Light propagation in local and linear media: Fresnel-Kummer wave surfaces with 16 singular points

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It is known that the Fresnel wave surfaces of transparent biaxial media have four singular points, located on two special directions. We show that, in more general media, the number of singularities can exceed 4. In fact, a highly symmetric linear material is proposed whose Fresnel surface exhibits 16 singular points. Because for every linear material the dispersion equation is quartic, we conclude that 16 is the maximum number of isolated singularities. The identity of Fresnel and Kummer surfaces, which holds true for media with a certain symmetry (zero skewon piece), provides an elegant interpretation of the results. We describe a metamaterial realization for our linear medium with 16 singular points. It is found that an appropriate combination of metal bars, split-ring resonators, and magnetized particles can generate the correct permittivity, permeability, and magnetoelectric moduli. Lastly, we discuss the arrangement of the singularities in terms of Kummer's 16_6 configuration of points and planes. An investigation parallel to ours, but in linear elasticity, is suggested for future research.

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I. INTRODUCTION

In the geometrical optics approximation, the Fresnel (wave) surface describes the propagation of light inside a transparent material. For biaxial media, such as aragonite [orthorhombic $CaCO_3$] and topaz [orthorhombic $Al_2SiO_4(F,OH)_2$], the surface consists of two shells that intersect at four locations [1]. Indeed, the property that the four singular points are aligned on two axes lends the name to this class of media. The singularities in the Fresnel surface of a biaxial crystal give rise to the phenomenon of internal conical refraction [2]. More precisely, a narrow light beam (in vacuum) striking a plate of crystal along the optical axis is refracted, upon incidence, into a cone and, upon exit, into a hollow cylinder. The geometry of wave propagation in biaxial media is a classical topic of research, first investigated by Fresnel [3] in 1821. As an aside, this explains why the wave surface is also termed Fresnel surface. Conical refraction in biaxial media was predicted as early as 1832 by Hamilton [4] on the basis of Fresnel's work. It was experimentally demonstrated just a year later by Lloyd, using a crystal of aragonite [5].

The rise of metamaterial technology [6] prompts new questions in the traditional analysis of Fresnel surfaces. By designing suitable artificial materials, it is possible to control a wide range of electromagnetic medium parameters. As a matter of fact, one can tune with remarkable accuracy not only the permittivity, but also the permeability and the magnetoelectric response. For comparison, it is worthwhile to recall that the familiar biaxial crystals have anisotropic permittivity but are not magnetic or magnetoelectric. Typically, more complicated media give rise to more elaborate Fresnel surfaces, whose properties are still largely unexplored. Our work intends to answer general questions with respect to the number and arrangement of the singularities in a Fresnel surface. Most notably, we put forward a highly symmetric medium that exhibits 16 singular points and discuss a metamaterial realization for it (Sec. II B and Sec. IV).

Two additional remarks are as follows: (i) Our analysis is restricted to isolated singular points (more correctly termed 'ordinary double points' [7]). This limitation is important, since, for example, all points on the Fresnel surface of an isotropic medium are singular. (ii) The papers [8–10] examine conical refraction in bianisotropic media and are complementary to our work. In particular, they establish that, for media with absorption, the singularities of the Fresnel surface are connected to a wealth of physical effects.

Biaxial medium

A good point of departure is the simple biaxial medium with electromagnetic response:

$$D^a = \varepsilon^{ab} E_b, \quad \varepsilon^{ab} = \mathrm{diag}(3,4,6),$$
$$H_a = \mu^{-1}_{ab} B^b, \quad \mu^{-1}_{ab} = \mathrm{diag}(1,1,1). \tag{1}$$

Here, ε^{ab} and μ^{-1}_{ab} are the permittivity and the inverse permeability. Throughout this article, Heaviside-Lorentz units are used, and the vacuum speed of light is normalized to 1, whereby $\varepsilon_0 = \mu_0 = c = 1$. Moreover, Latin indices range from 1 to 3, and Einstein's summation convention is employed. The medium (1) gives rise to the dispersion equation, cf. [11, (D.2.44)],

$$\begin{aligned} f(\omega,k_a) = {} & -72\omega^4 - 3k_1^4 - 4k_2^4 - 6k_3^4 \\ & + 30\omega^2 k_1^2 + 36\omega^2 k_2^2 + 42\omega^2 k_3^2 \\ & - 10k_2^2 k_3^2 - 9k_3^2 k_1^2 - 7k_1^2 k_2^2 = 0, \end{aligned} \tag{2}$$

with ω being the angular frequency and $k_a = (k_1,k_2,k_3)$ the spatial wave covector. Figure 1 illustrates the Fresnel surface

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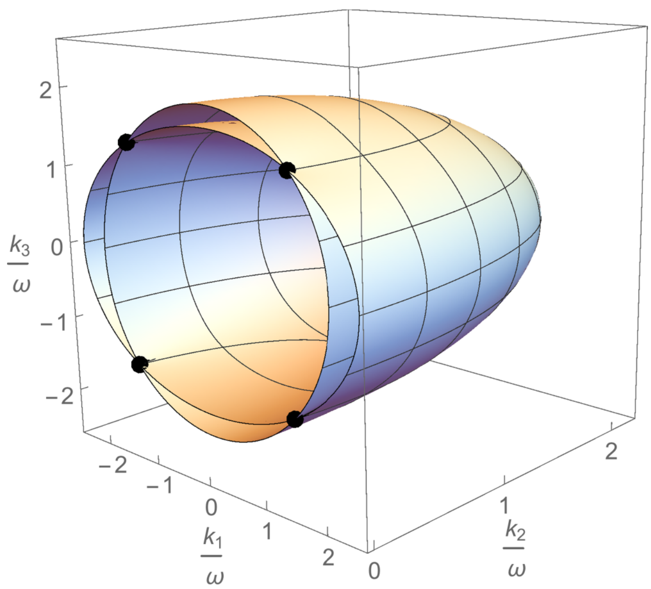

FIG. 1. Cross section of the Fresnel surface for the biaxial medium (1). The remaining half of the surface is obtained via symmetry.

corresponding to (2). This surface determines the inverse phase velocity k_a/ω of an electromagnetic wave traveling in a given direction. Since the medium is birefringent, there are typically two possible inverse phase velocities for each direction.

The solutions of the four simultaneous equations

$$\frac{\partial f(\omega,k_a)}{\partial \omega} = 0, \quad \frac{\partial f(\omega,k_b)}{\partial k_a} = 0, \tag{3}$$

are the singular points of the Fresnel surface [12, Sec. 99]. In a renowned monograph on Kummer surfaces, Hudson explains this concept as follows [13, Sec. 8]: an isolated singular point is "characterised geometrically by the fact that the tangent lines at it generate a quadric cone instead of a plane, and algebraically by the absence of terms of the first degree from the equation in point coordinates when the node [the singularity] is taken as origin."

We remark that the solutions of (3) can display $\omega = 0$, and therefore infinite k_a/ω. Hence, representing singular points through the inverse phase velocity is, at times, impractical. In this article, singular points are specified via the four-dimensional wave covector $q_\alpha = (-\omega,k_a)$, where $\alpha = 0,\ldots,3$. This representation does not suffer from the problem just mentioned. It is also worthwhile to observe that multiplying $q_\alpha = (-\omega,k_a)$ by a constant leaves the Fresnel surface unchanged. Thus, if a negative ω occurs, one is free to multiply the four-dimensional wave covector by -1 and retrieve a positive angular frequency. Because it is possible to take $\omega \geqslant 0$ systematically, one can introduce a sign convention and request that the angular frequency must be positive. We do not employ this convention, so that ω is a generic real number (positive, negative, or zero). Retaining the sign flexibility allows us to highlight a pattern in the singular points of Sec. II A, which are interlinked by cyclic permutations of the four-dimensional components.

As expected, the locations on the Fresnel surface in Fig. 1, where the two shells meet, are singular points. Indeed, they correspond to solutions of (3) with $f(\omega,k_a)$ being the function in (2). The locations, as represented through q_α, are

$$(1,+\sqrt{2},0,+\sqrt{2}), \quad (1,-\sqrt{2},0,-\sqrt{2}),$$
$$(1,+\sqrt{2},0,-\sqrt{2}), \quad (1,-\sqrt{2},0,+\sqrt{2}),$$

whereby it is easy to check that the singular points form two optical axes. In turn, this verifies that the electromagnetic medium (1) is biaxial.

The Fresnel surface of the medium (1) has, in addition to the four real singular points listed above, 12 complex ones, such as

$$(1,0,\mathrm{i}\sqrt{3},\sqrt{6}).$$

More generally, the solutions of (3) do not need to be real. The physical interpretation of complex singular points is quite unclear—they seem to describe waves that are partly evanescent and whose exponential decay is independent of polarization. A better understanding of complex singular points may be acquired by revisiting the boundary problem of a biaxial medium interfaced with vacuum. This idea is left as the subject of future work.

The discussion so far motivates a question: Can the Fresnel surface of a linear medium exhibit more than four real singular points?

II. SIXTEEN REAL SINGULAR POINTS

A. Four singular points at infinity

A medium is magnetoelectric if an applied electric field induces a nonzero magnetization and, similarly, an applied magnetic field induces a nonzero polarization. We consider the medium

$$\begin{aligned} D^a &= \varepsilon^{ab} E_b + \alpha^a{}_b B^b, \\ H_a &= \mu^{-1}_{ab} B^b - \alpha^b{}_a E_b, \end{aligned} \tag{4}$$

whose magnetoelectric tensor, permittivity, and permeability are

$$\begin{aligned} \alpha^a{}_b &= \mathrm{diag}\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right), \\ \varepsilon^{ab} &= \mathrm{diag}\left(1,1,-\frac{1}{2}\right), \\ \mu^{-1}_{ab} &= \mathrm{diag}\left(1,1,-\frac{1}{2}\right). \end{aligned} \tag{5}$$

The dispersion equation specified by (4) with (5), namely,

$$\begin{aligned} f(\omega,k_a) = -\tfrac{1}{2}\big(&\omega^4 + k_1^4 + k_2^4 + k_3^4 - \omega^2 k_1^2 - \omega^2 k_2^2 - \omega^2 k_3^2 \\ &- k_2^2 k_3^2 - k_3^2 k_1^2 - k_1^2 k_2^2\big) = 0, \end{aligned} \tag{6}$$

leads to a Fresnel surface with 16 real singular points. Moreover, (6) is the equation, in homogeneous coordinates, for a highly symmetric Kummer surface [14, p. 140]. The general relationship between Fresnel surfaces and Kummer surfaces is examined in Sec. III.

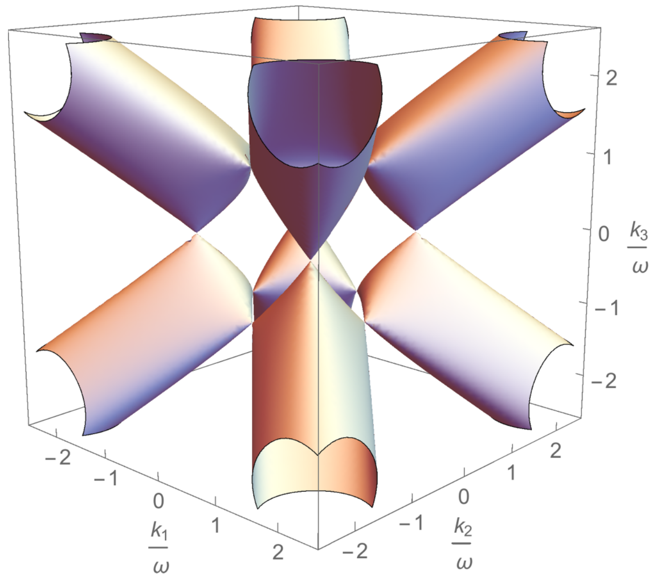

FIG. 2. The Fresnel surface of the magnetoelectric medium (4) with (5) has 12 singularities in the finite (and four at infinity).

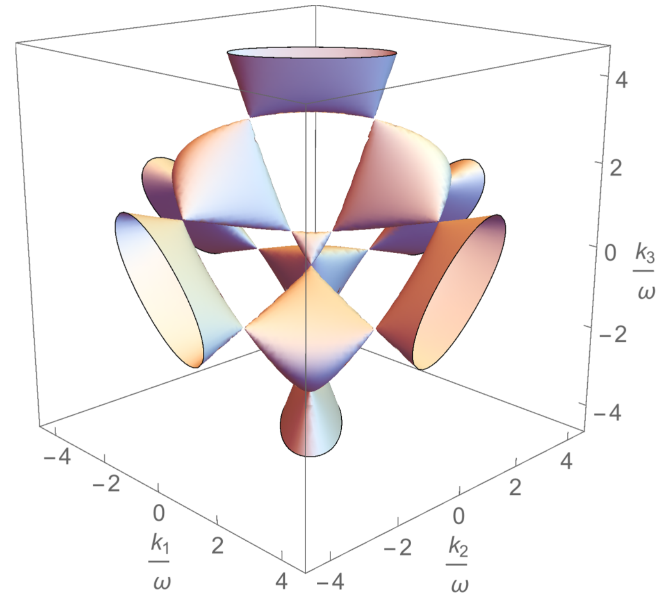

FIG. 3. The Fresnel surface given by the dispersion equation (9). Some readers will note that the plot illustrates a Kummer surface.

By importing $f(\omega,k_a)$ from (6) into (3), one arrives at

$$\begin{aligned}
\frac{\partial f}{\partial \omega} &= \omega\big(k_1^2 + k_2^2 + k_3^2 - 2\omega^2\big) = 0,\\
\frac{\partial f}{\partial k_1} &= k_1\big(\omega^2 - 2k_1^2 + k_2^2 + k_3^2\big) = 0,\\
\frac{\partial f}{\partial k_2} &= k_2\big(\omega^2 + k_1^2 - 2k_2^2 + k_3^2\big) = 0,\\
\frac{\partial f}{\partial k_3} &= k_3\big(\omega^2 + k_1^2 + k_2^2 - 2k_3^2\big) = 0.
\end{aligned} \tag{7}$$

The solutions to (7), that is, the 16 real singular points of the Fresnel surface for the medium (4) with (5) are as follows [14,15]:

1.	(0,1,1,1)	**9.**	(0,1, − 1,1)
2.	(1,0,1,1)	**10.**	(1,0,1, − 1)
3.	(1,1,0,1)	**11.**	(−1,1,0,1)
4.	(1,1,1,0)	**12.**	(1, − 1,1,0)
5.	(0, − 1,1,1)	**13.**	(0,1,1, − 1)
6.	(1,0, − 1,1)	**14.**	(−1,0,1,1)
7.	(1,1,0, − 1)	**15.**	(1, − 1,0,1)
8.	(−1,1,1,0)	**16.**	(1,1, − 1,0)

Figure 2 displays the Fresnel surface generated by (6). A careful inspection of the plot establishes that it includes only 12 singular points. This is because four singular points are located at infinity. In other words, the electromagnetic waves described by the covectors **1**,**5**,**9**, and **13** have $\omega = 0$. As a consequence, the inverse phase velocity k_a/ω for these is unbounded.

It is worthwhile to observe that the 12 singular points included in Fig. 2 lie on the edges of a cube, at the midpoints. The other four singularities are retrieved by extending the body diagonals of the cube to the plane at infinity. As explained in Sec. V below, this arrangement of 16 points has various remarkable properties.

The question arises: Can the Fresnel surface of a linear medium have 16 real singular points that are all finite?

B. No singular points at infinity

We consider the medium specified by (4) together with

$$\begin{aligned}
\alpha^a{}_b &= \tfrac{1}{4}\mathrm{diag}(3+\sqrt{3}, -3-\sqrt{3}, 0),\\
\varepsilon^{ab} &= \tfrac{1}{4}\mathrm{diag}(-1-\sqrt{3}, -1-\sqrt{3}, -4+2\sqrt{3}),\\
\mu^{-1}_{ab} &= \tfrac{1}{4}\mathrm{diag}(1+\sqrt{3}, 1+\sqrt{3}, 4-2\sqrt{3}),
\end{aligned} \tag{8}$$

and formulate the accompanying dispersion equation as

$$f(\omega,k_a) = -\tfrac{1}{2}(\psi^2 - 6abcd) = 0, \tag{9}$$

where ψ,a,b,c,d are polynomials in the angular frequency and the spatial wave covector,

$$\begin{aligned}
\psi(\omega,k_a) &= \tfrac{1}{2}\big(11\omega^2 - k_1^2 - k_2^2 - k_3^2\big),\\
a(\omega,k_a) &= \tfrac{1}{2}\big(-3\omega - k_1 - k_2 - k_3\big),\\
b(\omega,k_a) &= \tfrac{1}{2}\big(-3\omega - k_1 + k_2 + k_3\big),\\
c(\omega,k_a) &= \tfrac{1}{2}\big(-3\omega + k_1 - k_2 + k_3\big),\\
d(\omega,k_a) &= \tfrac{1}{2}\big(-3\omega + k_1 + k_2 - k_3\big).
\end{aligned}$$

The Fresnel surface of the medium determined by (4) and (8) has 16 real singular points that are all finite. Accordingly, the plot in Fig. 3 exhibits 16 singularities, which one can view by applying different rotations. The singular points, as represented via $q_\alpha = (-\omega,k_a)$, are as follows:

1.	$\frac{1}{2}(3,-1,-1,-1)$	**4.**	$\frac{1}{2}(3,1,1,-1)$
2.	$\frac{1}{2}(3,-1,1,1)$	**5.**	$\frac{1}{2}(1,-3,1,1)$
3.	$\frac{1}{2}(3,1,-1,1)$	**6.**	$\frac{1}{2}(1,1,-1,3)$

7.	$\frac{1}{2}(1,3,1,-1)$	**12.**	$\frac{1}{2}(1,-1,3,1)$
8.	$\frac{1}{2}(1,-1,-1,-3)$	**13.**	$\frac{1}{2}(1,1,1,-3)$
9.	$\frac{1}{2}(1,1,-3,1)$	**14.**	$\frac{1}{2}(1,-3,-1,-1)$
10.	$\frac{1}{2}(1,1,3,-1)$	**15.**	$\frac{1}{2}(1,-1,1,3)$
11.	$\frac{1}{2}(1,-1,-3,-1)$	**16.**	$\frac{1}{2}(1,3,-1,1)$

At these locations, the partial derivatives of the function $f(\omega,k_a)$ in (9) vanish, as is appropriate. Moreover, because **1–16** have $\omega \neq 0$, the singular points of the Fresnel surface are indeed all bounded; to put it differently, the inverse phase velocities of the electromagnetic waves, which correspond to the 16 singularities, are finite.

It is interesting to note that, although the surfaces in Figs. 1, 2, and 3 are dissimilar, the local geometry near each singular point is ultimately the same. Thus, in the geometrical optics limit, there is no fundamental difference between the 16 singularities above and the four singular points of a biaxial medium. The remark is consistent with algebraic geometry, which treats all isolated singularities on an equal footing.

Now, one may ask: Can the Fresnel surface of a linear medium exhibit more than 16 real singular points?

III. QUARTIC KUMMER SURFACES

The generic local and linear medium is

$$\begin{aligned} D^a &= \varepsilon^{ab} E_b + \alpha^a{}_b B^b, \\ H_a &= \mu^{-1}_{ab} B^b + \beta_a{}^b E_b, \end{aligned} \tag{10}$$

where $\{\varepsilon^{ab},\mu^{-1}_{ab},\alpha^a{}_b,\beta_a{}^b\}$ have real components and are independent of $\{\omega,k_a\}$, since the medium is nondispersive. No further conditions are imposed on these tensors. As a result, $\{\varepsilon^{ab},\mu^{-1}_{ab},\alpha^a{}_b,\beta_a{}^b\}$ contain $3\times 3 = 9$ parameters each, and the medium $4\times 9 = 36$ in total.

The dispersion equation for the generic local and linear response, as expressed in terms of three-dimensional quantities, is unwieldy. A more convenient formula is obtained with the help of four-dimensional electrodynamics. In this representation, the medium (10) becomes [11,16]

$$\mathcal{H}^{\alpha\beta} = \tfrac{1}{2}\chi^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{11}$$

where the excitation $\mathcal{H}^{\alpha\beta} = -\mathcal{H}^{\beta\alpha}$ and the field strength $F_{\alpha\beta} = -F_{\beta\alpha}$ summarize the fields (D^a,H_a) and (E_a,B^a), respectively. Greek indices are assumed to take the values $\{0,1,2,3\}$. Owing to the symmetries

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} = -\chi^{\alpha\beta\delta\gamma}, \tag{12}$$

the four-dimensional medium tensor has 36 independent components (see the remark above).

One can prove [11,17,18] that the dispersion equation of the generic local and linear medium is quartic in ω and k_a. As a matter of fact, in four dimensions, it takes the relativistic covariant form

$$\tilde{f}(q_\alpha) = \mathcal{G}^{\kappa\lambda\mu\nu}q_\kappa q_\lambda q_\mu q_\nu = 0, \tag{13}$$

with $q_\alpha = (-\omega,k_a)$ and $\tilde{f}(q_\alpha) = f(\omega,k_a)$. The Tamm-Rubilar tensor $\mathcal{G}^{\alpha\beta\gamma\delta}$ may be defined as [19]

$$\mathcal{G}^{\alpha\beta\gamma\delta} = \frac{1}{3!}\chi^{\kappa(\alpha\beta|\lambda}\,{}^\diamond\chi^\diamond{}_{\kappa\mu\lambda\nu}\,\chi^{\mu|\gamma\delta)\nu}. \tag{14}$$

In particular, the indices enclosed by round brackets are symmetrized. Moreover, ${}^\diamond\chi^\diamond{}_{\alpha\beta\gamma\delta} = \frac{1}{4}\epsilon_{\alpha\beta\kappa\lambda}\epsilon_{\gamma\delta\mu\nu}\chi^{\kappa\lambda\mu\nu}$, with $\epsilon_{\alpha\beta\gamma\delta}$ being the Levi-Civita symbol.

A surface that originates from a quartic homogeneous polynomial equation in four variables cannot have more than 16 isolated singular points [13]. The statement refers to the overall number of singular points—no distinction is made between those that are real and those that are not. In view of (13), one readily concludes that the Fresnel surface of a local and linear medium never exhibits more than 16 singularities.

By considering index symmetries, one can split the medium tensor into a principal part, a skewon part, and an axion part [11]:

$$\chi^{\alpha\beta\mu\nu} = {}^{(1)}\chi^{\alpha\beta\mu\nu} + {}^{(2)}\chi^{\alpha\beta\mu\nu} + {}^{(3)}\chi^{\alpha\beta\mu\nu}. \tag{15}$$

The decomposition is valid in all reference frames and is irreducible under the action of GL(4,$\mathbb{R}$). Moreover, the three elements contain, respectively, $20+15+1 = 36$ independent components. When

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta}, \tag{16}$$

so that ${}^{(2)}\chi^{\alpha\beta\mu\nu} = 0$, the medium is called skewon-free. Expressing (16) in three-dimensional form yields

$$\varepsilon^{ab} = \varepsilon^{ba}, \qquad \mu^{-1}_{ab} = \mu^{-1}_{ba}, \qquad \beta_a{}^b = -\alpha^b{}_a. \tag{17}$$

It is then simple to verify that the local linear media with the constitutive laws (1), (4) with (5), and (4) with (8) are all skewon-free. Furthermore, as the skewon part of (10) is arbitrary, this law includes two independent magnetoelectric tensors, $\alpha^a{}_b$ and $\beta_a{}^b$.

The Fresnel surfaces of skewon-free local and linear media are Kummer surfaces in the real projective space; conversely, every Kummer surface is the Fresnel surface of a local and linear medium with a vanishing skewon part [19,20]. We deduce that the Fresnel surface of a skewon-free medium has exactly 16 singular points, provided these are isolated. As a matter of fact, this property is known to be valid for the Kummer surfaces [14,21]. Notably, the surfaces discussed in earlier sections had 16 singularities (over the complex numbers), in agreement with the general theory.

If the electromagnetic response is linear but not local, $\{\varepsilon^{ab},\mu^{-1}_{ab},\alpha^a{}_b,\beta_a{}^b\}$ can be complex and depend on $\{\omega,k_a\}$. The impact of generalizing the medium parameters to be complex is limited. As the dispersion equation remains quartic, homogeneous, and polynomial in nature, the Fresnel surface exhibits no more than 16 singular points. By contrast, allowing $\{\varepsilon^{ab},\mu^{-1}_{ab},\alpha^a{}_b,\beta_a{}^b\}$ to depend on $\{\omega,k_a\}$, that is, allowing the material to be dispersive, makes general questions very difficult.

As an aside, a medium has a zero axion part [see (15)] if in three dimensions it satisfies $\alpha^a{}_a - \beta_b{}^b = 0$ (the indices a and b are summed over). One can easily check that the media (1), (4) with (5), and (4) with (8) are axion-free. Because the axion part ${}^{(3)}\chi^{\alpha\beta\gamma\delta}$ drops out from the Tamm-Rubilar tensor (14), and thus from the dispersion equation (13), it is irrelevant for the propagation of electromagnetic waves [11].

IV. METAMATERIAL REALIZATIONS

It appears likely that media with 12 or 16 real and finite singular points can be realized as metamaterials. More specifically, this section examines how (4) with (5), and (4) with (8) may be created as artificial materials.

The permittivity and permeability matrices in (5) and (8) have the structure $\varepsilon^{ab} = \mathrm{diag}(\varepsilon_\perp,\varepsilon_\perp,\varepsilon_\parallel)$ and $\mu^{ab} = \mathrm{diag}(\mu_\perp,\mu_\perp,\mu_\parallel)$, where the parameters $\{\varepsilon_\perp,\varepsilon_\parallel,\mu_\perp,\mu_\parallel\}$ are allowed to be negative. According to [22], permittivity and permeability matrices of this type can be achieved through a suitable arrangement of metal rods and split-ring resonators. As an aside, the metamaterial will reflect the property that ε^{ab} and μ^{ab} exhibit a preferred direction (they are uniaxial). Obtaining the numerical values for $\{\varepsilon_\perp,\varepsilon_\parallel,\mu_\perp,\mu_\parallel\}$, as given in (5) and (8), requires, almost certainly, a detailed optimization. More explicitly, the individual artificial particles must be tuned and their density adjusted. Because one takes advantage of resonant effects, the desired values of $\{\varepsilon_\perp,\varepsilon_\parallel,\mu_\perp,\mu_\parallel\}$ are expected to be achieved only in a narrow band of frequencies ω.

To realize magnetoelectric matrices of the type $\alpha^a{}_b = \mathrm{diag}(A, -A, 0)$, see (5) and (8), we suggest two alternative metamaterial designs. The first design relies on the properties of chromium sesquioxide (Cr_2O_3), a magnetoelectric crystal [23]. More in detail, a mixture of perpendicularly arranged Cr_2O_3 pieces can be used to generate a matrix $\alpha^a{}_b$ with the correct structure. The second design, which is practical at microwave frequencies, involves a suitable arrangement of magnetostatic wave resonators [24,25]. These are ferrite elements attached to metal wires and magnetized by a static external magnetic field. As in the case of the permittivity and the permeability, obtaining a specific numerical value for the magnetoelectric parameter A is likely to require accurate optimization. Moreover, one can perform a successful tuning across a narrow frequency band only.

A further warning is that interactions between the particles associated with different effects (electric, magnetic, and magnetoelectric) may not be negligible. These interactions generally produce unwanted bianisotropic terms in the medium law that are difficult to eliminate. Research on how to address the problem in magnetoelectric metamaterials is ongoing [26].

V. KUMMER'S 16_6 CONFIGURATION

As noted in Sec. II A, the singular points of the Fresnel surface generated by the medium (4) with (5) are very symmetric. In fact, it is possible to show [27,28] that the four-dimensional covectors $\boldsymbol{q} = (-\omega,\boldsymbol{k})$ representing these singular locations are mapped into each other by Dirac γ matrices:

$$\underline{\underline{\gamma}}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \underline{\underline{\gamma}}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{\underline{\gamma}}^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{\gamma}}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $\underline{\underline{\gamma}}^5 = \mathrm{i}\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^3$. Equivalently, one can start from the four-dimensional wave covector that describes a singular point and retrieve all other singularities by matrix multiplication with $\underline{\underline{\gamma}}^0,\ldots,\underline{\underline{\gamma}}^5$. For instance, if $\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{16}$ are the 16 singular locations listed in Sec. II A, we have that

$$\begin{aligned}
\boldsymbol{q}_1 &= (0,1,1,1), & \mathrm{i}\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^3\boldsymbol{q}_1 &= \boldsymbol{q}_2,\\
-\underline{\underline{\gamma}}^0\boldsymbol{q}_1 &= \boldsymbol{q}_5, & -\underline{\underline{\gamma}}^3\underline{\underline{\gamma}}^1\boldsymbol{q}_1 &= \boldsymbol{q}_{10},\\
\underline{\underline{\gamma}}^1\boldsymbol{q}_1 &= \boldsymbol{q}_{16}, & -\mathrm{i}\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^2\boldsymbol{q}_1 &= \boldsymbol{q}_9,\\
-\mathrm{i}\underline{\underline{\gamma}}^2\boldsymbol{q}_1 &= \boldsymbol{q}_8, & \underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^5\boldsymbol{q}_1 &= \boldsymbol{q}_7,\\
\underline{\underline{\gamma}}^3\boldsymbol{q}_1 &= \boldsymbol{q}_{15}, & -\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^5\boldsymbol{q}_1 &= \boldsymbol{q}_{14},\\
\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^1\boldsymbol{q}_1 &= \boldsymbol{q}_4, & \mathrm{i}\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^5\boldsymbol{q}_1 &= \boldsymbol{q}_6,\\
\mathrm{i}\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^2\boldsymbol{q}_1 &= \boldsymbol{q}_{12}, & -\underline{\underline{\gamma}}^3\underline{\underline{\gamma}}^5\boldsymbol{q}_1 &= \boldsymbol{q}_{13},\\
-\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^3\boldsymbol{q}_1 &= \boldsymbol{q}_{11}, & \underline{\underline{\gamma}}^5\boldsymbol{q}_1 &= \boldsymbol{q}_3.
\end{aligned} \tag{18}$$

Here, the extra factors of -1 and $\sqrt{-1}$ are immaterial, because the equations (7), which define the singular points, are invariant under scaling of $\boldsymbol{q} = (-\omega,\boldsymbol{k})$ by any nonzero constant, and the same holds true for the dispersion equation (6). The striking property (18) is a manifestation of the fact that the singularities give rise to a 16_6 configuration [14]. More explicitly, the singular points determine 16 planes, with each plane containing six points; in addition, the planes meet at the 16 singular points, with each point lying on six of the planes.

It is well known [13,15] that the singularities of a Kummer surface always identify a 16_6 configuration. Moreover, as explained in Sec. III, the Fresnel surfaces of skewon-free local and linear media are Kummer surfaces. We deduce that, for any medium in this wide class, isolated singular points give rise to a 16_6 configuration.

VI. CONCLUSION

We established that rather simple local and linear media can exhibit 16 real singular points. It was found that practical realizations of these media are within the current technical abilities. To achieve the required permittivity, permeability, and magnetoelectric moduli, metamaterial designs were put forward consisting of metal rods, split-ring resonators, and magnetized inclusions.

On the basis of the general dispersion equation, we discovered that the Fresnel surface of a local and linear medium cannot exhibit more than 16 singularities. It was in fact recognized that the Fresnel surfaces of media with no skewon part have exactly 16 singular points (assumed isolated). To reach this conclusion, we made use of an interesting link to the classical projective geometry of Kummer surfaces.

Finally, it was observed that, for the medium (4) with (5), the singular points are mapped into each other by the Dirac γ matrices. We related this property to the Kummer 16_6 configuration and described the generalization to all local and linear media with a vanishing skewon part.

An idea for future work is to conduct an investigation similar to the above in linear elasticity. For instance, it appears desirable to establish how many singularities can the wave surface of a linear anisotropic elastic medium have. Deriving a general answer to this question is likely to be more difficult than in electrodynamics. An upper bound on the number of singularities, however, can be readily attained. Because the dispersion equation for anisotropic elastic media is sextic [29], the resulting wave surface cannot display more than 65 singular points; this follows from algebraic geometry (see [30]).

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