Kinetic-energy-momentum tensor in electrodynamics

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We show that the Einstein-Laub formulation of electrodynamics is invalid since it yields a stress-energy-momentum (SEM) tensor that is not frame invariant. Two leading hypotheses for the kinetic formulation of electrodynamics (Chu and Einstein-Laub) are studied by use of the relativistic principle of virtual power, mathematical modeling, Lagrangian methods, and SEM transformations. The relativistic principle of virtual power is used to demonstrate the field dynamics associated with energy relations within a relativistic framework. Lorentz transformations of the respective SEM tensors demonstrate the relativistic frameworks for each studied formulation. Mathematical modeling of stationary and moving media is used to illustrate the differences and discrepancies of specific proposed kinetic formulations, where energy relations and conservation theorems are employed. Lagrangian methods are utilized to derive the field kinetic Maxwell's equations, which are studied with respect to SEM tensor transforms. Within each analysis, the Einstein-Laub formulation violates special relativity, which invalidates the Einstein-Laub SEM tensor.

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I. INTRODUCTION

The momentum of light in media remains one of the most controversial topics in physics [1–6]. The debate has continued for more than a century since Minkowski and Abraham formulated 4×4 energy-momentum tensors in the early 1900s [7–9]. Attention has focused on differing forms of the electromagnetic momentum density in media. Minkowski proposed $\bar{G}_M = \bar{D} \times \bar{B}$ to be the momentum density, where $\bar{D}$ is the electric displacement and $\bar{B}$ is the magnetic induction [7]. Soon after, Abraham suggested a more symmetric approach, yielding $\bar{G}_A = \bar{E} \times \bar{H}/c^2$ as the momentum density, where $\bar{E}$ and $\bar{H}$ are the electric and magnetic fields, respectively, and c is the speed of light in vacuum [8]. A number of experiments have been reported seemingly in favor of one form or the other [10–18].

In 2010, Barnett's resolution to the photon momentum controversy identified Abraham's momentum as the kinetic momentum and Minkowski's momentum as the canonical momentum [19]. Consequently, the Abraham momentum is responsible for the overall center-of-mass translation of a material, while the Minkowski momentum predicts translations within or with respect to the medium [4–6]. Equivalence between the two is shown by determining the total energy-momentum tensor since division into material and electromagnetic components is believed arbitrary [2]. It is the misinterpretation of such arbitrary divisions that can lead to erroneous predictions. Therefore, a complete resolution of the Abraham-Minkowski debate must include a complete description of the kinetic subsystem of electromagnetics which predicts center-of-mass translations due to electromagnetic fields [6]. However, the Barnett resolution only identified the kinetic- and canonical-momentum densities, whereas the original debate is in regard to the 4×4 energy-momentum tensor.

In light of this, another well-known stress-energy-momentum (SEM) tensor was also proposed in the first decade of the last century by Einstein and Laub [20]. The Einstein-Laub tensor, like the Abraham and Minkowski tensors, utilizes the Minkowski fields, and it shares the Abraham momentum density. Significant theoretical work has been presented over the last century pertaining to the electrodynamics of moving media, and the focus of the analyses has clearly been biased toward the relativistically invariant Minkowski, Chu (i.e., EH representation), and Amperian (i.e., EB representation) formulations [21–24]. The question of relativistic invariance of the Abraham formulation has only recently been answered [25–27], while the same question regarding the Einstein-Laub formulation has yet to be addressed. A cursory review of recent literature reveals that the Einstein-Laub formulation remains popular in scientific application [28–32]. This is, in part, due to the indistinguishability of the various formulations for computing total force and stress (i.e., material plus electromagnetic) in the quasistationary limit [24].

In this paper, we employ the fundamental tenets of special relativity to study the kinetic subsystem of macroscopic electromagnetics. Using the mathematical framework of the relativistic principle of virtual power (RPVP), we uniquely derive and review the associated stress tensor and momentum density from the shared-energy relations between the Einstein-Laub and Chu formulations. It is shown that the kinetic-momentum density and Chu stress tensor naturally derive from the energy relations, where the Einstein-Laub does not. The SEM tensors representing both formulations are analyzed using Lorentz transformation laws while further investigating the invariance properties with respect to field transformations. Mathematical models for stationary and moving media are derived, demonstrating the kinetic properties of the Chu, Einstein-Laub, and Abraham formulations with respect to energy relations and conservation theorems. Last, Lagrangian methods in conjunction with scalar and vector potentials allow for the derivation of the field kinetic subsystem, which is recast into matrix form and studied relativistically. In each of these demonstrations, it is shown that the Chu formulations transform relativistically between inertial reference frames.

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TABLE I. Leading formulations of electrodynamics.

	$\bar{G}(\bar{r},t)$	$\bar{\bar{T}}(\bar{r},t)$	$W(\bar{r},t)$	$\bar{S}(\bar{r},t)$
Chu[a]	$\epsilon_0\mu_0\bar{E}\times\bar{H}$	$\frac{1}{2}(\epsilon_0\bar{E}^2+\mu_0\bar{H}^2)\bar{\bar{I}}-\epsilon_0\bar{E}\bar{E}-\mu_0\bar{H}\bar{H}$	$\frac{1}{2}(\epsilon_0\bar{E}^2+\mu_0\bar{H}^2)$	$\bar{E}\times\bar{H}$
Einstein-Laub[b]	$\epsilon_0\mu_0\bar{E}\times\bar{H}$	$\frac{1}{2}(\epsilon_0\bar{E}^2+\mu_0\bar{H}^2)\bar{\bar{I}}-\bar{D}\bar{E}-\bar{B}\bar{H}$	$\frac{1}{2}(\epsilon_0\bar{E}^2+\mu_0\bar{H}^2)$	$\bar{E}\times\bar{H}$
Abraham[b]	$\epsilon_0\mu_0\bar{E}\times\bar{H}$	$\frac{1}{2}(\bar{D}\cdot\bar{E}+\bar{B}\cdot\bar{H})\bar{\bar{I}}-\bar{D}\bar{E}-\bar{B}\bar{H}$	$\frac{1}{2}(\bar{D}\cdot\bar{E}+\bar{B}\cdot\bar{H})$	$\bar{E}\times\bar{H}$

[a]Chu field transformations are used to describe moving contributions [21,22,36].
[b]Minkowski field transformations are used to describe moving contributions [21–23].

II. FRAMEWORK

In 1953, Balazs presented a simple gedanken experiment to determine the kinetic momentum of a pulse of light by considering center-of-mass-energy conservation [33]. This was accomplished by considering an optical pulse with initial free-space momentum $p_i = \mathfrak{E}/c$ incident upon an impedance-matched slab ($\sqrt{\mu/\epsilon} = \sqrt{\mu_0/\epsilon_0}$) of thickness d initially at rest. The optical pulse is slowed with respect to an alternate vacuum propagation path by the length $L = (n-1)d$ due to the reduced velocity within the slab, having refractive index $n = c\sqrt{\epsilon\mu}$. Conservation principles require that the slab acquire some linear momentum, giving rise to a material momentum $p_m = \frac{\mathfrak{E}}{c}(1-\frac{1}{n})$. Here, momentum conservation requires the momentum of the pulse of the light be the Abraham momentum $p = \frac{1}{n}\frac{\mathfrak{E}}{c}$, corresponding to the kinetic-momentum density $\bar{G}_{F_k} = \epsilon_0\mu_0\bar{E}\times\bar{H}$. Consequently, this result mathematically excludes other forms, such as Liven's momentum, $\bar{G}_L = \epsilon_0\bar{E}\times\bar{B}$, which is commonly tied to the Amperian formulation [6,23,34], and the Minkowski momentum, $\bar{G}_M = \bar{D}\times\bar{B}$, from being the kinetic momentum of light [4,6,19]. We emphasize that this assertion is only in regard to the interpretation of the Amperian and Minkowski SEM tensors and does not imply a lack of translational invariance.

Of the leading formulations [6], three formulations utilize the prescribed kinetic-momentum-density model: the Abraham formulation, the Einstein-Laub formulation, and the Chu formulation. Table I lists the leading field kinetic formulation candidates. However, when modeling the kinetic subsystem, it is unknown which formulation generates the true physics of the electromagnetic subsystem, thereby satisfying conservation theorems,

$$\bar{f}_{F_k} = \Box\cdot[\bar{\bar{\mathcal{T}}}_{F_k}, -ic\bar{G}_{F_k}], \tag{1a}$$

$$\varphi_{F_k} = \Box\cdot\left[-\frac{i}{c}\bar{S}_{F_k}, W_{F_k}\right], \tag{1b}$$

where $\bar{\bar{\mathcal{T}}} = -\bar{\bar{T}}$ as used in Refs. [6,23], $\Box = [\nabla, \frac{\partial}{\partial ict}]$, and subscript F_k denotes the field kinetic subsystem, rendering the field kinetic SEM tensor as

$$\mathfrak{T}_{F_k} = \begin{bmatrix} \bar{\bar{\mathcal{T}}}_{F_k} & -ic\bar{G}_{F_k} \\ -i\frac{\bar{S}_{F_k}}{c} & W_{F_k} \end{bmatrix}. \tag{2}$$

Tensors lacking relativistic invariance cannot be energy-momentum tensors. This fact is a fundamental tenet of modern physics. For example, consider a region of space occupied by ponderable media, which may be described locally by a mass density and a velocity field. Regardless of how a system of coordinates is assigned, the local momentum vector may vary with position and time. In typical optical manipulation experiments, this may be due to motion of a submerging fluid and/or the presence of multiple particles or cells. The inability to measure relativistic effects in any particular experiment by our present instrument capabilities, however, does not preclude the fundamental laws of physics from holding true. Relative motion within media and the laboratory frame will generally exist. We demonstrate such relative motion via mathematical modeling in Sec. IV, and we maintain that it is essential that the laws governing the physics of the system remain invariant.

Recent reports have addressed the lack on relativistic invariance while employing the Abraham energy-momentum tensor [25–27]. This, consequently, is due to both the Abraham and Minkowski energy-momentum tensors sharing field expressions for power flux, energy density, and stress tensor, while the momentum-density definitions differ. Simply stated, this imposes that both formulations utilize identical electromagnetic energy relations; however, each predicts independent force expressions within the subsystem. At least one of the two formulations cannot be a valid energy-momentum tensor, and due to previous research, the Abraham formulation demonstrates inconsistencies within relativistic transformations [25–27]. Thus, we can dismiss the Abraham formulation as a candidate for the kinetic SEM tensor.

Using the above rationale, we present a similar argument for the Einstein-Laub and Chu formulations, where both share definitions for the energy density W, power flux $\bar{S}$, and momentum density $\bar{G}_{F_k}$ but differ in the definition of both the stress tensor and electromagnetic field definitions. This stems from the interpretation of Maxwell's equations where both formulations possess the following relations:

$$\nabla\times\bar{H} - \epsilon_0\frac{\partial\bar{E}}{\partial t} = \bar{J}_e, \tag{3a}$$

$$\nabla\times\bar{E} + \mu_0\frac{\partial\bar{H}}{\partial t} = -\bar{J}_m, \tag{3b}$$

$$\epsilon_0\nabla\cdot\bar{E} = \rho_e, \tag{3c}$$

$$\mu_0\nabla\cdot\bar{H} = \rho_m. \tag{3d}$$

The Einstein-Laub interpretation directly divides the Minkowski formulation, composed of field values $\bar{D}_M$, $\bar{B}_M$, $\bar{E}_M$, and $\bar{H}_M$, into field and material components by field definitions $\bar{D}_M = \epsilon_0\bar{E}_M + \bar{P}_M$ and $\bar{B}_M = \mu_0(\bar{H}_M + \bar{M}_M)$, where subscript M denotes the Minkowski electromagnetic field values. As a result, the Einstein-Laub effective electric and magnetic current and charge densities are given as [32,35]

$$\bar{J}_e = \frac{\partial \bar{P}_M}{\partial t} + \bar{J}_f, \tag{4a}$$

$$\bar{J}_m = \mu_0 \frac{\partial \bar{M}_M}{\partial t}, \tag{4b}$$

$$\rho_e = -\nabla \cdot \bar{P}_M + \rho_f, \tag{4c}$$

$$\rho_m = -\mu_0 \nabla \cdot \bar{M}_M, \tag{4d}$$

where $\bar{J}_f$ and ρ_f represent the free current and charge density of the system. Thus, the electric and magnetic field components are represented by the Minkowski $\bar{E}_M$ and $\bar{H}_M$ fields, which comprise the Einstein-Laub electromagnetic formulation.

Alternatively, the Chu formulation, composed of field values $\bar{E}_C$, $\bar{H}_C$, $\bar{P}_C$, and $\bar{M}_C$, idealizes interrelated electric (E) and magnetic (H) fields, where material bodies contribute towards the prescribed electromagnetic fields by acting as source values [21,36]. Thus, the Chu effective electric and magnetic current and charge densities are given as

$$\bar{J}_e = \frac{\partial \bar{P}_C}{\partial t} + \nabla \times (\bar{P}_C \times \bar{v}) + \bar{J}_f, \tag{5a}$$

$$\bar{J}_m = \mu_0 \frac{\partial \bar{M}_C}{\partial t} + \mu_0 \nabla \times (\bar{M}_C \times \bar{v}), \tag{5b}$$

$$\rho_e = -\nabla \cdot \bar{P}_C + \rho_f, \tag{5c}$$

$$\rho_m = -\mu_0 \nabla \cdot \bar{M}_C, \tag{5d}$$

where subscript C denotes values from the Chu formulations. Comparing the two formulations, the following questions are raised:

(1) Which field and material interpretation is the true physical electromagnetic interpretation?

(2) Which stress tensor and momentum density is tied to the shared-energy relations prescribed by both formulations?

Consequently, this discrepancy clouds the idea of the true kinetic representation of light within media. Thus, we present a number of arguments to distinguish which of the two leading kinetic formulations, Chu or Einstein-Laub, is the valid interpretation of the electromagnetic system.

From here forward, we omit subscript notation for denoting the Chu and Minkowski field definitions and instead state when each field representation is used.

III. RELATIVISTIC ANALYSIS

A. Relativistic principle of virtual power

The relativistic principle of virtual power is derived from the fundamental tenets of the principle of virtual work, where the force of a system is derived via the amount of work put forth along the path of a particle [21]. Expanding these basic tenets, the relativistic principle of virtual power utilizes prior knowledge of a system's power flux, power density, and energy density to derive the associated system dynamics. The expressions for these relations must be valid even if the material is accelerating and/or deforming. By using valid transforms (i.e., Lorentz transformation) along with prescribed definition and manipulations of the power expressions, the force density, stress tensor, and momentum density are derived for the corresponding subsystem. The relativistic principle of virtual power is expressed mathematically as [21]

$$(\nabla \cdot \bar{S}^0)^0 + \frac{\bar{S}^0}{c} \cdot \left(\frac{\partial \bar{v}}{\partial t}\right)^0 + \left(\frac{\partial W^0}{\partial t}\right)^0 + W^0(\nabla \cdot \bar{v})^0 - \varphi^0$$
$$= -T^0 : (\nabla \bar{v})^0 - \bar{G}^0 \cdot \left(\frac{\partial \bar{v}}{\partial t}\right)^0, \tag{6}$$

where the superscript 0 denotes values within the arbitrary reference frame to the first-order velocity value [21]. Additionally, the operator : signifies the dyadic dot product, also known as the double-dot product with respect to the dyadic matrix.

As one would expect, transformation from one inertial frame to another requires prescribed Lorentz transformation laws. Here, we employ both Chu and Minkowski field transformations (1) to not assume a specific electromagnetic formulation and (2) to derive the system dynamics tied to the shared-energy relations in the moving frame,

$$\bar{S}^0 = \bar{E}^0 \times \bar{H}^0, \tag{7a}$$

$$W^0 = \tfrac{1}{2}(\epsilon_0 \bar{E}^0 \cdot \bar{E}^0 + \mu_0 \bar{H}^0 \cdot \bar{H}^0). \tag{7b}$$

Here, the question is which interpretation of the electromagnetic fields correctly describes the electromagnetic subsystem involving the given energy relations. In the following sections, we apply each respective field transformation to uniquely derive the electromagnetic stress tensor and momentum density associated with each field representation.

1. Minkowski field analysis

By use of Eqs. (7) and (1b), the Einstein-Laub energy relations are employed to derive the electrodynamic forces via the method of RPVP. Applying the first-order velocity vector field transforms [21],

$$\bar{E}^0 = \bar{E} + \bar{v} \times \bar{B}, \tag{8a}$$

$$\bar{H}^0 = \bar{H} - \bar{v} \times \bar{D}, \tag{8b}$$

$$\bar{B}^0 = \bar{B} - \frac{\bar{v} \times \bar{E}}{c^2}, \tag{8c}$$

$$\bar{D}^0 = \bar{D} + \frac{\bar{v} \times \bar{H}}{c^2}, \tag{8d}$$

the energy relations are rendered such that

$$\bar{S}^0 = \bar{E} \times \bar{H} + [(\bar{v} \times \bar{D}) \times \bar{E}] + [(\bar{v} \times \bar{B}) \times \bar{H}], \tag{9a}$$

$$W^0 = \tfrac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2) - \bar{v} \cdot [\epsilon_0 \bar{E} \times \bar{B} + \mu_0 \bar{D} \times \bar{H}], \tag{9b}$$

where higher-order velocity terms have been omitted. Substituting Eqs. (9) into the left-hand side of Eq. (6) results in the following relation:

$$Q^0 = [\nabla \cdot \{\bar{E} \times \bar{H} + [(\bar{v} \times \bar{D}) \times \bar{E}] + [(\bar{v} \times \bar{B}) \times \bar{H}]\}]^0$$
$$+ \frac{\bar{E}^0 \times \bar{H}^0}{c^2} \cdot \left(\frac{\partial \bar{v}}{\partial t}\right)^0 + \bar{E}^0 \cdot \left(\frac{\partial \epsilon_0 \bar{E}}{\partial t}\right)^0$$

$$+\bar{H}^0\cdot\left(\frac{\partial\mu_0\bar{H}}{\partial t}\right)^0-\left(\frac{\partial\bar{v}}{\partial t}\right)^0\cdot(\epsilon_0\bar{E}^0\times\bar{B}^0$$
$$+\mu_0\bar{D}^0\times\bar{H}^0)+\left(\frac{\epsilon_0}{2}\bar{E}^0\cdot\bar{E}^0+\frac{\mu_0}{2}\bar{H}^0\cdot\bar{H}^0\right)(\nabla\cdot\bar{v})^0$$
$$+\bar{E}^0\cdot\bar{J}_e^0+\bar{H}^0\cdot\bar{J}_m^0. \tag{10}$$

By use of vector properties in conjunction with Poynting's theorem, one finds Eq. (10) to render

$$\bar{\bar{T}}^0=\left[\bar{D}^0\cdot\bar{E}^0-\frac{\epsilon_0}{2}\bar{E}^0\cdot\bar{E}^0+\bar{B}^0\cdot\bar{H}^0-\frac{\mu_0}{2}\bar{H}\cdot\bar{H}^0\right]\bar{\bar{I}}$$
$$-\bar{D}^0\bar{E}^0-\bar{B}^0\bar{H}^0, \tag{11a}$$
$$\bar{G}^0=\epsilon_0\bar{E}^0\times\bar{B}^0+\mu_0\bar{D}\times\bar{H}-\epsilon_0\mu_0\bar{E}^0\times\bar{H}^0. \tag{11b}$$

The resulting stress tensor and momentum density do not transform to the Einstein-Laub formulation. Consequently, we are unaware of any reported stress-energy-momentum tensor that includes the stress-tensor and momentum-density relations presented in Eq. (11). Furthermore, the transformed momentum density is not consistent with the kinetic form of the momentum density and cannot be considered a kinetic representation of electrodynamics.

2. *Chu field analysis*

Here, we present an abbreviated analysis of the Chu field while employing the RPVP methods. This allows for comparison between the Chu and Einstein-Laub interpretations. However, the full treatment of the Chu analysis is given in Ref. [21].

Using the Chu transformation laws to the first-order velocity term [21],

$$\bar{E}^0=\bar{E}+\bar{v}\times\mu_0\bar{H}, \tag{12a}$$
$$\bar{H}^0=\bar{H}-\bar{v}\times\epsilon_0\bar{E}, \tag{12b}$$

the energy relations are rendered such that [21]

$$\bar{S}^0=\bar{E}\times\bar{H}+[(\bar{v}\times\epsilon_0\bar{E})\times\bar{E}]+[(\bar{v}\times\mu_0\bar{H})\times\bar{H}], \tag{13a}$$

$$W^0=\frac{1}{2}(\epsilon_0E^2+\mu_0H^2)-\frac{2\bar{v}}{c^2}\cdot[\bar{E}\times\bar{H}], \tag{13b}$$

where higher-order velocity terms have been omitted. Substituting Eqs. (13) into the left-hand side of Eq. (6) results in the following relation [21]:

$$Q^0=[\nabla\cdot\{\bar{E}\times\bar{H}+[(\bar{v}\times\epsilon_0\bar{E})\times\bar{E}]+[(\bar{v}\times\mu_0\bar{H})\times\bar{H}]\}]^0$$
$$+\frac{\bar{E}^0\times\bar{H}^0}{c^2}\cdot\left(\frac{\partial\bar{v}}{\partial t}\right)^0+\bar{E}^0\cdot\left(\frac{\partial\epsilon_0\bar{E}}{\partial t}\right)^0$$
$$+\bar{H}^0\cdot\left(\frac{\partial\mu_0\bar{H}}{\partial t}\right)^0-2\left(\frac{\partial\bar{v}}{\partial t}\right)^0\cdot\left(\frac{\bar{E}\times\bar{H}}{c^2}\right)^0$$
$$+\left(\frac{\epsilon_0}{2}\bar{E}^0\cdot\bar{E}^0+\frac{\mu_0}{2}\bar{H}^0\cdot\bar{H}^0\right)(\nabla\cdot\bar{v})^0$$
$$+\bar{E}^0\cdot\bar{J}_e^0+\bar{H}^0\cdot\bar{J}_m^0. \tag{14}$$

By use of vector manipulations and Poynting's theorem, one finds Eq. (14) to render

$$\bar{\bar{T}}^0=\left[\frac{\epsilon_0}{2}\bar{E}^0\cdot\bar{E}^0+\frac{\mu_0}{2}\bar{H}\cdot\bar{H}^0\right]\bar{\bar{I}}-\epsilon_0\bar{E}^0\bar{E}^0-\mu_0\bar{H}^0\bar{H}^0, \tag{15a}$$

$$\bar{G}^0=\epsilon_0\mu_0\bar{E}^0\times\bar{H}^0, \tag{15b}$$

which is the Chu stress tensor and momentum density within the arbitrarily moving reference frame. This indicates that the Chu energy-momentum tensor transforms correctly between inertial reference frames. However, further justification will be presented in the next section, where we provide relativistic analysis of the SEM tensors, further demonstrating the mathematical differences between the Chu and Einstein-Laub formulations.

B. Invariance of SEM components

It is well known that physical laws describing a system are relativistically invariant and transform between inertial reference frames. Here, we apply this concept to a generalized SEM tensor while considering two frames of reference, S and S', such that S' moves with velocity v along the x axis with respect to S. The SEM tensor, represented in the rest or moving frame S, takes the general form

$$\mathfrak{T}_{\alpha\beta}=\begin{bmatrix}T_{xx}&T_{xy}&T_{xz}&icG_x\\T_{yx}&T_{yy}&T_{yz}&icG_y\\T_{zx}&T_{zy}&T_{zz}&icG_z\\iS_x/c&iS_y/c&iS_z/c&W\end{bmatrix}. \tag{16}$$

Transformation from the S frame to the moving frame S' invokes the following relations:

$$T'_{xx}=\gamma^2[T_{xx}+\beta(S_x/c+cG_x)-\beta^2W], \tag{17a}$$
$$T'_{yy}=T_{yy}, \tag{17b}$$
$$T'_{zz}=T_{zz}, \tag{17c}$$
$$T'_{xy}=\gamma(T_{xy}+\beta S_y/c), \tag{17d}$$
$$T'_{yx}=\gamma(T_{xy}+c\beta G_yc), \tag{17e}$$
$$T'_{xz}=\gamma(T_{xz}+\beta S_z/c), \tag{17f}$$
$$T'_{zx}=\gamma(T_{xz}+c\beta G_z), \tag{17g}$$
$$T'_{yz}=T_{yz}, \tag{17h}$$
$$T'_{zy}=T_{zy}, \tag{17i}$$
$$iS'_x/c=-\gamma^2[icG_x+i\beta(T_{xx}-W)+i\beta^2S_x/c], \tag{17j}$$
$$icG'_x=-\gamma^2[iS_x/c+i\beta(T_{xx}-W)+ic\beta^2G_x], \tag{17k}$$
$$iS'_y/c=-i\gamma(S_y/c+\beta T_{xy}), \tag{17l}$$
$$icG'_y=-i\gamma(cG_y+\beta T_{xy}), \tag{17m}$$
$$iS'_z/c=-i\gamma(S_z/c+\beta T_{xz}), \tag{17n}$$
$$icG'_z=-i\gamma(cG_z+\beta T_{xy}), \tag{17o}$$
$$W'=\gamma^2[W-\beta(S_x/c+cG_x)-T_{xx}\beta^2], \tag{17p}$$

where $\beta=v/c$ and $\gamma=(1-\beta)^{-1/2}$. For the remainder of this section, we review the prospective Einstein-Laub and

Chu formulations with respect to the Lorentz transforms of the Chu and Minkowski equations. This will demonstrate the relativistic transformations of both formulations with regards to each field interpretation, which ultimately demonstrates the invariance of the force and power densities of each formulation, corresponding to the energy and momentum described by each formulation.

1. *Chu field analysis*

The Chu Lorentz transformation laws for field variables $\bar{E}$, $\bar{H}$, $\bar{P}$, and $\bar{M}$ from the moving frame to the laboratory frame are given by [21,36]

$$\bar{E}' = \bar{E}_{\parallel} + \gamma(\bar{E}_{\perp} + \bar{v} \times \mu_0 \bar{H}), \tag{18a}$$
$$\bar{H}' = \bar{H}_{\parallel} + \gamma(\bar{H}_{\perp} - \bar{v} \times \epsilon_0 \bar{E}), \tag{18b}$$
$$\bar{P}' = \bar{P}_{\parallel} + \gamma\left(\bar{P}_{\perp} - \frac{\bar{v} \times (\bar{P} \times \bar{v}^*)}{c^2}\right), \tag{18c}$$
$$\bar{M}' = \bar{M}_{\parallel} + \gamma\left(\bar{M}_{\perp} - \frac{\bar{v} \times (\bar{M} \times \bar{v}^*)}{c^2}\right), \tag{18d}$$

where $\bar{v}$ is the velocity of the unprimed frame and $\bar{v}^*$ is the velocity of the electromagnetic material. Here, defining $\bar{v} = \bar{v}^* = \hat{x}v$, the Lorentz transformations of the Chu field values are given as

$$E_x = E'_x, \tag{19a}$$
$$E_y = \gamma(E'_y + c\mu_0\beta H'_z), \tag{19b}$$
$$E_z = \gamma(E'_z - c\mu_0\beta H'_y), \tag{19c}$$
$$H_x = H'_x, \tag{19d}$$
$$H_y = \gamma(H'_y - c\epsilon_0\beta E'_z), \tag{19e}$$
$$H_z = \gamma(H'_z + c\epsilon_0\beta E'_y), \tag{19f}$$
$$P_x = P'_x, \tag{19g}$$
$$P_y = \gamma P'_y, \tag{19h}$$
$$P_z = \gamma P'_z, \tag{19i}$$
$$M_x = M'_x, \tag{19j}$$
$$M_y = \gamma M'_y, \tag{19k}$$
$$M_z = \gamma M'_z. \tag{19l}$$

Considering the Chu formulation, the components of the SEM tensor are

$$\bar{S} = \bar{E} \times \bar{H}, \tag{20a}$$
$$\bar{G} = \frac{\bar{S}}{c^2}, \tag{20b}$$
$$W = \tfrac{1}{2}(\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H}), \tag{20c}$$
$$T_{\alpha\beta} = \epsilon_0 E_\alpha E_\beta + \mu_0 H_\alpha H_\beta - \delta_{\alpha\beta} W. \tag{20d}$$

Due to the discrepancies between the stress tensors, we look to transform the field values within the moving frame back to the stationary frame, demonstrating the relativistic invariance (or lack thereof) of the stress tensor of each formulation. Thus, employing Eqs. (17a) and (20),

$$T'_{xx} = \gamma^2\bigg[\epsilon_0 E_x E_x + \mu_0 H_x H_x + 2\frac{\beta}{c}(\bar{E}\times\bar{H})_x - \frac{1}{2}(1+\beta^2)(\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H})\bigg]. \tag{21}$$

Expanding the scalar and vector products,

$$\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H} = \epsilon_0 E_x E_x + \epsilon_0 E_y E_y + \epsilon_0 E_z E_z + \mu_0 H_x H_x + \mu_0 H_y H_y + \mu_0 H_z H_z,$$
$$(\bar{E}\times\bar{H})_x = E_y H_z - E_z H_y,$$

and employing the Lorentz fields in Eqs. (19), we find

$$\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H} = \epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x + \gamma^2[(1+\beta^2)(\epsilon_0 E'_y E'_y + \epsilon_0 E'_z E'_z + \mu_0 H'_y H'_y + \mu_0 H'_z H'_z) + 4\epsilon_0\mu_0 c\beta(E'_y H'_z - E'_z H'_y)], \tag{22a}$$
$$(\bar{E}\times\bar{H})_x = \gamma^2[(1+\beta^2)(E'_y H'_z - E'_z H'_y) + c\beta(\epsilon_0 E'_y E'_y + \epsilon_0 E'_z E'_z + \mu_0 H'_y H'_y + \mu_0 H'_z H'_z)]. \tag{22b}$$

Substituting Eqs. (22) into Eq. (21) renders

$$T'_{xx} = \gamma^2\bigg[\frac{1}{2}(1-\beta^2)\epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x - \frac{\gamma^2}{2}(1-\beta^2)^2 \times (\epsilon_0 E'_y E'_y + \epsilon_0 E'_z E'_z + \mu_0 H'_y H'_y + \mu_0 H'_z H'_z)\bigg] = \epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x + \delta_{xx} W'. \tag{23}$$

As can be seen, the Chu stress tensor remains unchanged when transformed to the S' frame, demonstrating relativistic invariance. Here, we note that additional manipulation of the remaining expressions in Eqs. (17) provides the desired relativistic transformations.

For completeness, we repeat the previous derivation for the Einstein-Laub formulation while employing the Chu fields. However, we note that the Einstein-Laub formulation was originally formulated with Minkowski fields. Thus, the Einstein-Laub stress tensor under the Chu field representation is given as

$$T_{\alpha\beta} = (\epsilon_0 E + P)_\alpha E_\beta + \mu_0 (H + M)_\alpha H_\beta - \delta_{\alpha\beta} W, \tag{24}$$

where S_x, G_x, and W retain the form presented in Eqs. (20). Using standard field definitions along with the Einstein-Laub values renders Eq. (17a) as

$$T'_{xx} = \gamma^2\bigg[(\epsilon_0 E_x + P_x)E_x + \mu_0(H_x + M_x)H_x + 2\frac{\beta}{c} \times (\bar{E}\times\bar{H})_x - \frac{1}{2}(1+\beta^2)(\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H})\bigg]. \tag{25}$$

Plugging Eqs. (22) into Eq. (25) renders

$$T'_{xx} = \epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x + \delta_{xx} W' + \gamma^2(P'_x E'_x + \mu_0 M'_x H'_x). \tag{26}$$

It is easily seen that the additional material components included within the dyadic product of the Einstein-Laub stress tensor do not transform relativistically, thereby proving that the SEM tensor provided by the Einstein-Laub formulation is an invalid representation of electrodynamics while using the Chu fields.

2. *Minkowski field analysis*

The Lorentz transformation laws for Minkowski field values $\bar{E}$, $\bar{H}$, $\bar{D}$, and $\bar{B}$ are given as [21,37]

$$\bar{E}' = \bar{E}_{\parallel} + \gamma(\bar{E}_{\perp} + \bar{v} \times \bar{B}), \tag{27a}$$

$$\bar{H}' = \bar{H}_{\parallel} + \gamma(\bar{H}_{\perp} - \bar{v} \times \bar{D}), \tag{27b}$$

$$\bar{D}' = \bar{D}_{\parallel} + \gamma\left(\bar{D}_{\perp} + \frac{\bar{v} \times \bar{H}}{c^2}\right), \tag{27c}$$

$$\bar{B}' = \bar{B}_{\parallel} + \gamma\left(\bar{B}_{\perp} - \frac{\bar{v} \times \bar{E}}{c^2}\right). \tag{27d}$$

Asserting $\bar{v} = \hat{x}v$, the Lorentz transforms for the Minkowski fields are given as

$$E_x = E'_x, \tag{28a}$$

$$E_y = \gamma(E'_y + c\beta B'_z), \tag{28b}$$

$$E_z = \gamma(E'_z - c\beta B'_y), \tag{28c}$$

$$H_x = H'_x, \tag{28d}$$

$$H_y = \gamma(H'_y - c\beta D'_z), \tag{28e}$$

$$H_z = \gamma(H'_z + c\beta D'_y), \tag{28f}$$

$$D_x = D'_x, \tag{28g}$$

$$D_y = \gamma(D'_y + \beta H'_z/c), \tag{28h}$$

$$D_z = \gamma(E'_z - \beta H'_y/c), \tag{28i}$$

$$B_x = B'_x, \tag{28j}$$

$$B_y = \gamma(H'_y - \beta E'_z/c), \tag{28k}$$

$$B_z = \gamma(H'_z + \beta D'_y/c). \tag{28l}$$

Here, we utilize the previously defined SEM components (17a) and (20a)–(20c), along with

$$T_{\alpha\beta} = D_\alpha E_\beta + B_\alpha H_\beta - \delta_{\alpha\beta} W, \tag{29}$$

such that

$$T'_{xx} = \gamma^2\left[D_x E_x + B_x H_x + 2\frac{\beta}{c}(\bar{E} \times \bar{H})_x - \frac{1}{2}(1+\beta^2)(\epsilon_0 \bar{E}\cdot\bar{E} + \mu_0 \bar{H}\cdot\bar{H})\right]. \tag{30}$$

Expanding the scalar and vector products while employing the Lorentz field transformation (28), we find

$$\begin{aligned}\epsilon_0\bar{E}\cdot\bar{E} + \mu_0\bar{H}\cdot\bar{H} &= \epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x \\ &\quad + \gamma^2[(\epsilon_0 E'_y E'_y + \epsilon_0 E'_z E'_z + \mu_0 H'_y H'_y + \mu_0 H'_z H'_z) \\ &\quad + c^2\beta^2(\epsilon_0 B'_y B'_y + \epsilon_0 B'_z B'_z + \mu_0 D'_y D'_y + \mu_0 D'_z D'_z) \\ &\quad + 2c\beta\{(\epsilon_0\bar{E}'\times\bar{B}')_x + (\bar{D}'\times\mu_0\bar{H}')_x\}],\end{aligned} \tag{31a}$$

$$\begin{aligned}(\bar{E}\times\bar{H})_x &= \gamma^2[(\bar{E}'\times\bar{H}')_x + c^2\beta^2(\bar{D}'\times\bar{B}') \\ &\quad + c\beta(H'_y B'_y + H'_z B'_z + E'_y D'_y + E'_z D'_z)].\end{aligned} \tag{31b}$$

Plugging Eqs. (31) into Eq. (30) gives

$$\begin{aligned}T'_{xx} = \gamma^2\bigg[&E'_x D'_x + H'_x B'_x + \frac{2\gamma^2\beta}{c}\{(\bar{E}'\times\bar{H}')_x \\ &+ c^2\beta^2(\bar{D}'\times\bar{B}') + c\beta(H'_y B'_y + H'_z B'_z + E'_y D'_y \\ &+ E'_z D'_z)\} - \frac{1}{2}(1+\beta^2)\{\epsilon_0 E'_x E'_x + \mu_0 H'_x H'_x \\ &+ \gamma^2[(\epsilon_0 E'_y E'_y + \epsilon_0 E'_z E'_z + \mu_0 H'_y H'_y + \mu_0 H'_z H'_z) \\ &+ c^2\beta^2(\epsilon_0 B'_y B'_y + \epsilon_0 B'_z B'_z + \mu_0 D'_y D'_y + \mu_0 D'_z D'_z) \\ &+ 2c\beta\{(\epsilon_0\bar{E}'\times\bar{B}')_x + (\bar{D}'\times\mu_0\bar{H}')_x\}]\}\bigg],\end{aligned} \tag{32}$$

which does not transform to the Einstein-Laub formulation in the moving frame using the Minkowski field values. However, for the reader, it is easy to validate the expression given in Eq. (32) as the Einstein-Laub formulation by taking $\beta \to 0$, which, as one would expect, renders Eq. (29). Additionally, this result indicates that, while utilizing both the Chu and Minkowski field values, the Einstein-Laub formulation is an invalid interpretation of electromagnetics. This is due to untransformable SEM values, which constitute the force, power, energy, and momentum of the electromagnetic subsystem.

IV. MODELING

In this section, we utilize the Chu, Einstein-Laub, and Abraham formulations to demonstrate the electromagnetic force and power distributions, further illustrating the discrepancies within the prospective kinetic formulations. In doing so, we evaluate the electromagnetic interactions with a linear, lossless, nondispersive magnetodielectric material for both the stationary and moving material cases. This allows for discussion with respect to each electromagnetic formulation, where conservation theorems and the subsystem concept, as presented in Refs. [21,22], are utilized.

A. Stationary analysis

Consider an electromagnetic wave normally incident from vacuum onto a linear, lossless, nondispersive magnetodielectric half-space, as seen in Fig. 1. Here, the incident, reflected, and transmitted field values, along with the reflection and transmission coefficients, are presented in Appendix A. The material is stationary and is held at a constant velocity $v = 0$ by an external mechanical force. We note that for the stationary analysis, the field definitions for any representation of electrodynamics are equivalent. With this, we employ the time-average force and power relations via the subsystem concept for each respective formulation,

$$\langle\bar{F}_j\rangle = \iiint_V dV\langle\bar{f}_j\rangle = -\oint\!\!\oint_a d\bar{a}\cdot\langle\bar{\bar{T}}_j\rangle, \tag{33a}$$

$$\langle P_j\rangle = \iiint_V dV\langle\varphi_j\rangle = \oint\!\!\oint_a d\bar{a}\cdot\langle\bar{S}_j\rangle, \tag{33b}$$

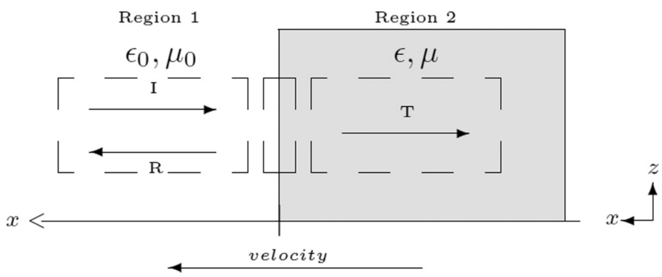

FIG. 1. A plane wave normally incident on a magnetodielectric half-space with refractive index $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$, moving with velocity $v = xt$.

where $\sum_j \bar{f} = 0$ and $\sum_j \varphi_j = 0$ for all system partitions j. Utilizing Table I, the Abraham force at the boundary interface is rendered as

$$\langle \bar{F}_{\text{Abr}} \rangle = -\hat{x}U_0[1 + R^2 - \epsilon_r T^2] = -\hat{x}\frac{2U_0(n^2 - 2n^2\mu_r + \mu_r^2)}{(n+\mu_r)^2}, \quad (34)$$

and the Einstein-Laub and Chu force is rendered as

$$\langle \bar{F}_{\text{Chu}} \rangle = \langle \bar{F}_{EL} \rangle = -\hat{x}U_0\left[1 + R^2 - \frac{1}{2}\left(1 + \frac{n^2}{\mu_r^2}\right)T^2\right] = \hat{x}0, \quad (35)$$

where ϵ_r and μ_r are the relative permittivity and permeability of the magnetodielectric medium, with $n = \sqrt{\epsilon_r\mu_r}$ and $U_0 = \frac{E_0^2}{2c^2\mu_0}$. Utilizing Eq. (33b) in a similar fashion yields

$$\langle P_{\text{Abr}} \rangle = \langle P_{\text{Chu}} \rangle = \langle P_{EL} \rangle = -cU_0\left[1 - R^2 - \frac{n}{\mu_r}T^2\right] = 0 \quad (36)$$

for each kinetic formulation. These results are consistent with previous related research [23,24,38] and global energy- and momentum-conservation laws. However, closer inspection of the stationary conservation statements reveals trivial results for each electromagnetic formulation. This can be shown by employing the standard power relation $\langle \bar{F}_e \rangle \cdot \bar{v} = P_e$ with $\bar{v} = \bar{0}$, validating any electromagnetic force expression, where by conservation theorems $\langle \bar{F}_{\text{mech}} \rangle = -\langle \bar{F}_e \rangle$. Simply stated, this relation holds because the stationary time-average electromagnetic power, as demonstrated in Eq. (36), always renders zero net-power flow, where the electromagnetic force can be arbitrarily defined and validate the system. This illustrates that stationary media analysis alone is insufficient for determining the kinetic subsystem [22,24]. Other conclusions from the present stationary analysis are further discussed in the later sections of this paper.

B. Moving analysis

Now, consider an electromagnetic wave normally incident from vacuum onto a moving, linear, lossless, nondispersive magnetodielectric half-space, as seen in Fig. 1. Here, the constitutive relations of the moving material are transformed from the moving frame to the laboratory frame, rendering bianisotropic material parameters [23]. Employing the moving material constitutive relations and the kDB system, wave vector $\bar{k}$ and the Minkowski fields are generated for relativistic analysis, where the methods are demonstrated in Refs. [22,23] and in Appendix B. Additionally, the Chu fields are generated by using field transformation laws, along with the derived Minkowski fields, and are expressed in Appendix B.

In evaluating the moving system, we employ the subsystem concept [21] in conjunction with the jump condition provided by kinematic theory [39]. This yields the time-average force and power relations for moving media as

$$\langle \bar{F}_j \rangle = -\oiint_a d\bar{a} \cdot \{\langle \bar{\bar{T}}_j \rangle - \bar{v}\langle \bar{G}_j \rangle\} \quad (37a)$$

$$\langle P_j \rangle = \oiint_a d\bar{a} \cdot \{\langle \bar{S}_j \rangle - \bar{v}\langle W_j \rangle\}, \quad (37b)$$

where $\sum_j \bar{f}_j = 0$ and $\sum_j \varphi_j = 0$ by conservation laws as before. Within the analysis, two subsystem are considered: electromagnetic and mechanical. Here, the electromagnetic subsystem is individually modeled by each leading kinetic formulation, where the mechanical subsystem retains a constant material velocity as per relativistic constraints. Now, application of Eqs. (37), formulation-specific SEM components, and the associated field definitions in Appendix B render the Abraham force and power as

$$\langle \bar{F}_{\text{Abr}} \rangle = -\hat{x}[\langle \bar{\bar{T}}_{\text{Abr}} \rangle] + v\langle \bar{G}_{\text{Abr}} \rangle] = -\hat{x}\frac{2U_0(1+\beta)}{(1-\beta)} \times \frac{\{n^2 - 2n\mu_r'[n + \beta(1-n^2)] + \mu_r'^2\}}{(n+\mu_r')^2}, \quad (38a)$$

$$\langle P_{\text{Abr}} \rangle = -[\langle \bar{S}_{\text{Abr}} \rangle] + v[\langle W_{\text{Abr}} \rangle] = \frac{2U_0(1+\beta)}{(1-\beta)} \times \frac{c\beta(n^2 - 2n\mu_r' + \mu_r'^2)}{(n+\mu_r')^2}, \quad (38b)$$

the Einstein-Laub force and power as

$$\langle \bar{F}_{EL} \rangle = -\hat{x}[\langle \bar{\bar{T}}_{EL} \rangle] + v[\langle \bar{G}_{EL} \rangle] = -\hat{x}\frac{2U_0\beta}{(1-\beta)^2(n+\mu_r')^2}\{[2\beta^2 n\mu_r'(1-n^2) + 2n(n^2 - \mu_r'(2-\mu_r'))] - \beta[n^4 + \mu_r'^2 + n^2(1 - 4\mu_r' + \mu_r'^2)]\}, \quad (39a)$$

$$\langle P_{EL} \rangle = -[\langle \bar{S}_{EL} \rangle] + v[\langle W_{EL} \rangle] = \frac{2U_0 c\beta}{(1-\beta)^2(n+\mu_r')^2}[(n^2\beta + \mu_r'^2\beta)(\beta + 2n - n^2\beta) - 2(n^2 + \mu_r'^2) - 2n\mu_r'(\beta - 2n + n^2\beta)], \quad (39b)$$

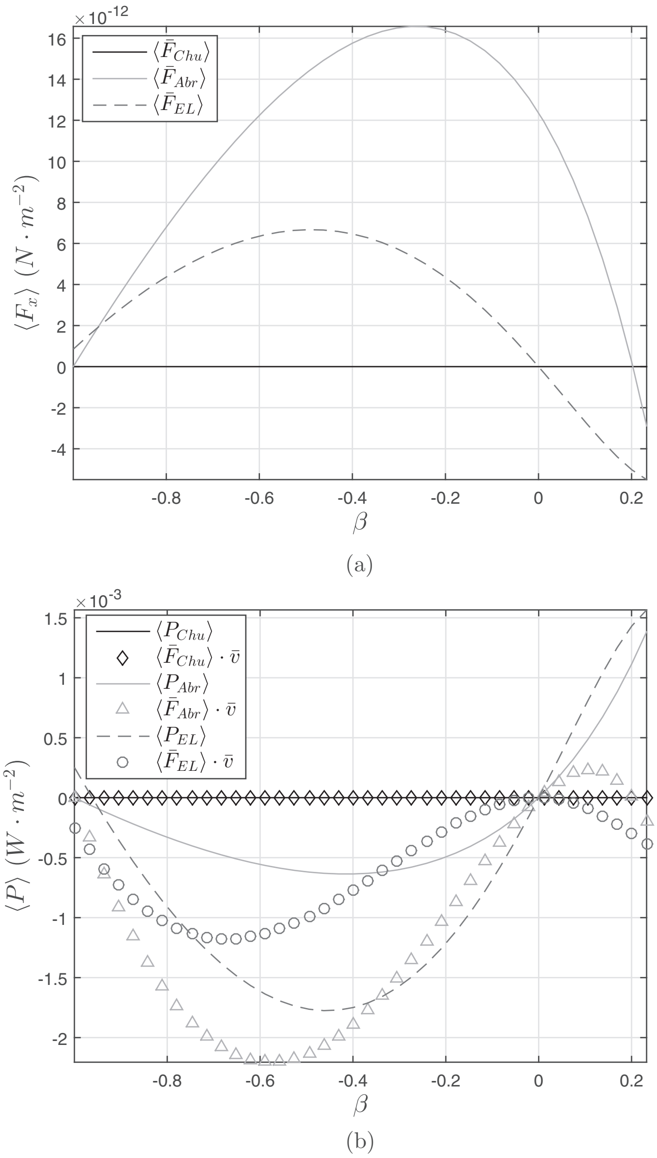

FIG. 2. The electromagnetic (a) force and (b) power versus velocity for the leading kinetic formulations are presented for a moving magnetodielectric half-space, with the normalized velocity ranging from -1 to $1/n$. Here, $\epsilon_r' = 5$, $\mu_r' = 3$, $n = \sqrt{\epsilon_r'\mu_r'}$, with $\beta = \frac{v}{c}$ being the normalized velocity.

and the Chu force as

$$\langle \bar{F}_{\text{Chu}} \rangle = \hat{x}[\langle \bar{\bar{T}}_{\text{Chu}} \rangle] + v[\langle \bar{G}_{\text{Chu}} \rangle] = \hat{x}0, \tag{40a}$$

$$\langle P_{\text{Chu}} \rangle = -[\langle \bar{S}_{\text{Chu}} \rangle] + v[\langle W_{\text{Chu}} \rangle] = 0, \tag{40b}$$

where a prime represents material parameters in the moving frame. Plots of the force and power distributions for each formulation are demonstrated in Fig 2. Here, the mechanical force and power used in maintaining the system are $\langle \bar{F}_{\text{mech}} \rangle = -\langle \bar{F}_e \rangle$ and $\langle P_{\text{mech}} \rangle = -\langle P_e \rangle$ by conservation theorems. This implies $\langle \bar{F}_e \rangle \cdot \bar{v} = \langle P_e \rangle$ must hold for each individual electromagnetic formulation. Applying this to each derived moving system, however, demonstrates neither the Abraham nor Einstein-Laub equations possess valid conservation expressions within the system, as can be seen in Fig. 2(b). At least one of the force or power expressions resulting from both the Abraham and Einstein-Laub expressions is incorrect, leading to invalid electrodynamic representations within the system. In contrast, the Chu formulation provides valid results for the kinetic force and power expressions, further demonstrating the correct electrodynamics in all inertial reference frames.

V. LAGRANGIAN

Hamilton's variational principle provides a systematic process for deriving the equations of motion and conservation laws for a physical system from a postulated Lagrangian density, where the use of a generalized Lagrangian produces consistent dynamics within a closed system. The Lagrangian density is formulated by [40,41]

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I + \mathcal{L}_M, \tag{41}$$

where $\mathcal{L}_F$, $\mathcal{L}_I$, and $\mathcal{L}_M$ are the electromagnetic field, field-matter interaction, and matter Lagrangian densities, respectively. The energy and coenergy functions are defined in terms of the electric and magnetic field vectors $\bar{E}$ and $\bar{H}$, consistent with the shared-energy and momentum relations, which leads to the invariant expression for the electromagnetic field Lagrangian density [21],

$$\mathcal{L}_F = \frac{\epsilon_0}{2}|\bar{E}|^2 - \frac{\mu_0}{2}|\bar{H}|^2. \tag{42}$$

To allow for accurate partitioning of field and material subsystems, we reserve discussion of the matter Lagrangian density $\mathcal{L}_M$ to a future publication. In general, however, $\mathcal{L}_M$ will depend upon the model used for the material, and examples of causal material models have been given for dielectrics [42] and magnetodielectrics [43]. With a vector potential structure previously applied to describe systems with magnetic monopoles [44] and to model radiation using the equivalence principle [45]

$$\bar{E} = -\nabla\Phi_e - \frac{1}{\epsilon_0}\nabla \times \bar{A}_e - \frac{\partial \bar{A}_m}{\partial t}, \tag{43a}$$

$$\bar{H} = -\nabla\Phi_m + \frac{1}{\mu_0}\nabla \times \bar{A}_m - \frac{\partial \bar{A}_e}{\partial t}, \tag{43b}$$

the field-matter interaction Lagrangian density is defined to include electric and magnetic interaction terms

$$\mathcal{L}_I = -\rho_e\Phi_e + \bar{J}_e \cdot \bar{A}_m + \rho_m\Phi_m + \bar{J}_m \cdot \bar{A}_e, \tag{44}$$

where $\bar{J}_e$ and $\bar{J}_m$ are the effective electric and magnetic current densities, ρ_e and ρ_m are the effective electric and magnetic charge densities, Φ_e and Φ_m are the electric and magnetic scalar potentials, and $\bar{A}_e$ and $\bar{A}_m$ are the electric and magnetic vector potentials.

The Euler-Lagrange equation [41] is given by the relation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_j} = \frac{\partial \mathcal{L}}{\partial x_j} - \frac{d}{dX_K}\frac{\partial \mathcal{L}}{\partial(\partial x_j/\partial X_K)}, \tag{45}$$

where the Lagrangian density is used to derive the dynamics of the subsystem. Thus, substituting the kinetic Lagrangian density,

$$\mathcal{L}_{F_k} = \frac{\epsilon_0}{2}\left(-\nabla\Phi_e - \frac{1}{\epsilon_0}\nabla\times\bar{A}_m - \frac{\partial\bar{A}_e}{\partial t}\right)^2 - \frac{\mu_0}{2}\left(-\nabla\Phi_m + \frac{1}{\mu_0}\nabla\times\bar{A}_e - \frac{\partial\bar{A}_m}{\partial t}\right)^2 - \rho_e\Phi_e + \bar{J}_e\cdot\bar{A}_e + \rho_m\Phi_m + \bar{J}_m\cdot\bar{A}_m, \quad (46)$$

into Eq. (45), the kinetic form of Maxwell's equations are derived. For the electric scalar potential, we find the Lagrangian with respect to Φ_e by

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\partial\Phi_e/\partial t)} = \frac{\partial\mathcal{L}}{\partial\Phi_e} - \frac{d}{dz_j}\frac{\partial\mathcal{L}}{\partial\Phi_{e,j}} \quad (47)$$

to yield

$$0 = -\rho_e + \epsilon_0\frac{d}{dz_j}\bar{E},$$
$$\rho_e = \epsilon_0\nabla\cdot\bar{E}. \quad (48)$$

Similarly, we find the Lagrangian with respect to the magnetic scalar potential Φ_m by yielding

$$0 = \rho_m - \mu_0\frac{d}{dz_j}\bar{H},$$
$$\rho_m = \mu_0\nabla\cdot\bar{H}. \quad (49)$$

Application of the Lagrangian with respect to the electric vector potential $\bar{A}_e$ renders

$$-\epsilon_0\frac{d}{dt}\bar{E} = J_{e_i} - (-\alpha_{kji})\frac{d}{dz_j}\bar{H},$$
$$\nabla\times\bar{H} - \epsilon_0\frac{\partial\bar{E}}{\partial t} = \bar{J}_e, \quad (50)$$

where the permutation symbol α_{klm} is used. Last, the Lagrangian in terms of the magnetic vector potential $\bar{A}_m$ is found such that

$$-\mu_0\frac{d}{dt}\bar{H} = J_{m_i} + (-\alpha_{kji})\frac{d}{dz_j}\bar{E},$$
$$\nabla\times\bar{E} + \mu_0\frac{\partial\bar{H}}{\partial t} = -\bar{J}_m. \quad (51)$$

Collection of Eqs. (48)–(51) yields Maxwell's equations in terms of the $\bar{E}$ and $\bar{H}$ fields, as seen in Eq. (3). Here, we must point out that the material contributions $\bar{J}_e$, $\bar{J}_m$, ρ_e, and ρ_m that uniquely define the electromagnetic subsystem have yet to be defined. This means that specific field definitions that render the physical interpretation and formulation of electrodynamics cannot be completely known. However, the electromagnetic framework has been derived, allowing for generalized field analysis of the electromagnetic subsystem.

Equations (3) are easily recast into matrix representation such that

$$\mathcal{F}_{\alpha\beta} = \begin{bmatrix} 0 & H_z & -H_y & -ic\epsilon_0E_x \\ -H_z & 0 & H_x & -ic\epsilon_0E_y \\ H_y & -H_x & 0 & -ic\epsilon_0E_z \\ ic\epsilon_0E_x & ic\epsilon_0E_y & ic\epsilon_0E_z & 0 \end{bmatrix}, \quad (52a)$$

$$\mathcal{G}_{\alpha\beta} = \begin{bmatrix} 0 & -E_z & E_y & -ic\mu_0H_x \\ E_z & 0 & -E_x & -ic\mu_0H_y \\ -E_y & E_x & 0 & -ic\mu_0H_z \\ ic\mu_0H_x & ic\mu_0H_y & ic\mu_0H_z & 0 \end{bmatrix}, \quad (52b)$$

where Greek subscripts α,β have their usual meaning [21]. Applying the generalized relativistic transformation matrix [21] such that S' is traveling with velocity $\bar{v} = \hat{x}v$ with respect to S, the field tensors transform to

$$\mathcal{F}'_{\alpha\beta} = \begin{bmatrix} 0 & H'_z & -H'_y & -ic\epsilon_0E'_x \\ -H'_z & 0 & H'_x & -ic\epsilon_0E'^{\dagger}_y \\ H'_y & -H'_x & 0 & -ic\epsilon_0E'^{\dagger}_z \\ ic\epsilon_0E'_x & ic\epsilon_0E'^{\dagger}_y & ic\epsilon_0E'^{\dagger}_z & 0 \end{bmatrix}, \quad (53a)$$

$$\mathcal{G}'_{\alpha\beta} = \begin{bmatrix} 0 & -E'_z & E'_y & -ic\mu_0H'_x \\ E'_z & 0 & -E'_x & -ic\mu_0H'_y \\ -E'_y & E'_x & 0 & -ic\mu_0H'_z \\ ic\mu_0H'_x & ic\mu_0H'^{\dagger}_y & ic\mu_0H'^{\dagger}_z & 0 \end{bmatrix}, \quad (53b)$$

with the primed matrix values given as

$$E'_x = E_x,$$
$$E'_y = \gamma(E_y - c\mu_0\beta H_z),$$
$$E'^{\dagger}_y = \gamma(c\mu_0H_z - \beta E_y),$$
$$E'_z = \gamma(E_z + c\mu_0\beta H_y),$$
$$E'^{\dagger}_z = \gamma(c\mu_0H_y + \beta E_z),$$
$$H'_x = H_x,$$
$$H'_y = \gamma(H_y + c\epsilon_0\beta E_z),$$
$$H'^{\dagger}_y = \gamma(c\epsilon_0E_z + \beta H_y),$$
$$H'_z = \gamma(H_z - c\epsilon_0\beta E_y),$$
$$H'^{\dagger}_z = \gamma(c\epsilon_0E_y - H_z\beta),$$

where the transformation made no assumption towards any field representation. In deriving the shared-energy relations and kinetic-momentum density, the relativistic SEM tensor is found such that

$$\mathfrak{T}'_{\alpha\beta} = \tfrac{1}{2}\mu_0\mathcal{F}'_{\alpha\beta}\mathcal{F}'_{\beta\gamma} + \tfrac{1}{2}\epsilon_0\mathcal{G}'_{\alpha\beta}\mathcal{G}'_{\beta\gamma}, \quad (54)$$

where Eq. (54) represents the combination of Eqs. (17), (18), and (20). Here, we note that this demonstrates the Chu formulation as the relativistically invariant electromagnetic system tied to Eqs. (3).

VI. DISCUSSION

In Secs. III–V, we studied the relativistic nature of two perspective kinetic formulations. Employing such analytic methods as the principle of virtual power and Lagrangian analysis, as well as utilizing Lorentz invariance arguments and relativistic modeling, both the Chu and Einstein-Laub electromagnetic subsystems were studied with reference to relativistic frameworks. In this section, we review our finding while discussing related contributions, thereby revealing the discrepancies between the two formulations.

Field kinetic research has utilized several leading formulations in modeling the center-of-mass translation with respect to electromagnetic material interactions [6]. Each prospective formulation attempts to reformulate or divide Maxwell's equations into pure field and material responses, which constitute the electromagnetic subsystem. Within the literature, two leading formulations used in modeling the kinetics of light are the Chu and Einstein-Laub formulations, where both formulations correspond to partitioning Maxwell's equations such that the material is modeled as electric and magnetic dipoles [20]. Although similar, the two formulations differ in the expressions of the force density, stress tensor, and electromagnetic field interpretations. A review of recent literature reveals both formulations are used in modeling the field kinetic subsystem for multiple computational experiments, as well as theoretical discussion [38,46–49]. The problem with this is that, for stationary media (which much of the literature models), both formulations simultaneously satisfy momentum conservation using different force-density expressions, where the power-density expressions remain equivalent and unchanged. This is a consequence of rearranging terms in the conservation equations

$$\bar{f}_{F_k} = -\nabla \cdot \bar{\bar{T}}_{F_k} - \frac{\partial}{\partial t}(\epsilon_0\mu_0 \bar{E} \times \bar{H}), \tag{55}$$

such that the force density and stress tensor are ambiguously defined, allowing for multiple mathematically valid force-density expressions representing the kinetic subsystem. To demonstrate, we employ vector calculus identities [24]

$$-(\nabla \cdot \bar{P})\bar{E} = (\bar{P} \cdot \nabla)\bar{E} - \nabla \cdot (\bar{P}\bar{E}), \tag{56a}$$

$$-\mu_0(\nabla \cdot \bar{M})\bar{H} = \mu_0(\bar{M} \cdot \nabla)\bar{H} - \mu_0\nabla \cdot (\bar{M}\bar{H}) \tag{56b}$$

to rearrange the Chu stress tensor and force-density values to yield the Einstein-Laub force density

$$\begin{aligned}[][\rho + (\bar{P} \cdot \nabla)]\bar{E} + (\mu_0\bar{M} \cdot \nabla)\bar{H} + \left(\bar{J} + \frac{\partial \bar{P}}{\partial t}\right) \times \mu_0 \bar{H} \\ - \mu_0 \frac{\partial \bar{M}}{\partial t} \times \epsilon_0 \bar{E} = -\frac{\partial}{\partial t}[\epsilon_0\mu_0 \bar{E} \times \bar{H}] \\ -\nabla \cdot \left[\frac{1}{2}(\epsilon_0 \bar{E} \cdot \bar{E} + \mu_0 \bar{H} \cdot \bar{H})\bar{\bar{I}} - \bar{D}\bar{E} - \bar{B}\bar{H}\right].\end{aligned} \tag{57}$$

This augmentation is similar to that used in describing the Abraham force, which alters the Minkowski formulation rendering the Abraham formulation while sharing Minkowski energy definitions [24]. However mathematically correct, such mathematical exercises should not be taken as a basis for defining physical systems and should instead be tested against known mathematical and physical constructs (one such being invariant forces).

To uniquely resolve the analytic electrodynamics expressions, one can employ RPVP while utilizing *a priori* energy relations along with respective field transformations to derive the related stress tensor and momentum density valid for all inertial reference frames. This was accomplished in Sec. III A, where both the Chu and Minkowski fields were applied to produce the associated stress-tensor and momentum-density values. Here, only one analysis, the Chu analysis, demonstrated in Eqs. (15), rendered invariant results for the force-density, stress-tensor, and momentum-density expressions, which corresponds to previous research on the kinetic subsystem [33]. In contrast, the Einstein-Laub analysis, demonstrated in Eqs. (11), did not produce the kinetic-momentum density and resulted in unreported expressions for both the stress tensor and momentum density. In Sec. III B, further analysis using relativistic SEM tensor and vector field transformations demonstrated relativistic invariance of the Chu stress tensor between inertial reference under Chu field transforms. The Einstein-Laub stress tensor, however, failed to transform under both Chu and Minkowski field definitions. This indicates that the Einstein-Laub formulation, in general, cannot constitute a valid electromagnetic subsystem. This is due to the prescribed electrodynamics changing from one inertial reference frame to another, which, according to special relativity, cannot be true.

In Sec IV, the Chu, Einstein-Laub, and Abraham formulations were used in modeling the field kinetic subsystem for stationary and moving media. First, considering the electromagnetic models in the stationary system, the Abraham momentum, as presented is Eq. (34), is equivalent to that of the Minkowski momentum Ref. [38], which requires the Abraham kinetic formulation to produce a pulling force equivalent to the canonical Minkowski force. This comes as no surprise because the stress-tensor definitions of both formulations, which, in the stationary frame, are used to derive the time-average force-density expression, are equivalent (i.e., $\bar{\bar{T}}_M = \bar{\bar{T}}_A$). Alternatively, the Chu and Einstein-Laub formulations demonstrate null quantities for the electromagnetic momentum normally incident at the boundary interface. This is due to both formulations modeling the transmitted electromagnetic momentum flow within the material and the free-space momentum flow outside the material as equal and opposite electromagnetic momentum contributions. For each case considered, the electromagnetic system satisfies global energy- and momentum-conservation laws, where the quasistationary approximation ensures conservation by limiting the materials' physical quantities (velocity $v \to 0$ and mass $m \to \infty$) such that the momentum vector remains nonzero [6,24]. For the moving media analysis, however, this is not the case. Instead one can use the electromagnetic energy flow of the incident, reflected, and transmitted waves to analyze the moving system, thus providing validation of global conservation laws [22,50]. Considering the moving electromagnetic models, the Abraham and Einstein-Laub formulations demonstrate a buildup of electromagnetic energy in front of the moving magnetodielectric slab, causing a material attraction or pulling force towards the incident light. This indicates that the mechanical force supplied to keep the material moving at a constant velocity must be balanced to sustain the system by the relation $\langle \bar{F}_{\text{mech}} \rangle \cdot \bar{v} = -\langle P_e \rangle$. By conservation theorems, the electromagnetic force and mechanical force must have equal and opposite contributions, allowing for the material to move at a constant velocity. The relations presented by Eqs. (38a) and (39a) demonstrate a pulling force but at different rates than what is prescribed by the mechanical force expression. This indicates that the Einstein-Laub and Abraham formulations predict incorrect force and/or power distributions for moving systems. In contrast, the Chu formulations retain the interpretation as presented in the stationary frame, where the free-space and transmitted contributions cancel, rendering invariant results in the moving frame.

In 1918, Dallenbach wrote Einstein about inherent problems within the Einstein-Laub force-density expression. After several exchanges, Einstein wrote in response to Dallenbach [35], "It has long been known that the values I had derived with Laub at the time are wrong; Abraham, in particular, was the one who presented this in a thorough paper. The correct strain tensor has incidentally already been pointed out by Minkowski." It has since been shown that additional torque terms $\bar{P} \times \bar{E} + \mu_0 \bar{M} \times \bar{H}$ must supplement the usual $\bar{r} \times \bar{f}$ form of the torque density in the Einstein-Laub formulation. Such an augmentation can be criticized on the basis that mathematical manipulation requires an equal augmentation to the interaction term to preserve the angular momentum continuity equation. Of course, this changes the division and interpretation of the subsystems. The same argument applies to the so-called hidden-momentum term which is often applied to the Amperian momentum to achieve the kinetic-field momentum [24]. In light of this, recent research [19] has shown a relation between the canonical and kinetic momenta. Thus, energy and momentum conservation demonstrates that the sum of the field and material contributions of the kinetic subsystem is equal to the field and material contributions of the canonical subsystem. This indicates that partitioning the total electrodynamic system leads to a material and kinetic subsystem; the kinetic subsystem demonstrates field and material responses and is consistent with global energy and momentum conservation.

VII. CONCLUSION

In conclusion, we have studied the Einstein-Laub and Chu formulations by use of the relativistic principle of virtual power, relativistic invariance, mathematical modeling, and Lagrangian methods. Within each analysis, both formulations are compared in determining which electromagnetic formulation is tied to the kinetic-momentum density and the shared-energy relations. The outcome of each analysis demonstrated inconsistencies within the Einstein-Laub formulation, which were revealed when transforming between inertial reference frames. Conversely, the Chu formulation presented invariant forms for RPVP and SEM transforms, as well as in mathematical modeling. In deriving field kinetic values independent of formulation-specific field definitions, Lagrangian methods were employed in which field values were replaced by prescribed scalar and vector potentials. Defining the Lagrangian density in energy and coenergy expressions [21], along with including any theoretically possible interaction values, the Lagrangian analysis yielded the field kinetic (E,H) form of Maxwell's equations. To validate the electrodynamics in all inertial reference frames, the field kinetic Maxwell's equations were recast into Minkowski space and transformed using a generalized Lorentz transformation. The energy-momentum tensor of the field kinetic subsystem was shown to be Lorentz invariant when deriving the kinetic-momentum density and shared-energy relations, where the transformed SEM tensor is consistent with the one derived by Chu [21,36]. Thus, the Chu formulation is relativistically invariant in a medium with a local velocity field $\bar{v}$ given that the charge and current densities describing material response are described by Eqs. (5). This was demonstrated in Sec. IV, where the Chu formulation, unlike the Einstein-Laub and Abraham formulations, demonstrated consistent and invariant results for the electromagnetic force and power within the moving system. The analysis presented herein demonstrates that the Chu formulation is the correct physical interpretation tied to Eqs. (3), thereby disproving the Einstein-Laub formulation of electromagnetics. Additionally, we note that the derivations, as presented in Secs. III A and V, made no *a priori* assumption for the force density, momentum density, or stress tensor, thereby strengthening our conclusions.

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APPENDIX A: STATIONARY FIELDS AND COEFFICIENTS

Consider an electromagnetic wave normally incident from vacuum onto a stationary ($v = 0$), linear, lossless, nondispersive magnetodielectric half-space, as seen in Fig. 1. The incident fields are rendered as

$$\bar{E}_i = \hat{z} E_0 e^{-ik_1 x}, \tag{A1a}$$

$$\bar{H}_i = \hat{y} \frac{E_0}{c\mu_0} e^{-ik_1 x}, \tag{A1b}$$

the reflected fields are rendered as

$$\bar{E}_r = \hat{z} E_0 R e^{ik_1}, \tag{A2a}$$

$$\bar{H}_r = -\hat{y} \frac{E_0}{c\mu_0} R e^{ik_1 x}, \tag{A2b}$$

and the transmitted fields are rendered as

$$\bar{E}_t = \hat{z} E_0 T e^{-ik_2 x}, \tag{A3a}$$

$$\bar{H}_t = \hat{y} \frac{E_0}{c\mu_0} \frac{n}{\mu_r} T e^{-ik_2 x}, \tag{A3b}$$

where the wave vectors and wave numbers for the respective regions are rendered as

$$k_1^2 = \omega^2 \epsilon_0 \mu_0,$$
$$k_2^2 = \omega^2 \epsilon \mu,$$
$$\bar{k}_i = -\hat{x} k_1,$$
$$\bar{k}_r = \hat{x} k_1,$$
$$\bar{k}_t = -\hat{x} k_2.$$

Application of the tangential boundary conditions, $\bar{E}_1 - \bar{E}_2 = 0$ and $\bar{H}_1 - \bar{H}_2 = \bar{J} = 0$, reveals the expressions for the reflection and transmission coefficients such that

$$R = \frac{\mu_r - n}{\mu_r + n}, \tag{A4a}$$

$$T = \frac{2\mu_r}{\mu_r + n}. \tag{A4b}$$

APPENDIX B: MOVING FIELDS AND COEFFICIENTS

Here, we present the field values for the transformed Minkowski and Chu formulations. Both field transformations take on bianisotropic material parameters, where the constitutive relations have been transformed from the moving frame to the laboratory frame such that [22,23]

$$\bar{\bar{C}} = \bar{\bar{L}}_6^{-1} \cdot \bar{\bar{C}}' \cdot \bar{\bar{L}}_6, \tag{B1}$$

where

$$\bar{\bar{C}}' = \begin{bmatrix} \bar{\bar{P}}' & \bar{\bar{L}}' \\ \bar{\bar{M}}' & \bar{\bar{Q}}' \end{bmatrix}, \tag{B2}$$

with $\bar{\bar{P}}' = c\epsilon'\bar{\bar{I}}$, $\bar{\bar{L}}' = \bar{\bar{M}}' = \bar{\bar{0}}$, $\bar{\bar{Q}} = \frac{\bar{\bar{I}}}{c\mu'}$, along with $\bar{\bar{L}}_6$ and $\bar{\bar{L}}_6^{-1}$ being the standard and inverse Lorentz transformation matrices [23]. Using constitutive relation transformations and the kDB system, the wave-vector relations along the $\hat{x}$ direction are [22,23]

$$\bar{k}^+ = \hat{x}\frac{n+\beta}{1+n\beta}\frac{\omega}{c}, \tag{B3a}$$

$$\bar{k}^- = -\hat{x}\frac{n-\beta}{1-n\beta}\frac{\omega}{c}, \tag{B3b}$$

where superscripts +,− denote the wave-vector solutions in the positive and negative directions within the material.

1. Minkowski representation

Consider a plane wave normally incident on a moving magnetodielectric half-space, as seen in Fig 1. The incident Minkowski fields in the laboratory frame are

$$\bar{E}_i = \hat{z}E_0 e^{-i(k_i x+\omega_i t)}, \tag{B4a}$$

$$\bar{H}_i = \hat{y}\frac{E_0}{c\mu_0}e^{-i(k_i x+\omega_i t)}, \tag{B4b}$$

where the incident wave vector is $\bar{k}_i = -\hat{x}\frac{\omega_i}{c}$. The reflected Minkowski fields in the laboratory frame are

$$\bar{E}_r = \hat{z}E_0 R e^{i(k_r x-\omega_r t)}, \tag{B5a}$$

$$\bar{H}_r = -\hat{y}\frac{E_0}{c\mu_0} R e^{i(k_r x-\omega_r t)}, \tag{B5b}$$

where the reflected wave vector is $\bar{k}_r = \hat{x}\frac{\omega_r}{c}$. The transmitted Minkowski fields observed from the laboratory frame are

$$\bar{E}_t = \hat{z}E_0 e^{-i(k_t x+\omega_i t)}, \tag{B6a}$$

$$\bar{B}_t = \hat{y}\frac{E_0}{c}\frac{n-\beta}{1-n\beta}T e^{-i(k_t x+\omega_i t)}, \tag{B6b}$$

$$\bar{D}_t = \hat{z}\frac{E_0}{c^2\mu_0}\frac{n(n-\beta)}{\mu_r'(1-n\beta)}T e^{-i(k_t x+\omega_i t)}, \tag{B6c}$$

$$\bar{H}_t = \hat{y}\frac{E_0}{c\mu_0}\frac{n}{\mu_r'}T e^{-i(k_t x+\omega_i t)}, \tag{B6d}$$

where the transmitted wave vector within the moving material is $\bar{k}_t = -\hat{x}n_t\frac{\omega_t}{c}$ and $n_t = \frac{(n-\beta)}{(1-n\beta)}$.

Employing moving tangential boundary conditions $\bar{E} + \bar{v}\times\bar{B} = 0$ and $\bar{H} - \bar{v}\times\bar{D} = \bar{J} = 0$, we derive the expressions

$$(1+\beta) + R(1-\beta) = T\left(\frac{1-\beta^2}{1-n\beta}\right),$$

$$(1+\beta) - R(1-\beta) = T\frac{n}{\mu_r'}\left(\frac{1-\beta^2}{1-n\beta}\right),$$

which, after manipulation, result in the reflection and transmission coefficients

$$R = \frac{(\mu_r'-n)}{(\mu_r'+n)}\frac{(1+\beta)}{(1-\beta)}, \tag{B7a}$$

$$T = \frac{2\mu_r'}{(\mu_r'+n)}\frac{(1-n\beta)}{(1-\beta)}. \tag{B7b}$$

2. Chu representation

Here, we transform the Minkowski fields presented to the Chu fields by the transformations [21]

$$\bar{E}_C = \bar{E}_M + \frac{\bar{v}\times\{[\bar{E}_M - (\frac{\bar{D}_M}{\epsilon_0})]\times\bar{v}\}}{c^2(1-\beta^2)} + \frac{\bar{v}\times(\bar{B}_M - \mu_0\bar{H}_M)}{(1-\beta^2)}, \tag{B8a}$$

$$\bar{H}_C = \bar{H}_M + \frac{\bar{v}\times\{[\bar{H}_M - (\frac{\bar{B}_M}{\mu_0})]\times\bar{v}\}}{c^2(1-\beta^2)} - \frac{\bar{v}\times(\bar{D}_M - \epsilon_0\bar{E}_M)}{(1-\beta^2)}, \tag{B8b}$$

$$\bar{P}_C = \bar{D}_M - \epsilon_0\bar{E}_M + \frac{\bar{v}\times\{(\bar{D}_M - \epsilon_0\bar{E}_M)\times\bar{v}\}}{c^2(1-\beta^2)} - \frac{\epsilon_0\bar{v}\times(\bar{B}_M - \mu_0\bar{H}_M)}{(1-\beta^2)}, \tag{B8c}$$

$$\mu_0\bar{M}_C = \bar{B}_M - \mu_0\bar{H}_M + \frac{\bar{v}\times\{(\bar{B}_M - \mu_0\bar{H}_M)\times\bar{v}\}}{c^2(1-\beta^2)} - \frac{\mu_0\bar{v}\times(\bar{D}_M - \epsilon_0\bar{E}_M)}{(1-\beta^2)}. \tag{B8d}$$

In vacuum, the field representations for the Chu and Minkowski fields are equivalent and are demonstrated in Eqs. (B4) and (B5). The transmitted Chu fields are

$$\bar{E}_{C_t} = \hat{z}\frac{\mu_r'-n\beta}{\mu_r'(1-n\beta)}E_0 T e^{-i(k_t x+\omega_t t)}, \tag{B9a}$$

$$\bar{H}_{C_t} = -\hat{y}\frac{n-\mu_r'\beta}{\mu_r'(1-n\beta)}\frac{E_0}{c\mu_0}T e^{-i(k_t x+\omega_t t)}, \tag{B9b}$$

$$\bar{P}_{C_t} = \hat{z}\frac{n^2-\mu_r'^2}{\mu_r'(1-n\beta)}\frac{E_0}{c^2\mu_0}T e^{-i(k_t x+\omega_t t)}, \tag{B9c}$$

$$\mu_0\bar{M}_{C_t} = -\hat{y}\frac{n(\mu_r'-1)}{\mu_r(1-n\beta)}\frac{E_0}{c}T e^{-i(k_t x+\omega_t t)}. \tag{B9d}$$

In determining the reflection and transmission coefficients, application of the Chu fields and boundary conditions results in a relation identical to that previously defined in Eqs. (B7).

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