Dynamical phase transitions as a resource for quantum enhanced metrology

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We consider the general problem of estimating an unknown control parameter of an open quantum system. We establish a direct relation between the evolution of both system and environment and the precision with which the parameter can be estimated. We show that when the open quantum system undergoes a first-order dynamical phase transition the quantum Fisher information (QFI), which gives the upper bound on the achievable precision of any measurement of the system and environment, becomes quadratic in observation time (cf. "Heisenberg scaling"). In fact, the QFI is identical to the variance of the dynamical observable that characterizes the phases that coexist at the transition, and enhanced scaling is a consequence of the divergence of the variance of this observable at the transition point. This identification makes it possible to establish the finite time scaling of the QFI. Near the transition the QFI is quadratic in time for times shorter than the correlation time of the dynamics. In the regime of enhanced scaling the optimal measurement whose precision is given by the QFI involves measuring both system and output. As a particular realization of these ideas, we describe a theoretical scheme for quantum enhanced estimation of an optical phase shift using the photons being emitted from a quantum system near the coexistence of dynamical phases with distinct photon emission rates.

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I. INTRODUCTION

The estimation of unknown parameters is a crucial task for quantum technology applications such as state tomography [1], system identification [2], and quantum metrology [3–5]. Enhancement in precision can be achieved by using entangled (highly correlated) quantum states which encode the unknown parameter, like the Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = |0\rangle^{\otimes N} + |1\rangle^{\otimes N}$ constructed of N qubits. With such a state as a resource an unknown parameter g can be encoded as $|\mathrm{GHZ}_g\rangle = |0\rangle^{\otimes N} + e^{-iNg}|1\rangle^{\otimes N}$. Since the parameter effectively encoded in the state is Ng, the estimation error on g scales as N^{-2} (referred to as Heisenberg scaling [6]) instead of the standard N^{-1} scaling for a noncorrelated state $(|0\rangle + e^{-ig}|1\rangle)^{\otimes N}$.

The key property that makes correlated states such as $|\mathrm{GHZ}\rangle$ useful for enhanced metrology is that they can be thought of as "bimodal", in the sense that the probability of an appropriate observable is peaked in two (or more) "phases" (the states $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ in the case of $|\mathrm{GHZ}\rangle$). This bimodality is reminiscent of what occurs near a (first-order) phase transition. In fact, enhanced parameter estimation can be achieved with pure states at quantum phase transitions [7]. For large N, highly correlated *pure* states are challenging to prepare in practice [8], either as the ground state of a closed many-body system or as the stationary state of some dissipative dynamics [9]. Typically, the latter requires careful system engineering, since generic open quantum systems have *mixed* rather than pure stationary states. In general, one therefore has to deal with mixed states. These, however, have an additional complication since the best possible measurement is difficult to formulate, in general, except for particular cases such as thermal states [10]. This means that with mixed states it is often difficult to compute the best possible precision of parameter estimation.

In this paper we show theoretically how to exploit the dynamics of open quantum systems (for example, driven atomic or molecular ensembles emitting photons [11] or quantum dots [12]) to generate states for quantum enhanced metrology. Our approach connects to recent work on parameter estimation with single stationary states of open quantum systems [20–22]. We overcome the problem of mixed states by considering the combined state of the system and output. This is a pure quantum state—actually a matrix product state (MPS) [13,14,16,17]—which encodes the state of the system as well as the record of emissions for the whole observation time. This allows us to find the best estimation precision using the system-output state as a resource.

This approach has several advantages. First, it provides improved precision due to the fact that the effective "size" of the system and output is now Nt, where t is the observation time and N is the system size. The second advantage arises from the fact that open systems can feature *dynamical* phase transitions (DPTs) [17–19], which, in contrast to static transitions, are characterized by singular changes in observables on the whole dynamical evolution and not just on the state of the system. We show that at a first-order DPT [18,19] the quantum Fisher information (QFI) of the system-and-output state may become quadratic in t giving rise to Heisenberg scaling. We also clarify the behavior away from the transition point. Here Heisenberg scaling of the QFI is present for times shorter than the correlation time of the dynamics, while asymptotically linear scaling is recovered. Moreover, due to the pure form of the system-output state we can always (formally) construct the optimal measurement. We discuss our ideas in a specific setting for quantum enhancement in estimation of optical phase shift using an intermittent system near a dynamical first-order transition as shown in Fig. 1(a).

II. ELEMENTS OF QUANTUM METROLOGY

We first review some essential aspects of quantum parameter estimation. Suppose that we wish to estimate a parameter g encoded in a quantum state ρ_g, by measuring an observable

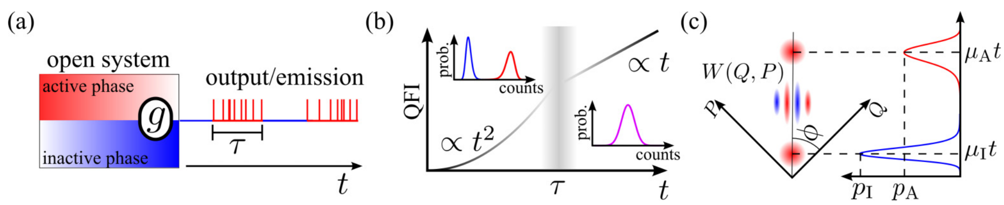

FIG. 1. (a) Open quantum system with a dynamics that features two dynamical phases of different activity and depends on the unknown parameter g. Near a first-order DPT the output (e.g., photons) shows strong intermittency where the temporal length of active and inactive periods is on average proportional to the correlation time τ. (b) In the vicinity of a DPT the QFI of the combined system-output state scales quadratically for observation times $t \ll \tau$. In the example of photon counting this regime features a bimodal count distribution; i.e., the two phases can be resolved. For $t \gg \tau$ this is no longer the case and the distribution becomes unimodal. Consequently, the QFI acquires a linear scaling with t. (c) Wigner distribution $W(Q,P)$ of the state (12) after being projected on an appropriate system state, e.g., $|\mathrm{I}\rangle + |\mathrm{A}\rangle$. The two peaks are located at radii that correspond to the count rates $\mu_{\mathrm{I,A}}$ of the inactive and active phases. The count distribution is not sensitive to the parameter ϕ, hence, counting is not an appropriate measurement for its estimation. Still, this state features an enhanced QFI with respect to changes in the parameter ϕ due to the highly oscillatory fringe pattern [with period $\propto t\,(\mu_{\mathrm{A}} - \mu_{\mathrm{I}})$] between the two peaks, which is characteristic for a Schrödinger cat state.

M. The estimation precision is given by the signal-to-noise ratio [23],

$$\mathrm{SNR}_g(M) = (d\langle M\rangle_g/dg)^2/\Delta_g^2 M,$$

where $\langle M\rangle_g = \mathrm{Tr}(\rho_g M)$ and $\Delta_g^2 M = \mathrm{Tr}(\rho_g M^2) - \langle M\rangle_g^2$ are the mean and variance of measuring M on ρ_g, respectively. The observable with the optimal SNR is (up to linear transformations) given by the so-called symmetric logarithmic derivative, $\mathcal{D}_g$, defined by the relation [24]

$$\frac{d\rho_g}{dg} = \frac{1}{2}(\mathcal{D}_g\rho_g + \rho_g\mathcal{D}_g). \tag{1}$$

Except for very particular forms of ρ_g, the optimal measurement $\mathcal{D}_g$ is difficult to engineer. Nevertheless, the SNR for this observable is given by the QFI [24], $F(\rho_g)$, which bounds the precision of any measurement that can be performed in practice. This bound is, in fact, given by the variance of $\mathcal{D}_g$, i.e., $F(\rho_g) = \Delta_g^2\mathcal{D}_g$.

In general, the QFI is hard to compute, but for a pure state, $|\psi_g\rangle$, it can be obtained from the fidelity $\langle\psi_{g_1}|\psi_{g_2}\rangle$ [7,20] according to

$$F(|\psi_g\rangle) = 4\partial_{g_1}\partial_{g_2}\ln\langle\psi_{g_1}|\psi_{g_2}\rangle\big|_{g_1=g_2=g}. \tag{2}$$

A situation which is relevant for what follows is when the parameter g is encoded in a unitary transformation on a pure state, $|\psi_g\rangle = e^{-igG}|\psi\rangle$. Here the fidelity $\langle\psi_{g_1}|\psi_{g_2}\rangle$ is the characteristic function of G at $g_1 - g_2$, and the QFI is given by its variance, $F(|\psi_g\rangle) = 4\Delta_g^2 G$. Note that while the QFI is given by the variance of both $\mathcal{D}_g$ and G, these two operators play very different roles. The optimal measurement to recover the parameter g is $\mathcal{D}_g$, and its SNR is maximal, $\mathrm{SNR}_g(\mathcal{D}_g) = F(|\psi_g\rangle)$. In contrast, G encodes g in the quantum state, but measuring it provides no information about g since $\mathrm{SNR}_g(G) = 0$.

For example, for the state $|\mathrm{GHZ}_g\rangle$ the generator is $G = \sum_j(1+\sigma_z^{(j)})/2$ and the optimal measurement $\mathcal{D}_g = \bigotimes_j e^{-igG}\sigma_y^{(j)}e^{igG}$, where $\sigma_a^{(j)}$ are Pauli operators acting on qubit j. The QFI for the GHZ state then obeys Heisenberg scaling, $F(|\mathrm{GHZ}_g\rangle) = N^2$. This is related to the fact that the distributions of both G and $\mathcal{D}_g$ are bimodal. In contrast, the QFI of the uncorrelated state is standard, $F((|0\rangle + e^{-ig}|1\rangle)^{\otimes N}) = N$, given by the fact that the corresponding distributions are unimodal. Below we show that an analogous change from bimodal to unimodal also accompanies a change in the scaling with time of the QFI when approaching a first-order DPT.

III. OPEN DYNAMICS AND MATRIX PRODUCT STATES

Our goal is to explore open quantum systems as resources for parameter estimation. We consider systems whose reduced dynamics, after tracing out the environment, is given by a Markovian master equation [25],

$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + \sum_{j=1}^{k}\left(L_j\rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j,\rho\}\right), \tag{3}$$

where H is the system's Hamiltonian and L_j are the jump operators ($j = 1,\ldots,k$). In the input-output formalism [26], the *joint system and output state* is given by a continuous MPS (CMPS) [13–15,17]. For clarity, we discretize time by δt, and the CMPS is approximated by a regular MPS (see [13–15,17] and Appendix A),

$$|\Psi(t)\rangle = \sum_{j_n,\ldots,j_1=0}^{k} K_{j_n}\cdots K_{j_1}|\chi\rangle \otimes |j_1,\ldots,j_n\rangle,$$

where $n = t/\delta t$ and

$$K_0 = e^{-i\delta t H}\sqrt{1 - \delta t\sum_j L_j^\dagger L_j},$$

$$K_{j>0} = e^{-i\delta t H}\sqrt{\delta t}L_j,$$

where $|\chi\rangle$ is the initial state of the system. The output state $|j_1,\ldots,j_n\rangle$ describes the time record of emissions into the environment, as sketched in Fig. 1(a).

IV. RELATION BETWEEN DPTS AND QFI

We now assume that the dynamics depends on the parameter g to be estimated; see Fig. 1(a). This means that the Hamiltonian, H_g, and jump operators, $L_{j,g}$, and consequently the master operator, $\mathcal{L}_g$ [Eq. (3)], may depend on g. It follows then that the MPS, $|\Psi_g(t)\rangle$, also depends on g. When varying a parameter g in $\mathcal{L}_g$, the state $|\Psi_g(t)\rangle$ can have a singular change. This could correspond to either a static phase transition in the stationary state of the system or a DPT in the system and output. Both kinds of transitions are captured by discontinuities in the average, or a higher cumulant, of an observable that acts on the whole of $|\Psi_g(t)\rangle$, or more abstractly by the fidelity [7], which also depends on the parameter g,

$$\langle\Psi_{g_1}(t)|\Psi_{g_2}(t)\rangle = \mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\}, \tag{4}$$

where $\mathcal{L}_{g_1,g_2}$ is a deformation of the master operator [see Appendix A],

$$\mathcal{L}_{g_1,g_2}\ \rho = -iH_{g_1}\rho + i\rho H_{g_2} + \sum_{j=1}^{k}\Big[L_{j,g_1}\rho L_{j,g_2}^\dagger - \frac{1}{2}\big(L_{j,g_1}^\dagger L_{j,g_1}\rho + \rho L_{j,g_2}^\dagger L_{j,g_2}\big)\Big]. \tag{5}$$

Thus, in the long time limit the QFI of $|\Psi_g(t)\rangle$ is related to the largest eigenvalue $\lambda_1(g_1,g_2)$ of $\mathcal{L}_{g_1,g_2}$,

$$\lim_{t\to\infty} t^{-1}F(|\Psi_g(t)\rangle) = 4\partial_{g_1}\partial_{g_2}\lambda_1(g_1,g_2)|_{g_1=g_2=g}. \tag{6}$$

One can already see that something interesting will occur as the system approaches a DPT, so that the gap between the two leading eigenvalues of $\mathcal{L}_g$ closes at some g, see also Appendix B 2.

When the gap is small, for example close to a DPT, there is a time regime where the QFI is quadratic in time,

$$F(|\Psi_g(t)\rangle) = 4t^2\partial_{g_1}\partial_{g_2}\mathrm{ReTr}\{\mathcal{L}_{g_1,g}\mathcal{P}\mathcal{L}_{g,g_2}\mathcal{P}|\chi\rangle\langle\chi|\}_{g_1=g_2=g} - \big|2t\partial_{g_1}\mathrm{Tr}\{\mathcal{L}_{g_1,g}\mathcal{P}|\chi\rangle\langle\chi|\}_{g_1=g}\big|^2 + t^2O(t\lambda_2) + O(t), \tag{7}$$

where $\mathcal{P}$ is a projection onto the first two eigenvectors of $\mathcal{L}_g$ corresponding to the two eigenvalues with the largest real part, $(\lambda_1=0,\lambda_2)$. The gap is given by $-\mathrm{Re}\ \lambda_2$. This approximation of Eq. (7) is valid for $\tau'\ll t\ll\tau$, where τ is the correlation time given by the gap, $\tau\equiv(-\mathrm{Re}\ \lambda_2)^{-1}$, while τ' is the longest time scale associated with the rest of the spectrum, $\tau'\equiv(-\mathrm{Re}\ \lambda_3)^{-1}$. The quadratic time dependence of the QFI (7) is a consequence of time-correlations in the system-output MPS. Furthermore, at a DTP $\lambda_2\to 0$ and the asymptotic scaling of Eq. (6) is no longer valid. Instead, the QFI is quadratic in time and this *Heisenberg scaling* is given exactly by the Eq. (7) for all $t\gg\tau'$ (see Appendix B 3).

V. INTERMITTENCY AND ENHANCED ESTIMATION OF AN OPTICAL PHASE SHIFT

We now use the ideas above for the case of a system with intermittent dynamics [12,19,27] used as a resource for parameter estimation; see Fig. 1(a). The parameter here is an optical phase shift $g=\phi$ encoded in the jump operator L_1 by defining $L_{1,\phi}=e^{-i\phi}L_1$. For concreteness, note that the quantum jump associated with L_1 is the emission of a photon. This means that a parameter ϕ is imprinted on each outgoing photon. As we now show, if the system displays intermittent photon emission associated with a (first-order) DPT in counting statistics [17–19], then it will be an efficient resource for quantum metrology. With the above choice, the master operator is independent of ϕ, $\mathcal{L}_\phi=\mathcal{L}$. In turn, the deformed generator $\mathcal{L}_{\phi,\phi'}$, Eq. (5), from which the QFI is obtained, reads ($\Delta\phi=\phi-\phi'$)

$$\mathcal{L}_{\phi,\phi'}\rho = \mathcal{L}\rho + (e^{-i\Delta\phi}-1)L_1\rho L_1^\dagger. \tag{8}$$

With these definitions there is a direct connection to a photon counting problem [18,26]. The parameter ϕ is encoded in a unitary transformation of the MPS with generator $G=\Lambda(t)$, where $\Lambda(t)$ is the operator that counts the number of photons emitted up to time t, so that $|\Psi_\phi(t)\rangle = e^{-i\phi\Lambda(t)}|\Psi(t)\rangle$. The fidelity $\langle\Psi_\phi(t)|\Psi_{\phi'}(t)\rangle$ is the characteristic function of $\Lambda(t)$, the logarithm of which encodes all its cumulants. The cumulants are also encoded in the cumulant generating function (CGF),

$$\Theta_t(s) = \ln\sum_\Lambda e^{-s\Lambda}P(\Lambda,t),$$

where $P(\Lambda,t)$ is the probability of observing Λ photons in time t. The CGF can be related to a deformation of the master operator,

$$\Theta_t(s) = \mathrm{Tr}\{e^{t\mathcal{L}(s)}|\chi\rangle\langle\chi|\},$$

where $\mathcal{L}(s)$ is the same as (8) with $\Delta\phi=-is$. The long time limit of the CGF, $\theta(s)=\lim_{t\to\infty}t^{-1}\Theta(s,t)$, plays the role of a dynamical free energy for the ensemble of trajectories of photon emissions [18]. A singularity of $\theta(s)$ at some s_c is an indication of a phase transition in the ensemble of quantum jump trajectories, and when $s_c=0$ we have what we term a DPT, i.e., a singular change in the actual dynamics of the open system associated with a vanishing of the spectral gap λ_2 [17,18].

The asymptotic QFI (6) becomes

$$\lim_{t\to\infty} t^{-1}F(|\Psi_g(t)\rangle) = 4\partial_s^2\theta(s)|_{s=0}. \tag{9}$$

When the function $\theta(s)$ has a first-order singularity at some $|s_c|\gtrsim 0$, i.e., we are near a DPT, Eq. (9) will be large at $s=0$. In this a case the system will display an intermittent dynamics that switches between long periods with very distinct emission characteristics. Such a situation can be understood in terms of the coexistence of dynamical phases with significantly different photon count rates [18]; see Fig. 1(a). The QFI of $|\Psi_\phi(t)\rangle$ is proportional to the variance of the photon counting generator $G=\Lambda(t)$. For times shorter than the correlation time τ the system is mostly in one of the two phases, the distribution of the photon count is approximately bimodal, and the dynamics displays large fluctuations in the total photon emission; see Fig. 1(b). This implies a quadratic increase of the QFI with time, with Eq. (7) reducing to

$$F(|\Psi_g(t)\rangle) \approx 4t^2p_\mathrm{A}p_\mathrm{I}(\mu_\mathrm{A}-\mu_\mathrm{I})^2 + O(t). \tag{10}$$

Here μ_A and μ_I are the average counting rates, $\langle\Lambda(t)\rangle/t$, in the two phases (which we term "active" and "inactive" as we assume $\mu_\mathrm{A}>\mu_\mathrm{I}$), while p_A and p_I are the probability of the initial state $|\chi\rangle$ being in either phase. The above approximation holds for $t<\tau$, and becomes valid for all times at a DPT; see Appendix B 4. For times longer than τ, dynamics switches between the two phases, giving rise to intermittent behavior and eventual normal (unimodal) distribution of the photon count around the overall average (9); see Fig. 1(b).

The above shows that an intermittent system near a DPT can be used as a photon source for quantum enhanced estimation of an optical phase shift. The situation is then similar to that of GHZ states: The total photon count distribution is bimodal for times up to the correlation time τ and imprints an effective macroscopic parameter $t(\mu_A - \mu_I)\phi$ between the active and inactive dynamical phases; see discussion after Eq. (12).

VI. ENHANCED METROLOGY AND DPT IN GENERAL

We now extend the above discussion to the case where the dynamics has an arbitrary dependence on the parameter g to be estimated. In this case, g is encoded in the action of a "generator" $G_g(t)$,

$$G_g(t)|\Psi_g(t)\rangle = i\partial_g|\Psi_g(t)\rangle, \tag{11}$$

where $G_g(t)$ is the time integral of a local-in-time observable, just like $\Lambda(t)$ in the photon counting case. In terms of $G_g(t)$, the fidelity reads

$$\langle\Psi_{g_1}(t)\big|\Psi_{g_2}(t)\rangle = \langle\Psi_g(t)|\mathcal{T}e^{-i\int_{g_1}^{g_2} dg' G_{g'}(t)}|\Psi_g(t)\rangle,$$

where $\mathcal{T}$ is the g-ordering (cf. time-ordering) operator; see also [28,29]. The QFI is then the variance of $G_g(t)$. It follows that if we have a system which displays a first-order DPT where the dynamical phases are characterized by $G_g(t)$, then, in the $\tau' \ll t \ll \tau$ time regime, the QFI follows Eq. (10), where $\mu_{A,I}$ are the averages of $G_g(t)$ per unit time in the two coexisting dynamical phases [30]. Again, this emphasizes the connection between dynamical bimodality and enhanced quantum sensitivity.

The t^2 behavior of the QFI is an intrinsically quantum feature. This behavior cannot occur in systems for which the associated MPS is real and therefore the parameter cannot be encoded in unitary transformations of this state. Note that this includes all classical systems. In such a case, the average of $G_g(t)$ is zero [cf. Eq. (11)], and only terms linear in t will survive in Eq. (10).

VII. MEASUREMENT SCHEMES

We have shown that near a DPT the system-output state can have a large QFI. However, to exploit this, and achieve quantum enhanced sensitivity, it is necessary to measure an appropriately chosen observable. The optimal observable is known to be the symmetric logarithmic derivative $\mathcal{D}_g$ defined by (1), which for pure states can be written explicitly as $\mathcal{D}_g = 2\partial_g|\psi_g\rangle\langle\psi_g|$. However, the measurement of $\mathcal{D}_g$ will be difficult to engineer in most practical situations. One needs therefore to find an alternative which is both practical and whose SNR is as close as possible to the QFI.

Despite the fact that the intricacy of the optimal measurement makes it impractical, we can still formulate general characteristics for a measurement that achieves enhanced precision. The first consideration is whether the measurement should be on the system or output or both. In fact, in the regime of enhanced scaling the optimal measurement whose precision is given by the QFI involves measuring both system and output. The reason is that the precision achievable by measuring only the output is bounded by $p_A F(|\Psi_A(t)\rangle) + p_I F(|\psi_I(t)\rangle)$, which scales linearly in time. Here $|\Psi_{A,I}(t)\rangle$ are the MPS states associated with the individual active and inactive stationary states and $p_{A,I}$ are their probabilities; see Eq. (10). This last result is the precision of an idealized protocol given by a first measurement of the system to project onto one of the subspaces associated with the competing stationary states, followed by an optimal measurement of the conditioned system-output state $|\Psi_{A,I}(t)\rangle$. The second consideration is what should be the time extension t of a single measurement run. Here we imagine that the total time available to the experiment is T and one performs $n = T/t$ independent repetitions of an efficient system-output measurement of the state $|\Psi_g(t)\rangle$. This corresponds to a measurement of the joint state $|\Psi_g(t)\rangle^{\otimes n}$, and the optimal time t is that which maximizes the QFI of the joint state, $F(|\Psi_g(t)\rangle^{\otimes n}) = nF(|\Psi_g(t)\rangle)$. Equation (7) tells us that this optimal time is of the order of the correlation time, $t = O(\tau)$.

For the case of optical phase-shift estimation at a DPT, the bimodality of the system-output state means that it is essentially of the form of a "Schrödinger cat" state. Assuming for simplicity that the competing stationary states are pure, it reads

$$|\Psi_\phi(t)\rangle = \sqrt{p_I}|I\rangle \otimes |\alpha_I(\phi)\rangle + \sqrt{p_A}|A\rangle \otimes |\alpha_A(\phi)\rangle, \tag{12}$$

where $|\alpha_A(\phi)\rangle$ are coherent states with amplitudes $\alpha_{I,A}(\phi) = e^{i\phi}\sqrt{t\mu_{I,A}}$, where $\mu_{I,A}$ are the photon emission rates of the dynamical phases; see Eq. (10) and Fig. 1(c). In fact, as shown in Ref. [31], the state (12) is approximately a GHZ state with an effective parameter $t(\mu_A - \mu_I)\phi$. Note that for (12) neither counting nor homodyne measurements achieve Heisenberg scaling, which highlights the general challenge of identifying optimal measurements. However, one might think of instead employing interferometric protocols, related to the ones put forward in Refs. [31–33] for superpositions of coherent states, in order to exploit the enhanced precision scaling.

VIII. CONCLUSIONS

We have shown that, close to a DPT, the output of an open quantum system can be seen as a resource for quantum metrology applications. For times of the order of the correlation time, the system-output QFI scales quadratically with time, while in the long time limit the QFI scales linearly in time with a rate which diverges when the spectral gap closes, as in a DPT. It remains an open issue how to exploit in a general and systematic way the large QFI of the system-output state close to a DPT.

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APPENDIX A: FIDELITY AND QFI OF MPS STATES

In this Appendix we prove Eqs. (2) and (4). We have

$$\begin{aligned}\partial_{g_1}\partial_{g_2}\ln\langle\psi_{g_1}|\psi_{g_2}\rangle\big|_{g_1=g_2=g}\\ &=\frac{\langle\psi'_g|\psi'_g\rangle}{\langle\psi_g|\psi_g\rangle}-\frac{\langle\psi'_g|\psi_g\rangle\langle\psi_g|\psi'_g\rangle}{\langle\psi_g|\psi_g\rangle^2}\\ &=\langle\psi'_g|\psi'_g\rangle-|\langle\psi_g|\psi'_g\rangle|^2,\end{aligned}$$

where $|\psi'_g\rangle:=\partial_{g_1}|\psi_{g_1}\rangle|_{g_1=g}$ and we have used the normalization of the state, $\langle\psi_g|\psi_g\rangle=1$.

On the other hand, for a family of pure states $\rho_g=|\psi_g\rangle\langle\psi_g|$ the symmetric logarithmic derivative is $\mathcal{D}_g=2(|\psi_g\rangle\langle\psi'_g|+|\psi'_g\rangle\langle\psi_g|)$. Therefore,

$$\begin{aligned}F(|\psi_g\rangle)&=\mathrm{Tr}\big(\rho_g\mathcal{D}_g^2\big)\\ &=4(\langle\psi'_g|\psi'_g\rangle+\langle\psi'_g|\psi_g\rangle\langle\psi_g|\psi'_g\rangle\\ &\quad+\langle\psi'_g|\psi_g\rangle^2+\langle\psi_g|\psi'_g\rangle^2)\\ &=4(\langle\psi'_g|\psi'_g\rangle-|\langle\psi'_g|\psi_g\rangle|^2)\\ &=4\partial_{g_1}\partial_{g_2}\ln\big|\langle\psi_{g_1}|\psi_{g_2}\rangle\big|_{g_1=g_2=g},\end{aligned}$$

where we used $\langle\psi'_g|\psi_g\rangle=-\langle\psi_g|\psi'_g\rangle$, resulting from differentiating $\langle\psi_g|\psi_g\rangle=1$.

In order to prove Eq. (4), let us consider the discretization of the master dynamics described below Eq. (2). We have

$$\begin{aligned}&\langle\Psi_{g_1}(t)|\Psi_{g_2}(t)\rangle\\ &=\mathrm{Tr}\{|\Psi_{g_2}(t)\rangle\langle\Psi_{g_1}(t)|\}\\ &=\mathrm{Tr}_S\left\{\sum_{j_n,\ldots,j_1=0}^{k}K_{j_n,g_2}\cdots K_{j_1,g_2}|\chi\rangle\langle\chi|K^\dagger_{j_n,g_1}\cdots K^\dagger_{j_1,g_1}\right\},\end{aligned}\tag{A1}$$

where $n=t/\delta t$, $|\chi\rangle$ is the initial pure state of the system and

$$K_{0,g}=e^{-i\delta tH_g}\sqrt{1-\delta t\sum_{j=1}^{k}L^\dagger_{j,g}L_{j,g}},$$

$$K_{j>0,g}=e^{-i\delta tH_g}\sqrt{\delta t}L_{j,g}.$$

Thus, in the limit $\delta t\to0$, the fidelity becomes

$$\langle\Psi_{g_1}(t)|\Psi_{g_2}(t)\rangle=\mathrm{Tr}_S\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\},$$

where $\mathcal{L}_{g_1,g_2}$ is a modified master operator defined in Eq. (5). This happens analogously to the convergence of discretization in the master dynamics given by $\mathcal{L}_g$:

$$\begin{aligned}\rho_g(n)&=\sum_{j_n,\ldots,j_1=0}^{k}K_{j_n,g}\cdots K_{j_1,g}|\chi\rangle\langle\chi|K^\dagger_{j_n,g}\cdots K^\dagger_{j_1,g}\\ &\xrightarrow[\delta t\to0]{}\rho_g(t)=e^{t\mathcal{L}_g}|\chi\rangle\langle\chi|.\end{aligned}$$

The same result has already been discussed in [21,22] and can be derived similarly to the discretized version (A1) by using the CMPS which describes the state of the system and the output in continuous time [14],

$$\begin{aligned}|\Psi(t)\rangle=&\sum_{m=0}^{\infty}\sum_{j_1,\ldots,j_m=1}^{k}\int_0^t dt_1\int_{t_1}^t dt_2\cdots\int_{t_{m-1}}^t dt_m\\ &\times\big(e^{-i(t-t_m)H^{\rm eff}}L_{j_m}e^{-i(t_m-t_{m-1})H^{\rm eff}}\\ &\cdots L_{j_2}e^{-i(t_2-t_1)H^{\rm eff}}L_{j_1}e^{-it_1H^{\rm eff}}|\chi\rangle\big)\\ &\otimes|(j_1,t_1),(j_1,t_2),\ldots,(j_m,t_m)\rangle,\end{aligned}\tag{A2}$$

where $H^{\rm eff}=H-i\sum_{j=1}^{k}L_j^\dagger L_j$ is the effective Hamiltonian, $|(j_1,t_1),(j_1,t_2),\ldots,(j_m,t_m)\rangle$ is a state of the output with m emissions $\{j_1,\ldots,j_m\}$ at times $\{t_1,\ldots,t_m\}$, and the term $m=0$ in the first sum corresponds to the no-emission event when the output state is the vacuum.

APPENDIX B: TIME DEPENDENCE OF QFI

In this section we first discuss the general dependence of the QFI of the MPS state $|\Psi(t)\rangle$ on time t. This enables us to prove the asymptotic linear behavior of the QFI for dynamics with a unique stationary state; see Eq. (6). Using the general time dependence of the QFI, we then prove the existence of a quadratic scaling regime of the QFI [cf. Eq. (7)] for dynamics near a first-order DPT. Finally, for a system displaying a first-order DPT in photon emissions, we argue how the quadratic scaling of the QFI for phase estimation with emitted photons can be related to difference in photon emission rates between two dynamical phases; cf. Eq. (10).

1. General time dependence of the QFI

In order to express the QFI of the MPS state $|\Psi(t)\rangle$, we use Eqs. (2) and (4) and obtain

$$\begin{aligned}F(|\Psi_g(t)\rangle)=4\partial_{g_1}\partial_{g_2}\ln\mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\}_{g_1=g_2=g}=&-4\left|\mathrm{Tr}\left\{\int_0^t dt'\partial_{g_1}\mathcal{L}_{g_1,g}\rho_g(t')\right\}_{g_1=g}\right|^2\\ &+4\mathrm{Tr}\left\{\int_0^t dt'\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}\rho_g(t')\right\}_{g_1=g_2=g}\\ &+8\mathrm{Re}\mathrm{Tr}\left\{\int_0^t dt'\int_0^{t-t'}dt''\partial_{g_1}\mathcal{L}_{g_1,g}e^{t''\mathcal{L}_g}\partial_{g_2}\mathcal{L}_{g,g_2}\rho_g(t')\right\}_{g_1=g_2=g},\end{aligned}\tag{B1}$$

where $\rho_g(t):=e^{t\mathcal{L}_g}|\chi\rangle\langle\chi|$, $|\chi\rangle$ is an initial pure state of the system, $\mathcal{L}_g$ is the master operator [see Eq. (2)], and $\mathcal{L}_{g_1,g_2}$ is the modified master operator [see Eq. (5)]. Tr refers to the trace over the system from now on. Equation (B1) above results from the following calculations. First,

$$\partial_{g_1}\partial_{g_2}\ln \mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\}_{g_1=g_2=g} = -\left|\partial_{g_1}\mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g}}|\chi\rangle\langle\chi|\}_{g_1=g}\right|^2 + \partial_{g_1}\partial_{g_2}\mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\}_{g_1=g_2=g}.$$

Second,

$$\partial_{g_1}\mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g}}|\chi\rangle\langle\chi|\}_{g_1=g} = \mathrm{Tr}\left\{\int_0^t dt' e^{(t-t')\mathcal{L}_{g_1,g}}\partial_{g_1}\mathcal{L}_{g_1,g}e^{t'\mathcal{L}_{g_1,g}}|\chi\rangle\langle\chi|\right\}_{g_1=g} = \mathrm{Tr}\left\{\int_0^t dt' \partial_{g_1}\mathcal{L}_{g_1,g}\rho_g(t')\right\}_{g_1=g},$$

where the third line results from the operator $e^{t\mathcal{L}_g}$ being trace preserving. Similarly, the second and third line in Eq. (B1) correspond to $\partial_{g_1}\partial_{g_2}\mathrm{Tr}\{e^{t\mathcal{L}_{g_1,g_2}}|\chi\rangle\langle\chi|\}_{g_1=g_2=g}$.

For clarity of further presentation, we assume that $\mathcal{L}_g$ can be diagonalized with right and left eigenvectors $\{\rho_k\}_{k=1}^{d^2}$, $\{l_k\}_{k=1}^{d^2}$, ordered such that the corresponding eigenvalues $0=\lambda_1 > \mathrm{Re}\lambda_2 \geqslant \mathrm{Re}\lambda_3 \geqslant \cdots \geqslant \mathrm{Re}\lambda_{d^2}$ and normalized $\mathrm{Tr}\{l_j^\dagger \rho_k\} = \delta_{jk}$, $j,k=1,\ldots,d^2$, where d is the dimension of the system Hilbert space $\mathcal{H}$ and we have explicitly assumed one stationary state $\rho_1=\rho_{ss}$. Note that the eigenvectors are matrices acting on $\mathcal{H}$. For convenience, apart from the standard matrix notation, they will be also denoted as vectors $\{\|\rho_k\rangle\rangle\}_{k=1}^{d^2}$, $\{\langle\langle l_k\|\}_{k=1}^{d^2}$ in the space of matrices, with the scalar product $\langle\langle l\|\rho\rangle\rangle := \mathrm{Tr}\{l^\dagger\rho\}$. Note the contrast to vectors (pure states) $|\chi\rangle$ in $\mathcal{H}$. One can now simply write $\mathcal{L}_g = 0\|\rho_{ss}\rangle\rangle\langle\langle 1\| + \sum_{k=2}^{d^2}\lambda_k\|\rho_k\rangle\rangle\langle\langle l_k\|$. The discussion below will be similar for a general Jordan decomposition of $\mathcal{L}_g$.

As Eq. (B1) involves integrals of $e^{t\mathcal{L}_g}$, we need to consider the 0 eigenspace of $\mathcal{L}_g$, i.e., the stationary state ρ_{ss}, separately from the rest of eigenvectors whose eigenvalues differ from 0. We introduce the projection $\mathcal{P}_1 := \sum_{k=2}^{d^2}\|\rho_k\rangle\rangle\langle\langle l_k\|$ on the complement of ρ_{ss} and denote the restriction of an operator $\mathcal{X}$ to this complement by $[\mathcal{X}]_{\mathcal{P}_1} := \mathcal{P}_1\mathcal{X}\mathcal{P}_1$.

We now express the finite time behavior of QFI using derivatives of the modified master operator and the diagonal decomposition of the original master operator $\mathcal{L}_g$. From Eq. (B1) it follows that

$$\begin{aligned}
F(|\Psi_g(t)\rangle) = &-4\left|t\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\rho_{ss}\} + \mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\left[\frac{e^{t\mathcal{L}_g}-\mathcal{I}}{\mathcal{L}_g}\right]_{\mathcal{P}_1}|\chi\rangle\langle\chi|\right\}\right|^2_{g_1=g} \\
&+4\left(t\mathrm{Tr}\{\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}\rho_{ss}\} + \mathrm{Tr}\left\{\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}\left[\frac{e^{t\mathcal{L}_g}-\mathcal{I}}{\mathcal{L}_g}\right]_{\mathcal{P}_1}|\chi\rangle\langle\chi|\right\}\right)_{g_1=g_2=g} \\
&+4t^2\left|\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\rho_{ss}\}\right|^2 + 8\mathrm{Re}\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\rho_{ss}\}\mathrm{Tr}\left\{\partial_{g_2}\mathcal{L}_{g,g_2}\left[\frac{e^{t\mathcal{L}_g}-\mathcal{I}-t\mathcal{L}_g}{\mathcal{L}_g^2}\right]_{\mathcal{P}_1}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g} \\
&+8\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\left[\frac{e^{t\mathcal{L}_g}-\mathcal{I}-t\mathcal{L}_g}{\mathcal{L}_g^2}\right]_{\mathcal{P}_1}\partial_{g_2}\mathcal{L}_{g,g_2}\rho_{ss}\right\}_{g_1=g_2=g} \\
&-8\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}[\mathcal{L}_g^{-1}]_{\mathcal{P}_1}\partial_{g_2}\mathcal{L}_{g,g_2}\left[\frac{e^{t\mathcal{L}_g}-\mathcal{I}}{\mathcal{L}_g}\right]_{\mathcal{P}_1}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g} \\
&+8\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\left[\frac{e^{t\mathcal{L}_g}}{\mathcal{L}_g}\left(\int_0^t dt' e^{-t'\mathcal{L}_g}\partial_{g_2}\mathcal{L}_{g,g_2}e^{t'\mathcal{L}_g}\right)\right]_{\mathcal{P}_1}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g},
\end{aligned} \tag{B2}$$

and one can show that

$$\left[\frac{e^{t\mathcal{L}_g}}{\mathcal{L}_g}\left(\int_0^t dt' e^{-t'\mathcal{L}_g}\partial_{g_2}\mathcal{L}_{g,g_2}e^{t'\mathcal{L}_g}\right)\right]_{\mathcal{P}_1} = t\sum_{k=2}^{d^2}\frac{e^{t\lambda_k}}{\lambda_k}\langle\langle l_k\|\partial_{g_2}\mathcal{L}_{g,g_2}\|\rho_k\rangle\rangle\|\rho_k\rangle\rangle\langle\langle l_k\| + \sum_{j\neq k,j,k>1}^{d^2}\frac{e^{t\lambda_j}-e^{t\lambda_k}}{\lambda_k(\lambda_j-\lambda_k)}\langle\langle l_k\|\partial_{g_2}\mathcal{L}_{g,g_2}\|\rho_j\rangle\rangle\|\rho_k\rangle\rangle\langle\langle l_j\|.$$

The first line and the second line in Eq. (B2) correspond to the first line and the second line in Eq. (B1), respectively. All other terms in Eq. (B2) correspond to the third line in Eq. (B1). We see that the quadratic contribution $t^2|\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\rho_{ss}\}|^2$ cancels out and for one stationary state there is no explicit quadratic behavior. Equation (B2) will be further used for investigating the asymptotic and the quadratic time regime of QFI in the next sections.

We note that as an alternative route, one can use the eigendecomposition of the modified master operator $\mathcal{L}_{g_1,g_2}$ defined in Eq. (5):

$$e^{t\mathcal{L}_{g_1,g_2}} = \sum_{k=1}^{d^2} e^{t\lambda_k(g_1,g_2)}\|\rho_k(g_1,g_2)\rangle\rangle\langle\langle l_k(g_1,g_2)\|.$$

From Eqs. (2) and (4), we obtain for a single stationary state

$$F(|\Psi_g(t)\rangle) = -4\left[t^2\left|\partial_{g_1}\lambda_1(g_1,g)\right|^2 + 2t\mathrm{Re}\partial_{g_1}\lambda_1(g_1,g)\sum_{k=1}^{d^2} e^{t\lambda_k}\partial_{g_2}p_k(g,g_2) + \sum_{j,k=1}^{d^2} e^{t(\lambda_k+\lambda_j)}\partial_{g_1}p_j(g_1,g)\partial_{g_2}p_k(g,g_2)\right]_{g_1=g_2=g}$$
$$+4\left[t^2\left|\partial_{g_1}\lambda_1(g_1,g)\right|^2 + t\partial^2_{g_1g_2}\lambda_1(g_1,g_2) + \sum_{k=1}^{d^2} e^{t\lambda_k}\partial^2_{g_1g_2}p_k(g_1,g_2) + 2t\mathrm{Re}\sum_{k=1}^{d^2} e^{t\lambda_k}\partial_{g_1}p_k(g_1,g)\partial_{g_2}\lambda_k(g,g_2)\right]_{g_1=g_2=g}, \tag{B3}$$

where $p_k(g_1,g_2) = \langle\langle l_k(g_1,g_2)\|\,|\chi\rangle\langle\chi|\rangle\rangle \times \mathrm{Tr}\{\rho_k(g_1,g_2)\}$ and $p_k = p_k(g,g)$, $\lambda_k = \lambda_k(g,g)$, and $k = 1,\ldots,d^2$. The first line corresponds to the first line of Eq. (B2) and the second to the rest of terms in Eq. (B2). We see again that quadratic terms $t^2|\partial_{g_1}\lambda_1(g_1,g)|^2$ cancel out and there is no explicit quadratic behavior. In derivation of Eq. (B3) we have used that, for a single stationary state, $p_1(g,g) = 1$ and $p_k(g,g) = 0$, $k = 2,\ldots,d^2$, which follows from the orthogonality and normalization of the $\mathcal{L}_g$ eigenbasis, $\langle\langle l_i\|\rho_j\rangle\rangle = \delta_{i,j}$, and as $l_1 = I_{\mathcal{H}}$, we have $\mathrm{Tr}\{\rho_k(g,g)\} = \delta_{1,k}$.

2. Asymptotic QFI for the case of a unique stationary state

When the stationary state is unique, the second eigenvalue of the master operator $\mathcal{L}_g$ is different from 0, $\lambda_2 \neq 0$. As $\lim_{t\to\infty}[e^{t\mathcal{L}_g}]_{\mathcal{P}_1} = 0$, from Eq. (B2) we obtain an asymptotic *linear* behavior of the QFI:

$$\lim_{t\to\infty} t^{-1}F(|\Psi_g(t)\rangle) = 4\mathrm{Tr}\{\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}\rho_{ss}\} - 8\mathrm{Re}\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}[\mathcal{L}_g^{-1}]_{\mathcal{P}_1}\partial_{g_2}\mathcal{L}_{g,g_2}\rho_{ss}\}_{g_1=g_2=g}. \tag{B4}$$

This result was also obtained using different methods in [22]. We see that Eq. (B4) can diverge at a first-order DPT when $\lambda_2 \to 0$ for $g \to g_c$, as $[\mathcal{L}_g^{-1}]_{\mathcal{P}_1}$ has then a diverging eigenvalue λ_2^{-1}.

The asymptotic linear behavior of the QFI can be also obtained from Eq. (B3) as

$$\lim_{t\to\infty} t^{-1}F(|\Psi_g(t)\rangle) = 4\partial_{g_1}\partial_{g_2}\lambda_1(g_1,g_2)|_{g_1=g_2=g}; \tag{B5}$$

cf. Eq. (6). By comparing Eqs. (B4) and (B5), we see that when the gap closes at g_c, $\lambda_2 = 0$, the maximal eigenvalue of $\mathcal{L}_{g_1,g_2}$ can be *nonanalytic* at $g_1 = g_2 = g_c$.

3. Quadratic time regime of QFI

In this section we describe the quadratic regime in the QFI scaling with time, which can be present for systems at and near a first-order DPT.

Quadratic behavior near a DPT. For a system near a DPT the gap is much smaller than the gap associated with the rest of the spectrum. For simplicity, we consider only one low-lying eigenvalue, i.e., $(-\mathrm{Re}\lambda_2) \ll (-\mathrm{Re}\lambda_3)$, but the discussion is similar for the general case of several low-lying eigenvalues. Note that the eigenvalues of $\mathcal{L}_g$ come in conjugate pairs because $\mathcal{L}_g$ preserves the Hermiticity of a matrix ρ, and thus $(-\mathrm{Re}\lambda_2) \ll (-\mathrm{Re}\lambda_3)$ implies $\lambda_2 \in \mathbb{R}$.

The separation in the eigenvalues introduces the intermediate time regime $(-\mathrm{Re}\lambda_3)^{-1} = \tau' \ll t \ll \tau = (-\lambda_2)^{-1}$. In this regime we expect the second eigenvector ρ_2 of $\mathcal{L}_g$ to be almost stationary and determine, with the stationary state ρ_{ss}, dominant terms in the QFI in Eq. (B2), whereas other eigenvectors not to play any significant role. We introduce the projection $\mathcal{P} := \|\rho_{ss}\rangle\rangle\langle\langle 1\| + \|\rho_2\rangle\rangle\langle\langle l_2\|$ on the subspace spanned by the ρ_{ss} and ρ_2. We also introduce the projection on their complement $\mathcal{P}_2 := \mathcal{I} - \mathcal{P} = \sum_{k=3}^{d^2}\|\rho_k\rangle\rangle\langle\langle l_k\|$ and denote by $[X]_{\mathcal{P}_2} = (\mathcal{I}-\mathcal{P})X(\mathcal{I}-\mathcal{P})$ the restriction of an operator X to this complement.

The general behavior of the QFI in Eq. (B2) simplifies to

$$F(|\Psi_g(t)\rangle) = -4\left|t\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\mathcal{P}|\chi\rangle\langle\chi|\} - \mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}[\mathcal{L}_g^{-1}]_{\mathcal{P}_2}|\chi\rangle\langle\chi|\}\right|^2_{g_1=g} + 4\big(t\mathrm{Tr}\{\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}\mathcal{P}|\chi\rangle\langle\chi|\}$$
$$- \mathrm{Tr}\{\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}[\mathcal{L}_g^{-1}]_{\mathcal{P}_2}|\chi\rangle\langle\chi|\}\big)_{g_1=g_2=g} + 4t^2\mathrm{Re}\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}\mathcal{P}\partial_{g_2}\mathcal{L}_{g,g_2}\mathcal{P}|\chi\rangle\langle\chi|\}_{g_1=g_2=g}$$
$$- 8\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\mathcal{P}\partial_{g_2}\mathcal{L}_{g,g_2}\left[\frac{\mathcal{I}+t\mathcal{L}_g}{\mathcal{L}_g^2}\right]_{\mathcal{P}_2}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g} - 8\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\left[\frac{\mathcal{I}+t\mathcal{L}_g}{\mathcal{L}_g^2}\right]_{\mathcal{P}_2}\partial_{g_2}\mathcal{L}_{g,g_2}\mathcal{P}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g}$$
$$+ 8\mathrm{Re}\mathrm{Tr}\{\partial_{g_1}\mathcal{L}_{g_1,g}[\mathcal{L}_g^{-1}]_{\mathcal{P}_2}\partial_{g_2}\mathcal{L}_{g,g_2}[\mathcal{L}_g^{-1}]_{\mathcal{P}_2}|\chi\rangle\langle\chi|\}_{g_1=g_2=g} + t^2 O(\lambda_2 t)O[c_2(c_2+1)C_1^2]$$
$$+ t\left\{O(\lambda_2 t)[O(c_2C_1^2C_2) + O(c_2C_3)] + O[(1+c_2)C_1^2C_2\|[e^{t\mathcal{L}_g}]_{\mathcal{P}_2}\|_1] + O(c_2C_1^2C_2)O\left(\frac{\lambda_2}{\lambda_3}\right)\right\}$$
$$+ O(\lambda_2 t)O(c_2C_1^2C_2^2) + O[(1+c_2)C_1^2C_2^2\|[e^{t\mathcal{L}_g}]_{\mathcal{P}_2}\|_1] + O(C_2C_3\|[e^{t\mathcal{L}_g}]_{\mathcal{P}_2}\|_1)$$
$$+ O(c_2C_1^2C_2^2)O\left(\frac{\lambda_2}{\lambda_3}\right) + O\left[C_1\left\|\sum_{j\neq k,j,k>2}^{d^2}\frac{e^{t\lambda_j}-e^{t\lambda_k}}{\lambda_k(\lambda_j-\lambda_k)}\|\rho_k\rangle\rangle\langle\langle l_k\|\partial_{g_2}\mathcal{L}_{g,g_2}\|\rho_j\rangle\rangle\langle\langle l_j\|\right\|_1\right]_{g_2=g}, \tag{B6}$$

where corrections in the approximation are given by $c_2 = \| \|\rho_2\rangle\rangle\langle\langle l_2\| \|_1$, $C_1 = \|\partial_{g_1}|_{g_1=g}\mathcal{L}_{g_1,g}\|_1$, $C_2 = \|[\mathcal{L}_g^{-1}]_{\mathcal{P}_2}\|_1$, and $C_3 = \|\partial_{g_1}\partial_{g_2}|_{g_1=g_2=g}\mathcal{L}_{g_1,g_2}\|_1$. The above-introduced norm $\|\mathcal{X}\|_1$ is an operator norm for $\mathcal{X}$ acting on matrices ρ on the system Hilbert space $\mathcal{H}$, induced by the trace-norm of the matrices, $\|\rho\|_1 = \mathrm{Tr}\{\sqrt{\rho^\dagger\rho}\}$. We note that estimate of the approximation error in Eq. (B6) is very rough and implies strong conditions on the master dynamics near a DPT, i.e., when the corrections are negligible. For a given model one should check the approximation by comparing to the exact results in Eq. (B2).

Assuming that the corrections in Eq. (B6) are negligible, there are *quadratic*, linear, and constant terms in Eq. (B6). In particular, the quadratic terms in Eq. (B6) correspond to Eq. (7).

Let us note that using Eq. (B3) does not provide clear results for the regime $\tau' \ll t \ll \tau$. From comparing Eq. (B5) to Eq. (B4), we see that when $\lambda_2 \to 0$, many terms in Eq. (B3) diverge. Thus, in order to simplify (B3) when $(-\lambda_2) \ll (-\mathrm{Re}\lambda_3)$, one needs to go back to the operators $\partial_{g_1}\mathcal{L}_{g_1,g}|_{g_1=g}$ and $\partial_{g_1}\partial_{g_2}\mathcal{L}_{g_1,g_2}|_{g_1=g_2=g}$ and to Eqs. (B2) and (B6).

Quadratic behavior at a first-order DPT. At a first-order DPT we have $\lambda_2 = 0$ and the considered time-regime is infinitely long, $\tau = \infty$. Moreover, in the limit of long time t all the corrections in Eq. (B6) are 0. Therefore, Eq. (B6) gives asymptotic *quadratic* behavior of the QFI [see also Eq. (7) in the main text]:

$$\lim_{t\to\infty} t^{-2} F(|\Psi_g(t)\rangle) = -4\left|\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\mathcal{P}|\chi\rangle\langle\chi|\right\}\right|^2_{g_1=g} + 4\mathrm{Re}\mathrm{Tr}\left\{\partial_{g_1}\mathcal{L}_{g_1,g}\mathcal{P}\partial_{g_2}\mathcal{L}_{g,g_2}\mathcal{P}|\chi\rangle\langle\chi|\right\}_{g_1=g_2=g}. \quad \text{(B7)}$$

4. Quadratic behavior and bimodality

Here we consider a system at and near a first-order DPT in photon emissions. We show how the quadratic behavior of the QFI emerges from the bimodality of the distribution of the emitted photons number. This relation is given in the main text by Eq. (10). A similar relation holds in the case of an arbitrary parameter, but this will be discussed in later work [30].

For the optical phase shift, $g = \phi$, encoded on photons emitted by a system we have $|\Psi_\phi(t)\rangle = e^{-i\phi\Lambda(t)}|\Psi(t)\rangle$, where $\Lambda(t)$ is the operator of the number of photons emitted up to time t. On the level of the master operator, this corresponds to defining $L_{1,\phi} := e^{-i\phi}L_1$, where L_1 is a jump operator of the master operator $\mathcal{L}$; see Eq. (3). Note that, as the parameter ϕ is encoded on the output, the master operator itself does not depend on the ϕ, $\mathcal{L}_\phi = \mathcal{L}$.

System at a DPT. We consider the system at a first-order DPT and, for simplicity, with twofold degeneracy of the zero eigenvalue of $\mathcal{L}$. There exist two stationary states, ρ_{A} and ρ_{I}, supported within orthogonal subspaces $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{I}}$, so that $\mathcal{H} = \mathcal{H}_{\mathrm{A}} \oplus \mathcal{H}_{\mathrm{I}}$. Moreover, the jump and Hamiltonian operators have a block diagonal form in this decomposition $H = H^{\mathrm{A}} \oplus H^{\mathrm{I}}$, $L_{1,\phi} = L^{\mathrm{A}}_{1,\phi} \oplus L^{\mathrm{I}}_{1,\phi}$, and $L_j = L^{\mathrm{A}}_j \oplus L^{\mathrm{I}}_j$, $j = 2,\ldots,k$, acting on $\mathcal{H} = \mathcal{H}_{\mathrm{A}} \oplus \mathcal{H}_{\mathrm{I}}$. Let $\mathcal{P}_{\mathcal{H}_{\mathrm{A}}}$, $\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}$ denote orthogonal projections on $\mathcal{H}_{\mathrm{A}}, \mathcal{H}_{\mathrm{I}}$, respectively. It simply follows that $\partial_{\phi_2}|_{\phi_2=\phi}\mathcal{L}_{\phi,\phi_2}\rho = iL_1\rho L_1^\dagger$ preserves the decomposition too and, thus, $\mathrm{Tr}\{\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}L_1\rho_{\mathrm{A}}L_1^\dagger\} = 0 = \mathrm{Tr}\{\mathcal{P}_{\mathcal{H}_{\mathrm{A}}}L_1\rho_{\mathrm{I}}L_1^\dagger\}$. This simplifies Eq. (B7) as

$$\begin{aligned}\lim_{t\to\infty} t^{-2} F(|\Psi_\phi(t)\rangle) &= -4(p_{\mathrm{A}}\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{A}}\} + p_{\mathrm{I}}\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{I}}\})^2 \\ &\quad + 4p_{\mathrm{A}}(\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{A}}\})^2 + 4p_{\mathrm{I}}(\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{I}}\})^2 \\ &= 4p_{\mathrm{A}}p_{\mathrm{I}}(\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{A}}\} - \mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{I}}\})^2,\end{aligned}$$

where $p_{\mathrm{A}} = \mathrm{Tr}\{\mathcal{P}_{\mathcal{H}_{\mathrm{A}}}|\chi\rangle\langle\chi|\}$ and $p_{\mathrm{I}} = \mathrm{Tr}\{\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}|\chi\rangle\langle\chi|\}$ are the probabilities of finding the evolved state (at any time) in subspaces $\mathcal{H}_{\mathrm{A}}$, $\mathcal{H}_{\mathrm{I}}$, respectively. In particular, these are the probabilities with which the system evolves asymptotically into the the stationary states ρ_{A}, ρ_{I}.

Below we show that $\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{A}}\}$, $\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{I}}\}$ correspond to the photon emission rates μ_{A}, μ_{I} for two *dynamical* phases, and we have

$$\lim_{t\to\infty} t^{-2} F(|\Psi_g(t)\rangle) = 4p_{\mathrm{A}}p_{\mathrm{I}}(\mu_{\mathrm{A}} - \mu_{\mathrm{I}})^2; \quad \text{(B8)}$$

see also Eq. (7). This confirms that the diverging variance of photon number and the quadratic behavior of the QFI are due to the fact that the distribution of $\Lambda(t)$ is a *mixture*, with probabilities p_{A} and p_{I}, of two distributions that have different means and both their means and variances are asymptotically *linear* in time.

Note that in order to ensure quadratic scaling of the QFI, $F(|\Psi_\phi(t)\rangle)$, the initial pure state $|\chi\rangle$ of the system needs to be a superposition of states inside $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{I}}$, so that both $p_{\mathrm{A}}, p_{\mathrm{I}} > 0$. Moreover, the two stationary states must differ in the photon emission rate, $\mu_{\mathrm{A}} \neq \mu_{\mathrm{I}}$. Otherwise, the asymptotic distribution of photon counts $\Lambda(t)$ is unimodal for large t and its variance scales linearly with time.

Proof of Eq. (B8). As $|\Psi_\phi(t)\rangle = e^{-i\phi\Lambda(t)}|\Psi(t)\rangle$, we have

$$\partial_{\phi_2}\mathrm{Tr}\{e^{t\mathcal{L}_{\phi,\phi_2}}|\chi\rangle\langle\chi|\}_{\phi_2=\phi} = \partial_{\phi_2}\langle\Psi_\phi(t)|\Psi_{\phi_2}(t)\rangle_{\phi_2=\phi} = -i\langle\Psi(t)|\Lambda(t)|\Psi(t)\rangle,$$

and thus,

$$\lim_{t\to\infty} t^{-1}\langle\Psi(t)|\Lambda(t)|\Psi(t)\rangle = i\mathrm{Tr}\left\{\partial_{\phi_2}\mathcal{L}_{\phi,\phi_2}\mathcal{P}|\chi\rangle\langle\chi|\right\}_{\phi_2=\phi} = -\mathrm{Tr}\{L_1^\dagger L_1\mathcal{P}|\chi\rangle\langle\chi|\}.$$

Choosing the initial state of the system as $|\chi_{\mathrm{A}}\rangle \in \mathcal{H}_{\mathrm{A}}$ or $|\chi_{\mathrm{I}}\rangle \in \mathcal{H}_{\mathrm{A}}$, we arrive at $-\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{A}}\} = \mu_{\mathrm{A}}$ and $-\mathrm{Tr}\{L_1^\dagger L_1\rho_{\mathrm{I}}\} = \mu_{\mathrm{I}}$, where μ_{A}, μ_{I} are the asymptotic rates of $\Lambda(t)$ when the system is initially in the state $|\chi_{\mathrm{A}}\rangle \in \mathcal{H}_{\mathrm{A}}$, $|\chi_{\mathrm{I}}\rangle \in \mathcal{H}_{\mathrm{I}}$. Let us assume $\mu_{\mathrm{A}} > \mu_{\mathrm{I}}$. We see that in the asymptotic limit $t \gg \tau'$ we can define two *dynamical phases* corresponding to active (A) and inactive (I) modes in total photon count $\Lambda(t)$ distribution, to be any MPS states $|\Psi^{\mathrm{A}}(t)\rangle$, $|\Psi^{\mathrm{I}}(t)\rangle$, which after tracing out the output are supported only on $\mathcal{H}_{\mathrm{A}}$, $\mathcal{H}_{\mathrm{I}}$, respectively. We therefore arrive at Eq. (B8).

Quadratic behavior and approximate bimodality for an intermittent system. Near a DPT in photon emissions, the system dynamics is intermittent and it switches between long time intervals of active and inactive behavior. The typical length of those intervals is given by the correlation time $\tau = (-\lambda_2)^{-1}$. In the regime, $\tau' \ll t \ll \tau$, when the QFI is quadratic in time, dynamics appears stationary and we would like to construct two approximately stationary states. Although the master operator $\mathcal{L}$ has only one stationary state ρ_{ss} and its second eigenvector ρ_2 fulfills $\mathrm{Tr}\rho_2 = 0$ (due to orthogonality of $\mathcal{L}$ eigenvectors), $\mathcal{L}$ is degenerate up to order λ_2 and below we sketch a construction of two approximately stationary states as linear combinations of ρ_{ss} and ρ_2. The construction closely follows the theory of classical nonequilibrium first-order phase transitions [34]. We leave rigorous proofs and discussion of Eq. (B8) in that case for later work [30].

Let us start with the case when a first-order DPT can be approached by changing parameters in the master operator $\mathcal{L}$. When approaching the DPT, the first two eigenvectors converge to ρ_1 and ρ_2, such that $\rho_1 \geqslant 0$, $\mathrm{Tr}\rho_1 = 1$, and $\mathrm{Tr}\rho_2 = 0$. One can show that in that case $\rho_1 = p\rho_{\mathrm{A}} + (1-p)\rho_{\mathrm{I}}$, $\rho_2 = \rho_{\mathrm{A}} - \rho_{\mathrm{I}}$, and $l_2 = (1-p)\mathcal{P}_{\mathcal{H}_{\mathrm{A}}} - p\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}$, where $0 < p < 1$, ρ_{A}, ρ_{I} are the stationary states supported on orthogonal subspaces $\mathcal{H}_{\mathrm{A}}$, $\mathcal{H}_{\mathrm{I}}$, respectively, and $\mathcal{P}_{\mathcal{H}_{\mathrm{A}}}$, $\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}$ are the orthogonal projections on these subspaces. In the general case of the system near a DPT, i.e., with small gap $-\lambda_2 \ll -\mathrm{Re}(\lambda_3)$ the construction of approximately stationary states is as follows. The master operator $\mathcal{L}$ can be shown to act almost block diagonally, i.e., H, L_j, $j = 1,\ldots,k$ are approximately block diagonal with respect to a splitting into some orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For pure initial states $|\chi^{(1)}\rangle$, $|\chi^{(2)}\rangle$ supported in these subspaces, the corresponding evolved states $\rho^{(1)}(t)$, $\rho^{(2)}(t)$ will thus stay supported approximately within $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, for times $t \ll \tau$. Moreover, those states will be well approximated by linear combinations of ρ_{ss} and ρ_2 for times $\tau' \ll t \ll \tau$.

We now sketch how to define the orthogonal subspaces $\mathcal{H}_1$, $\mathcal{H}_2$, and the initial states $|\chi^{(1)}\rangle$, $|\chi^{(2)}\rangle$ using the master operator $\mathcal{L}$. Due to $\mathcal{L}$ preserving Hermiticity of matrices, $\lambda_2 \in \mathbb{R}$ and both ρ_2 and l_2 are Hermitian matrices on $\mathcal{H}$; i.e., they diagonalize and their spectra are real. First, inspired by the form of the second eigenvector at a first-order DPT, $l_2 = (1-p)\mathcal{P}_{\mathcal{H}_{\mathrm{A}}} - p\mathcal{P}_{\mathcal{H}_{\mathrm{I}}}$, we define the subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ in the following way. $\mathcal{H}_1$ is spanned by the eigenvectors of l_2 which correspond to positive eigenvalues close to the maximal eigenvalue of l_2, while $\mathcal{H}_2$ is spanned by eigenvectors of l_2 corresponding to negative eigenvalues close to the minimal eigenvalue of l_2. Next, the initial states $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$ are chosen to be the eigenvectors corresponding to maximal and minimal eigenvalue of l_2, respectively. Finally, these states evolve into the states $\rho^{(1)}(t)$, $\rho^{(2)}(t)$ which are approximately stationary in the regime $\tau' \ll t \ll \tau$ and well approximated by linear combinations of ρ_{ss} and ρ_2 [30]. Moreover, two approximate dynamical phases in photon emissions can be defined as any MPS states which after tracing out the output are supported mostly on $\mathcal{H}_1$, $\mathcal{H}_2$, respectively. Using the above decomposition, it can be shown that the quadratic behavior of the QFI in the regime $\tau' \ll t \ll \tau$ is again related to the two modes in the counting distribution corresponding to the two approximate dynamical phases; see Eq. (7).

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