Scheme for accelerating quantum tunneling dynamics

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We propose a scheme of the exact fast forwarding of standard quantum dynamics for a charged particle. The present idea allows the acceleration of both the amplitude and the phase of the wave function throughout the fast-forward time range and is distinct from that of Masuda and Nakamura [Proc. R. Soc. A **466**, 1135 (2010)], which enabled acceleration of only the amplitude of the wave function on the way. We apply the proposed method to the quantum tunneling phenomena and obtain the electromagnetic field to ensure the rapid penetration of wave functions through a tunneling barrier. Typical examples described here are (1) an exponential wave packet passing through the δ-function barrier and (2) the opened Moshinsky shutter with a δ-function barrier just behind the shutter. We elucidate the tunneling current in the vicinity of the barrier and find a remarkable enhancement of the tunneling rate (tunneling power) due to the fast forwarding. In the case of a very high barrier, in particular, we present the asymptotic analysis and exhibit a suitable driving force to recover a recognizable tunneling current. The analysis is also carried out on the exact acceleration of macroscopic quantum tunneling with use of the nonlinear Schrödinger equation, which accommodates a tunneling barrier.

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I. INTRODUCTION

One of the most fascinating phenomena in quantum mechanics is quantum tunneling through a barrier. The tunneling shows up in Zener tunneling in biased semiconductors, quantum devices like diodes, scanning tunneling microscopy, α decay in heavy nuclei, etc. In general, however, the tunneling rate (tunneling power) is very small or the tunneling time is very long. Even in the case of resonant tunneling of electrons through heterostructures [1], there is room for research on the tunneling time [2]. Therefore, it is desirable to invent a protocol for accelerating the tunneling.

Masuda and Nakamura [3–5] investigated a way to accelerate quantum dynamics with the use of a characteristic driving potential determined by the additional phase of a wave function. One can accelerate a given quantum dynamics to obtain a target state in any desired short time. This kind of acceleration is called the fast forward [6] of quantum dynamics, which constitutes one of the promising means to the shortcut to adiabaticity [7–13]. The relationship between the fast forward and the shortcut to adiabaticity is nowadays clear [14,15].

The idea of tunneling seems to be incompatible with that of fast forward. But here one can combine these two ideas; that is, one can conceive a theory to accelerate the tunneling dynamics through the high barrier and complete, in any desired short time, the tunneling phenomenon which originally needed a long tunneling time.

Before embarking upon the main part of the text, we briefly summarize the previous theory of the fast forward of quantum dynamics by Masuda and Nakamura [3]. The Schrödinger equation in standard time with a nonlinearity constant c_0 (appearing in macroscopic quantum dynamics) is represented as

$$\imath\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_0 + V(\mathbf{x},t)\psi_0 - c_0|\psi_0|^2\psi_0. \quad (1.1)$$

$\psi_0 \equiv \psi_0(\mathbf{x},t)$ is a known function of space $\mathbf{x}$ and time t under a given potential $V(\mathbf{x},t)$ and is called a standard state. For any long time T as a standard final time, we choose $\psi_0(t = T)$ as a target state that we are going to generate.

Let $\tilde{\psi}_0(\mathbf{x},t)$ be a fast-forwarded state of $\psi_0(\mathbf{x},t)$ as defined by

$$\tilde{\psi}_0(\mathbf{x},t) \equiv \psi_0(\mathbf{x},\Lambda(t)) \quad (1.2)$$

with

$$\Lambda(t) = \int_0^t \alpha(t')\,dt', \quad (1.3)$$

where t is a new time variable distinct from the standard one. $\alpha(t)$ is a magnification time-scale factor defined by

$$\begin{aligned} &\alpha(0) = 1,\\ &\alpha(t) > 1 \quad (0 < t < T_{FF}),\\ &\alpha(t) = 1 \quad (t \geqslant T_{FF}). \end{aligned} \quad (1.4)$$

We consider the fast-forward dynamics with a new time variable which reproduces the target state $\psi_0(T)$ in a shorter final time $T_{FF}(<T)$ defined by

$$T = \int_0^{T_{FF}} \alpha(t)dt. \quad (1.5)$$

Since the generation of $\tilde{\psi}_0$ requires an anomalous mass reduction, $\tilde{\psi}_0$ as it stands cannot be a candidate for the fast-forward state [3]. But one can obtain the target state by considering a fast-forwarded state $\psi_{FF} = \psi_{FF}(\mathbf{x},t)$ which differs from $\tilde{\psi}_0$ by an extra phase as

$$\psi_{FF}(t) = e^{if}\tilde{\psi}_0(t) = e^{if}\psi_0(\Lambda(t)), \quad (1.6)$$

where $f \equiv f(\mathbf{x},t)$ is a real function of $\mathbf{x}$ and t and is called the additional phase. With use of a new time variable t that appeared in Eq. (1.3), the Schrödinger equation for the fast-forward state ψ_{FF} is supposed to be given as

$$i\hbar\frac{\partial\psi_{FF}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_{FF} + V_{FF}\psi_{FF} - c_0|\psi_{FF}|^2\psi_{FF} \quad (1.7)$$

with the driving scalar potential $V_{FF} = V_{FF}(\mathbf{x},t)$. If we choose $\alpha(t)$ as in Eq. (1.4), the additional phase can vanish at the final time of the fast forward (T_{FF}) and we can obtain the exact target state

$$\psi_{FF}(T_{FF}) = \psi_0(T). \quad (1.8)$$

The explicit expression for $\alpha(t)$ in the fast-forward range ($0 \leqslant t \leqslant T_{FF}$) is proposed by Masuda and Nakamura [3–5] as

$$\alpha(t) = (\bar{\alpha} - 1)\cos\left(\frac{2\pi}{T/\bar{\alpha}}t + \pi\right) + \bar{\alpha}, \quad (1.9)$$

where $\bar{\alpha}$ is the mean value of $\alpha(t)$ and is given by $\bar{\alpha} = T/T_{FF}$. Besides the time-dependent scaling factor in Eq. (1.9) in the fast-forward range, we also have recourse to the uniform scaling factor

$$\alpha(t) = \bar{\alpha} \quad (0 \leqslant t \leqslant T_{FF}), \quad (1.10)$$

which is useful in the quantitative analysis of fast forward. Substituting Eqs. (1.1), (1.3), and (1.6) into Eq. (1.7) and taking its real and imaginary parts, we obtain a pair of equations for f and V_{FF}, which are solvable.

While the above idea guarantees the exact target state at $t = T_{FF}$, in the intermediate time range $0 \leqslant t \leqslant T_{FF}$ it accelerates only the amplitude of the wave function and fails to accelerate its phase because of the nonvanishing additional phase f in Eq. (1.6) on the way. If one wishes to accelerate the time-dependent current, one must innovate the theory to recover the phase exactly in the intermediate time range until $t = T_{FF}$, which is done below.

Our theory holds to both quantum dynamics described by the Schrödinger equation and macroscopic quantum dynamics described by the nonlinear Schrödinger equation. Section II is concerned with a framework of the exact fast forward of both quantum and macroscopic quantum dynamics. Sections III and IV are devoted to application to quantum tunneling ($c_0 = 0$) and Sec. V treats the macroscopic quantum tunneling which includes the nonlinearity ($c_0 \neq 0$).

II. APPROACH TO FAST-FORWARD THEORY

The Schrödinger equation for the wave function $\psi_0 \equiv \psi_0(\mathbf{x},t)$ of a charged particle in the presence of the scalar potential V is the same as in Eq. (1.1). For any long time T called a standard final time, we choose $\psi_0(\mathbf{x},T)$ as a target state that we are going to generate. In contrast to the previous works [3–5], the fast-forward wave function here does not include the additional phase factor throughout the fast-forwarding time range until T_{FF} in Eq. (1.5) and is given by

$$\psi_{FF}(\mathbf{x},t) = \psi_0(\mathbf{x},\Lambda(t)) = \tilde{\psi}_0(\mathbf{x},t). \quad (2.1)$$

Here $\Lambda(t)$ is the same as in Eq. (1.3). We try to realize ψ_{FF} by applying the electromagnetic field $\mathbf{E}_{FF}$ and $\mathbf{B}_{FF}$.

Let us assume ψ_{FF} to be the solution of the Schrödinger equation for a charged particle in the presence of an additional vector $\mathbf{A}_{FF}(\mathbf{x},t)$ and scalar $V_{FF}(\mathbf{x},t)$ potentials, as given by

$$\begin{aligned} i\hbar\frac{\partial\psi_{FF}}{\partial t} &= \left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}_{FF}\right)^2 + qV_{FF} + V\right)\psi_{FF} \\ &\quad - c_0|\psi_{FF}|^2\psi_{FF} \\ &= -\frac{\hbar^2}{2m}\nabla^2\psi_{FF} + \frac{i\hbar q}{2mc}(\nabla\cdot\mathbf{A}_{FF})\psi_{FF} \\ &\quad + \frac{i\hbar q}{mc}\mathbf{A}_{FF}\cdot\nabla\psi_{FF} + \frac{q^2\mathbf{A}_{FF}^2}{2mc^2}\psi_{FF} \\ &\quad + (qV_{FF} + V)\psi_{FF} - c_0|\psi_{FF}|^2\psi_{FF}. \end{aligned} \quad (2.2)$$

For simplicity, however, we hereafter employ the unit of velocity of light, $c = 1$, and the prescription of a positive unit charge, $q = 1$. Note that V in Eq. (2.2) is introduced independently from a given potential V, in contrast to the one in Eq. (1.7) which included V. The driving electromagnetic field is related by

$$\begin{aligned} \mathbf{E}_{FF} &= -\frac{\partial\mathbf{A}_{FF}}{\partial t} - \nabla V_{FF}, \\ \mathbf{B}_{FF} &= \nabla\times\mathbf{A}_{FF}. \end{aligned} \quad (2.3)$$

Substituting Eqs. (1.1) and (2.1) into Eq. (2.2) and taking its real and imaginary parts, we obtain a pair of equations,

$$\nabla\cdot\mathbf{A}_{FF} + 2\mathrm{Re}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right]\mathbf{A}_{FF} + \hbar(\alpha-1)\mathrm{Im}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] = 0 \quad (2.4)$$

and

$$\begin{aligned} V_{FF} = &-(\alpha-1)\frac{\hbar^2}{2m}\mathrm{Re}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] + \frac{\hbar}{m}\mathbf{A}_{FF}\mathrm{Im}\left[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}\right] \\ &-\frac{1}{2m}\mathbf{A}_{FF}^2 + (\alpha-1)V - (\alpha-1)c_0|\tilde{\psi}_0|^2. \end{aligned} \quad (2.5)$$

Now we write $\tilde{\psi}_0$ as

$$\tilde{\psi}_0 = \rho e^{i\eta} \quad (2.6)$$

with use of the real amplitude ρ and phase η defined by

$$\begin{aligned} \rho &\equiv \rho(\mathbf{x},\Lambda(t)), \\ \eta &\equiv \eta(\mathbf{x},\Lambda(t)). \end{aligned} \quad (2.7)$$

Then, using in Eqs. (2.4) the equalities $\mathrm{Re}[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}] = \frac{\nabla\rho}{\rho}$, $\mathrm{Im}[\frac{\nabla\tilde{\psi}_0}{\tilde{\psi}_0}] = \nabla\eta$, $\mathrm{Re}[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}] = \frac{\nabla^2\rho}{\rho} - (\nabla\eta)^2$, and $\mathrm{Im}[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}] = 2\frac{\nabla\rho}{\rho}\nabla\eta + \nabla^2\eta$, one finds that

$$\mathbf{A}_{FF} = -\hbar(\alpha-1)\nabla\cdot\eta \quad (2.8)$$

satisfies Eq. (2.4). Thanks to Eq. (1.1) with the variable t being replaced by $\Lambda(t)$, $\mathrm{Re}[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}]$ can be reexpressed as

$$\mathrm{Re}\left[\frac{\nabla^2\tilde{\psi}_0}{\tilde{\psi}_0}\right] = \frac{2m}{\hbar}\left(\frac{\partial\eta}{\partial\Lambda(t)} + \frac{1}{\hbar}V\right) - \frac{2mc_0}{\hbar^2}\rho^2. \quad (2.9)$$

Then V_{FF} can be expressed only with use of η as

$$V_{FF} = -(\alpha-1)\hbar\frac{\partial\eta}{\partial\Lambda(t)} - \frac{\hbar^2}{2m}(\alpha^2-1)(\nabla\eta)^2. \quad (2.10)$$

With use of the driving vector $\mathbf{A}_{FF}$ and scalar V_{FF} potentials in Eqs. (2.8) and (2.10), we can obtain the fast-forwarded state $\psi_{\rm FF}$ in Eq. (2.1) which is now free from the additional phase factor f in Eq. (1.6) used in Masuda and Nakamura's framework [3–5]. Logically, $\mathbf{A}_{\rm FF}$ and $V_{\rm FF}$, which both prove to be independent of the amplitude ρ, serve to compensate the additional phase f in Eq. (1.6) in their framework. The electromagnetic field introduced in Refs. [5,16] is designed to guarantee the equality in Eq. (1.8) at $t = T_{\rm FF}$ and fails in removing the additional phase in the fast-forward time range $0 < t < T_{\rm FF}$.

Two points should be noted: (1) the above driving potentials do not explicitly depend on the nonlinearity coefficient c_0 [i.e., Eqs. (2.8) and (2.10) work for the nonlinear Schrödinger equation as well] and (2) the magnetic field $\mathbf{B}_{\rm FF}$ is vanishing, because a combination of Eqs. (2.3) and (2.8) leads to $\mathbf{B}_{\rm FF} = \nabla\times\mathbf{A}_{\rm FF} = 0$. Therefore, only the electric field $\mathbf{E}_{\rm FF}$ is required to accelerate a given dynamics. With use of Eqs. (2.3), (2.8), and (2.10), $\mathbf{E}_{\rm FF}$ is given explicitly by

$$\mathbf{E}_{\rm FF} = \hbar\dot{\alpha}\nabla\eta + \hbar\frac{\alpha^2-1}{\alpha}\partial_t\nabla\eta + \frac{\hbar^2}{2m}(\alpha^2-1)\nabla(\nabla\eta)^2. \quad (2.11)$$

A remarkable issue of the present scheme is the enhancement of the current density $\mathbf{j}_{\rm FF}$. Using a generalized momentum which includes a contribution from the vector potential in Eq. (2.8), we see

$$\mathbf{j}_{\rm FF}(\mathbf{x},t) \equiv \mathrm{Re}\left[\psi_{\rm FF}^*(\mathbf{x},t)\frac{1}{m}\left(\frac{\hbar}{i}\nabla - \mathbf{A}_{\rm FF}\right)\psi_{\rm FF}(\mathbf{x},t)\right] = \frac{\hbar}{m}\alpha(t)\rho^2(\mathbf{x},\Lambda(t))\nabla\eta(\mathbf{x},\Lambda(t)), \quad (2.12)$$

under the prescription of a positive unit charge. Noting the current density in the standard dynamics,

$$\mathbf{j}(\mathbf{x},t) \equiv \mathrm{Re}\left[\psi_0^*(\mathbf{x},t)\frac{\hbar}{im}\nabla\psi_0(\mathbf{x},t)\right] = \frac{\hbar}{m}\rho^2(\mathbf{x},t)\nabla\eta(\mathbf{x},t), \quad (2.13)$$

we find

$$\mathbf{j}_{\rm FF}(\mathbf{x},t) = \alpha(t)\mathbf{j}(\mathbf{x},\Lambda(t)). \quad (2.14)$$

Thus, the standard current density becomes both squeezed and magnified by a time-scaling factor $\alpha(t)$ in Eq. (1.9) or Eq. (1.10) as a result of the exact fast forwarding which enables acceleration of both amplitude and phase of the wave function throughout the time evolution.

Finally in this section, we evaluate the expectation of the energy of a particle in fast-forward dynamics and compare it with the corresponding expectation in standard dynamics. We can formally rewrite the Schrödinger equations in Eq. (1.1) and Eq. (2.2), respectively, as

$$i\hbar\frac{\partial\psi_0}{\partial t} = \hat{H}_0\psi_0, \quad (2.15)$$

$$i\hbar\frac{\partial\psi_{\rm FF}}{\partial t} = \hat{H}_{\rm FF}\psi_{\rm FF}, \quad (2.16)$$

where $\hat{H}_0$ and $\hat{H}_{\rm FF}$ are taken as corresponding Hamiltonian operators. We can write the expectation of energy in two cases as

$$\mathcal{E}_0(t) = \int \psi_0^*(\mathbf{x},t)\hat{H}_0\psi_0(\mathbf{x},t)d\mathbf{x} \quad (2.17)$$

and

$$\mathcal{E}_{\rm FF}(t) = \int \psi_{\rm FF}^*(\mathbf{x},t)\hat{H}_{\rm FF}\psi_{\rm FF}(\mathbf{x},t)d\mathbf{x}, \quad (2.18)$$

where the integration is over the full space [$\mathbf{x} \equiv (x,y,z)$]. Substituting Eqs. (2.15) and (2.16) into Eqs. (2.17) and (2.18), respectively, we obtain

$$\mathcal{E}_0(t) = i\hbar\int \psi_0^*(\mathbf{x},t)\frac{\partial\psi_0(\mathbf{x},t)}{\partial t}d\mathbf{x} \quad (2.19)$$

and

$$\mathcal{E}_{\rm FF}(t) = i\hbar\int \psi_{\rm FF}^*(\mathbf{x},t)\frac{\partial\psi_{\rm FF}(\mathbf{x},t)}{\partial t}d\mathbf{x} = i\hbar\int \psi_0^*(\mathbf{x},\Lambda)\frac{\partial\psi_0(\mathbf{x},\Lambda)}{\partial\Lambda}\frac{\partial\Lambda}{\partial t}d\mathbf{x} = i\hbar\alpha(t)\int \psi_0^*(\mathbf{x},\Lambda)\frac{\partial\psi_0(\mathbf{x},\Lambda)}{\partial\Lambda}d\mathbf{x}. \quad (2.20)$$

Here $\alpha(t)$ comes from $\frac{\partial\Lambda}{\partial t}$ in Eq. (1.3). Comparing Eq. (2.20) with Eq. (2.19), we have the relation between the expectations of energy between standard and fast-forward dynamics:

$$\mathcal{E}_{\rm FF}(t) = \alpha(t)\mathcal{E}_0(\Lambda(t)), \quad (2.21)$$

which is similar to Eq. (2.14) and plays a vital role in the fast forward of quantum tunneling.

Now we apply the present scheme to several tunneling phenomena in quantum mechanics. As for $\alpha(t)$, we choose a nonuniform factor in Eq. (1.9) in the fast-forward time region, except when stated otherwise.

III. FAST FORWARD OF TUNNELING OF WAVE-PACKET DYNAMICS

Confining to one-dimensional (1D) motion, we now investigate the time evolution of a localized wave packet when it runs through the δ-function barrier. The initial wave packet centered at $x = -x_0$ and having width β^{-1} and momentum k is expressed as

$$\psi^{(0)}(x,0) = \sqrt{\beta}e^{-\beta|x+x_0|}e^{ik(x+x_0)}. \quad (3.1)$$

$\psi^{(0)}(x,0)$ satisfies the normalization condition $\int_{-\infty}^{\infty}|\psi^{(0)}(x,0)|^2 dx = 1$. Therefore, $\langle x\rangle = -x_0$ and $\langle p\rangle = k$ at $t = 0$. While the wave function in Eq. (3.1) is nondifferentiable at $x = -x_0$, it does not generate a discontinuity in physical quantities like probability amplitude, current density, energy density, etc.

The time-dependent Schrödinger equation with δ-function barrier at $x = 0$ is given by

$$\left[i\hbar\partial_t + (\hbar^2/2m)\partial_x^2\right]\psi_0(x,t) = V(x)\psi_0(x,t) \quad (3.2)$$

with $V(x) = V_0\delta(x)$. In order to simplify the notation, we use "natural units" ($\hbar = m = 1$) from now on.

The time evolution of ψ_0 for $t > 0$ follows from

$$\psi_0(x,t) = \int_{-\infty}^{\infty} dx' K_0(x,t|x',0)\psi^{(0)}(x',0) - V_0\int_{-\infty}^{\infty} dx' M(|x| + |x'|; -iV_0; t)\psi_0(x',0). \quad (3.3)$$

The first term on the right-hand side of Eq. (3.3) describes the time evolution of the free ($V_0 = 0$) wave packet. Here K_0 is the free-particle propagator given by

$$K_0(x,t,|x',0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left(i\frac{m(x-x')^2}{2\hbar t}\right). \quad (3.4)$$

$M(x;k;t)$ is the Moshinsky function [17,18] defined in terms of the complementary error function by

$$M(x;k;t) = \frac{1}{2}e^{i(kx-k^2t/2)}\mathrm{erfc}\left(\frac{x-kt}{\sqrt{2it}}\right), \quad (3.5)$$

which is interpreted as the wave function of a monochromatic particle that is confined to the left half space $x \leqslant 0$ at $t = 0$.

The explicit solution $\psi_0(x,t)$ for $t > 0$ was given by Elberfeld and Kleber [19] as

$$\begin{aligned}\psi_0(x,t) = {} & \sqrt{\beta}[M(x+x_0;k-i\beta;t) \\ & + M(-x-x_0;-k-i\beta;t)] \\ & + V_0\sqrt{\beta}[S(x_0,\lambda^*;t) - S(x_0,-\lambda;t) \\ & + e^{-\lambda x_0}[S(0,-\lambda;t) + S(0,\lambda;t)]], \end{aligned} \quad (3.6)$$

where $\lambda = \beta - ik$ and $S(\xi,\lambda;t)$ is defined by

$$\begin{aligned}S(\xi,\lambda;t) = {} & [1/(V_0-\beta)][M(|x|+\xi;-iV_0;t) \\ & - M(|x|+\xi;-i\beta;t)]. \end{aligned} \quad (3.7)$$

The tunneling current evaluated just behind the barrier at $x = 0$ is

$$j(+0,t) = \mathrm{Im}[\psi_0^*(x,t)\partial_x\psi_0(x,t)]_{x=+0}. \quad (3.8)$$

The current is continuous, i.e., $j(+0,t) = j(-0,t)$, owing to a nonabsorbing potential barrier. These are results of the standard tunneling, i.e., tunneling dynamics on standard time scale.

Now we analyze the fast forward of tunneling of the wave packet and find the current density. By extracting the space-time-dependent phase η of the wave function in Eq. (3.6) in standard tunneling, one can obtain both vector and scalar potentials in Eqs. (2.8) and (2.10). Here η is available only numerically because ψ_0 in Eq. (3.6) is a linear combination of special functions. Under these driving potentials, one can generate the fast-forward state of tunneling of the wave packet through the barrier:

$$\begin{aligned}\psi_{\mathrm{FF}}(x,t) \equiv {} & \psi_0(x,\Lambda(t)) \\ = {} & \sqrt{\beta}[M(x+x_0;k-i\beta;\Lambda(t)) \\ & + M(-x-x_0;-k-i\beta;\Lambda(t))] \\ & + V_0\sqrt{\beta}[S(x_0,\lambda^*;\Lambda(t)) - S(x_0,-\lambda;\Lambda(t)) \\ & + e^{-\lambda x_0}[S(0,-\lambda;\Lambda(t)) + S(0,\lambda;\Lambda(t))]], \end{aligned} \quad (3.9)$$

which accelerates both amplitude and phase of Eq. (3.6) exactly. It should be noted that, without having recourse to η, ψ_{FF} in Eq. (3.9) is obtained from ψ_0 by the definition itself in Eq. (2.1). From Eq. (2.14), the current density evaluated at $x = +0$ for the fast-forward 1D tunneling phenomenon is

$$j_{\mathrm{FF}}(+0,t) = \alpha(t)j(+0,\Lambda(t)). \quad (3.10)$$

In our numerical analysis in Secs. III and IV, we use typical space and time units like $L = 10^{-2}$ times the *linear dimension of a device* and $\tau = 10^{-2}$ times the *phase coherent time*, in addition to the natural unit ($\hbar = m = 1$). Then, any length, wave number, and time are scaled by L, L^{-1}, and τ, respectively. In this scaling, we choose $x_0 = 2$, $k = 2$, and $\beta = 1$. We show the standard dynamics up to the standard final time $T = 2.5$ and its fast-forward version up to the shortened final time $T_{\mathrm{FF}} \equiv \frac{T}{\bar{\alpha}} = 0.5$ with the mean acceleration factor $\bar{\alpha} = 5$.

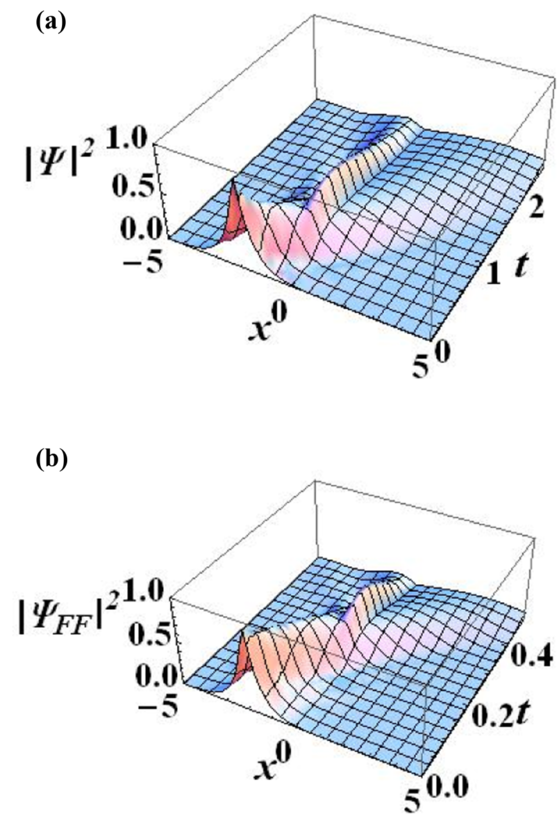

FIG. 1. Three-dimensional (3D) plot of the density distribution for the tunneling wave function starting from the exponential wave packet in Eq. (3.1). In the scaling described below Eq. (3.10), $x_0 = 2$, $k = 2$, $\beta = 1$, $T = 2.5$, $T_{\mathrm{FF}} = 0.5$, and $\bar{\alpha} = 5$. The barrier height here is $V_0 = 1$: (a) standard dynamics given by $|\psi_0(x,t)|^2$ for $0 \leqslant t \leqslant T$ and (b) fast-forward dynamics given by $|\psi(x,\Lambda(t))|^2$ for $0 \leqslant t \leqslant T_{\mathrm{FF}}$. In Figs. 2–7, the same scaling as in this figure is used.

In Fig. 1, we see the exponential wave function partly go through the barrier and it is partly reflected back after its collision with the barrier. The dynamics up to T on the standard time scale is reproduced in the fast-forward dynamics up to T_{FF}: The phenomenon in the latter is just the squeezing (along the time axis) of the one in the former. The minor discrepancy in the similarity of figures between upper and lower panels in time axis is due to the nonuniform time-scaling factor in Eq. (1.9), while the wave function amplitude with respect to to space axis is exactly reproduced at each time of the fast-forward dynamics.

Figure 2 shows the tunneling current at the position just behind the barrier ($x = +0$) as a function of time t. Here we choose $T = 5$, $T_{\mathrm{FF}} = 1$, and $\bar{\alpha} = 5$. We find the temporal behavior of the current j in the standard tunneling is both

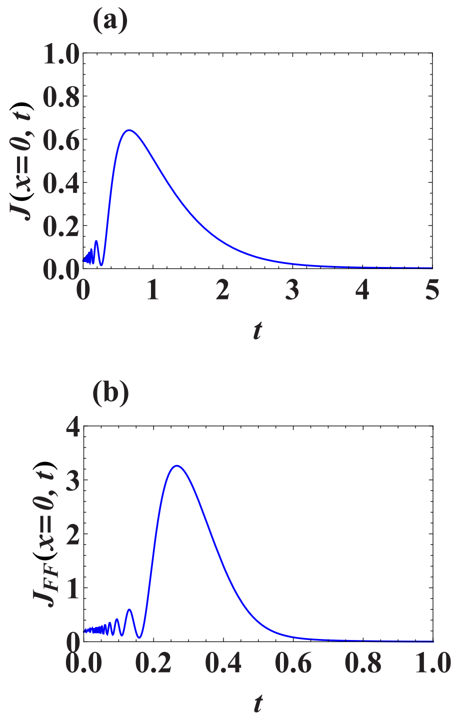

FIG. 2. Temporal behavior of tunneling current density at the position just behind the barrier ($x = +0$) for the tunneling wave function starting from the exponential wave packet in Eq. (3.1). $T = 5$, $T_{\rm FF} = 1$, and $\bar{\alpha} = 5$. The barrier height is $V_0 = 1$: (a) $j(t)$ and (b) $j_{\rm FF}(t)$. Note: Scales of the horizontal and vertical axes differ between upper and lower panels.

squeezed and amplified in that of the current $j_{\rm FF}$ in the fast-forward tunneling.

Let us define the tunneling rate (tunneling power) as

$$\Gamma = \frac{\int_0^T j(x = +0,t)dt}{T}, \tag{3.11}$$

where T here means the final time that the tunneling phenomenon is almost completed. The corresponding rate for the fast-forwarded case is

$$\Gamma_{\rm FF} = \frac{\int_0^{T_{\rm FF}} j_{\rm FF}(x = +0,t)dt}{T_{\rm FF}}. \tag{3.12}$$

In Fig. 2 we see $T = 5$ and $T_{\rm FF} = 1$ in the standard and fast-forward tunnelings, respectively. Noting Eq. (3.10), we find the numerator of the right-hand side of Eq. (3.12) is equal to that of Eq. (3.11). Then one can conclude

$$\Gamma_{\rm FF} = \frac{T}{T_{\rm FF}}\Gamma = \bar{\alpha}\Gamma. \tag{3.13}$$

Thus, the fast forwarding lets the standard tunneling rate be enhanced by a factor of the mean magnification time scale. This is a great advantage of the fast forward of quantum tunneling.

Figure 3 shows a 1D version of the driving electric field $\mathbf{E}_{\rm FF}$ in Eq. (2.11) to generate the exact fast forward of the tunneling dynamics, which is given by

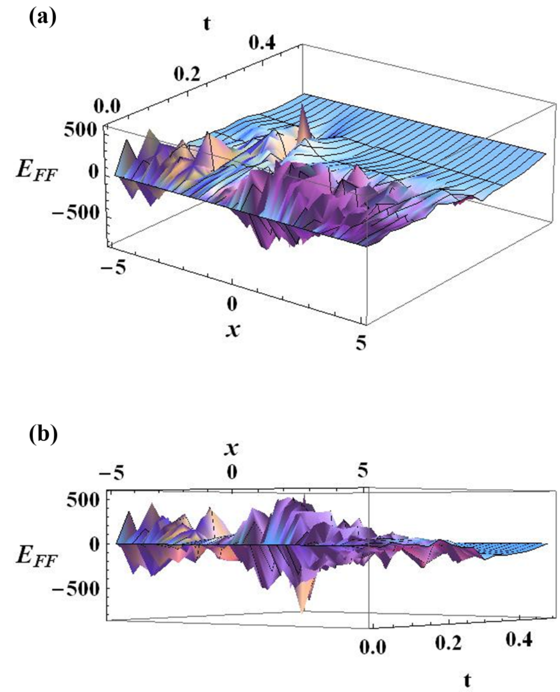

FIG. 3. 3D plot of electric field $E_{\rm FF}$ as a function of x and t: (a) top view and (b) side view.

$$E_{\rm FF} = \hbar\dot{\alpha}\partial_x\eta + \hbar\frac{\alpha^2 - 1}{\alpha}\partial_t\partial_x\eta + \frac{\hbar^2}{m}(\alpha^2 - 1)\partial_x\eta \cdot \partial_x^2\eta. \tag{3.14}$$

Figures 3(a) and 3(b) show 3D plots of $E_{\rm FF}$ on the x-t plane from different perspectives. In SI units for the electric field, our dimensionless $E_{\rm FF}$ corresponds to $E_{\rm SI}^{\rm FF} = \frac{m_e c\omega}{e} \times E_{\rm FF} \sim \frac{10^6}{\lambda}E_{\rm FF}$, where m_e, e, c, ω, and λ are electron mass, electron charge, velocity of light, frequency of laser light, and its wavelength, respectively. The typical value $E_{\rm FF} = 100$ in ordinates in Fig. 3 in the case of ir lasers of wavelength $\sim 1\ \mu$m means $E_{\rm SI}^{\rm FF} = 10^{14}$. The driving electric field shown in Fig. 3 can be implemented using, for instance, a rapidly moving laser beam to create a possibly dynamic time-averaged optical dipole potential [20].

Before closing this section, we should note the standard tunneling here is autonomous (i.e., $V_0 = \text{const}$) and thereby the total energy of the electron $\mathcal{E}_0 = \text{const}$, while the corresponding fast-forward tunneling is nonautonomous with the time-dependent total energy $\mathcal{E}_{\rm FF}(t)$. $\mathcal{E}_{\rm FF}(t)$ satisfies $\mathcal{E}_{\rm FF}(t) = \alpha(t)\mathcal{E}_0$ as a special case of Eq. (2.21). In the tunneling phenomenon of standard dynamics, we always see the inequality $\mathcal{E}_0 < V_0$ (barrier height). Then we can define $\alpha_{\max} \equiv \frac{V_0}{\mathcal{E}_0}(>1)$ and choose the time scaling factor $\alpha(t)$ as

$$1 \leqslant \alpha(t) < \alpha_{\max}, \tag{3.15}$$

which guarantees the inequality $\mathcal{E}_{\rm FF}(t) = \alpha(t)\mathcal{E}_0 < V_0$. In conclusion, so long as Eq. (3.15) is satisfied, the fast-forwarded dynamics here is also the tunneling phenomenon keeping the particle's energy $\mathcal{E}_{\rm FF}$ below the barrier height V_0 throughout the time evolution, and the time scaling works more effectively for the particle with lower incident energy. The same assertion as above holds in the following sections.

IV. FAST-FORWARD TUNNELING DYNAMICS FROM MOSHINSKY SHUTTER

Now let us investigate the dynamics of a monochromatic beam of noninteracting particles of mass m and energy $\hbar^2k^2/2m$ moving parallel to the x axis from the left to the right. Until $t < 0$, the beam is assumed to be stopped by the shutter at $x = 0$ perpendicular to the beam. If at $t = 0$ the shutter is opened, the transient particle current is observed at a distance x from the shutter. This problem was first solved by Moshinsky [17] and then received renewed attention from Elberfeld and Kleber [19], who introduced a δ-function barrier with a finite height at $x = 0$ and considered the tunneling through it.

The shutter acts as a perfect absorber. Then, the wave function that represents a particle of the beam is initially given by

$$\psi^{(0)}(x,t=0) = \Theta(-x)e^{ikx} \tag{4.1}$$

with the step function $\Theta(x) = 0$ and 1 for $x < 0$ and for $x > 0$, respectively. The time-dependent Schrödinger equation with a δ-function barrier is the same as in Eq. (3.2). By applying the same method as in Eq. (3.3), the solution satisfying the initial

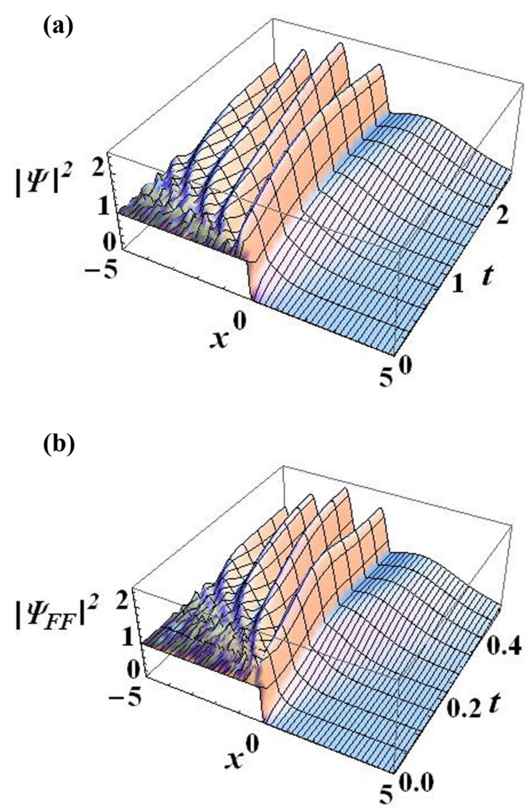

FIG. 4. 3D plot of the density distribution (vertical direction) as a function of x and t. The density distribution illustrates the tunneling dynamics of the semi-infinite wave train ($k = 2$) penetrating through the δ barrier with the height $V_0 = 1$ located at $x = +0$: (a) standard tunneling dynamics until the final time $T = 2.5$ and (b) fast-forward tunneling dynamics until $T_{\rm FF} = 0.5$ with the average time scaling factor $\bar{\alpha} = 5$.

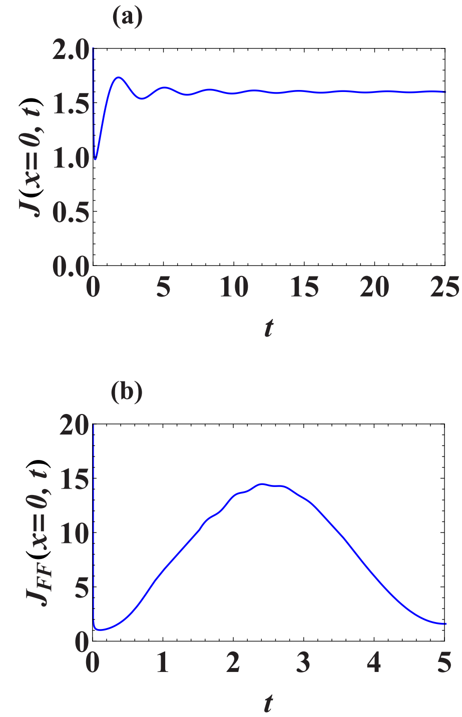

FIG. 5. Tunneling current densities for an initial semi-infinite wave train in the case of $k = 2$, $V_0 = 1$, and $\bar{\alpha} = 5$: (a) $j(+0,t)$ with $T = 25$ and (b) $j_{\rm FF}(+0,t)$ with $T_{\rm FF} = \frac{T}{\alpha} = 5$. Note: Scales of the horizontal and vertical axes differ between upper and lower panels.

condition in Eq. (4.1) was obtained [19] as

$$\psi_0(x,t) = M(x;k;t) + [V_0/(V_0 - ik)] \times [M(|x|;-iV_0;t) - M(|x|;k;t)]. \tag{4.2}$$

The tunneling current just behind the barrier is evaluated using Eq. (3.8).

A. Fast forward of Moshinsky shutter in the presence of δ-function barrier

We now analyze the fast-forwarded tunneling for the Moshinsky shutter. One can evaluate the phase η of the wave-function solution in Eq. (4.2), which is used to determine both driving vector and scalar potentials in Eqs. (2.8) and (2.10). By applying these driving potentials, we obtain the exact fast-forwarded state, which is given by replacing t by $\Lambda(t)$ in Eq. (1.3) as

$$\begin{aligned}\psi_{\rm FF}(x,t) &\equiv \psi_0(x,\Lambda(t)) \\ &= M(x;k;\Lambda(t)) + [V_0/(V_0 - ik)] \\ &\quad\times[M(|x|;-iV_0;\Lambda(t)) - M(|x|;k;\Lambda(t))].\end{aligned} \tag{4.3}$$

Concerning the relation between $\psi_{\rm FF}$ and η, one should recall the notion just before and after Eq. (3.9). The fast-forwarded tunneling current is given by Eq. (3.10).

Using the same units as described below Eq. (3.10), we choose $k = 2$, $V_0 = 1$, $\bar{\alpha} = 5$, $T = 2.5$, and $T_{\rm FF} = 0.5$ and show in Fig. 4 the density profiles of wave functions in cases of both the standard and the fast-forward (or accelerated) dynamics. We find that the time evolution of $|\psi_0(x,t)|^2$ is

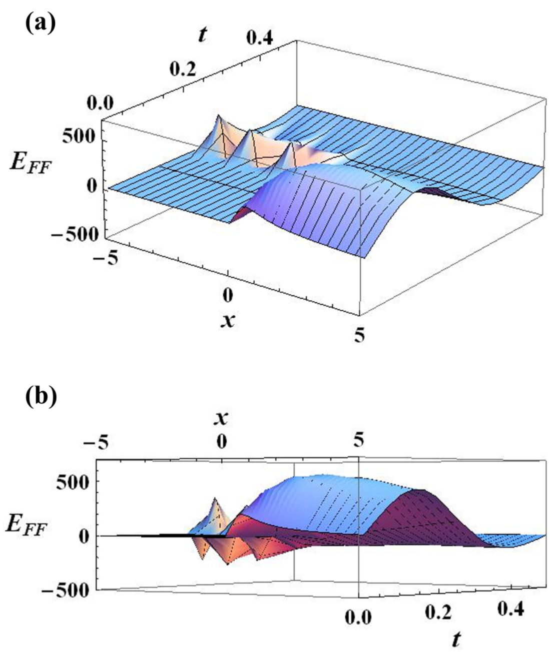

FIG. 6. 3D plot of electric field $E_{\rm FF}$ as a function of x and t: (a) top view and (b) side view.

squeezed in $|\psi_{\rm FF}(x,t)|^2$ by the factor $\frac{1}{\bar{\alpha}}$. Interference between the incoming and reflected waves leads to an oscillatory profile for $x < 0$. For $x > 0$ the outgoing wave shows a rather smoothly varying density profile.

In the case of $V_0 = 1, \bar{\alpha} = 5$, we show in Fig. 5 the tunneling current just behind the barrier in a wider time range with use of $T = 25$ and $T_{\rm FF} = 5$. The temporal behavior of the current $j(+0,t)$ in the standard dynamics, which shows a very slow decrease with respect to t, is squeezed and enhanced in that of $j_{\rm FF}(+0,t)$ in the fast-forward dynamics. As noted in the previous section, Eq. (3.10) leads to the equality

$$\int_0^{T_{\rm FF}} j_{\rm FF}(x=+0,t)dt = \int_0^T j(x=+0,t)dt. \tag{4.4}$$

In the case of Fig. 5, a rough estimate for the right-hand side of Eq. (4.4) is 25×1.6, which agrees with $5 \times \bar{J}_{\rm FF}$ (average fast-forward current) evaluated for the left-hand side. Therefore $\bar{J}_{\rm FF} \sim 8$, as seen in Fig. 5. In general, the tunneling current in the standard dynamics greatly decreases when the barrier height V_0 becomes much larger than unity. But, a suitable fast-forward mechanism recovers the current for the case of $V_0 = 1$, which is described by asymptotic argument in the next section.

The fast-forward state in Eq. (4.3) can be generated as a solution of the time-dependent Schrödinger equation in Eq. (2.2) for the charged particle in the presence of vector $A_{\rm FF}(x,t)$ and scalar $V_{\rm FF}(x,t)$ potentials. Figure 6 shows the corresponding electric field $E_{\rm FF}$ evaluated by Eq. (3.14) as a function of x and t.

B. Asymptotic approach in the case of a very high barrier

In the limit of a very high barrier, the tunneling current becomes negligibly small. But by a suitable choice of the time scaling $\alpha(t)$, one can recover the standard magnitude of the tunneling current, which we show below.

We first rewrite the wave-function solution on the right-hand side of the barrier (i.e., for $x > 0$) in Eq. (4.2) as

$$\psi_0(x,t) = -\frac{ik}{V_0 - ik}M_1 + \frac{V_0}{V_0 - ik}M_2. \tag{4.5}$$

Here

$$M_1 \equiv M(x;k;t), \quad M_2 \equiv M(x;-iV_0;t), \tag{4.6}$$

where $M(x;k;t)$ is the Moshinsky function defined in Eq. (3.5). With use of a new variable

$$z = \frac{x + iV_0 t}{\sqrt{2t}}e^{-i\frac{\pi}{4}}, \tag{4.7}$$

we see $(-iV_0)x - \frac{(-iV_0)^2 t}{2} = -iz^2 + \frac{x^2}{2t}$ and

$$M_2 = \tfrac{1}{2}e^{z^2 + \frac{x^2}{2t}i}{\rm erfc}(z). \tag{4.8}$$

We now concentrate on the asymptotic region given by $V_0 \gg 1$ with $k = O(1)$, $t = O(1)$, and $x \ll 1$, which leads to $|z| \gg 1$. Then we find ${\rm erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}z}$ and

$$M_2 \sim \frac{1}{2\sqrt{\pi}z}e^{\frac{x^2}{2t}i}. \tag{4.9}$$

Noting $z|_{x=0} = V_0\sqrt{\frac{t}{2}}e^{i\frac{\pi}{4}}$ and $\frac{\partial z}{\partial x}|_{x=0} = \frac{1}{\sqrt{2t}}e^{-i\frac{\pi}{4}}$, we have $M_2|_{x=0} = \frac{1}{V_0\sqrt{2\pi t}}e^{-i\frac{\pi}{4}}$ and $\frac{\partial M_2}{\partial x}|_{x=0} = \frac{1}{V_0^2\sqrt{2\pi t^3}}e^{i\frac{\pi}{4}}$, which results in

$$M_2^*\frac{\partial M_2}{\partial x}\bigg|_{x=0} = \frac{i}{2\pi t^2 V_0^3}\left(1 + O\left(\frac{1}{V_0^2}\right)\right). \tag{4.10}$$

In a similar way, we find

$$M_1^*\frac{\partial M_2}{\partial x}\bigg|_{x=0} = M^*(0;k;t)\frac{1}{\sqrt{2\pi t^3}V_0^2}e^{i\frac{\pi}{4}}\left(1 + O\left(\frac{1}{V_0^2}\right)\right),$$
$$M_2^*\frac{\partial M_1}{\partial x}\bigg|_{x=0} = \frac{1}{\sqrt{2\pi t}V_0}e^{i\frac{\pi}{4}}\frac{\partial M(x;k;t)}{\partial x}\bigg|_{x=0}\left(1 + O\left(\frac{1}{V_0^2}\right)\right). \tag{4.11}$$

Using the decomposition in Eq. (4.5), the standard current just behind the barrier can be expressed as

$$\begin{aligned} j(x=+0,t) &= {\rm Im}\left[\psi^*(x=+0,t)\frac{\partial\psi}{\partial x}\bigg|_{x=+0}\right] \\ &= \frac{k^2}{V_0^2 + k^2}{\rm Im}(M_1^*\partial_x M_1)\bigg|_{x=0} \\ &\quad + \frac{kV_0}{V_0^2 + k^2}{\rm Re}(M_1^*\partial_x M_2)\bigg|_{x=0} \\ &\quad + \frac{kV_0}{V_0^2 + k^2}{\rm Re}(M_2^*\partial_x M_1)\bigg|_{x=0} \\ &\quad + \frac{V_0^2}{V_0^2 + k^2}{\rm Im}(M_2^*\partial_x M_2)\bigg|_{x=0}. \end{aligned} \tag{4.12}$$

Noting the asymptotics in Eqs. (4.10) and (4.11), one sees that the first and third terms give dominant contributions of $O(\frac{1}{V_0^2})$ and other terms give minor contributions of $O(\frac{1}{V_0^3})$.

Then $j(x=+0,t)$ becomes asymptotically

$$j(x=+0,t) = \frac{k^2}{V_0^2}\text{Im}(M^*(0;k;t)\partial_x M(x;k;t)\Big|_{x=0} + \frac{k}{V_0^2}\text{Re}\left(\frac{1}{\sqrt{2\pi t}}e^{i\frac{\pi}{4}}\partial_x M(x;k;t)\Big|_{x=0}\right) + O\left(\frac{1}{V_0^3}\right). \tag{4.13}$$

The tunneling current in the standard dynamics has proved to be of $O(\frac{1}{V_0^2})$, which is very small. However, the idea of fast forward can recover the current in the case of $V_0 = O(1)$. In fact, by applying the driving vector and scalar potentials in Eqs. (2.8) and (2.10), we can realize the exact fast-forward state in the time domain $0 < t < T_{FF}$ with T_{FF} in Eq. (1.5), and its corresponding fast-forward current is given by Eq. (3.10). Therefore, if we use a large enough magnification time-scaling factor $\alpha(t)$ with its mean value $\bar{\alpha} = O(V_0^2)$, the current in the case of the small barrier ($V_0 = O(1)$) will be recovered.

To make the quantitative argument, let us employ the uniform scaling factor in Eq. (1.10). Then the fast-forward current is given by

$$\begin{aligned} j_{FF}(x=+0,t) &= \bar{\alpha}\, j(x=+0,\Lambda(t)) \\ &= \frac{k^2\bar{\alpha}}{V_0^2}\text{Im}(M^*(0;k;\Lambda(t))\partial_x M(x;k;\Lambda(t))\Big|_{x=0} \\ &\quad + \frac{k\bar{\alpha}}{V_0^2}\text{Re}\left(\frac{1}{\sqrt{2\pi\Lambda(t)}}e^{i\frac{\pi}{4}}\partial_x M(x;k;\Lambda(t))\Big|_{x=0}\right). \end{aligned} \tag{4.14}$$

Noting $\Lambda(t) = O(1)$ in the fast-forward time domain, Eq. (4.14) shows that if we choose

$$\bar{\alpha} = V_0^2, \tag{4.15}$$

the tunneling current in the high-barrier case ($V_0 \gg 1$) will recover the value in the low-barrier case [$V_0 = O(1)$].

Figure 7 shows that the fast forwarding with use of the driving electric field makes a negligible tunneling current for the case of a very high barrier with $V_0 \gg 1$ increased to the value for the case of a standard barrier with $V_0 = O(1)$. In fact, in the case of $V_0 = 50$, $j = O(10^{-3})$ in Fig. 7(a), but $j_{FF} = O(10)$ in Figs. 7(b) and 7(c) in the fast-forward time region $0 \leqslant t \leqslant T_{FF}(=5)$.

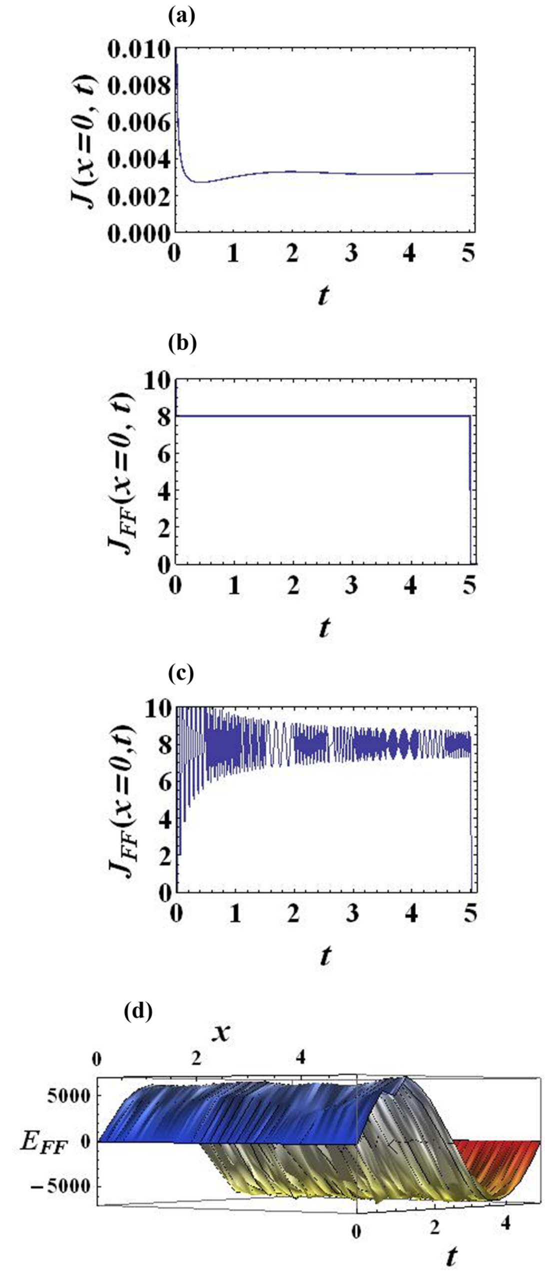

FIG. 7. Standard and fast-forwarded tunneling current of a semi-infinite wave train in the case of $V_0 = 50$ and $k = 2$. In the fast-forwarded case, the uniform time scaling with $\bar{\alpha} = V_0^2 = 2500$ is employed during the period between zero and T_{FF} with $T_{FF} = \frac{T}{\bar{\alpha}} = \frac{12500}{2500} = 5$. The time range depicted is $0 \leqslant t \leqslant 5$. (a) Standard current available from Eqs. (3.8) and (4.2), (b) fast-forwarded exact current in Eqs. (3.10) and (4.3), and (c) fast-forwarded asymptotic current in Eq. (4.14). The rapid oscillation in the fast-forward time region comes from the second term in the last expression of Eq. (4.14) which includes an uncanceled factor $e^{-ik^2\Lambda(t)/2}$ of the Moshinsky function in Eq. (3.5). (d) 3D plot of the driving electric field to realize the fast-forwarded exact current in case (b).

V. FAST FORWARD OF MACROSCOPIC TUNNELING

The theory of fast forward can also be applied to macroscopic quantum mechanics. We consider the fast-forwarded tunneling of a solitonic wave packet in 1D Bose-Einstein condensates (BECs) governed by the nonlinear Schrödinger equation in Eq. (1.1) with the barrier at origin, $V(x) = V_0\delta(x)$. The standard dynamics for ψ_0 is described by

$$i\hbar\partial_t\psi_0 = -\frac{\hbar^2}{2m}\partial_x^2\psi_0 + V_0\delta(x)\psi_0 - c_0|\psi_0|^2\psi_0. \tag{5.1}$$

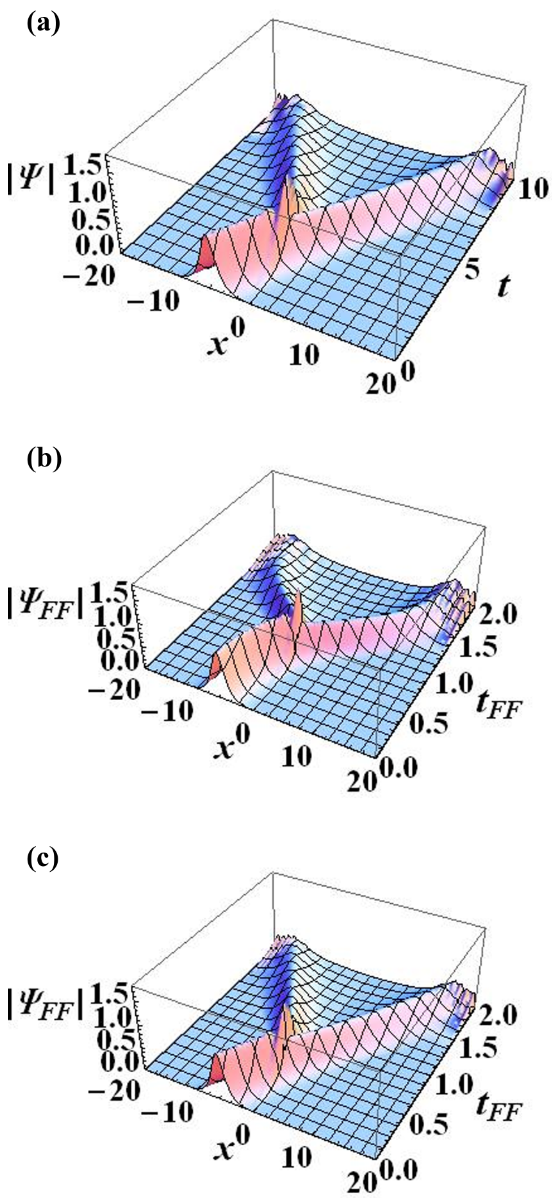

FIG. 8. 3D plot of $|\psi(x,t)|$ (vertical direction) as a function of x and t: (a) standard tunneling dynamics of the soliton where $\psi_0(x,t)$ satisfies Eq. (5.1) with $V_0 = 30$, $v = 2.25$, and $x_0 = 6$; (b) fast-forward tunneling dynamics of the soliton where $\psi_{\rm FF}(x,t)$ satisfies Eq. (5.2) under the nonuniform time-scaling factor $\alpha(t)$ with its mean $\bar{\alpha} = 5$ in Eq. (1.9) [V_0, v, x_0 are the same as in (a)]; and (c) the same as in (b) except for the uniform time scaling $\alpha(t) = \bar{\alpha} = 5$. In Figs. 9 and 10, the same space and time units as in this figure are used.

On the other hand, the governing equation for the fast-forward function $\psi_{\rm FF}$ is given by [see the notice just below Eq. (2.2)]

$$\imath\hbar\partial_t\psi_{\rm FF} = \left(\frac{1}{2m}\left(\frac{\hbar}{i}\partial_x - A_{\rm FF}\right)^2 + V_{\rm FF} + V_0\delta(x)\right)\psi_{\rm FF} - c_0|\psi_{\rm FF}|^2\psi_{\rm FF}. \quad (5.2)$$

Below, in addition to the natural units ($\hbar = m = 1$) we employ the same units as used in the previous sections on microscopic quantum dynamics. Namely, space and time are scaled by $L = 10^{-2}$ *times the system size* and $\tau = 10^{-2}$ *times the dissipation time*, respectively, and we put the nonlinearity constant c_0 (scaled by $L\tau^{-1}$) = 1. Then Eqs. (5.1) and (5.2) become dimensionless, which we analyze. If there is no barrier, the solution of Eqs. (5.1) is a traveling Zakharov-Shabat soliton [21] given by

$$\psi^{(0)}(x,t) = A\,{\rm sech}[(A(x - vt)]e^{i\phi_0 + ivx + i(A^2 - v^2)t/2} \quad (5.3)$$

where A and v are the amplitude and propagation velocity, respectively.

In the presence of the barrier, we numerically solve Eq. (5.1) with the use of an initial profile:

$$\psi^{(0)}(x,t) = {\rm sech}(x + x_0)e^{ivx}, \quad (5.4)$$

which stands for the soliton with $A = 1$ and initial position $x = -x_0$ ($x_0 \gg 1$) for center of mass.

In Fig. 8(a), we show the amplitude $|\psi(x,t)|$ of the soliton as a function of x and t in standard time. The soliton located at $x = -x_0$ moves to the right and after the time $t = \frac{x_0}{v}$ it collides with the barrier at $x = 0$. Then it splits into two parts: reflected and transmitted ones which are moving to left and right, respectively. The result accords with the one by Holms *et al.* [22]. Increase of the barrier height (V_0) diminishes the transmitted part; namely, it decreases the tunneling rate.

According to the idea of the fast forward, the same wave-function patterns as seen in the time domain $0 < t < T$ can be realized in the shortened time domain $0 < t < T_{\rm FF}$ [see Eq. (1.5)] with the use of $A_{\rm FF}$ and $V_{\rm FF}$ in Eq. (2.8) and Eq. (2.10), respectively.

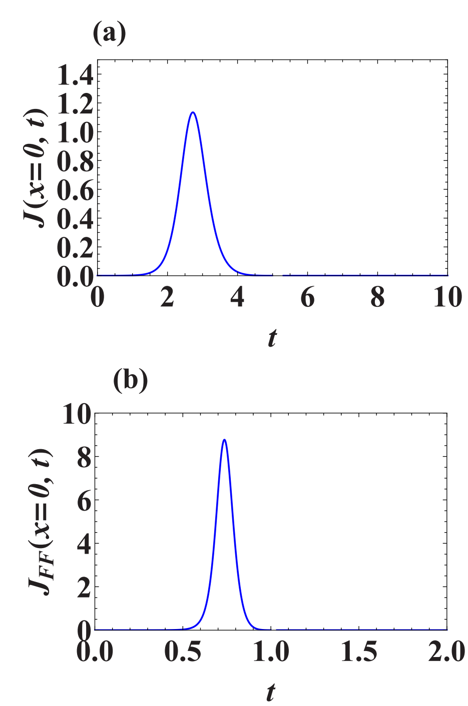

FIG. 9. Tunneling current densities at $x = +0$: (a) standard tunneling current and (b) fast-forward tunneling current. Note: Scales of the horizontal and vertical axes differ between upper and lower panels.

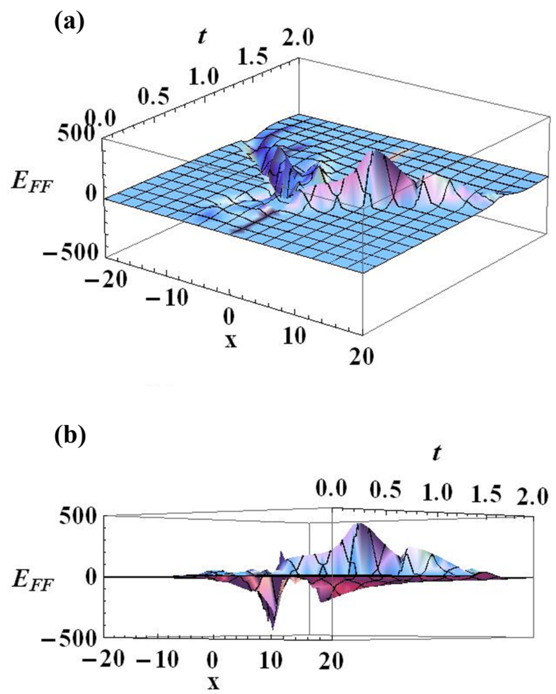

FIG. 10. 3D plot of the electric field $E_{\rm FF}$ as a function of x and t: (a) top view and (b) side view.

Using the nonuniform time-scaling factor $\alpha(t)$ with its mean $\bar{\alpha} = 5$ in Eq. (1.9), we have solved Eq. (5.2). In Fig. 8(b), the amplitude $|\psi_{\rm FF}(x,t)|$ of the soliton is shown. At the shortened time $t \sim \frac{x_0}{\bar{\alpha} v}$ the soliton collides with the δ barrier at $x = 0$ and the splitting process is also shortened. In Fig. 8(c), we show $|\psi_{\rm FF}(x,t)|$ by solving Eq. (5.2) with the use of the uniform scaling factor $\bar{\alpha} = 5$ in Eq. (1.10). In this case we can see the exact time-squeezed version of the soliton dynamics in Fig. 8(a). The soliton reaches the barrier at $t = \frac{x_0}{\bar{\alpha} v}$ and transmitted and reflected patterns are shortened by the constant time scaling $\bar{\alpha}$.

Now we compute the tunneling current $j(+0,t)$ and $j_{\rm FF}(+0,t)$ at $x = +0$ in Eqs. (3.8) and (3.10), respectively. Figure 9 shows standard and fast-forward (with a nonuniform time-scaling factor with its mean $\bar{\alpha} = 5$) cases, respectively. The standard tunneling current has a peak at $t = t_0 \sim \frac{x_0}{v}$, when the soliton almost reaches the barrier. The fast-forward tunneling current is a squeezed and enhanced version of the standard one. Since the soliton arrives at the barrier earlier than the standard arrival time, the peak of the current is realized at time $t_{0\rm FF} = \frac{x_0}{\bar{\alpha} v}$. Figure 9 also shows the enhancement of the tunneling rate by $\bar{\alpha} = 5$ as indicated by Eq. (3.13). In Fig. 10 the driving electric field $E_{\rm FF}$ necessary for the fast forwarding of the soliton is evaluated by Eq. (3.14) and is depicted as a function of x and t.

VI. CONCLUSION

We developed a theory of fast forwarding of quantum dynamics for charged particles, which exactly accelerates both amplitude and phase of the wave function throughout the fast-forward time range. We elucidated the nature of the driving electromagnetic field together with vector and scalar potentials to guarantee the exact fast forwarding. The theory is applied to the tunneling phenomena through a tunneling barrier. Typical examples described here are (1) the initially exponential wave packet moving through the δ-function barrier and (2) the opened Moshinsky shutter with a δ-function barrier just behind the shutter. Standard (nonaccelerated) dynamics in these examples is known to be exactly solvable. We see the remarkable squeezing and enhancement of the tunneling current density, caused by the fast forwarding of quantum tunneling. We find that, even if the barrier height is increased, one can generate a recognizable tunneling current by using a large enough time-scaling factor $\alpha(t)$. At the same time, we have shown that, so long as $\alpha(t)$ is less than $\alpha_{\rm max} \equiv \frac{V_0}{\mathcal{E}_0}(>1)$ with the barrier height V_0 and incident energy $\mathcal{E}_0$ in the standard tunneling, the corresponding fast-forwarded dynamics is also the tunneling phenomenon keeping the particle's energy $\mathcal{E}_{\rm FF}(t)$ below the barrier height V_0 throughout the time evolution, and the time scaling works more effectively for the particle with lower incident energy. The analysis is also carried out on the acceleration of macroscopic quantum tunneling with the use of the nonlinear Schrödinger equation which accommodates a δ-function barrier.

Finally we should note that this work is inside a broader concept to enhance the visibility of quantum transient phenomena (postexponential decay [23,24], quantum backflow [25], diffraction in time [26,27], as well as quantum tunneling), which are predictable by quantum mechanics but hardly detectable because the detection number of particles is very small. The general theory in Sec. II will be an alternative vehicle to optimize the visibility parameters to improve those feeble observations.

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[6] This terminology means to reproduce a series of events or a history of matters in a shortened time scale, like a rapid projection of movie films on the screen. The fast forward contains as special limits the shortcut to adiabaticity and acceleration of particle motions.

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