Lambda_t(\rho_E)$ is mixed (see Theorem III). If $\lambda_-(\rho)$ is the smallest eigenvalue of the partial transposition of ρ, we have that $\lambda_-(\rho_E) = -\frac{1}{2}$ and $\lambda_-[\Lambda(\rho_E)] = \delta > -\frac{1}{2}$ (see Ref. [8]). We can choose $0 < p < 1$ such that $\lambda_-[p\rho_E + (1-p)\frac{\mathbb{1}}{4}] = p(-\frac{1}{2} - \frac{1}{4}) + \frac{1}{4} < 0$ and $\lambda_-[p\Lambda(\rho_E) + (1-p)\frac{\mathbb{1}}{4}] = p(\delta - \frac{1}{4}) + \frac{1}{4} > 0$. That is, the initial state $p\rho_E + (1-p)\frac{\mathbb{1}}{4}$ is entangled but its time evolved state at $\bar{t}$, $p\Lambda(\rho_E) + (1-p)\frac{\mathbb{1}}{4}$, is in the interior of the set of separable states. By Proposition 1, we have FTD.

(ii) Finally, if this is the case, the continuity of the family of maps allows us to conclude the existence of a $0 < \bar{t} < t$ where $\Lambda_{\bar{t}}$ fits in either case (iii) or (iv), since the set of CPTP maps induced by such unitaries is disjoint from the set induced by local unitaries (a continuous path between two disjoint sets must necessarily pass through the complement of them). ■

C. Example: Markovian dynamics

A Markovian dynamics [9] is distinguished by a semigroup property satisfied by the family of CPTP maps

$$\Lambda_{t+t'} = \Lambda_t \circ \Lambda_{t'} \tag{2}$$

for all $t,t' \geqslant 0$. It holds then [10] that the dynamics can be equivalently described by a differential equation (a Lindblad equation)

$$\frac{d\rho(t)}{dt} = -i[H,\rho] + \sum_{i=1}^{N}\left(A_i\rho A_i^* - \frac{1}{2}\{A_i^*A_i,\rho\}\right), \tag{3}$$

where H is self-adjoint while A_i are linear operators. Lindbladian equations can describe a plethora of physical phenomena such as the dissipation of electromagnetic field modes of a cavity, spontaneous emission of atoms, and spin dephasing due to a random magnetic field. Therefore, despite the fact that the semigroup condition is somewhat restrictive, it is satisfied by many relevant quantum systems. The first term on the right-hand side (rhs) generates a unitary evolution and can usually be interpreted as the Hamiltonian evolution of the isolated system. The term involving the operators A_i is usually called a dissipator, being responsible for the contractive part of the dynamics.

When an operator A_i is proportional to the identity it does not contribute to the dynamics. Moreover, the dynamics will preserve the purity of initial states if and only if all operators A_i are of such kind (that is, the dynamics is Hamiltonian).

Lemma 1. For $\rho(t)$, a solution of Eq. (3) with initial condition $|\psi\rangle\langle\psi|$, it holds that $\lim_{t\to 0}\frac{d\,\mathrm{Tr}[\rho^2(t)]}{dt} = 0$ for all $|\psi\rangle$ if and only if $A_i = \lambda_i I$ for $i = 1,\dots,N$.

Proof. Indeed, for $t > 0$,

$$\frac{d\,\mathrm{Tr}[\rho^2]}{dt} = 2\,\mathrm{Tr}\left(\frac{d\rho}{dt}\rho\right) = 2\,\mathrm{Tr}\left(-i[H,\rho]\rho + \sum_{i=1}^{N} A_i\rho A_i^*\rho - \frac{1}{2}\{A_i^*A_i,\rho\}\rho\right).$$

Since $\lim_{t\to 0}\rho = |\psi\rangle\langle\psi|$, it follows that

$$\lim_{t\to 0}\frac{d\,\mathrm{Tr}[\rho^2(t)]}{dt} = 2\sum_{i=1}^{N}(|\langle\psi|A_i|\psi\rangle|^2 - ||A_i|\psi\rangle||^2). \tag{4}$$

By the Cauchy-Schwarz inequality

$$|\langle\psi|A_i|\psi\rangle|^2 \leqslant ||\,|\psi\rangle||^2||A_i\,|\psi\rangle||^2 = ||A_i|\psi\rangle||^2,$$

we can conclude the rhs of Eq. (4) is zero if and only if all terms in the sum are zero and $|\psi\rangle \propto A_i\,|\psi\rangle$ for every $i = 1,\dots,N$. These proportionality relations holds for all $|\psi\rangle$ if and only if all A_i are proportional to the identity operator. ■

The above lemma shows that, for every $t > 0$, the CPTP map defined by Eq. (3) is not induced by a unitary operation. It is also easy to check that every CPTP map given by Eq. (3) is unital as long as $\sum_{i=1}^{N}(A_iA_i^* - A_i^*A_i) = 0$. With this in hand, by Theorem 2, we can state the following.

Corollary 1. If a bipartite system with Hilbert space $\mathcal{H}_{AB}$, where $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$, has a dynamics described by Eq. (3), where some A_i is not a multiple of the identity and $\sum_{i=1}^{N}(A_iA_i^* - A_i^*A_i) = 0$, there is FTD.

III. UNITAL PURE STATE PRESERVING MAPS AND PRODUCT PRESERVING UNITARIES

In this section we prove some results about CPTP maps, such as the characterization of unital and pure state preserving ones, which were used in the Sec. II.

Consider a bipartite quantum system with finite-dimensional Hilbert space $\mathcal{H}$. We say that a CPTP map Λ, acting on the set of all density operators $\mathcal{D}(\mathcal{H})$, is pure state preserving if $\Lambda(|\psi\rangle\langle\psi|)$ is a pure state for every pure state $|\psi\rangle$. Trivial examples of such maps are those induced by unitary operations [$\Lambda(\rho) = U\rho U^\dagger$ for a unitary U acting on $\mathcal{H}$] and the constant maps $\Lambda(\rho) = |\phi_0\rangle\langle\phi_0|$ where $|\phi_0\rangle$ is a fixed state. Moreover, a CPTP map is said to be unital if it maps the maximally mixed state on itself.

Theorem 3. Every pure state preserving unital map $\Lambda : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$, where $\dim(\mathcal{H}) = d < \infty$, is induced by a unitary operation.

Proof. Take a Naimark dilation of Λ, that is, a unitary U acting on a larger space $\mathcal{H}\otimes\mathcal{R}$ and a fixed vector $|R\rangle \in \mathcal{R}$ such that $\Lambda(\rho) = \mathrm{Tr}_{\mathcal{R}}[U(\rho\otimes|R\rangle\langle R|)U^*]$ for all $\rho \in \mathcal{D}(\mathcal{H})$.

It must be the case that $U\,|\phi\rangle\otimes|R\rangle$ is a product vector for all $|\phi\rangle \in \mathcal{H}$ since otherwise $\mathrm{Tr}_{\mathcal{R}}[U(|\phi\rangle\langle\phi|\otimes|R\rangle\langle R|)U^*]$ would not be a one-dimensional projector and Λ would not preserve pure states.

Now, if $\{|\phi_j\rangle\}_{j=1}^{d}$ is an orthonormal basis, we have that $\Lambda(|\phi_j\rangle\langle\phi_j|) = P_j$ for some one-dimensional projectors P_j. From Λ being unital, it holds that $\Lambda(\sum_{j=1}^{d}|\phi_j\rangle\langle\phi_j|) = \sum_{j=1}^{d}P_j = I$, so the projectors P_j must be mutually orthogonal.

With the preceding two paragraphs in mind, it must be true that, for $j = 1,\dots,d$, there are normalized vectors $|\psi_j\rangle \in \mathcal{H}$ and $|R_j\rangle \in \mathcal{R}$ such that $U\,|\phi_j\rangle\otimes|R\rangle = |\psi_j\rangle\otimes|R_j\rangle$. Moreover, the set $\{|\psi_j\rangle\}_{j=1}^{d}$ must be orthonormal. On the other hand, for $j = 2,\dots,d$,

$$U(|\phi_1\rangle + |\phi_j\rangle)\otimes|R\rangle = |\psi_1\rangle\otimes|R_1\rangle + |\psi_j\rangle\otimes|R_j\rangle.$$

For the vectors on the rhs of this equation being products, given that $|\phi_1\rangle$ is orthogonal to $|\phi_j\rangle$, it must hold that $|R_j\rangle = z_j\,|R_1\rangle$ for some $z_j \in \mathbb{C}$ of unity modulus. If we define a unitary V acting on $\mathcal{H}$ by $V\,|\phi_j\rangle = z_j\,|\psi_j\rangle$ for $j = 1,\ldots,d$, we get $\Lambda(\rho) = V\rho V^*$ for all density operators ρ. ■

Lemma 2. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two two-dimensional Hilbert spaces. If $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\phi\rangle + e^{i\theta}\,|\psi\rangle$ is a product vector for all $\theta \in \mathbb{R}$, then $|\phi\rangle$ and $|\psi\rangle$ are products too.

Proof. Let $|\psi\rangle = a\,|00\rangle + b\,|11\rangle$ be a Schmidt decomposition for $|\psi\rangle$ and $|\phi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$ be the expression for $|\phi\rangle$ with respect to the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. For arbitrary $z \in \mathbb{C}$, we can define the family of vectors

$$\begin{aligned}|z\rangle &= |\phi\rangle + z\,|\psi\rangle\\ &= (az+\alpha)\,|00\rangle + (bz+\delta)\,|11\rangle + \beta\,|01\rangle + \gamma\,|10\rangle.\end{aligned}$$

For each z, the above state factorizes if and only if the following determinant is zero:

$$D = \begin{vmatrix}(az+\alpha) & \beta\\ \gamma & (bz+\delta)\end{vmatrix} = abz^2 + (a\delta + b\alpha)z + \alpha\delta + \beta\gamma.$$

If $a,b \neq 0$ (i.e., $|\psi\rangle$ is entangled), D cannot be identically zero for all values of z. Therefore, $|\psi\rangle$ must be a product. By similar reasoning, we conclude that $|\phi\rangle$ is also a product. ■

Lemma 3. Let $\mathcal{H}_A,\mathcal{H}_B$ be two Hilbert spaces with dimension $d \geqslant 2$. If $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\phi\rangle + e^{i\theta}\,|\psi\rangle$ is a product state for all $\theta \in \mathbb{R}$, then $|\phi\rangle$ and $|\psi\rangle$ are products too.

Proof. Let us argue by contradiction. Suppose that $|\psi\rangle$ is entangled; thus in the Schmidt decomposition $|\psi\rangle = \sum_{l=1}^{d} \psi_l\,|ll\rangle$ there are at least two indices l_1 and l_2 such that $\psi_{l_1},\psi_{l_2} \neq 0$. Writing $|\phi\rangle = \sum_{k,j} \phi_{k,j}\,|kj\rangle$ in the same basis as $|\psi\rangle$ and defining $\psi_{k,j} = \psi_k\delta_{k,j}$ we get

$$|\theta\rangle = |\psi\rangle + e^{i\theta}\,|\phi\rangle = \sum_{k,j}(\psi_{k,j} + e^{i\theta}\phi_{k,j})\,|kj\rangle\ \forall\theta \in \mathbb{R}.$$

Therefore, $|\theta\rangle$ is a product, by hypothesis, for all $\theta \in \mathbb{R}$. Projecting $|\theta\rangle$ at the subspace generated by $\{|l_1l_1\rangle, |l_1l_2\rangle, |l_2l_1\rangle, |l_2l_2\rangle\}$, we obtain

$$|\xi_\theta\rangle = \sum_{k,j\in\{l_1,l_2\}}(\psi_{k,j} + e^{i\theta}\phi_{k,j})\,|kj\rangle.$$

Since $|\xi_\theta\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is a product for all values of θ, we can apply Lemma 2 and obtain the desired contradiction. ■

Theorem 4. If Λ is a unital map acting on $\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and preserves the purity of maximally entangled states, then Λ is induced by a unitary operation.

Proof. Take a representation of Λ in terms of a unitary U acting on a larger space $\mathcal{H}_{AB} \otimes \mathcal{H}_R$ such that

$$\Lambda(\rho) = \mathrm{Tr}_R[U(\rho \otimes |R\rangle\langle R|)U^*],$$

where $|R\rangle \in \mathcal{H}_R$. With $U(|00\rangle \otimes |R\rangle) = |\psi\rangle$ and $U(|11\rangle \otimes |R\rangle) = |\phi\rangle$, we have, for all $\theta \in \mathbb{R}$,

$$(|00\rangle + e^{i\theta}\,|11\rangle) \otimes |R\rangle \overset{U}{\mapsto} |\psi\rangle + e^{i\theta}\,|\phi\rangle.$$

As Λ preserves the purity of $(|00\rangle + e^{i\theta}\,|11\rangle)$, the state $|\psi\rangle + e^{i\theta}\,|\phi\rangle$ is a product for all θ, with respect to $\mathcal{H}_{AB} \otimes \mathcal{H}_R$. Lemma 3 implies that $|\psi\rangle$ and $|\phi\rangle$ are both products, that is,

$$|00\rangle \otimes |R\rangle \overset{U}{\mapsto} |\psi_{00}\rangle \otimes |R_{00}\rangle, \tag{5a}$$

$$|11\rangle \otimes |R\rangle \overset{U}{\mapsto} |\psi_{11}\rangle \otimes |R_{11}\rangle. \tag{5b}$$

Let $\mathfrak{B} = \{|\Psi_\pm\rangle, |\Phi_\pm\rangle\}$ be the Bell basis in $\mathcal{H}_{AB}$. The map Λ satisfies

$$\begin{aligned}\mathbb{1} = \Lambda(\mathbb{1}) = \Lambda(&|\Phi_+\rangle\langle\Phi_+|\\ &+|\Phi_-\rangle\langle\Phi_-| + |\Psi_+\rangle\langle\Psi_+| + |\Psi_-\rangle\langle\Psi_-|).\end{aligned}$$

Since the images $\Lambda(|\Phi_\pm\rangle\langle\Phi_\pm|)$ and $\Lambda(|\Psi_\pm\rangle\langle\Psi_\pm|)$ are four one-dimensional projectors (Λ preserves the purity of maximally entangled states) that sum up to the identity, they must be mutually orthogonal.

Observe that the combinations $(|\psi_{00}\rangle \otimes |R_{00}\rangle) \pm (|\psi_{11}\rangle \otimes |R_{11}\rangle)$ must be products with respect to $\mathcal{H}_{AB} \otimes \mathcal{H}_R$ because they are images of $|\Phi_\pm\rangle \otimes |R\rangle$ under U. We state that

$$|R_{00}\rangle = e^{i\gamma}\,|R_{11}\rangle.$$

Otherwise $|\psi_{00}\rangle \propto |\psi_{11}\rangle$ and then $\Lambda(|\Phi_+\rangle\langle\Phi_+|) = |\Psi_{00}\rangle\langle\Psi_{00}| = \Lambda(|\Phi_-\rangle\langle\Phi_-|)$, contradicting the fact that $\Lambda(|\Phi_\pm\rangle\langle\Phi_\pm|)$ are mutually orthogonal. Again, from

$$|01\rangle \otimes |R\rangle \overset{U}{\mapsto} |\psi_{01}\rangle \otimes |R_{01}\rangle,$$

$$|10\rangle \otimes |R\rangle \overset{U}{\mapsto} |\psi_{10}\rangle \otimes |R_{10}\rangle,$$

we derive that $|R_{01}\rangle = e^{i\delta}\,|R_{10}\rangle$. Now define $|\xi\rangle = a\,|\Phi_+\rangle + b\,|\Phi_-\rangle + c\,|\Psi_+\rangle + d\,|\Psi_-\rangle$ for a suitable choice of constants $a,b,c,d \neq 0$ such that $|\xi\rangle$ is maximally entangled. Therefore,

$$\begin{aligned}U(|\xi\rangle \otimes |R\rangle) = (a\,|\psi_{00}\rangle + be^{-i\gamma}\,|\psi_{11}\rangle) \otimes |R_{00}\rangle\\ + (c\,|\psi_{01}\rangle + de^{-i\delta}\,|\psi_{10}\rangle) \otimes |R_{01}\rangle\end{aligned}$$

and then $|R_{00}\rangle = e^{i\beta}\,|R_{01}\rangle$. We can define a unitary operator V, acting on $\mathcal{H}_{AB}$, given by

$$|00\rangle \overset{V}{\mapsto} |\psi_{00}\rangle, \tag{6a}$$

$$|11\rangle \overset{V}{\mapsto} e^{-i\gamma}\,|\psi_{11}\rangle, \tag{6b}$$

$$|01\rangle \overset{V}{\mapsto} e^{i(\delta-\beta-\gamma)}\,|\psi_{01}\rangle, \tag{6c}$$

$$|10\rangle \overset{V}{\mapsto} e^{-i(\delta+\gamma)}\,|\psi_{10}\rangle. \tag{6d}$$

With this definition, we have $\Lambda(\cdot) = V(\cdot)V^*$. ■

When $\mathcal{H}_A = \mathcal{H}_B$, we can define the so-called SWAP operator S by $S(|\phi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\phi\rangle$. If the Hilbert spaces are not the same but have the same dimension, we can take any isomorphism $\Psi : \mathcal{H}_A \to \mathcal{H}_B$ between them and define the operators $S_\Psi = (\Psi^{-1} \otimes I_B) \circ S \circ (\Psi \otimes I_B)$, where I_B is the identity operator on $\mathcal{H}_B$, i.e., $S_\Psi\,|\phi\rangle \otimes |\psi\rangle = \Psi^{-1}(|\psi\rangle) \otimes \Psi(|\phi\rangle)$, which we will also denote by SWAP.

The following theorem characterizes unitary operations acting on composite Hilbert spaces that preserve product vectors.

Theorem 5. Let U be a unitary operation acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_{A\,(B)}$ has finite dimension $d_{A\,(B)} \geqslant 2$. Then U is product preserving if and only if it is a local unitary operation or, for the case $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$, a composition of a local unitary operation with a SWAP operator.

Proof. Consider an orthonormal basis in each space $\{|j\rangle_A\}_{j=0}^{\dim(\mathcal{H}_A)-1}$ and $\{|k\rangle_B\}_{k=0}^{\dim(\mathcal{H}_B)-1}$. The unitary operation must map states $|j\rangle_A \otimes |k\rangle_B$ into elements $|\psi_{jk}\rangle_A \otimes |\phi_{jk}\rangle_B$, which are mutually orthogonal. Since the images of the product vectors $(|j\rangle_A + |j'\rangle_A) \otimes |k\rangle_B$, that is, $|\psi_{jk}\rangle_A \otimes |\phi_{jk}\rangle_B + |\psi_{j'k}\rangle_A \otimes |\phi_{j'k}\rangle_B$, are also product vectors, we must have one of two options

$$|\psi_{jk}\rangle_A \perp |\psi_{j'k}\rangle_A\,, \quad |\phi_{jk}\rangle_B \propto |\phi_{j'k}\rangle_B \tag{7a}$$

or

$$|\phi_{jk}\rangle_B \perp |\phi_{j'k}\rangle_B\,, \quad |\psi_{jk}\rangle_A \propto |\psi_{j'k}\rangle_A\,. \tag{7b}$$

For a fixed k, if one of the options is valid for a pair j and j', it must be valid for all such pairs. Indeed, suppose that the first option is valid for, say, $j=0$ and $j'=1$ and the second for $j=0$ and $j'=2$. The image of the product vector $(|1\rangle_A + |2\rangle_A) \otimes |k\rangle_B$, given by $|\psi_{1k}\rangle_A \otimes |\phi_{1k}\rangle_B + |\psi_{2k}\rangle_A \otimes |\phi_{2k}\rangle_B$, would be an entangled vector since we would have $|\psi_{1k}\rangle_A \perp |\psi_{0k}\rangle_A$, $|\psi_{2k}\rangle_A \propto |\psi_{0k}\rangle_A$, $|\phi_{1k}\rangle_B \propto |\phi_{0k}\rangle_B$, and $|\phi_{2k}\rangle_B \perp |\phi_{0k}\rangle_B$. Therefore, $|\psi_{1k}\rangle_A \perp |\psi_{2k}\rangle$ and $|\phi_{1k}\rangle_B \perp |\phi_{2k}\rangle_B$.

(i) Assume that (7a) is true. That means that the vectors $|\phi_{jk}\rangle_B$ are proportional to each other for fixed k, while the vectors $|\psi_{jk}\rangle_A$, also for fixed k, form an orthonormal basis. We can write then $U\,|j\rangle_A \otimes |k\rangle_B = e^{i\theta_{jk}}\,|\psi_{jk}\rangle_A \otimes |\phi_{0k}\rangle_B$.

If we consider the image of the vectors $|j\rangle_A \otimes (|k\rangle_B + |k'\rangle_B)$, we deduce that we have the following options:

$$|\phi_{jk}\rangle_B \perp |\phi_{jk'}\rangle_B\,, \quad |\psi_{jk}\rangle_A \propto |\psi_{jk'}\rangle_A \tag{8a}$$

or

$$|\psi_{jk}\rangle_A \perp |\psi_{jk'}\rangle_A\,, \quad |\phi_{jk}\rangle_B \propto |\phi_{jk'}\rangle_B\,. \tag{8b}$$

Again, similarly to what we have above, if one of the option is valid for a pair k and k', for fixed j, it must be valid for all such pairs. However, given that (7a) is true, now only (8a) can also be true. Indeed, if (8b) were true, we would have, for example, the subspace generated by the vectors $\{|j\rangle_A \otimes |0\rangle_B\,, |0\rangle_A \otimes |k\rangle_B\}$, of dimension $\dim(\mathcal{H}_A) + \dim(\mathcal{H}_B) - 1$, mapped to the subspace $\mathcal{H}_A \otimes |\phi_{00}\rangle$, of dimension $\dim(\mathcal{H}_A)$, contradicting the fact U is unitary.

Since we have that (8a) is true, we can write $U\,|j\rangle_A \otimes |k\rangle_B = e^{i\theta_{jk}}\,|\psi_{j0}\rangle_A \otimes |\phi_{0k}\rangle_B$. Using this expression and demanding that the states $(|j\rangle_A + |j'\rangle_A) \otimes (|k\rangle_B + |k'\rangle_B)$ are of the product form for all pairs j,j' and k,k', we obtain $e^{i(\theta_{jk}+\theta_{j'k'})} = e^{i(\theta_{jk'}+\theta_{j'k})}$. In particular, if $k'=j'=0$, we get $\theta_{jk} = \theta_{j0} + \theta_{0k} (\mathrm{mod}\ 2\pi)$, since $\theta_{00}=0$ by construction. Finally, we have $U = U_A \otimes U_B$ with $U_A\,|j\rangle_A = e^{i\theta_{j0}}\,|\psi_{j0}\rangle_A$ and $U_B\,|k\rangle_B = e^{i\theta_{0k}}\,|\phi_{0k}\rangle_B$.

(ii) Assume that (7b) is true. Note first that it is necessary to have $\dim(\mathcal{H}_A) \geqslant \dim(\mathcal{H}_B)$ since, for fixed k, we are varying over $\dim(\mathcal{H}_A)$ orthonormal vectors on A, which therefore give rise to a set of orthonormal vectors $|\phi_{jk}\rangle_B$ in $\mathcal{H}_B$. So $U(|j\rangle_A \otimes |k\rangle_B) = e^{i\tilde{\theta}_{jk}}\,|\psi_{0k}\rangle_A \otimes |\phi_{jk}\rangle_B$. Now only the option (8b) can be true, so again we have $\dim(\mathcal{H}_B) \geqslant \dim(\mathcal{H}_A)$, and therefore $\dim\mathcal{H}_A = \dim\mathcal{H}_B$, which allows us to write $U(|j\rangle_A \otimes |k\rangle_B) = e^{i\tilde{\theta}_{jk}}\,|\psi_{0k}\rangle_A \otimes |\phi_{j0}\rangle_B$. Considering again that the image of the states $(|j\rangle_A + |j'\rangle_A) \otimes (|k\rangle_B + |k'\rangle_B)$ must be product vectors, we have $\tilde{\theta}_{jk} = \tilde{\theta}_{j0} + \tilde{\theta}_{0k} (\mathrm{mod}\ 2\pi)$. In other words, $U = (U_A \otimes U_B) \circ S_\Psi$, where $U_A\,|j\rangle_A = e^{i\tilde{\theta}_{0j}}\,|\psi_{0j}\rangle$, $U_B = e^{i\tilde{\theta}_{k0}}\,|\phi_{k0}\rangle_B$, and $\Psi\,|k\rangle_A = |k\rangle_B$. ■

Putting these results together we have the following.

Corollary 2. If U is a unitary operator acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_{A\,(B)}$ has finite dimension and preserves entangled states, then it is a local unitary operation or, for the case $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$, a composition of a local unitary operation with a SWAP operator.

Proof. If U preserves entangled states, its inverse U^{-1} preserves product states. From Theorem 5, there are unitaries V_A and V_B acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that $U^{-1} = V_A \otimes V_B$ or $U^{-1} = S \circ V_A \otimes V_B$, therefore $U = U_A \otimes U_B$ or $U = U_A \otimes U_B \circ S$, with $U_A = V_A^{-1}$ and $U_B = V_B^{-1}$. ■

IV. DISCUSSION

Although we could not prove Conjecture 1 in its full generality, we manage to do it for some large and important families of quantum dynamics. They include all possible dynamics for a bipartite closed system, whatever interaction the parts might have and whatever time variation their Hamiltonian may have. For qubits, a much larger class of dynamics was considered, only requiring a technical condition (unitality) on CPTP maps describing the time evolution. Since the proof for qubits seems quite technical and the geometric ingredients are the same for other finite dimensions, the conjecture that the only class of bipartite dynamics not to show FTD is the local unitaries must hold, but still demands a final proof.

The requirement of finite-dimensional Hilbert spaces seems to be essential. Indeed, the geometric insight is based on the fact that the set of separable states has a nonempty interior, which ceases to be true whenever one of the Hilbert spaces is of infinite dimension [11]. Of course, even in that case, where generically one does not expect FTD, many physically relevant dynamics actually can show it, such as those preserving Gaussian states [6].

Another situation where topology changes, and consequently the entanglement dynamics changes, is when one is restricted to pure states. There, the set of separable states (indeed, product states) has an empty interior. For these systems, FTD can only happen if tailored. For example, starting from an entangled state, some family of global unitaries is applied up to a time when the state is a product; from this time on, only local unitaries are applied. This is clearly not generic in the set of dynamics.

As a final comment, it is natural to recall that for practical implementations of quantum information processing, it is important to fight against FTD. Our results about the generic nature of FTD do not make this fight impossible. Even for dynamics where FTD does happen, is is natural to search for initial states where it can be avoided, or at least delayed [2,3,12,13].

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