Generalized interaction-free evolutions

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A thorough analysis of the evolutions of bipartite systems characterized by the "effective absence" of interaction between the two subsystems is reported. First, the connection between the concepts underlying interaction-free evolutions (IFE) and decoherence-free subspaces (DFS) is explored, showing intricate relations between these concepts. Second, starting from this analysis and inspired by a generalization of DFS already known in the literature, we introduce the notion of generalized IFE (GIFE), also providing a useful characterization that allows one to develop a general scheme for finding GIFE states.

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I. INTRODUCTION

Quantum systems are intrinsically subject to relaxation and dephasing phenomena caused by their unavoidable coupling with the surroundings [1,2]. A lot of effort has been made over the last decade in order to protect quantum systems from the detrimental effects of the interaction with their environment [3–7]. This research area involves basic concepts of quantum dynamics of realistic systems [8,9] but undoubtedly the great deal of attention dedicated to such issues may be traced back to the growing interest toward the implementation of reliable nanodevices where the miniaturization obliges one to investigate their performance treating them as open quantum systems. It is well known indeed that simple quantum systems can be ideal candidates to speed up and improve computational operations [10]. However, if it is true that solving problems with the use of quantum algorithms is a revolutionary change in the theory of computational complexity, one has to deal with the fact that decoherence poses a serious obstacle causing information loss from the system to its environment. Thus the possibility of having different ways to bypass detrimental effects due to decoherence, or, generally speaking, the capability of systematically envisaging states which preserve coherence properties, is an appealing research topic. In this context, *subradiant* [11–17] as well *decoherence-free* (DF) states [18–20] have gained widespread attention leading to unitary system dynamics. Several papers indeed have appeared in the last 20 years concerning the preparation of such states immune from decoherence in different physical contexts [21–24]. At the same time, starting from the idea that decoherence can be avoided remaining inside special subspaces that are protected from the interaction with the environment, the theory of DF subspaces and subsystems has been developed (see, for example, the review of Lidar and Whiley, and references therein [25–27]).

Very recently a new class of states for a closed system, namely, *interaction-free evolving* (IFE) states, has been introduced [28] also in the cases wherein the system is governed by a time-dependent Hamiltonian [29]. By definition an IFE state of a composed system $A+B$ is a state that evolves as if the interaction between the two parts A and B were absent, thus implying a unitary evolution for both systems A and B. As pointed out in Refs. [28–31] the concept of an IFE state is somehow related to that of a decoherence-free state even if the two concepts are still different in many aspects.

The aim of this paper is to explore in depth the class of dynamics of a compound system characterized by the fact that the interaction between the two subsystems is seemingly not effective. On the one hand, this analysis leads us to an in-depth study of the connection between the already known interaction-free evolutions and the already known decoherence-free evolutions. On the other hand, and more importantly, inspired by the notion of generalized decoherence-free subspaces (DFS), we are brought to the definition of a new and extended class of IFE, that we call generalized interaction-free evolutions (GIFE). This extension is based on the idea that there could be states of the total system $\mathrm{A}+\mathrm{B}$ that evolve under the action of the total Hamiltonian $H=H_\mathrm{A}+H_\mathrm{B}+H_\mathrm{I}$ as $e^{-i\widetilde{H}_\mathrm{A}t}\otimes e^{-i\widetilde{H}_\mathrm{B}t}$, where $\widetilde{H}_\mathrm{A}$ and $\widetilde{H}_\mathrm{B}$ may in general differ from H_A and H_B, respectively. We emphasize that in such a case these GIFE states are sensitive to the interaction Hamiltonian H_I, nevertheless they do evolve via local unitary operator $U_\mathrm{A}(t)\otimes U_\mathrm{B}(t)$. Therefore, for proper GIFE states (i.e., GIFE which are not IFE) one of the subsystems (or even both of them) evolves as under the effect of a certain dressed local Hamiltonian that somehow keeps some information about the interaction but does not imply nonlocal actions. The effect of dressing $H_\mathrm{A}\to\widetilde{H}_\mathrm{A}$ (and similarly for H_B) is the key idea leading to GIFE states. This fact is of conceptual relevance itself. Moreover, it can be important for applications in the field of quantum information, since in what follows GIFE states are proved to be entanglement-preserving states. Beyond the definition of GIFE, we provide a characterization of the new class of evolutions (which, of course, contains the previously known IFE) in terms of conservation of some functionals, in the sense that a quantum state is a GIFE state if and only if during its evolution some functionals (we will clarify which ones) maintain their initial values. On this basis, we are also in a condition to formulate a recipe that gives the possibility of finding the GIFE states for a given Hamiltonian.

The paper is structured as follows. In the next section we first recall the definitions of IFE and DFS, and then start the discussion about the connection of the two relevant concepts. In Sec. III we introduce the notion of generalized IFE. In the subsequent two sections we try to characterize such new class of evolutions. In particular, in Sec. IV we describe a class of Hamiltonian operators that admit GIFE states, while in Sec. V we prove some general properties of GIFE states, in particular the fact that during their evolution some functionals (for example, any measure of entanglement) are conserved and on this basis we provide a recipe to find, in principle, all GIFE states for any given Hamiltonian. Finally, in Sec. VI we summarize the results of this paper and give some conclusive remarks.

II. IFE VS DFS—HAMILTONIAN FORMULATION

In this section we analyze the connection between IFE and DFS. To this end, let us first of all recall the two relevant definitions.

IFE. Consider a system whose dynamics is governed by a Hamiltonian which is the sum of an unperturbed term H_0 and an interaction term $H_{\rm I}$: $H = H_0 + H_{\rm I}$ (we start by considering time-independent operators). We will say that a state $|\chi\rangle$ undergoes an interaction-free evolution if its evolution is essentially governed by the unperturbed Hamiltonian H_0 up to a phase factor:

$$U(t)|\chi\rangle = e^{-iat}U_0(t)|\chi\rangle, \tag{1a}$$

where

$$U(t) = e^{-i(H_0+H_{\rm I})t}, \qquad U_0(t) = e^{-iH_0 t}, \tag{1b}$$

and "a" is a real number. In particular, if we consider a composite system "system (S) + environment (E)" living in $\mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm E}$ governed by the time-independent Hamiltonian

$$H = H_{\rm S}\otimes\mathbb{I}_{\rm E} + \mathbb{I}_{\rm S}\otimes H_{\rm E} + H_{\rm I} = H_0 + H_{\rm I}, \tag{2}$$

then $|\chi\rangle$ is IFE if

$$U(t)|\chi\rangle = e^{-iat}e^{-iH_{\rm S}t}e^{-iH_{\rm E}t}|\chi\rangle, \tag{3}$$

which, for a product state $|\psi\rangle\otimes|\phi\rangle$, becomes

$$U(t)|\psi\rangle\otimes|\phi\rangle = e^{-iiat}e^{-iH_{\rm S}t}|\psi\rangle\otimes e^{-iH_{\rm E}t}|\phi\rangle. \tag{4}$$

DFS. A decoherence-free subspace of a system (S) interacting with its environment (E) is a subspace $\mathcal{C}_{\rm DFS}\subset\mathcal{H}_{\rm S}$ such that the reduced dynamics

$$\rho_{\rm S}(t) = {\rm tr}[U(t)\rho_{\rm S}\otimes\rho_{\rm E}U^\dagger(t)] = U_{\rm S}(t)\rho_{\rm S}U_{\rm S}^\dagger(t), \tag{5}$$

for any $\rho_{\rm S}$ supported on $\mathcal{C}_{\rm DFS}$ (i.e., $\rho_{\rm S} = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ and $|\psi_k\rangle\in\mathcal{C}_{\rm DFS}$). Note, that the evolution of $\rho_{\rm S}$ does not depend upon the initial state $\rho_{\rm E}$ of the environment. It means that for any $|\psi\rangle\in\mathcal{C}_{\rm DFS}$ and arbitrary $|\phi\rangle\in\mathcal{H}_{\rm E}$ one has

$$U(t)|\psi\rangle\otimes|\phi\rangle = \exp(-iH_{\rm S}t)|\psi\rangle\otimes\exp\big(-iH_{\rm E}^{\rm eff}t\big)|\phi\rangle, \tag{6}$$

where $H_{\rm E}^{\rm eff}$ denotes an effective environment Hamiltonian. Now, if in addition one has $H_{\rm I} = \sum_\alpha S_\alpha\otimes E_\alpha$, then necessarily $\sum_\alpha S_\alpha\otimes E_\alpha|\psi\rangle\otimes|\phi\rangle = |\psi\rangle\otimes\sum_\alpha c_\alpha E_\alpha|\phi\rangle$ (in Ref. [32] it is proven that the necessary and sufficient conditions to have a DFS are that (1) for all the states in the DFS it is $S_\alpha|\psi\rangle = c_\alpha|\psi\rangle$, and (2) DFS is invariant under the action of $H_{\rm S}$. These conditions imply the previous condition (see also [25,27]) and hence

$$H_{\rm E}^{\rm eff} = H_{\rm E} + \sum_\alpha c_\alpha E_\alpha. \tag{7}$$

It is therefore clear that if $\mathcal{C}_{\rm E}\subset\mathcal{H}_{\rm E}$ satisfies

$$e^{-iH_{\rm E}^{\rm eff}t}\big|_{\mathcal{C}_{\rm E}} = e^{-iat}e^{-iH_{\rm E}}\big|_{\mathcal{C}_{\rm E}}, \tag{8}$$

then $\mathcal{C}_{\rm DFS}\otimes\mathcal{C}_{\rm E}$ is IFE in $\mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm E}$. (In the previous equation we have introduced the notation $O|_{\mathcal{C}} = O\Pi_{\mathcal{C}}$, where $\Pi_{\mathcal{C}}$ denotes a projector onto $\mathcal{C}$). For example, it happens when $\mathcal{C}_{\rm E}$ is a common eigenspace of $H_{\rm E}^{\rm eff}$ and $H_{\rm E}$. In the very special case where $H_{\rm E}$ and $H_{\rm E}^{\rm eff}$ differ for a global shift, that is, $\sum_\alpha c_\alpha E_\alpha = c\mathbb{I}_{\rm E}$, there is a huge IFE subspace $\mathcal{C}_{\rm DFS}\otimes\mathcal{H}_{\rm E}$. Nevertheless, if the two operators do not commute, there are states of the environment evolving in a way which is significantly different from the evolution induced by $H_{\rm E}$. This means that the small system S evolves as if the interaction with the environment were absent, but the environment somehow "feels" the presence of the small system.

On the other hand, if there is a collection of IFE subspaces which involves all the states of a given subspace of $\mathcal{H}_{\rm S}$ and all possible states of the environment, this clearly implies that the small system evolves as if the environment were absent, singling out the presence of a DFS. Stated another way, if $\mathcal{C}\otimes\mathcal{H}_{\rm E}^{(\alpha)}$ is a collection of IFE subspaces labeled by α and if $\oplus\mathcal{H}_{\rm E}^{(\alpha)} = \mathcal{H}_{\rm E}$, then $\mathcal{C}$ is a DFS. But it is evident that this last condition implies the presence of an effective environment Hamiltonian which commutes with $H_{\rm E}$.

All these facts show in a very clear way that the two concepts of IFE states and DFS are somehow related and that under some specific hypotheses each of them implies the other. Nevertheless, there are a variety of situations, which form the biggest class of possible situations, wherein one can have IFE states but no DFS (consider the case of a collection of IFE states whose environmental parts do not span the whole Hilbert space of the environment) and vice versa (when there is no common eigenstate of $H_{\rm E}$ and $H_{\rm E}^{\rm eff}$).

So far, the analysis has been developed in a way that fits well with time-independent Hamiltonians, but we can make analogous considerations in a way that fits also when the Hamiltonian is time dependent.

To this end, let us analyze the evolution of the composed system in the interaction picture, that is, let $\tilde{H}_{\rm I}(t) = U_0(t)H_{\rm I}U_0^\dagger(t)$ denote the interaction Hamiltonian in the interaction picture with respect to the free evolution governed by $H_0 = H_{\rm S} + H_{\rm E}$.

A subspace $\mathcal{C}_{\rm DFS}\subset\mathcal{H}_{\rm S}$ is DFS if and only if $\tilde{H}_{\rm I}(t)|_{\mathcal{C}_{\rm DFS}\otimes\mathcal{H}_{\rm E}} = \Pi_{\rm DFS}\otimes H_{\rm E}^{\rm eff}(t)$, where $\Pi_{\rm DFS}$ is the projector to the subspace $\mathcal{C}_{\rm DFS}$. On the other hand, a subspace $\mathcal{C}\otimes\mathcal{H}_{\rm E}^{(\alpha)}\subset\mathcal{H}_{\rm S}\otimes\mathcal{H}_{\rm E}$ is IFE if and only if $\tilde{H}_{\rm I}(t)|_{\mathcal{C}\otimes\mathcal{H}_{\rm E}^{(\alpha)}} = \alpha(t)\Pi_{\mathcal{C}}\otimes\Pi_{{\rm E},\alpha}$, where $\Pi_{\mathcal{C}}$ and $\Pi_{{\rm E},\alpha}$ are the projectors to $\mathcal{C}$ and $\mathcal{H}_{\rm E}^{(\alpha)}$, respectively. Now, if $\bigoplus_\alpha\mathcal{H}_{\rm E}^{(\alpha)} = \mathcal{H}_{\rm E}$, then $\mathcal{C}$ is DFS, being $H^{\rm eff}(t) = \bigoplus_\alpha\alpha(t)\Pi_{{\rm E},\alpha}$. Moreover, if $\mathcal{C}\otimes\mathcal{H}_{\rm E}$ is IFE, then $\tilde{H}_{\rm I}(t)|_{\mathcal{C}\otimes\mathcal{H}_{\rm E}} = \alpha(t)\Pi_{\mathcal{C}}\otimes\mathbb{I}_{\rm E}$.

These last two assertions clarify very well the connection between IFE and DFS.

III. GENERALIZED IFE

It is worth noting that Eq. (6) shows that the presence of a DFS implies that the small system (S) evolves according to its free Hamiltonian, while the environment evolves through an effective Hamiltonian, which may commute or not with the environment-free Hamiltonian. It somehow resembles an IFE evolution, where the two systems do not interact, though the Hamiltonian of one of the two systems (the environment, in this case) is not the free one.

As another important fact, we mention that the notion of DFS can be generalized, according to the analysis in Ref. [25], in the following way. Suppose a quantum system interacting with its environment is describable by the Hamiltonian, as in Eq. (2). Then a given subspace of the Hilbert space of the system, say $\mathcal{C}_{\rm GDFS} \subset \mathcal{H}_{\rm S}$, is a generalized DFS if, whatever the state $|\psi\rangle$ of the environment, the system prepared in a state $|\phi\rangle \in \mathcal{C}_{\rm GDFS}$ evolves as if it was not interacting with the environment, even if its dynamics is not governed by $H_{\rm S}$ but it is determined by an effective system Hamiltonian $H_{\rm S}^{\rm eff} \neq H_{\rm S}$.

Both of these facts suggest a possible extension of the concept of IFE. Consider a bipartite system $A + B$, whose dynamics is governed by

$$H = H_{\rm A} \otimes \mathbb{I}_{\rm B} + \mathbb{I}_{\rm A} \otimes H_{\rm B} + H_{\rm I}. \tag{9}$$

We can define generalized IFE (GIFE) as those evolutions where each of the two subsystems undergoes an evolution *seemingly independent* from the other subsystem. This means that each of the two subsystems evolves under the action of a Hamiltonian $H_k^{\rm eff}(t)$, with $k = A, B$, not necessarily coincident with H_k. More precisely, a state $|\chi\rangle \in \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B}$ is a GIFE state if there exist two operators $H_{\rm A}^{\rm eff}(t)$ and $H_{\rm B}^{\rm eff}(t)$ such that the following set of equations can be satisfied:

$$\begin{aligned} U(t)|\chi\rangle &= U_{\rm A}^{\rm eff}(t) \otimes U_{\rm B}^{\rm eff}(t)|\chi\rangle, \\ i\dot{U}_k^{\rm eff} &= H^{\rm eff}(t)U_k^{\rm eff}(t), \qquad k = {\rm A,B}. \end{aligned} \tag{10}$$

Note, that if

$$|\chi\rangle = \sum_\alpha \chi_{\alpha\beta}\, |e_\alpha\rangle \otimes |f_\beta\rangle \tag{11}$$

with $\{e_\alpha\}$ and $\{f_\beta\}$ being orthonormal basis in $\mathcal{H}_{\rm A}$ and $\mathcal{H}_{\rm B}$, respectively, then

$$|\chi(t)\rangle = \sum_\alpha \chi_{\alpha\beta}\, U_{\rm A}^{\rm eff}(t)|e_\alpha\rangle \otimes U_{\rm B}^{\rm eff}(t)|f_\beta\rangle. \tag{12}$$

It should be clear that GIFE states are nothing but IFE states with respect to a suitable effective interaction Hamiltonian. Following Ref. [29] one finds that $|\chi\rangle$ defines GIFE state if and only if

$$\breve{H}_{\rm I}^{\rm eff}(t)|\chi\rangle = 0, \tag{13a}$$

where

$$\breve{H}_{\rm I}^{\rm eff}(t) \equiv \breve{H}(t) - \breve{H}_{\rm A}^{\rm eff}(t) - \breve{H}_{\rm B}^{\rm eff}(t), \tag{13b}$$

and the new interaction picture is defined as follows:

$$\breve{O}(t) = U_{\rm A}^{\rm eff\dagger}(t) \otimes U_{\rm B}^{\rm eff\dagger}(t)\, O\, U_{\rm A}^{\rm eff}(t) \otimes U_{\rm B}^{\rm eff}(t). \tag{13c}$$

One could think of replacing the condition in Eq. (13a) with the seemingly more general condition $\breve{H}_{\rm I}^{\rm eff}(t)|\chi\rangle = \alpha(t)|\chi\rangle$, resembling what we have found in Ref. [29]. However, since in the case of GIFE we have to find also the two effective unperturbed Hamiltonian operators, the constant term $\alpha(t)I$ can be included in such operators, which makes the two problems essentially equivalent.

Of course, the standard case of IFE is included as a special case of GIFE. (We will use the expression "proper GIFE" to talk about GIFE states which are not IFE states.)

An example. In order to better illustrate the notion of GIFE, we will analyze a specific physical situation where both IFE and GIFE arise. Consider the multispin system interacting with a bosonic field (see, for example, Ref. [27]). The relevant Hamiltonian is given by

$$H_{\rm S} = \sum_k \Omega_k \sigma_z^{(k)}, \tag{14a}$$

$$H_{\rm E} = \sum_j \omega_j a_j^\dagger a_j, \tag{14b}$$

$$H_{\rm I} = \left(\sum_k \sigma_z^{(k)}\right) \otimes \sum_j g_j(a_j + a_j^\dagger). \tag{14c}$$

Since $\sum_k \sigma_z^{(k)}$ is nothing but the total pseudospin (let us call it J_z), and $[H_{\rm S}, J_z] = 0$, we find that each eigenspace of J_z is decoherence-free. When S is prepared in an eigenstate of J_z with eigenvalue m, the environment evolves according to

$$H_{\rm E}^{\rm eff} = \sum_k \left[\omega_k a_k^k a_k + m g_k(a_k + a_k^\dagger)\right], \tag{15}$$

and we have a GIFE subspace, unless $m = 0$, in which case we have an IFE subspace.

Now comes the crucial question: *how does one characterize bipartite Hamiltonians giving rise to GIFE states?* In the next two sections, we will make some efforts in this direction.

IV. A CLASS OF HAMILTONIANS THAT ADMITS GIFE

The previous example suggests a structure of Hamiltonians that admit GIFE.

Consider the following time-independent Hamiltonian in $\mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B}$:

$$H = H_{\rm A} \otimes \mathbb{I}_{\rm B} + \mathbb{I}_{\rm A} \otimes H_{\rm B} + \sum_k P_k \otimes B_k, \tag{16}$$

where $P_k = |k\rangle\langle k|$ are projectors into the computational basis vectors $|k\rangle$ in $\mathcal{H}_{\rm A}$ and B_k are Hermitian operators in $\mathcal{H}_{\rm B}$. Now, assuming that $H_{\rm A} = \sum_k \epsilon_k P_k$ one finds

$$H = \sum_k P_k \otimes Z_k, \tag{17}$$

where $Z_k = \epsilon_k \mathbb{I}_{\rm B} + H_{\rm B} + B_k$. Such Hamiltonian leads to a pure decoherence of the density operator $\rho_{\rm A}$ of subsystem A:

$$\rho_{\rm A}(t) = {\rm tr}_{\rm B}(e^{-iHt}\rho_{\rm A} \otimes \rho_{\rm B} e^{iHt}) = \sum_{k,l} c_{kl}(t) P_k \rho_{\rm A} P_l, \tag{18}$$

with $c_{kl}(t) = {\rm tr}(e^{-iZ_k t}\rho_{\rm B} e^{iZ_l t})$. It is clear that each one-dimensional subspace in $\mathcal{H}_{\rm A}$ spanned by $|k\rangle$ defines DFS. Note, that $|k\rangle \otimes |\phi_{\rm B}\rangle$, where $|\phi_{\rm B}\rangle$ is an arbitrary vector from $\mathcal{H}_{\rm B}$, defines GIFE but not IFE. Indeed, one has

$$e^{-iHt}|k\rangle \otimes |\phi_{\rm B}\rangle = e^{-iH_{\rm A}t}|k\rangle \otimes e^{-i(H_{\rm B}+Z_k)t}|\phi_{\rm B}\rangle. \tag{19}$$

It is clear that one may replace $H_{\rm B}$ and B_k by time-dependent operators.

The previous Hamiltonian structure gives rise to evolutions which are IFE for one subsystem and GIFE for the other one. In the following we give a more general structure for the Hamiltonians that give rise to GIFE evolutions for both subsystems. Consider what follows.

Let us recall [28] that $|\chi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ satisfying

$$H_I|\chi\rangle = 0 \tag{20}$$

is IFE for the Hamiltonian (9) if and only if

$$H_I H_0^n|\chi\rangle = 0, \qquad n = 1,2,3,\ldots, \tag{21}$$

where $H_0 = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B$ is the "free" part of H. Note that one can always rewrite the total Hamiltonian performing the following "corrections" of H_A and H_B:

$$H = (H_A + \Delta_A) \otimes \mathbb{I}_B + \mathbb{I}_A \otimes (H_B + \Delta_B) + H_I^{\text{eff}}, \tag{22}$$

with

$$H_I^{\text{eff}} = H_I - [\Delta_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \Delta_B]. \tag{23}$$

Hence the question of the existence of GIFE states is equivalent to the existence of suitable operators Δ_A and Δ_B such that

$$H_A^{\text{eff}} = H_A + \Delta_A, \qquad H_B^{\text{eff}} = H_B + \Delta_B \tag{24}$$

satisfies conditions (10). Here we propose the following class of Hamiltonians admitting GIFE subspaces: let S_A and S_B be linear subspaces in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Moreover, let Π_A and Π_B be the corresponding orthogonal projectors, that is,

$$S_A = \Pi_A \mathcal{H}_A, \qquad S_B = \Pi_B \mathcal{H}_B. \tag{25}$$

We construct a class of bipartite Hamiltonians such that any $|\chi\rangle \in S_{AB} = S_A \otimes S_B$ is a GIFE state. Let H_A and H_B be Hamiltonians of systems A and B, respectively, such that

$$[H_A,\Pi_A] = 0, \qquad [H_B,\Pi_B] = 0. \tag{26}$$

Let Δ_A and Δ_B be two "corrections" satisfying the same commutation relations, i.e.,

$$[\Delta_A,\Pi_A] = 0, \qquad [\Delta_B,\Pi_B] = 0. \tag{27}$$

Consider now the interaction part

$$H_I = \Delta_A \otimes \Pi_B + \Pi_A \otimes \Delta_B + \Delta^\perp \tag{28}$$

with $\Delta^\perp$ an arbitrary bipartite operator such that

$$\Delta^\perp \Pi_A \otimes \Pi_B = \Pi_A \otimes \Pi_B \Delta^\perp = 0. \tag{29}$$

It is clear that taking $|\chi\rangle \in S_{AB}$ one finds in general $H_I|\chi\rangle \neq 0$, and hence condition (20) is not satisfied. However, correcting H_A and H_B as in (22) one finds

$$\begin{aligned} H &= H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B + H_I \\ &= H_A^{\text{eff}} \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B^{\text{eff}} + H_I^{\text{eff}}, \end{aligned} \tag{30}$$

where

$$H_I^{\text{eff}} = \Delta_A \otimes \Pi_B^\perp + \Pi_A^\perp \otimes \Delta_B + \Delta^\perp, \tag{31}$$

with

$$\Pi_A^\perp = \mathbb{I}_A - \Pi_A, \qquad \Pi_B^\perp = \mathbb{I}_B - \Pi_B. \tag{32}$$

It is therefore clear that

$$H_I^{\text{eff}}|\chi\rangle = 0, \tag{33}$$

for any $|\chi\rangle \in S_{AB}$. Moreover, one easily checks

$$H_I^{\text{eff}}\left(H_0^{\text{eff}}\right)^n|\chi\rangle = 0, \qquad n = 1,2,\ldots, \tag{34}$$

where $H_0^{\text{eff}} = H_A^{\text{eff}} \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B^{\text{eff}}$. Hence, conditions (33) and (34) for new effective Hamiltonians are exactly the same as (20) and (21) for the original H_0 and H_I. This proves that $|\chi\rangle$ defines the GIFE state.

Interestingly, in the special case when $[H_A,\Delta_A] = 0$ and $[H_B,\Delta_B] = 0$, the subspace S_{AB} can be decomposed into common eigenspaces of H_A and Δ_A, for the subsystem A (call them $\mathcal{C}_A^\alpha$, where α is the relevant eigenvalue of Δ_A), and into common eigenspaces of H_B and Δ_B, for the subsystem B (call them $\mathcal{C}_B^\beta$, where β is the relevant eigenvalue of Δ_B):

$$S_{AB} = \bigoplus_{\alpha,\beta} \mathcal{C}_A^\alpha \otimes \mathcal{C}_B^\beta. \tag{35}$$

The tensor product $\mathcal{C}_A^\alpha \otimes \mathcal{C}_B^\beta$ of each two of such common eigenspaces corresponds to a proper IFE subspace, where the dynamics differs from the unperturbed one by the phase factor $\exp(\int_0^t[\alpha(s) + \beta(s)]ds)$.

V. CHARACTERIZATION OF GIFE

In this section we provide general properties of generalized interaction-free evolutions and derive a general scheme for finding GIFE states in principle for any given Hamiltonian.

A. General properties of GIFE

Let us first of all briefly discuss the relation between GIFE and entangled states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Let us observe that if $\mathcal{E}$ is a genuine entanglement measure, then for any GIFE state $|\chi\rangle$ one has $\frac{d}{dt}\mathcal{E}(|\chi(t)\rangle) = 0$, i.e., every GIFE state is an entanglement-preserving state. Indeed, GIFE states evolve as if they were under the action of two local (effective) Hamiltonians, and then, whatever is the entanglement measure considered, the amount of entanglement does not change in the evolution of a GIFE state. In particular, the entropy of entanglement $S(\text{tr}_A|\chi(t)\rangle\langle\chi(t)|) = S(\text{tr}_B|\chi(t)\rangle\langle\chi(t)|)$ and the linear entropy $S_L = 1 - \text{tr}_B(\text{tr}_A|\chi(t)\rangle\langle\chi(t)|)^2$ do not depend on time. Moreover, any function of the eigenvalues of the two reduced density operators, either $\rho_B = \text{tr}_A|\chi(t)\rangle\langle\chi(t)|$ or $\rho_A = \text{tr}_B|\chi(t)\rangle\langle\chi(t)|$, does not depend on time.

Now we are ready to provide the necessary and sufficient condition for a state to be a GIFE state.

Theorem. If $\min\{n_A,n_B\} = n$ (where $n_{A/B} = \dim\mathcal{H}_{A/B}$), then the state $|\chi\rangle$ is GIFE if and only if

$$\frac{d}{dt}\text{tr}_B(\text{tr}_A|\chi(t)\rangle\langle\chi(t)|)^k = 0, \tag{36}$$

for $k = 1,2,\ldots,n$.

Proof. Given the set of eigenvalues $p_1,\ldots,p_n$ of the reduced density operator $\rho_B = \text{tr}_A|\chi\rangle\langle\chi|$ (we here assume $n_B = n$, otherwise we use ρ_A), the set of equations in Eq. (36) turns out to be equivalent to

$$\sum_{l=1}^n p_l^k(t) = s_k, \qquad k = 1,\ldots,n, \tag{37}$$

where s_k's are n real positive numbers. By the way, the condition corresponding to $k = 1$ is trivial, being nothing but the normalization.

From the comments we have made just above it is clear that any GIFE state satisfies all such equations. We then only need to prove that if a state satisfies Eq. (37), then it is GIFE. If the set of algebraic equations in Eq. (37) (which is solvable through the use of Newton-Girard identities) is the same at any time, then it also admits the same solutions (p_l's) at any time. Now, given the set of p_l's, it is well known that the pure state describing the total system can be put in the following form:

$$|\chi(t)\rangle = \sum_{l=1}^{n} \sqrt{p_l}|\phi_l(t)\rangle_{\rm A} \otimes |\psi_l(t)\rangle_{\rm B}, \tag{38}$$

where the coefficients $\sqrt{p_l}$, which are nothing but the Schmidt coefficients of $|\chi\rangle$, in this case are time independent. Moreover, the states $|\phi_l(t)\rangle_{\rm A}$ and $|\psi_l(t)\rangle_{\rm B}$, though time dependent, are two sets of orthonormal states at every time. Since it is clear that there are two unitary operators, $U_{\rm A}^{\rm eff}(t)$ and $U_{\rm B}^{\rm eff}(t)$, that map $|\phi_k(0)\rangle$ into $|\phi_k(t)\rangle$ and $|\psi_k(0)\rangle$ into $|\psi_k(t)\rangle$, then we can consider the state $|\chi\rangle$ as if it evolves unitarily: $|\chi(t)\rangle = U_{\rm A}^{\rm eff}(t) \otimes U_{\rm B}^{\rm eff}(t)|\chi\rangle$.

B. Recipe to find GIFE states

By exploiting the previous results we propose a strategy to check whether a given Hamiltonian admits GIFE evolutions. Let us restrict our analysis to the case of time-independent Hamiltonians. Using

$$H|\lambda_i\rangle = \lambda_i|\lambda_i\rangle, \tag{39}$$

we can write the general solution of the relevant Schrödinger problem in the following way:

$$|\chi(t)\rangle = \sum_i c_i e^{-i\lambda_i t}|\lambda_i\rangle. \tag{40}$$

Now, since ${\rm tr}_{\rm A}|\chi(t)\rangle\langle\chi(t)|$ can be cast in the following form,

$${\rm tr}_{\rm A}|\chi(t)\rangle\langle\chi(t)| = \sum_{ij} c_i c_j^* e^{-i(\lambda_i-\lambda_j)t}{\rm tr}_{\rm A}|\lambda_i\rangle\langle\lambda_j|, \tag{41}$$

conditions in Eq. (36) assume the following form:

$$k = 1: \quad \frac{d}{dt}\sum_i |c_i|^2 = 0, \tag{42a}$$

$$k = 2: \quad -i\sum_{i_1 j_1 i_2 j_2} c_{i_1}c_{j_1}^* c_{i_2}c_{j_2}^* e^{-i(\lambda_{i_1}-\lambda_{j_1}+\lambda_{i_2}-\lambda_{j_2})t}(\lambda_{i_1}-\lambda_{j_1}+\lambda_{i_2}-\lambda_{j_2}){\rm tr}_{\rm B}\big({\rm tr}_{\rm A}|\lambda_{i_1}\rangle\langle\lambda_{j_1}|{\rm tr}_{\rm A}|\lambda_{i_2}\rangle\langle\lambda_{j_2}|\big) = 0, \tag{42b}$$

...

$$k = n: \quad (-i)^{n-1}\sum_{i_1 j_1\cdots i_n j_n}\left(\prod_{s=1}^{n} c_{i_s}\right)\left(\prod_{s=1}^{n} c_{j_s}^*\right)e^{-i(\sum_{s=1}^n\lambda_{i_s}-\sum_{s=1}^n\lambda_{j_s})t}\left(\sum_{s=1}^{n}\lambda_{i_s}-\sum_{s=1}^{n}\lambda_{j_s}\right)^{n-1}{\rm tr}_{\rm B}\left(\prod_{s=1}^{n}{\rm tr}_{\rm A}|\lambda_{i_s}\rangle\langle\lambda_{j_s}|\right) = 0. \tag{42c}$$

The condition in Eq. (42a) is essentially the preservation of the normalization condition at any time t, which is trivial because c_i's are time independent. The condition in Eq. (42b) expresses the conservation of the linear entropy at every time. Let us analyze this condition more carefully. Note that, due to the linear independence of the exponential functions we can simplify condition in Eq. (42b) as follows: let us call two sets of indices $\{i_1,j_1,i_2,j_2\}$ and $\{i'_1,j'_1,i'_2,j'_2\}$ equivalent if and only if

$$\lambda_{i_1} - \lambda_{j_1} + \lambda_{i_2} - \lambda_{j_2} = \lambda_{i'_1} - \lambda_{j'_1} + \lambda_{i'_2} - \lambda_{j'_2},$$

and denote the class indices equivalent to $\{i_1,j_1,i_2,j_2\}$ by $[i_1,j_1,i_2,j_2]$. Now, Eq. (42b) implies the following condition: for any $\{\underline{i_1},\underline{j_1},\underline{i_2},\underline{j_2}\}$ such that $\lambda_{i_1} + \lambda_{i_2} \neq \lambda_{j_1} + \lambda_{j_2}$ one has

$$\sum_{\{i_1,j_1,i_2,j_2\}\in[\underline{i_1},\underline{j_1},\underline{i_2},\underline{j_2}]} c_{i_1}c_{j_1}^* c_{i_2}c_{j_2}^*\,{\rm tr}_{\rm B} \times \big({\rm tr}_{\rm A}|\lambda_{i_1}\rangle\langle\lambda_{j_1}|{\rm tr}_{\rm A}|\lambda_{i_2}\rangle\langle\lambda_{j_2}|\big) = 0. \tag{43}$$

It is easy to verify that when H does not contain any interaction term, then these conditions are automatically satisfied. Indeed, if $H = H_{\rm A} + H_{\rm B}$, then there exists a set of eigenvectors which are nothing but products of states of $\mathcal{H}_{\rm A}$ and $\mathcal{H}_{\rm B}$: $|\lambda_k\rangle = |\phi_k\rangle_{\rm A} \otimes |\psi_k\rangle_{\rm B}$, which implies

$${\rm tr}_{\rm B}\big({\rm tr}_{\rm A}|\lambda_{i_1}\rangle\langle\lambda_{j_1}|{\rm tr}_{\rm A}|\lambda_{i_2}\rangle\langle\lambda_{j_2}|\big) = \delta_{i_1 j_2}\delta_{i_2 j_1}, \tag{44}$$

and hence either ${\rm tr}_{\rm B}({\rm tr}_{\rm A}|\lambda_{i_1}\rangle\langle\lambda_{j_1}|{\rm tr}_{\rm A}|\lambda_{i_2}\rangle\langle\lambda_{j_2}|) = 0$ or $\lambda_{i_1} + \lambda_{i_2} = \lambda_{j_1} + \lambda_{j_2}$, and this ensures that Eq. (42b) is satisfied.

For the generic k one gets: for any $\{i_1,j_1,\ldots,i_k,j_k\}$ such that $\lambda_{i_1} + \cdots + \lambda_{i_k} \neq \lambda_{j_1} + \cdots + \lambda_{j_k}$ one has

$$\sum_{\{i_1,j_1,\ldots,i_k,j_k\}\in[\underline{i_1},\underline{j_1},\ldots,\underline{i_k},\underline{j_k}]} c_{i_1}c_{j_1}^* c_{i_2}c_{j_2}^*\cdots c_{i_k}c_{j_k}^*\,{\rm tr}_{\rm B} \times \big({\rm tr}_{\rm A}|\lambda_{i_1}\rangle\langle\lambda_{j_1}|{\rm tr}_{\rm A}|\lambda_{i_2}\rangle\langle\lambda_{j_2}|\ldots\,{\rm tr}_{\rm A}|\lambda_{i_k}\rangle\langle\lambda_{j_k}|\big) = 0, \tag{45}$$

where now the equivalence of indexes $\{i_1,j_1,\ldots,i_k,j_k\}$ and $\{i'_1,j'_1,\ldots,i'_k,j'_k\}$ is defined by

$$\sum_{l=1}^{k}(\lambda_{i_l} - \lambda_{j_l}) = \sum_{l=1}^{k}\big(\lambda_{i'_l} - \lambda_{j'_l}\big).$$

Again, one immediately verifies that when H does not contain any interaction term, then these conditions are automatically satisfied.

It is also the case to point out that all such conditions, for all values of k, are automatically satisfied if all c_i's are zero but one: $c_i = \delta_{ip}$ for a given p. This corresponds to the trivial result that all the eigenstates of the Hamiltonian are GIFE states.

An example. In order to illustrate our strategy for finding the GIFE states of a given Hamiltonian, let us consider the very simple example of two interacting two-level systems described by the following Hamiltonian:

$$H = \omega_A \sigma_z^{(A)} + \omega_B \sigma_z^{(B)} + \gamma(\sigma_-^{(A)}\sigma_+^{(B)} + \sigma_-^{(A)}\sigma_+^{(B)}). \quad (46)$$

One finds for the eigenvalues

$$\lambda_1 = \omega_A + \omega_B,$$
$$\lambda_2 = -\omega_A - \omega_B,$$
$$\lambda_3 = -\sqrt{\gamma^2 + (\omega_B - \omega_A)^2},$$
$$\lambda_4 = \sqrt{\gamma^2 + (\omega_B - \omega_A)^2}$$

together with the corresponding eigenvectors

$$|\lambda_1\rangle = |+\rangle_A|+\rangle_B,$$
$$|\lambda_2\rangle = |-\rangle_A|-\rangle_B,$$
$$|\lambda_3\rangle = N_3[(\omega_B - \omega_A - \sqrt{\gamma^2 + (\omega_B - \omega_A)^2})|-\rangle_A|+\rangle_B + \gamma|+\rangle_A|-\rangle_B],$$
$$|\lambda_4\rangle = N_4[(\omega_B - \omega_A + \sqrt{\gamma^2 + (\omega_B - \omega_A)^2})|-\rangle_A|+\rangle_B + \gamma|+\rangle_A|-\rangle_B],$$

with N_3 and N_4 being suitable normalization factors. It is now straightforward to obtain conditions for coefficients c_k in $|\chi\rangle = \sum_k c_k|\lambda_k\rangle$, which guarantee the preservation of linear entropy and hence provide GIFE states:

$$c_1 = c_3 = 0, \quad \text{and arbitrary} \;\; c_2, c_4; \quad (47a)$$
$$c_2 = c_3 = 0, \quad \text{and arbitrary} \;\; c_1, c_4; \quad (47b)$$
$$c_3 = c_4 = 0, \quad \text{and arbitrary} \;\; c_1, c_2; \quad (47c)$$
$$c_1 = c_4 = 0, \quad \text{and arbitrary} \;\; c_2, c_3; \quad (47d)$$
$$c_2 = c_4 = 0, \quad \text{and arbitrary} \;\; c_1, c_3. \quad (47e)$$

These solutions show in a clear way what we have anticipated, that all the eigenstates of the Hamiltonian are GIFE states. Some of the conditions we have found for the coefficients give IFE states: for example, Eq. (47c) gives rise to IFE states, as is quite easy to see. On the contrary, other conditions, such as Eq. (47a), for example, give rise to GIFE, since the state $c_2 e^{-i\lambda_2 t}|\lambda_2\rangle + c_4 e^{-i\lambda_4 t}|\lambda_4\rangle$ can never be considered as essentially evolving according to the free Hamiltonian of the system, unless $\omega_A = \omega_B$. Indeed, for example, the complete evolution shows that the states $|-\rangle_A|+\rangle_B$ and $|+\rangle_A|-\rangle_B$ accumulate the same phase, while the free evolution alone would give to them different phases, when $\omega_A \neq \omega_B$. An example of two possible effective Hamiltonians is given by $H_A^{\text{eff}} = \tilde{\omega}\sigma_z^{(A)}$ and $H_B^{\text{eff}} = \tilde{\omega}\sigma_z^{(B)}$, with $\tilde{\omega} = (\lambda_4 - \lambda_2)/2$.

VI. CONCLUSIONS

In this paper we have explored the connection between the two concepts of interaction-free evolutions and decoherence-free subspaces, bringing to light the similarities and differences. The very first difference between IFE and DFS is given by the context and the class of systems they refer to, in the sense that talking about DFS requires that one of the two subsystems is the environment and that the dynamics of the small system is unitary for all possible states of the environment; such restrictions do not apply to IFE. Therefore, when the system is made of a small system and its environment, both IFE and DFS are in principle possible. Since one could think that the existence of one of the two classes of states implies the existence of the other one, we have explored this possible connection, pointing out some general statements. We have brought to light the fact that, although the two concepts are both related to the idea that somehow the interaction between the two subsystems is not felt, the two concepts are quite independent. In fact, it can happen that DFS are present but no IFE, that IFE are present but no DFS, and that both IFE and DFS are present. This independence has been discussed in Sec. II.

The dynamics of a system prepared in a DFS is essentially governed by the free Hamiltonian of the system, as if the interaction were not present, but this notion has been generalized in the literature including the case where the system undergoes a unitary evolution even if such an evolution is generated by an effective Hamiltonian that differs from the free one (in this case we talk about generalized DFS, i.e., GDFS). On this basis, we have extended the concept of IFE introducing the idea of GIFE and provided a characterization of such class of evolutions. In particular, starting from noting that in such evolutions the amount of entanglement between the two subsystems must be preserved, we have found that a set of functionals can be considered that are necessarily preserved during any kind (properly generalized or not) of interaction-free evolution. On this basis, we have developed a strategy for systematically obtaining all possible GIFE states for any given Hamiltonian. Then we have applied such strategy on a specific (simple) situation.

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