# Nonlocality with ultracold atoms in a lattice

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We study the creation of nonlocal states with ultracold atoms trapped in an optical lattice. We show that these states violate Bell inequality by measuring one- and two-body correlations. Our scheme only requires beam-splitting operations and global phase shifts, and can be realized within the current technology, employing single-site addressing. This proposal paves the way to study multipartite nonlocality and entanglement in ultracold-atomic systems.

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## I. INTRODUCTION

Bell's theorem [1] unveils the incompatibility between quantum mechanics and a generic local hidden-variable theory via a set of inequalities that can be experimentally tested. The notion of Bell nonlocality (see [2] for a recent review) was originally proposed and mainly developed in the case of two parties. It is now recognized as a crucial resource for several types of quantum information processing, such as secure communication [3,4] and certified random-number generation [5]. Violations of Bell's inequalities have been reported in a variety of bipartite physical systems, giving strong evidence that nature is nonlocal [6]. Experiments have been mostly performed with photons [7–11] and, in recent years, also with ions [12–14] and atom-photon hybrid systems [15,16]. Quantum nonlocality in many-partite systems is, by far, more complex and less developed than in the bipartite case. One crucial obstacle is that multipartite Bell's inequalities [17,18] usually involve high-order correlations that should be measured for $N$ parties in order to demonstrate nonlocality. This poses serious experimental—but also theoretical—challenges for large $N$. Presently, the experimental violation of multipartite Bell's inequalities has been achieved with three [19–21] and four [22,23] photons and trapped ions [24], whereas most of these are limited to Greenberger-Horne-Zeilinger (GHZ) states. Much less is known about the possibility to investigate quantum nonlocality for ultracold neutral atoms. These systems offer the practical advantage of large detection efficiencies and a variety of control techniques, including single-site addressing and fine manipulation of internal and external degrees of freedom. Recent progress has evidenced that ultracold atoms are optimal candidates for the creation of entangled states with a large number of particles [25–29]. However, while entanglement is generally a prerequisite for the violation of Bell's inequalities [30], to date, there are only a few proposals [31–33] to demonstrate quantum nonlocality in these many-body systems.

Here we propose a realistic experimental protocol to create and observe Bell's nonlocality of an arbitrarily large number of neutral atoms trapped in a homogeneous one-dimensional optical lattice. The crucial ingredient of the proposal is the coupling between hyperfine states of atoms in neighboring well. This coupling can be accomplished by resonant light interaction and a moving state-dependent lattice potential, as outlined in Fig. 1. We notice that the combination of internal Rabi splitting and the coherent transport of atoms in a spin-dependent optical lattice was first experimentally demonstrated in [34]. This capability has been further used to generate entangling gates [35,36], superexchange interaction [37], quantum random walks [38], and spin-squeezed states [27]. Our protocol for the violation of Bell inequalities consists of four steps. A Bose gas is initially prepared with one atom per lattice site [see Fig. 1(a)], with each atom having two internal levels, indicated as $a$ and $b$. This can be accomplished by starting from a superfluid gas and then raising the potential barriers in such a way as to enter the Mott-insulator phase [39], eventually suppressing tunneling between neighboring wells. The system is thus prepared in the state $|\psi_0\rangle = \otimes_l |a\rangle_l$, where $|k\rangle_l$ indicates an atom in the internal state $k = a,b$ at lattice

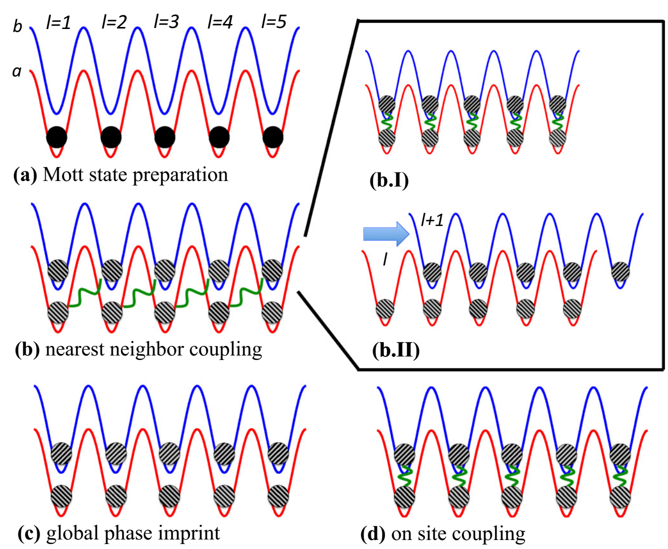


FIG. 1. Quantum nonlocality of neutral atoms in an optical lattice (here limited, for illustration sake, to few lattice sites). The protocol consists of four operations involving two internal (hyperfine) states and is implemented by a state-selective optical lattice. Here, $a$ and $b$ indicate hyperfine levels, and $l$ is the lattice site. (a) Preparation of a Mott state with one atom per lattice site in level $a$, with each atom being schematically represented by a dot. (b) Balance pulse coupling neighboring wells: this is made of a $\pi/2$ Rabi pulse, schematically represented by a green waving line (b.I), followed by a relative shift of the state-selective lattice (b.II). (c) Collective phase shift $\theta$ or $\varphi$; see text. (d) Balance pulse coupling the internal levels of each atom. The protocol ends with the measurement of the number of particles in the internal ground level.

 

site $l = 1,2,\ldots,N$. As a second step, each atom is placed in the coherent superposition $|a\rangle_l \to (|a\rangle_l + |b\rangle_{l+1})/\sqrt{2}$; see Fig. 1(b). This operation (linear in the $N$ particles) creates correlations between lattice sites, which are our parties. It will be shown that these correlations are nonlocal as they are responsible for the violation of Bell's inequalities. Nearest-neighbor coupling can be experimentally realized following two stages: (I) a $\pi/2$ Rabi pulse for atoms at each lattice site, $|a\rangle_l \to (|a\rangle_l + |b\rangle_l)/\sqrt{2}$ [Fig. 1(b.I)], followed by (II) a shift of the state-dependent lattice potential $|b\rangle_l \to |b\rangle_{l+1}$ [Fig. 1(b.II)] that can be realized by changing the polarization angle of two linearly polarized and counterpropagating laser fields [34,40]. As a third step, we apply a local phase shift $e^{i\theta_l \hat{n}_{g,l}}$ at each lattice site [Fig. 1(c)], where $\hat{n}_{g,l}$ is the number operator and $\theta_l$ are a set of local phases. Below we show that the optimal choice for the phase shifts involves imprinting the same phase to each atom. This is an important simplification for the experimental implementation of the protocol. Phase imprinting should happen on a time scale much faster than the interaction time scale between the $|g\rangle_l$ and $|e\rangle_{l+1}$ atoms. We point out that differently from the creation of entangling gates [40,41], our protocol does not involve interaction between atoms. The final operation is an on-site coupling pulse $|a\rangle_l \to (|a\rangle_l + |b\rangle_l)/\sqrt{2}$ and $|b\rangle_l \to (|b\rangle_l - |a\rangle_l)/\sqrt{2}$ [Fig. 1(d)], formally described as $e^{-i\frac{\pi}{2}J_y}$ with $J_y = \frac{1}{2}\sum_{l=1}^N \hat{\sigma}_y^l$ and $\hat{\sigma}_y^{(l)}$ the Pauli matrix for the $l$th site (notice that we could alternatively implement the rotation $e^{-i\frac{\pi}{2}J_x}$ or any balanced rotation in the $x$-$y$ plane). The protocol ends by counting the total number of particles in the internal ground state, $N_g = \sum_{l=1}^N n_{g,l}$, from which we obtain a dichotomic measurement given by the parity $P_g = (-1)^{N_g}$. We emphasize that the whole protocol does not involve vibrational states of the well.

Our criterion for witnessing quantum nonlocality of the ultracold-atomic system is based on the violation of a class of multipartite Bell's inequalities introduced in [42] and involving only one- and two-body correlations. These inequalities are

$$B_N(\theta_i,\varphi_i) = \alpha\left(S_0 + \frac{S_1}{N}\right) + \frac{\gamma}{2}S_{00} + \frac{N}{2}S_{01} - \frac{S_{11}}{2} \geqslant -\beta_c^{(N)}, \tag{1}$$

where

$$S_k = \sum_{i=1}^N \langle \mathcal{M}_k^{(i)}\rangle, \quad S_{kl} = \sum_{\substack{i,j=1\\ i\neq j}}^N \langle \mathcal{M}_k^{(i)}\mathcal{M}_l^{(j)}\rangle \tag{2}$$

denote the one- and two-body correlations, with $\mathcal{M}_k^{(i)}$ ($k = 0,1$) representing the two different local measurement observables realized in well $l$:

$$\begin{aligned}\mathcal{M}_0^{(l)} &= e^{-i\theta_l n_{g,l}} e^{i\frac{\pi}{2}J_y}(-1)^{N_g} e^{-i\frac{\pi}{2}J_y} e^{i\theta_l n_{g,l}},\\ \mathcal{M}_1^{(l)} &= e^{-i\varphi_l n_{g,l}} e^{i\frac{\pi}{2}J_y}(-1)^{N_g} e^{-i\frac{\pi}{2}J_y} e^{i\varphi_l n_{g,l}}.\end{aligned} \tag{3}$$

The two measurements differ by local phase shifts, $\theta_l$ and $\varphi_l$, applied to the ground state. These correspond to the two different detector settings in the Clauser-Horne-Shimony-Holt (CHSH) scheme [43]. We choose $\alpha = N(N-1)(\lceil\frac{N}{2}\rceil - \frac{N}{2})$ and $\gamma = \frac{N(N-1)}{2}$, in accordance with the parameters used in [42] to trace nonlocality of Dicke states, for which the classical limit is set by $\beta_c^{(N)} = \frac{N(N-1)}{2}\lceil\frac{N+2}{2}\rceil$. To illustrate our proposal, we will first focus on the simple case of two atoms trapped in two wells and then generalize the discussion to many atoms and many wells.

## II. TWO-WELLS CASE

For two and three wells, our proposal reduces to the scheme first proposed by Yurke and Stoler [44,45] for optical systems. In the double-well case, the final state is $|\psi_{\text{fin}}\rangle = (e^{i(\theta_1+\theta_2)/2}|\psi_1\rangle + \sqrt{2}|\psi_2\rangle)/4$, where

$$|\psi_1\rangle = c(|1010\rangle + |0101\rangle) + is(|1001\rangle + |0110\rangle), \tag{4}$$

$$|\psi_2\rangle = e^{i\theta_1}(|0200\rangle - |2000\rangle) + e^{i\theta_2}(|0002\rangle - |0020\rangle), \tag{5}$$

$c = 2\cos\frac{\theta_1+\theta_2}{2}$, $s = 2\sin\frac{\theta_1+\theta_2}{2}$, and $|n_{g,1}n_{e,1}n_{g,2}n_{e,2}\rangle$ represents the state of occupation numbers in each mode of the system, $0 \leqslant n_{g,l} \leqslant 2$. From Eq. (5), we can analytically obtain the probability of the possible measurement results. Similarly to Ref. [44], let us denote with $P(\epsilon_1,\epsilon_2)$ the probability that an event $\epsilon_m$ occurs in the well $m$. As we consider only bosonic atoms, these events are elements of the set $[0,G,E,G^2,E^2]$, where $G$ represents one particle in its ground state and $E$ one particle in its excited state. We can distinguish three subsets of events obtained by the measurement of the population of each atomic state. These subsets are, first, the case where both wells are populated with atoms in the same internal state, $A = \{GG,EE\}$, the second one represents the case where both wells are populated but with atoms in different internal states, $B = \{GE,EG\}$, and, finally, we have the case where only one well is populated with the two atoms in the same state, $C = \{G^20,E^20,0G^2,0E^2\}$. The corresponding probabilities are

$$P(\epsilon_1,\epsilon_2) = \begin{cases}\frac{1}{4}\cos^2(\frac{\theta_1+\theta_2}{2}) & \text{if} \quad \epsilon_1,\epsilon_2 \in A,\\ \frac{1}{4}\sin^2(\frac{\theta_1+\theta_2}{2}) & \text{if} \quad \epsilon_1,\epsilon_2 \in B,\\ \frac{1}{8} & \text{if} \quad \epsilon_1,\epsilon_2 \in C.\end{cases} \tag{6}$$

We can see here that, independently of the local phases, if one measures the atomic population in one well, one can predict whether the total results belong to the set $A \cup B$ or $C$. This is coherent with local realism as it is a simple consequence of the conservation of the total number of particles in the system. As a consequence, we can directly ignore the part of the state belonging to the set $C$ and test the nonlocality for the reduced state $|\psi_1\rangle/2$, which we propose to study using the criterion (1). We stress here that for two particles, the class of Bell's inequalities (1) reduces to the well-known CHSH inequality [43]:

$$-2 \leqslant \frac{S_{00} - S_{11}}{2} + S_{01} \leqslant 2. \tag{7}$$

Using the set of measurements described previously leads to the following expression for the mean value of the Bell's operator $B_2(\theta_1,\theta_2,\varphi_1,\varphi_2)$:

$$\begin{aligned}B_2(\theta_1,\theta_2,\varphi_1,\varphi_2) = &\cos(\theta_1+\theta_2) - \cos(\varphi_1+\varphi_2)\\ &+ \cos(\varphi_1+\theta_2) + \cos(\theta_1+\varphi_2).\end{aligned} \tag{8}$$

The optimal choice for the phase shifts can be rewritten as $\theta_1 + \theta_2 = -(\varphi_1 + \theta_2) = -(\theta_1 + \varphi_2) = \omega$ and $-(\varphi_1 + \varphi_2) =$

$3\omega$, as in Ref. [31]. Expression (7) becomes $-2 \leqslant 3\cos(\omega) - \cos(3\omega) \leqslant 2$, which is maximal for $\omega = \pi/4$, and we find $B_2^{(\max)} \simeq 2\sqrt{2}$. This violation corresponds to the maximal violation predicted by the Tsirelson's bound [46]. It is worthwhile to note here that a violation of the CHSH inequality is also observed without the reduction of the state (5). For the same set of values of $\theta_1$, $\theta_2$, $\varphi_1$, and $\varphi_2$ we find a smaller violation, i.e., $B_2^{(\max)} \simeq 2.41$, in accordance with Refs. [31] and [44].

## III. MANY-WELLS CASE

We now generalize the discussion to the many-wells case. In particular, we emphasize that the case $N > 2$ requires postselection of the output results and therefore single-site imaging of the optical lattice, a technique which has been successfully demonstrated experimentally [47]. In the two-wells case, we showed that postselection of the state allows a stronger violation of the Bell's inequalities, but is not compulsory to observe it. After the last beam-splitting operation of the protocol outlined in Fig. 1, we obtain [see, for instance, Eq. (5) for the double-well case] all wells populated by only one atom, with a probability $1/2^{N-1}$, or at least one empty well, with a probability $1 - 1/2^{N-1}$. In the latter case, observing the number of particles in $N - 1$ wells allows one to know with certainty the population of the last one, independently of the local detectors settings $\{\theta_k, \varphi_k\}$. As a consequence, the correlations involving states with at least one empty well are classical correlations. A violation of Eq. (1) requires to ignore this case. It should be noticed that the same postselection has been considered and proven necessary elsewhere [31,45] in order to observe the GHZ contradiction with three particles [48]. After postselection, we can express the mean value of the Bell's operator in Eq. (1) as

$$B_N(\theta_k,\varphi_k) = \sum_{k=1}^{N} \alpha\cos\theta_k + \beta\cos\varphi_k + \sum_{\substack{k,l=1\\ k\neq l}}^{N} \frac{\gamma}{2}\cos(\theta_k+\theta_l) + \delta\cos(\theta_k+\varphi_l) + \frac{\epsilon}{2}\cos(\varphi_k+\varphi_l). \tag{9}$$

We further minimize this expression as a function of $\theta_k, \varphi_k$ using a genetic algorithm [49]. In Fig. 2, we plot the maximum quantum violation of Eq. (1) normalized by the classical bound, $\xi_N \equiv \beta_c^{(N)}/\min_{\theta_k,\varphi_k}[B_N(\theta_k,\varphi_k)]$, as a function of the number of parties $N$ of the system ($\xi_N < 1$ implies nonlocality). The figure compares $\xi_N$ (crosses) with $\xi_N^{\text{glob}} \equiv \beta_c^{(N)}/\min_{\theta,\varphi}[B_N(\theta,\varphi)]$ (orange line) obtained with a global optimization assuming that all the local phases are equal. We thus conclude that the case $\theta_k = \theta$ and $\varphi_k = \varphi$ is very close to being an optimal choice of local phase. This is of particular interest from the experimental point of view as it is far easier to shift the whole system with the same phase for each well instead of applying a different phase shift to each party. Furthermore, a simple fit of our results shows that $B_N \sim 1/N$, similar to the result obtained in Ref. [42] for Dicke states. Figure 3 shows the behavior of our relative quantum violation as a function of the phase shifts $\theta$ and $\varphi$ for different number of parties. There, we plot only the values of $\xi_N^{\text{glob}} < 1$, i.e., the colored regions represent the region where we observe a violation. We see here that as the number of parties increases, the robustness of the violation against the fluctuations of the phase shifts $\theta$ and $\varphi$ increases as well.

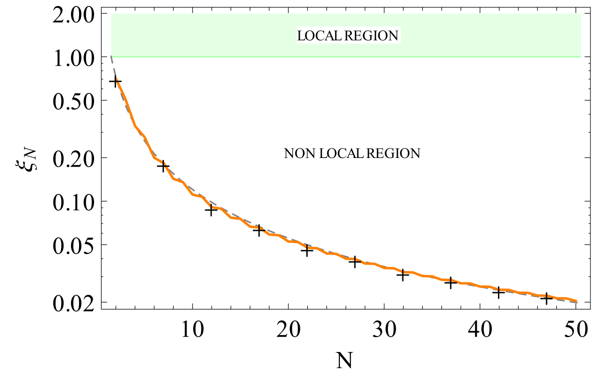


FIG. 2. Maximal relative violation $\xi_N$ of Eq. (1), optimized over different local phases, as a function of the number of parties $N$ (crosses). The orange line is $\xi_N^{\text{glob}}$, where all local phases are equal, $\theta_k = \theta$ and $\varphi_k = \varphi$. The dashed gray line is a power-law fit $1.5/N$.

## IV. DISCUSSION

Having in sight the possibility to realize experimentally the proposed scheme, some technical issues must be studied further. First, let us emphasize that the above discussion, focused on Bose gases, can be easily generalized to fermions. Furthermore, we do not need to assume the particles originate from a common source (an initial superfluid, for instance): interestingly, the protocol works even if the particles have never seen each other [31,44,45].

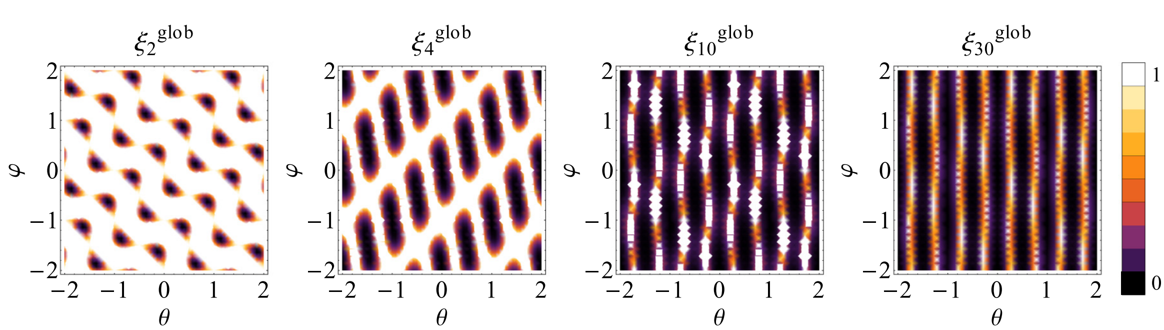


FIG. 3. Relative quantum violation $\xi_N^{\text{glob}}$ taking the same local phase shift $\theta$ and $\varphi$ expressed in $\pi$ rad, at each lattice site. Different panels refer to $N = 2,4,10,30$ (from left to right). The color scale is cut at $\xi_N^{\text{glob}} = 1$ and violation of Bell's inequality ($\xi_N^{\text{glob}} < 1$) is obtained in the colored regions.

A possible obstacle to the observation of nonlocality is given by interactions between atoms which can affect the first beam-splitting operations. To evaluate how the interactions impact the violation of the inequality (1), we generalize the scheme of Fig. 1, replacing the linear beam-splitter operation $e^{-i\frac{\pi}{2}J_y}$ in Eq. (3) with a nonlinear one $U = e^{-i\frac{\pi}{2}(J_y+\chi J_z^2)}$ [50], where $J_z = \frac{1}{2}\sum_{l=1}^{N}\hat{\sigma}_z^{(l)}$. The coefficient $\chi$ represents the relative strength of interaction with respect to the coupling between wells. We then estimate the role played by the interactions on the Bell operator $B_2(\theta_1,\theta_2,\varphi_1,\varphi_2)$ (for simplicity, we consider only the two-wells case here), for optimized phase shifts. This calculation can be performed analytically for two wells; see Appendix. In the Rabi regime, $\chi \ll 1$, we obtain

$$B_2 \simeq 2\sqrt{2} - \frac{\chi^2}{\sqrt{2}}, \tag{10}$$

where the first order in $\chi$ cancels exactly. The maximal violation of Bell's inequalities is decreased by interaction, to the second order in $\chi$. In the Appendix, we further demonstrate that this result holds also for more than two wells. This suggests that interactions among particles should not be a key problem for the violation of Bell's inequalities in our system.

## V. CONCLUSIONS

Neutral atoms are currently playing a key role for the creation of many-particle entanglement [25–29]. Yet, the investigation of nonlocality in these systems has been scarcely considered. In this manuscript, we have proposed a scheme to observe quantum nonlocality in an ultracold gas trapped in an optical lattice, requiring only linear operations (balanced coupling between the internal state of neighboring lattice sites), collective phase shifts, and involving an arbitrarily large number of neutral atoms. Nonlocality can be demonstrated via the violation of a set of Bell's inequalities recently proposed in [42] and involving only one- and two-body correlations. For two wells, the protocol reduces to the one first proposed by Yurke and Stoler in [44]. For many wells, the relative violation goes as $1/N$ as a function of the number of atoms involved in the measurement. The violation of the Bell inequality requires postselection, as is typical [31,45]. Our work paves the way toward the demonstration of quantum nonlocality with ultracold atoms—involving a large number of parties—and their use for quantum information protocols.

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## APPENDIX: EFFECT OF INTERPARTICLE INTERACTIONS

Here we report the calculation of the optimized Bell operator for the two- and three-wells case, including the effect of interactions between atoms during the balanced beam splitter used to couple neighboring wells in our system. In particular, we show that the first order in the intensity of interaction exactly cancels. By symmetry, since our protocol involves coupling between neighboring wells, we expect that the result holds for any number of wells.

### 1. Two-wells case

The interactions between atoms during the first beam splitter can be modeled by a nonlinear beam splitter as

$$\mathrm{BS}_1 = e^{-i\frac{\pi}{2}(J_y+\chi J_z^2)}, \tag{A1}$$

where $J_y = \frac{1}{2i}(c_{g,1}^\dagger c_{e,2} - c_{e,2}^\dagger c_{g,1} + c_{g,2}^\dagger c_{e,1} - c_{e,1}^\dagger c_{g,2})$ and $J_z = \frac{1}{2}(c_{g,1}^\dagger c_{g,1} - c_{e,1}^\dagger c_{e,1} + c_{g,2}^\dagger c_{g,2} - c_{e,2}^\dagger c_{e,2})$. The coefficient $\chi$ represents the relative strength of the interaction with respect to the stimulated tunneling between wells. We diagonalize Eq. (A1) on the basis of all permutations of two particles in two wells with two internal levels per well:

$$\begin{aligned}&\{|2000\rangle,|0200\rangle,|0020\rangle,|0002\rangle,|1010\rangle,\\&|1001\rangle,|0110\rangle,|0101\rangle,|1100\rangle,|0011\rangle\}.\end{aligned} \tag{A2}$$

For $\chi \ll 1$, we can write the state after the beam splitter as

$$\begin{aligned}\mathrm{BS}_1|\psi_0\rangle \simeq\ & \frac{1}{2}e^{-i\frac{\pi\chi}{2}}(|1010\rangle + |0101\rangle)\\ &+ \frac{i\chi}{4}e^{-i\frac{\pi\chi}{4}}(|0101\rangle - |1010\rangle)\\ &+ \frac{1}{2}e^{-i\frac{\pi\chi}{4}}(|1100\rangle + |0011\rangle).\end{aligned} \tag{A3}$$

We can now come back to our sequence of local transformations and measurement and apply it to this state (A3). We now describe this sequence involved in the calculation of $S_{00}$ as an example. First of all, we apply the phase shift on the ground atomic state:

$$\begin{aligned}|\tilde{\psi}_{00}\rangle =\ & \frac{1}{2}e^{-i\pi\frac{\chi}{2}}(e^{i(\theta_1+\theta_2)}|1010\rangle + |0101\rangle)\\ &+ i\frac{\chi}{4}e^{-i\pi\frac{\chi}{4}}(|0101\rangle - e^{i(\theta_1+\theta_2)}|1010\rangle)\\ &+ \frac{1}{2}e^{-i\pi\frac{\chi}{4}}(e^{i\theta_1}|1100\rangle + e^{i\theta_2}|0011\rangle).\end{aligned} \tag{A4}$$

Then, we apply the beam splitter in each well separately so that the state becomes

$$\begin{aligned}|\psi_{00}\rangle =\ & (|1010\rangle + |0101\rangle)\Big[\frac{e^{-i\pi\frac{\chi}{2}}}{4}(e^{i(\theta_1+\theta_2)} + 1)\\ &+ i\frac{\chi e^{-i\pi\frac{\chi}{4}}}{8}(1 - e^{i(\theta_1+\theta_2)})\Big] + (|1001\rangle + |0110\rangle)\\ &\times\Big[\frac{e^{-i\pi\frac{\chi}{2}}}{4}(e^{i(\theta_1+\theta_2)} - 1) - i\frac{\chi e^{-i\pi\frac{\chi}{4}}}{8}(1 + e^{i(\theta_1+\theta_2)})\Big]\\ &+ \frac{e^{-i\pi\frac{\chi}{4}}}{2\sqrt{2}}[e^{i\theta_1}(|0200\rangle - |2000\rangle) + e^{i\theta_2}(|0002\rangle - |0020\rangle)].\end{aligned} \tag{A5}$$

As a consequence, the correlation $S_{00}$ is written as after selection of the useful part of the state,

$$\begin{aligned}S_{00} &= 2\langle\psi_{00}|(-1)^{n_g^{(1)}+n_g^{(2)}}|\psi_{00}\rangle\\ &= \frac{1}{4}\Big[-2(4-\chi^2)\cos(\theta_1+\theta_2) + 8\chi\cos\frac{\pi\chi}{2}\sin(\theta_1+\theta_2)\Big].\end{aligned} \tag{A6}$$

Proceeding in the same way for the terms $S_{11}$ and $S_{01}$, we finally obtain the expression for the Bell's operator,

$$B(\theta_1,\theta_2,\theta_1',\theta_2') = \frac{-4+\chi^2}{4}[\cos(\theta_1+\theta_2) - \cos(\theta_1'+\theta_2') + \cos(\theta_1'+\theta_2) + \cos(\theta_1+\theta_2')] + \chi\cos\frac{\pi\chi}{2}[\sin(\theta_1+\theta_2) - \sin(\theta_1'+\theta_2') + \sin(\theta_1'+\theta_2) + \sin(\theta_1+\theta_2')]. \tag{A7}$$

To better understand the role of the interatomic interactions on the violations of Bell's inequalities, we look at the expression (A7) when we set the optimal $\theta_i,\theta_i'$ reported in the main text. With these values, we then obtain

$$|B_{\mathrm{opt}}| = \frac{4-\chi^2}{\sqrt{2}} = 2\sqrt{2} - \frac{\chi^2}{\sqrt{2}}. \tag{A8}$$

For $\chi \ll 1$, this expression is always greater than 2 and violates the Bell's inequalities.

#### 2. Three-wells case

We proceed the same way for the three-wells case, reusing the expression (A1) with $J_y$ and $J_z$ corresponding to the three-wells case. Similarly to the two-wells case, we have to diagonalize the Hamiltonian $J_y + \chi J_z^2$ on the base of all permutations of three particles in three wells with two internal levels per well. However, this base is huge and difficult to deal with analytically. In order to reduce the size of this basis, we notice that our initial state presents only one atom per well in its ground state, $|\psi_0\rangle = |101010\rangle$, and that only the $J_y$ part of the beam splitter is mixing the states. We can then reduce to the eight states,

$$\{|101010\rangle,|001011\rangle,|110010\rangle,|010011\rangle, |101100\rangle,|010101\rangle,|001101\rangle,|110100\rangle\}. \tag{A9}$$

Writing the Hamiltonian in this reduced base and diagonalizing it allows one to find the effect of our nonlinear beam splitter on the initial state. We then apply the set of local transformations and measurements on this state in order to compute the behavior of our Bell operator with the first order of $\chi$, which is given by

$$\tilde{B}_3(\theta,\varphi) = \frac{2}{\mathcal{N}(\chi)}\left(2 + \frac{81\pi^2\chi^2}{512}\right)B_3(\theta,\varphi), \tag{A10}$$

with $\mathcal{N}(\chi) \simeq 4 + \frac{81\pi^2\chi^2}{256}$. The coefficient in (A10) is then equal to 1, showing that the first order in $\chi$ is canceled for three wells. As a consequence, symmetry of the system allows us to conclude that for any number of wells, the first order of interaction is always canceled.


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