Atom and quantum oscillator coupled by the vacuum field: Radiation pattern, emission spectrum, and decay dynamics

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We study the dynamics of a system consisting of a two-level atom and a quantum oscillator coupled by the vacuum radiation field. The oscillator and the atom are assumed to have the same resonance frequency. In contrast to the two-atom or two-oscillator case, the atom-oscillator system leads to an *unclosed* hierarchy of equations for the operators that determine the radiation pattern emitted by the atom-oscillator system. Instead of using the operator equations, we formulate the problem in terms of density-matrix equations which can be solved once the initial conditions are specified. As an example, we looked at the radiation pattern, the decay dynamics, and the spectrum for an initial condition in which the oscillator is in its n = 1 state and the atom is in its excited state. Dressed states (dressed by the static interaction between the atom and oscillator) prove to be especially useful for interpreting the results and comparing them with the two-atom and two-oscillator systems. All calculations are carried out using the resonance or rotating-wave approximation. The relationship of our work to damped Jaynes-Cummings models is noted.

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I. INTRODUCTION

A problem of fundamental interest in quantum optics is the coupling of two atoms or two oscillators via the transverse vacuum field. By its very nature, this coupling includes all effects related to retardation of the fields exchanged by the two atoms or oscillators. For separations of the atoms or oscillators that are less than a wavelength, one recovers interaction energies that can be obtained using an alternative theory in which retardation is neglected and the atoms or oscillators are assumed to interact by the longitudinal fields of their static charge distributions, producing effects such as the van der Waals interaction. With increasing separation of the atoms or oscillators, retardation begins to play a role and the calculations become somewhat more complicated, especially when either or both of the atoms or oscillators are prepared in their excited states [1].

For stationary atoms, the inclusion of retardation leads to differential equations for state amplitudes that show explicitly how the time rate of change of a given state amplitude depends on other state amplitudes evaluated at retarded times [2]. For example, the time rate of change of the state amplitude to have one atom (atom A) excited and the other (atom B) in its ground state at time t will depend on the amplitude to have atom B excited and atom A in its ground state evaluated at t - R/c, where R is the interatomic separation. However, in the limit that

$$\Gamma R/c \ll 1,\tag{1}$$

where Γ is an excited-state decay rate, the solution simplifies considerably since effects related to retardation can be neglected except insofar as they appear in phase factors.

Lehmberg [3] obtained equations from which the atomic or oscillator dynamics could be calculated for arbitrary initial conditions, under the assumption that Eq. (1) holds. He then went on to study the radiated field intensity for several different initial conditions, corresponding to excitation of each or both of the atoms or oscillators with the equivalent of $\pi/2$ or π pulses. In both the atomic and oscillator cases, it is possible to obtain time-evolution equations for the relevant Heisenberg operators that form a *closed* system; that is, one does not find an open-ended hierarchy of equations for the operators.

On the other hand, if we replace the two-atom or twooscillator system with one consisting of a two-level atom and a quantum oscillator, it appears that the equations for the operators are no longer closed for arbitrary initial conditions. As a consequence, the coupled atom-quantum-oscillator system can exhibit fundamental properties not found in atom-atom or oscillator-oscillator systems. For this reason alone, there is sufficient motivation to understand the atom-oscillator dynamics. From a somewhat more practical viewpoint, the coupling of an atom to a quantum oscillator is a topic of current research in the field of metamaterials [4]. There is considerable interest in coupling atoms or quantum dots to nanoantennas to increase the decay rate of the atoms [5]. If the antennas are modeled as point quantum oscillators, we have exactly the physical system we propose to study in this paper. Of course, real nanoantennas are not point dipoles, and there can be important effects arising from the excitation of surface plasmons; such effects are absent in our atom-oscillator system.

In Sec. II we introduce the basic formalism and obtain general expressions for the radiation pattern and the energy in the atom-oscillator system. A dressed atom-oscillator approach is introduced in Sec. III to allow one to easily view the relaxation processes and level splittings that occur and to compare them with the corresponding results for two oscillators or two atoms. Using this dressed-atom approach in a secular approximation, we obtain the spectra associated with the atom-oscillator decay. In the Appendix, we review the results for the two-atom and two-oscillator systems. All calculations are carried out using a rotating-wave approximation (RWA).

At first glance, it would seem that our calculation is closely linked to damped Jaynes-Cummings models. In these models a two-level atom is coupled to a single-mode cavity field whose energy levels are identical to those of an oscillator. When damping of the cavity field [6] or both the cavity field and atoms [7] are taken into account, the physical systems appear to be the same. It is true that the dressed states of the Jaynes-Cummings model and our atom-oscillator system are identical. On the other hand, there is no coupling of the atom and the oscillator via spontaneous emission in these damped Jaynes-Cummings models, while such coupling plays a critical role in our model. In addition, the cavity in the Jaynes-Cummings model is not a point-dipole radiator and does not contribute to the type of radiation patterns considered in this work. There is also a discussion of atom-oscillator coupling in the paper of Eberlein and Janowicz [8], but they are concerned mainly with level shifts without making the RWA approximation; moreover, their Green's-function approach differs considerably from ours.

II. RADIATION PATTERN AND ENERGY

The physical system under consideration consists of an atom located at $\mathbf{R}_1 = -(X_0/2)\mathbf{i}$ and an oscillator at $\mathbf{R}_2 = (X_0/2)\mathbf{i}$ (Fig. 1). At t = 0 the atom and the oscillator are prepared in some arbitrary initial state of the combined atom-oscillator system. It is assumed that the oscillator is constrained to move in the *z* direction and that the atom has a J = 0 angular momentum ground state and J = 1 angular momentum excited state. This allows one to treat the atom as a two-level atom provided its $m = \pm 1$ excited state sublevels are initially unpopulated. Physically, this type of initial condition ensures that the field emitted by the atom is *z* polarized at the position of the oscillator and the field emitted by the oscillator is *z* polarized at the position of the zero-point energy of the oscillator, the Hamiltonian for the system is

$$H_{ao} = \hbar\omega_0 \sigma_{ee} + \hbar\omega_0 b^{\dagger} b + \hbar \sum_{\mathbf{k},\lambda} \omega_k a^{\dagger}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \hbar \sum_{\mathbf{k},\lambda} (g_{\mathbf{k}} \sigma_+ a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}_1} + g^*_{\mathbf{k}} a^{\dagger}_{\mathbf{k}\lambda} \sigma_- e^{-i\mathbf{k}\cdot\mathbf{R}_1}) + \hbar \sum_{\mathbf{k},\lambda} (g'_{\mathbf{k}} b^{\dagger} a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}_2} + g'^*_{\mathbf{k}} a^{\dagger}_{\mathbf{k}\lambda} b e^{-i\mathbf{k}\cdot\mathbf{R}_2}), \quad (2)$$



FIG. 1. Geometry of the problem. The atom is located at position 1, and the oscillator is at position 2. The electronic motion of the atom and the motion of the oscillator are constrained to the *z* direction. The atom and the oscillator are separated by a distance X_0 .

where $\sigma_{ee} = |e\rangle \langle e|, |e\rangle$ and $|g\rangle$ are the eigenkets of the atom, b^{\dagger} and b are raising and lowering operators for the oscillator, σ_{+} and σ_{-} are raising and lowering operators for the atom, $a_{\mathbf{k}\lambda}$ is the annihilation operator for a photon having momentum **k** and polarization λ ,

$$g_{\mathbf{k}} = ie\sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}} z_{ge} \sin\theta_{\mathbf{k}},\tag{3a}$$

$$g'_{\mathbf{k}} = ie \sqrt{\frac{\omega_k}{2\hbar\epsilon_0 V}} \sqrt{\frac{\hbar}{2m\omega_0}} \sin\theta_{\mathbf{k}},$$
 (3b)

 $\omega_k = kc$, and θ_k is the angle between **k** and the *z* axis. The quantity z_{ge} is the matrix element (assumed to be real) of the operator *z* between the atomic states, and $\sqrt{\hbar/2m\omega_0}$ is the matrix element between the n = 0 and n = 1 states of the oscillator.

Since the Hamiltonian (2) is written in the RWA, it does not include contributions from any virtual transitions induced by the vacuum field. Such virtual transitions would lead to position-independent Lamb shifts, as well as van der Waals shifts of the atom-oscillator system that are proportional to X_0^{-6} . In order to neglect the shifts that are proportional to X_0^{-6} , we must assume that

$$\frac{\gamma_a}{\omega_0} \frac{1}{\xi^3}, \quad \frac{\gamma_o}{\omega_0} \frac{1}{\xi^3} \ll 1, \tag{4}$$

where

$$\gamma_a = \frac{\Gamma_a}{2} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2 z_{ge}^2 \omega_0^3}{\hbar c^3},$$
 (5a)

$$\gamma_o = \frac{\Gamma_o}{2} = \frac{1}{4\pi\epsilon_0} \frac{1}{3} \frac{e^2 \omega_0^2}{mc^3}$$
(5b)

are (half) decay rates associated with the atom and oscillator, $k_0 = \omega_0/c$, and

$$\xi = k_0 X_0. \tag{6}$$

Inequality (4) corresponds to the requirement that the shifts of the *resonantly* coupled levels of the atom and the oscillator are much less than the unperturbed level spacings of the atom and the oscillator.

The objective is to calculate the radiation pattern for some arbitrary initial condition of the system, as well as the system dynamics. Using a source-field expression [9], we can write the electric-field operator in the radiation zone as

$$\mathbf{E}_{+}(\mathbf{R},t) \approx \left(\frac{\omega_{0}^{2}e\sin\theta}{4\pi\epsilon_{0}c^{2}R}\right)\mathbf{u}_{\theta}\left[z_{ge}\sigma_{-}\left(t-\frac{|\mathbf{R}-\mathbf{R}_{1}|}{c}\right) + \sqrt{\frac{\hbar}{2m\omega_{0}}}b\left(t-\frac{|\mathbf{R}-\mathbf{R}_{2}|}{c}\right)\right],\tag{7}$$

where \mathbf{u}_{θ} is a unit vector in the direction of increasing θ . Assuming that, in the radiation zone,

$$\sigma_{-}\left(t - \frac{|\mathbf{R} - \mathbf{R}_{1}|}{c}\right) \approx \sigma_{-}\left(t - \frac{R}{c}\right)e^{-ik_{0}\mathbf{u}_{R}\cdot\mathbf{R}_{1}}, \quad (8a)$$

$$b\left(t - \frac{|\mathbf{R} - \mathbf{R}_2|}{c}\right) \approx b\left(t - \frac{R}{c}\right)e^{-ik_0\mathbf{u}_R\cdot\mathbf{R}_2},$$
 (8b)

where \mathbf{u}_R is a unit vector in the **R** direction, we find that the differential time-averaged (over a cycle of the field) power is

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}(\tau) \rangle \cdot \mathbf{u}_R = 2\gamma_a \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta \\ \times \left[\langle \sigma_+(\tau) \sigma_-(\tau) \rangle + \frac{\gamma_o}{\gamma_a} \langle b^{\dagger}(\tau) b(\tau) \rangle \right. \\ \left. + 2\sqrt{\frac{\gamma_o}{\gamma_a}} \operatorname{Re}(\langle \sigma_+(\tau) b(\tau) \rangle e^{-i\alpha}) \right], \qquad (9)$$

where $\langle S(\tau) \rangle$ is the time-averaged Poynting vector of the radiated field,

$$\alpha = \xi \sin \theta \cos \varphi \tag{10}$$

is the phase responsible for interference effects, and

$$\tau = t - \frac{R}{c} \tag{11}$$

is a retarded time.

We can get equations of motion for the Heisenberg operators appearing in Eq. (8) and for the field creation and annihilation operators using the Hamiltonian (2). It is then possible to formally solve for the field creation and annihilation operators, substitute the results in the equations for the atomic and oscillator operators, and take a quantum-mechanical average using the Weisskopf-Wigner approximation (and neglecting any position-independent Lamb shifts). In this manner [1], for quantities defined by

$$M(\tau) = \langle \sigma_{+}(\tau)\sigma_{-}(\tau)\rangle + \langle b^{\dagger}(\tau)b(\tau)\rangle = \langle \sigma_{ee}(\tau)\rangle + \langle n(\tau)\rangle,$$
(12a)

$$P(\tau) = \langle \sigma_{+}(\tau)b(\tau) \rangle + \langle b^{\dagger}(\tau)\sigma_{-}(\tau) \rangle, \qquad (12b)$$

$$D(\tau) = \langle \sigma_{+}(\tau)\sigma_{-}(\tau)\rangle - \langle b^{\dagger}(\tau)b(\tau)\rangle = \langle \sigma_{ee}(\tau)\rangle - \langle n(\tau)\rangle,$$

$$Y(\tau) = i(\langle b^{\dagger}(\tau)\sigma_{-}(\tau)\rangle - \langle \sigma_{+}(\tau)b(\tau)\rangle), \qquad (12d)$$

$$T(\tau) = \langle \sigma_{+}(\tau)\sigma_{-}(\tau)b^{\dagger}(\tau)b(\tau)\rangle = \langle \sigma_{ee}(\tau)n(\tau)\rangle, \qquad (12e)$$

with

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$$\sigma_{ee} = |e\rangle\langle e|, \quad n(\tau) = b^{\dagger}(\tau)b(\tau), \tag{13}$$

we can arrive at the following equations:

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$$\frac{dM}{d\tau} = -(\gamma_a + \gamma_o)M - (\gamma_a - \gamma_o)D - 2r\sqrt{\gamma_a\gamma_o}P, \quad (14a)$$

$$\frac{dP}{d\tau} = -(\gamma_a + \gamma_o)P - 2r\sqrt{\gamma_a\gamma_o}M + 4r\sqrt{\gamma_a\gamma_o}T, \quad (14b)$$

$$\frac{d\tau}{d\tau} = -2(\gamma_a + \gamma_o)T - (r - is)\sqrt{\gamma_a\gamma_o}\langle\sigma_+(\tau)n(\tau)b(\tau)\rangle$$

$$-(r+is)\sqrt{\gamma_a\gamma_o}\langle b^{\dagger}(\tau)n(\tau)\sigma_{-}(\tau)\rangle, \qquad (14c)$$

$$\frac{dD}{d\tau} = -(\gamma_a + \gamma_o)D - (\gamma_a - \gamma_o)M + 2s\sqrt{\gamma_a\gamma_o}Y, \quad (14d)$$

$$\frac{dY}{d\tau} = -(\gamma_a + \gamma_o)Y - 2s\sqrt{\gamma_a\gamma_o}D + 4s\sqrt{\gamma_a\gamma_o}T, \quad (14e)$$

where

$$r = \frac{3}{2} \left(\frac{\sin \xi}{\xi} + \frac{\cos \xi}{\xi^2} - \frac{\sin \xi}{\xi^3} \right), \tag{15a}$$

$$s = \frac{3}{2} \left(-\frac{\cos \xi}{\xi} + \frac{\sin \xi}{\xi^2} + \frac{\cos \xi}{\xi^3} \right).$$
 (15b)

The initial values at $\tau = 0$ are denoted by M_0 , P_0 , T_0 , D_0 , and Y_0 .

Unfortunately, Eqs. (14) are not closed, in contrast to the analogous equations for two atoms or two oscillators (see the Appendix). We still need differential equations for $\langle \sigma_+(\tau)n(\tau)b(\tau)\rangle$ and $\langle b^{\dagger}(\tau)n(\tau)\sigma_-(\tau)\rangle$. These new equations will bring in extra factors of $n(\tau)$, and the equations will never close [10].

In terms of the variables defined in Eqs. (12), the radiation pattern (power radiated per unit solid angle) is

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}(\tau) \rangle \cdot \mathbf{u}_R = 2\gamma_a \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta$$
$$\times \left[\frac{M(\tau) + D(\tau)}{2} + \frac{\gamma_o}{\gamma_a} \frac{M(\tau) - D(\tau)}{2} + \sqrt{\frac{\gamma_o}{\gamma_a}} P(\tau) \cos \alpha + \sqrt{\frac{\gamma_o}{\gamma_a}} Y(\tau) \sin \alpha\right].$$
(16)

The time-integrated radiation pattern is defined as

$$\frac{dW}{d\Omega} = \int_0^\infty d\tau \frac{d^2 W(\Omega, \tau)}{d\tau d\Omega}$$
(17)

and the spatially integrated radiated power is defined as

$$\frac{dW}{d\tau} = \int \frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} d\Omega$$

$$= 2\gamma_a \hbar \omega_0 \left[\frac{M(\tau) + D(\tau)}{2} + \frac{\gamma_o}{\gamma_a} \frac{M(\tau) - D(\tau)}{2} + \sqrt{\frac{\gamma_o}{\gamma_a}} r P(\tau) \right],$$
(18)

where, to arrive at this result, we used

$$\int d\Omega\left(\frac{3}{8\pi}\right)\sin^2\theta\cos\alpha = r,$$
(19a)

$$\int d\Omega\left(\frac{3}{8\pi}\right)\sin^2\theta\sin\alpha = 0.$$
(19b)

The total energy in the field at time t is given by

$$W(t) = \int_0^t d\tau \frac{dW}{d\tau}$$
(20)

and it follows from Eqs. (14a) and (18) that the sum of the energies of the atom, oscillator, and field is a constant of the motion, that is,

$$\hbar\omega_0 M(t) + W(t) = \hbar\omega_0 M_0. \tag{21}$$

Since Eqs. (14) are not closed, they cannot be used to calculate the values of $M(\tau)$, $P(\tau)$, and $Y(\tau)$ needed in

Eqs. (16)–(18). To get a solvable set of equations from which these quantities can be evaluated, we go over to density-matrix equations. In terms of density-matrix elements $\rho_{ao:a'o'}$, the variables $M(\tau)$, $P(\tau)$, $D(\tau)$, $Y(\tau)$, and $T(\tau)$ can be expressed as

$$M(\tau) = \langle \sigma_{ee}(\tau) \rangle + \langle n(\tau) \rangle$$

= $\sum_{n} \rho_{en;en}(\tau) + \sum_{j=g,e} \sum_{n} n \rho_{jn;jn}(\tau),$ (22a)
 $P(\tau) = 2 \operatorname{Re} \langle \sigma_{+}(\tau) b(\tau) \rangle$

$$= 2 \operatorname{Re} \sum_{n} \sqrt{n+1} \rho_{g,n+1;en}(\tau), \qquad (22b)$$

$$D(\tau) = \langle \sigma_{ee}(\tau) \rangle - \langle n(\tau) \rangle$$

= $\sum_{n} \rho_{en;en}(\tau) - \sum_{j=g,e} \sum_{n} n \rho_{jn;jn}(\tau),$ (22c)

$$Y(\tau) = 2 \operatorname{Im} \langle \sigma_+(\tau) b(\tau) \rangle = 2 \operatorname{Im} \sum_n \sqrt{n+1} \rho_{g,n+1;en}(\tau),$$

(22d)

$$T(\tau) = \langle \sigma_{ee}(\tau)n(\tau) \rangle = \sum_{n} n\rho_{en;en}(\tau), \qquad (22e)$$

where the g and e indices refer to atomic states and the n indices refer to oscillator states [11]. The time evolution of the density matrix equations can be obtained from equations for state amplitudes if "in terms" are added into the resulting densitymatrix equations. That is, in an interaction representation, the equations for the state amplitudes $c_{\alpha n}$ ($\alpha = g, e; n =$ (0, 1, 2, ...), obtained in a manner similar to the one employed in Ref. [12], are

$$\dot{c}_{g0} = 0, \qquad (23a)$$

$$\dot{c}_{en} = -(\gamma_a + n\gamma_o)c_{en} - (r + is)\sqrt{n+1}\sqrt{\gamma_a\gamma_o}c_{g,n+1}, \qquad (23b)$$

$$\dot{c}_{gn} = -(\gamma_a + n\gamma_o)c_{gn} - (r + is)\sqrt{n}\sqrt{\gamma_a\gamma_o}c_{e,n-1}, n > 0. \qquad (23c)$$

Special care must be given to the inclusion of the in terms resulting from spontaneous emission in getting the densitymatrix equations. Such in terms are calculated in the usual manner [9] by tracing over field states, that is,

$$\frac{d\rho_{\alpha n;\beta m}^{in}}{dt} = \sum_{\mathbf{k}} \frac{d\rho_{\alpha n\mathbf{k};\beta m\mathbf{k}}}{dt}.$$
(24)

In this manner, forming density-matrix elements $\dot{\rho}_{\alpha n;\beta m} =$ $\dot{c}_{\alpha n}c^*_{\beta m} + c_{\alpha n}\dot{c}^*_{\beta m}$ using Eqs. (23) and adding in the terms obtained using Eq. (24), we find

$$\frac{d\rho_{en;em}}{dt} = -[2\gamma_a + (n+m)\gamma_o]\rho_{en;em}$$
$$-\sqrt{n+1}(r+is)\sqrt{\gamma_a\gamma_o}\rho_{g,n+1;em}$$
$$-\sqrt{m+1}(r-is)\sqrt{\gamma_a\gamma_o}\rho_{en;g,m+1}$$
$$+ \frac{2\gamma_o\sqrt{n+1}\sqrt{m+1}\rho_{e,n+1;e,m+1}}, \qquad (25a)$$

$$\frac{d\rho_{gn;gm}}{dt} = -(n+m)\gamma_o\rho_{gn;gm} - \sqrt{n}(r+is)\sqrt{\gamma_a\gamma_o}\rho_{e,n-1;gm} -\sqrt{m}(r-is)\sqrt{\gamma_a\gamma_o}\rho_{gn;e,m-1} + \frac{2\gamma_a\rho_{en;em}}{2\gamma_o\sqrt{m+1}\sqrt{n+1}\rho_{g,n+1;g,m+1}} + \frac{2\gamma_o\sqrt{m+1}\sqrt{n+1}\rho_{en;g,m+1} + \sqrt{n+1}\rho_{g,n+1;em}]}{(25b)},$$

$$\frac{d\rho_{gn;em}}{dt} = -[\gamma_a + (n+m)\gamma_o]\rho_{gn;em}
- \sqrt{n}(r+is)\sqrt{\gamma_a\gamma_o}\rho_{e,n-1;em}
- \sqrt{m+1}(r-is)\sqrt{\gamma_a\gamma_o}\rho_{gn;g,m+1}
+ 2\gamma_o\sqrt{m+1}\sqrt{n+1}\rho_{g,n+1;e,m+1}
+ \frac{2r\sqrt{\gamma_a\gamma_o}\sqrt{m+1}\rho_{en;e,m+1}},$$
(25c)

$$\rho_{en;gm} = \rho_{gm;en}^*. \tag{25d}$$

The in terms, underlined in Eqs. (25), give rise to interesting exchanges between the atom and the oscillator. For example, if the atom in its excited state and the oscillator in a superposition of two neighboring states, as indicated in Fig. 2, spontaneous emission can result in a "coherence swap," in which the atom acquires the dipole coherence of the oscillator, while the oscillator transitions into the lower of its two initially excited states. In other words, $d\rho_{g0:e0}/dt$ is driven by an in term of the form $2r\sqrt{\gamma_a\gamma_o}\rho_{e0;e1}$.

Equations (16), (22), and (25) can be used to calculate the radiation pattern and system dynamics for arbitrary initial conditions. As a specific example we solve for initial conditions in which the atom is inverted and the oscillator is in its n = 1 state, $M_0 = 2$, $T_0 = 1$, $D_0 = Y_0 = P_0 = 0$. We then compare the results with those for two-atom (both atoms excited initially) and two-oscillator (both oscillators initially in their n = 1 states) cases for a specific value of ξ . With the



FIG. 2. A relaxation process in which coherence is transferred from the oscillator to the atom.

initial conditions as given, the maximum n possible is 2, and we arrive at nine coupled equations for the variables,

$$x_1 = \rho_{e0;e0},$$
 (26a)

$$x_2 = \rho_{e1;e1},$$
 (26b)
 $x_3 = \rho_{a0;a0},$ (26c)

$$x_3 = \rho_{g0;g0},$$
 (26c)
 $x_4 = \rho_{g1;g1},$ (26d)

$$x_5 = \rho_{g2;g2},$$
 (26e)

$$x_6 = \rho_{g1;e0},$$
 (26f)

$$x_7 = \rho_{g2;e1},\tag{26g}$$

$$x_8 = \rho_{e0;e1},$$
 (26h)

$$r_0 = 0 + c_0 \qquad (26i)$$

$$x_9 = \rho_{e1;g2},$$
 (201)

with initial condition $x_2(0) = 1$ and all others equal to zero. In the limit that $\gamma_a = \gamma_o = \gamma$, the equations of motion are

$$\dot{x}_1 = -2\gamma x_1 + 2\gamma x_2 - \gamma (r+is)x_6 - \gamma (r-is)x_8,$$
 (27a)

$$\dot{x}_2 = -4\gamma x_2 - \sqrt{2\gamma(r+is)x_7} - \sqrt{2\gamma(r-is)x_9},$$
 (27b)

$$\dot{x}_3 = 2\gamma x_1 + 2\gamma x_4 + 2r\gamma (x_6 + x_8),$$
 (27c)

$$\dot{x}_4 = 2\gamma x_2 - 2\gamma x_4 + 4\gamma x_5 - \gamma (r - is)x_6$$
 (27d)

$$+2\sqrt{2r\gamma(x_7+x_9)}-\gamma(r+is)x_8,$$
 (27e)

$$\dot{x}_5 = -4\gamma x_5 - \sqrt{2}\gamma (r - is)x_7 - \sqrt{2}\gamma (r + is)x_9, \quad (27f)$$

$$\dot{x}_6 = -\gamma(r+is)x_1 + 2r\gamma x_2 - \gamma(r-is)x_4$$

$$-2\gamma x_6 + 2\sqrt{2\gamma x_7}, \qquad (27g)$$

$$\dot{x}_7 = -\sqrt{2\gamma(r+is)x_2 - 4\gamma x_7 - \sqrt{2\gamma(r-is)x_5}},$$
 (27h)

$$x_{8} = -\gamma(r - is)x_{1} + 2r\gamma x_{2} - \gamma(r + is)x_{4}$$

$$-2\gamma x_8 + 2\sqrt{2\gamma} x_9, \qquad (2/1)$$

$$\dot{x}_9 = -\sqrt{2\gamma(r-is)x_2} - \sqrt{2\gamma(r+is)x_5} - 4\gamma x_9.$$
 (27j)

These equations can be solved analytically for a given value of ξ .

In terms of these variables,

$$M = x_1 + 2x_2 + x_4 + 2x_5, \tag{28a}$$

$$P = x_6 + \sqrt{2}x_7 + x_8 + \sqrt{2}x_9, \tag{28b}$$

$$T = x_2, \tag{28c}$$

$$D = x_1 - x_4 - 2x_5, \tag{28d}$$

$$Y = i(x_6 - \sqrt{2}x_7 - x_8 + \sqrt{2}x_9).$$
(28e)

In Fig. 3, we plot the time-integrated radiation pattern given in Eq. (17) as a function of φ for $\theta = \pi/2$ and $\xi = 1.5$ [r + is = 0.601 + 0.626i] and compare it with that for two atoms and two quantum oscillators when the atoms are both inverted and the oscillators are both in their n = 1 states at t = 0 (see the Appendix). The curves are normalized to the corresponding result for two independent dipole emitters. The radiation pattern of the atom-oscillator system differs from both the two-atom case and the two-oscillator case. For two atoms, the time-integrated signal does not exhibit any interference phenomena, a result found by Lehmberg [3].





FIG. 3. Time-integrated radiation pattern for $\theta = \pi/2$ and $\xi = 1.5$. The solid blue line is for two atoms (there is no time-integrated interference), the long-dashed green curve is for two oscillators, and the short-dashed red curve is for the atom-oscillator system. The results are expressed in dimensionless units, normalized to the emission pattern of two independent dipole emitters. In this case, $W_0 = 2\hbar\omega_0 M_0$. The curves for two atoms and two oscillators are drawn using Eqs. (A9) and (A21) of the Appendix.

Interference effects are seen for the two-oscillator and atomoscillator systems. Other initial conditions and atom-oscillator separations can be explored using the numerical solutions of Eq. (25).

III. DRESSED STATES, DECAY DYNAMICS, AND SPECTRA

It is convenient to introduce dressed states to explain the decay dynamics and to calculate the spectrum of the radiation emitted by the atom-oscillator system. If we look at the amplitude equations (23) in the absence of decay, they take the form

$$\dot{c}_{g0} = 0, \tag{29a}$$

$$i\dot{c}_{e,n-1} = g_n c_{g,n},\tag{29b}$$

$$i\dot{c}_{gn} = g_n c_{e,n-1}, \quad n > 0,$$
 (29c)

where

$$g_n = \sqrt{n} \sqrt{\gamma_a \gamma_o} s. \tag{30}$$

These equations are those encountered in the Jaynes-Cummings model for a two-level atom interacting with a single-mode radiation field. Thus, in the absence of decay, the atom-oscillator problem in *free space* maps into the Jaynes-Cummings model for a two-level atom interacting with a single-mode *cavity* field. In contrast to the Jaynes-Cummings model, however, the coupling constant depends on the product of the decay rates of the atom and the oscillator; that is, spontaneous emission is responsible for the atom-oscillator coupling. Moreover, the decay terms appearing in Eqs. (23) have a structure similar to the coupling terms and must be included for consistency. Nevertheless, as in the Jaynes-Cummings model, we can use the coupled atom-oscillator dressed states as our basis states to investigate the nature of decay rates and the emission spectrum. This approach will be especially useful only in the so-called secular approximation, $|s| \gg r$, since the coherence between the dressed states in a

given manifold (en; g, n + 1) oscillates rapidly on a time scale $(\gamma_a \gamma_o)^{-1/2}$ and averages to zero. The condition $|s| \gg r$ will be satisfied if $\xi = \omega_0 R/c \ll 1$ since $r + is \sim 1 + 3i/(2\xi^3)$ in this limit [see Eqs. (15)]. If $|s| \leq r$, different components of the emission spectrum are not resolved, and there is little advantage to using the dressed basis.

In this section we set

$$\gamma_o = \gamma_a \equiv \gamma = \Gamma/2. \tag{31}$$

The dressed-state eigenkets and eigenenergies are given by

$$|0\rangle = |g0\rangle, \quad E_{g0} = 0, \tag{32a}$$

$$|n_{\pm}\rangle = \frac{|e,n-1\rangle \pm |gn\rangle}{\sqrt{2}}, \quad E_{n_{\pm}} = \hbar(\omega_0 \pm \sqrt{n\gamma}s).$$
(32b)

The density-matrix elements in the two bases are related by [9]

$$\rho_{n_{+}n_{+}} = \frac{1}{2}(\rho_{e,n-1;e,n-1} + \rho_{gn;gn} + \rho_{e,n-1;gn} + \rho_{gn;e,n-1}),$$
(33a)
$$\rho_{n_{-}n_{-}} = \frac{1}{2}(\rho_{e,n-1;e,n-1} + \rho_{gn;gn} - \rho_{e,n-1;gn} - \rho_{gn;e,n-1}),$$
(33b)

$$\rho_{n+n-} = \rho_{n-n+}^{-}$$

= $\frac{1}{2}(\rho_{e,n-1;gn} - \rho_{gn;e,n-1} + \rho_{e,n-1;e,n-1} - \rho_{gn;gn}).$
(33c)

It is now a relatively easy task to use Eqs. (25), Eqs. (33), and the inverse of Eqs. (33) to get equations for the time development of density-matrix elements in the dressed basis. In the limit $|s| \gg r \approx 1$, we can neglect the off-diagonal density-matrix elements in the dressed basis and obtain rate equations for the dressed-state populations given by

$$\begin{split} \dot{\rho}_{n+n_{+}} &= -\Gamma(n+\sqrt{n}r)\rho_{n+n_{+}} \\ &+ \frac{\Gamma}{2}[(\sqrt{1+n}+r)(\sqrt{1+n}+\sqrt{n})]\rho_{(n+1)_{+}(n+1)_{+}} \\ &+ \frac{\Gamma}{2}[(\sqrt{1+n}-r)(\sqrt{1+n}-\sqrt{n})]\rho_{(n+1)_{-}(n+1)_{-}}, \end{split}$$
(34a)

$$\begin{split} \dot{\rho}_{n_{-}n_{-}} &= -\Gamma(n-\sqrt{n}r)\rho_{n_{-}n_{-}} \\ &+ \frac{\Gamma}{2}[(\sqrt{1+n}+r)(\sqrt{1+n}-\sqrt{n})]\rho_{(n+1)_{+}(n+1)_{+}} \\ &+ \frac{\Gamma}{2}[(\sqrt{1+n}-r)(\sqrt{1+n}+\sqrt{n})]\rho_{(n+1)_{-}(n+1)_{-}}, \end{split}$$

$$\dot{\rho}_{00} = \Gamma(1+r)\rho_{1+1+} + \Gamma(1-r)\rho_{1-1-}, \qquad (34c)$$

where Eqs. (34a) and (34b) are valid for $n \ge 1$. If the maximum value of *n* is n_{max} , the problem in the secular approximation is reduced to solving a set of $(2n_{\text{max}} + 1)$ coupled, linear differential equations.

As a specific example, we use the dressed atom-oscillator approach to calculate the decay dynamics and the spectrum for initial conditions in which the atom is inverted and the oscillator is in its n = 1 state corresponding to $M_0 = 2$, $T_0 = 1$, $D_0 = Y_0 = P_0 = 0$. Moreover, we compare the results with the corresponding results for two atoms prepared in their excited states and two oscillators prepared in their n = 1 states. For this initial condition, Eqs. (34) reduce to

$$\dot{\rho}_{2_+2_+} = -\Gamma_{2_+}\rho_{2_+2_+},\tag{35a}$$

$$\dot{\rho}_{2_{-}2_{-}} = -\Gamma_{2_{-}}\rho_{2_{-}2_{-}},\tag{35b}$$

$$\dot{\rho}_{1_{+}1_{+}} = -\Gamma_{1_{+}}\rho_{1_{+}1_{+}} + \Gamma_{2_{+},1_{+}}\rho_{2_{+}2_{+}} + \Gamma_{2_{-},1_{+}}\rho_{2_{-}2_{-},} \quad (35c)$$

$$\dot{\rho}_{1-1-} = -\Gamma_{1-}\rho_{1+1+} + \Gamma_{2+,1-}\rho_{2+2+} + \Gamma_{2-,1-}\rho_{2-2-}, \quad (35d)$$

$$\dot{\rho}_{00} = \Gamma_{1+,0}\rho_{1+1+} + \Gamma_{1-,0}\rho_{1-1-}, \qquad (35e)$$

where the decay rates are given by

$$\Gamma_{2_+} = \Gamma(2 + \sqrt{2}r), \tag{36a}$$

$$\Gamma_{2_{-}} = \Gamma(2 - \sqrt{2}r), \tag{36b}$$

$$\Gamma_{1\pm} \equiv \Gamma_{\pm} = \Gamma(1\pm r), \qquad (36c)$$

and the repopulation rates are given by

$$\Gamma_{2_+,1_+} = \frac{\Gamma}{2}(\sqrt{2}+1)(\sqrt{2}+r), \qquad (37a)$$

$$\Gamma_{2_+,1_-} = \frac{\Gamma}{2}(\sqrt{2}-1)(\sqrt{2}+r),$$
 (37b)

$$\Gamma_{2_{-},1_{+}} = \frac{\Gamma}{2}(\sqrt{2} - 1)(\sqrt{2} - r), \qquad (37c)$$

$$\Gamma_{2_{-},1_{-}} = \frac{\Gamma}{2}(\sqrt{2}+1)(\sqrt{2}-r),$$
 (37d)

$$\Gamma_{1\pm,0} = \Gamma_{\pm}.\tag{37e}$$

The initial condition $\rho_{e1;e1} = 1$ in the "bare" basis translates into $\rho_{2+2_+} = \rho_{2-2_-} = \rho_{2+2_-} = \rho_{2-2_+} = 1/2$ in the dressed basis; the atom-oscillator system is prepared in a coherent superposition of its two excited dressed states; however, we are concerned here only with dressed-state populations. Equations (35) can be solved exactly, but the solution is somewhat unwieldy and is not given here.

A. Decay dynamics

In Fig. 4 we plot the energy remaining in the atom-oscillator system in dimensionless units, namely,

$$E(t) = \frac{M(t)}{M_0},\tag{38}$$

along with the corresponding curves for (a) two noninteracting atoms prepared in their excited states, (b) two interacting atoms prepared in their excited states (see the Appendix), and (c) two interacting quantum oscillators prepared in their n = 1 excited states (see the Appendix). The curves are drawn for $\xi = 0.2$ (r + is = 0.992 + 8.2i). For this value of ξ , the exact value of M(t) given by Eq. (28a) which is obtained by solving Eqs. (26) differs insignificantly from the value

$$E_d(t) = \rho_{2_+2_+}(t) + \rho_{2_-2_-}(t) + [\rho_{1_+1_+}(t) + \rho_{1_-1_-}(t)]/2 \quad (39)$$

obtained by solving the approximate dressed-state population equations (35). For two noninteracting atoms, E(t) decays



FIG. 4. Normalized dimensionless energy E(t) with $\xi = 0.2$ (r + is = 0.992 + 8.2i) for (a) two independent atoms, both initially excited, (b) two coupled atoms, both initially excited, (c) two coupled oscillators both initially in their n = 1 states, and (d) the coupled atom and oscillator, with the atom initially excited and the oscillator initially in its n = 1 state.

simply as $\exp(-2\gamma t)$. The decay is more rapid for the two interacting atoms, characteristic of superradiant decay. However, there are slowly decaying components for both the two-oscillator and the atom-oscillator systems.

These results can be explained qualitatively using the dressed-state diagrams shown in Fig. 5, drawn for $\xi \ll 1$. The solid arrows indicate allowed decay paths, while the dashed arrow indicate pathways that are suppressed owing to a cancellation of matrix elements when $\xi \ll 1$ ($|s| \gg r \approx 1$). For the noninteracting atoms, each excited atomic state popu-



FIG. 5. Decay channels for (a) two independent atoms, both initially excited, (b) two coupled atoms, both initially excited, (c) two coupled oscillators both initially in their n = 1 states, and (d) the coupled atom and oscillator, with the atom initially excited and the oscillator initially in its n = 1 state. The levels shown are the dressed states of the coupled systems. Dashed arrows indicate transitions that are suppressed for $r \sim 1$. The eigenkets $|\pm\rangle$ for two atoms and $|1_{\pm}\rangle^{o}, |2_{\pm}\rangle^{o}$ for two oscillators are given in the Appendix, while the repopulation rates $\Gamma_{i,j}$ are given by Eqs. (37). All curves are drawn for $\gamma_{o} = \gamma_{a} = \Gamma/2$.

lation decays exponentially with rate Γ [for two noninteracting oscillators, E(t) also decays exponentially with rate Γ]. For the two interacting atoms [Fig. 5(b)], the dominant decay channel is through the symmetric intermediate state. The decay rates in this channel are enhanced by a factor of 2 when $r \sim 1$, leading to the simplest example of superradiant decay. For the two interacting oscillators [Fig. 5(c)], the oscillators are prepared in an equal superposition of the dressed oscillator states shown in the figure by the large dots. One of these states is metastable when $r \sim 1$; half of the population is almost completely trapped in this state. From the exact solution for M(t) = 2E(t) given by Eq. (A19a) in the Appendix [which is identical to that calculated from the decay rates shown in Fig. 5(c)], one finds

$$E(t) = \frac{1}{2} [e^{-\Gamma_{+}t} + e^{-\Gamma_{-}t}].$$
(40)

The slowly decaying component seen in Fig. 4, curve (c), corresponds to the second term in Eq. (40). Finally, for the atom-oscillator system the initial state is a superposition of the n = 2 symmetric and antisymmetric dressed states shown in Fig. 5(d). Both of these states decay to the n = 1 symmetric and antisymmetric dressed states. The n = 1 antisymmetric state is metastable when $r \sim 1$, decaying slowly to the ground state with rate Γ_- . The total decay rate out of the n = 2 dressed states is 4Γ , and $(2 + r)\Gamma/4$ of the decay is via the symmetric intermediate state and $(2 - r)\Gamma/4$ is via the antisymmetric one. This explains the slow decay component having amplitude 1/4 and decay rate Γ_- seen in Fig. 4, curve (d).

Explicitly, in the limit that $r \approx 1$ but inequality (4) is still respected, we find

$$E(t) \sim e^{-\Gamma t}$$
 (noninteracting atoms), (41a)

$$E(t) \sim e^{-2\Gamma t} (1 + \Gamma t) \quad \text{(two atoms)}, \tag{41b}$$

$$E(t) \sim (1 + e^{-2\Gamma t})/2$$
 (two oscillators), (41c)

$$E(t) \sim \frac{1}{4} + \frac{e^{-2\Gamma t}}{2} + \frac{(1+\sqrt{2})}{8}e^{-\Gamma(2-\sqrt{2})t} + \frac{(1-\sqrt{2})}{8}e^{-\Gamma(2+\sqrt{2})t} \quad \text{(atom-oscillator)} \quad (41d)$$

for the initial conditions depicted in Fig. 5. In each case, $E(t) \sim (1 - \Gamma t)$ for $\Gamma t \ll 1$.

Figure 6 shows the decay dynamics if the oscillator is prepared initially in state n_0 ($n_0 = 0, 1, 2, 3, 4$) and the atom is prepared in its excited state, in the limit that $r \approx 1$. These curves exhibit two interesting features. First, the energy trapped in the system decreases with increasing n_0 , owing to the fact that decay via the symmetric states is favored over those into the antisymmetric states. As the population cascades downwards from the initially excited state via spontaneous emission, the population trapped in state $|1_-\rangle^o$ is $1/[2(n_0 + 1)]$ as *r* approaches unity. The various decay rates of the cascade emission are shown in Fig. 7. Second, although the decay rate increases with increasing n_0 , there is never a superradiant effect since the energy always decays more slowly than $e^{-\Gamma t}$ (an $e^{-\Gamma t}$ decay is indicated by the dashed black curve in Fig. 6).



FIG. 6. Normalized dimensionless energy E(t) with $r \sim 1$ for initial conditions in which the atom is excited and the oscillator is in the $n = n_0$ state. The solid curves from top to bottom correspond to $n_0 = 0, 1, 2, 3, 4$, while the dashed curve is the simple exponential decay, $e^{-\Gamma t}$.

B. Spectrum

In the secular approximation and for the initial conditions depicted in Fig. 5(d), the spectrum for the atom-oscillator system consists of four lines associated with transitions from the second excited manifold to the first excited manifold and two additional lines from the first excited manifold to the ground state. The width of each line is one half the sum of the widths of initial and final states of each transition,

$$\gamma_{ij} = (\Gamma_i + \Gamma_j)/2, \tag{42}$$

and the relative strength of each line is proportional to the branching ratio for the given decay channel. With the various decay rates Γ_i defined by Eqs. (36) and repopulation rates $\Gamma_{i,j}$



FIG. 7. Decay rates in the dressed-state basis for $\xi \ll 1$ ($r \sim 1$). In this limit, state $|1_{-}\rangle$ is metastable. As can be seen, decay via the symmetric channels is favored. The energy-level splitting $\Delta E_n = \hbar \sqrt{n} \Gamma s$ between the dressed states in a given manifold has been suppressed.

defined by Eqs. (37) and with $|s| \gg r$, we find the spectrum is given by a sum of six terms,

$$I(2_{+} \to 1_{+}) = \frac{S_{2_{+},1_{+}}}{\pi} \frac{\gamma_{2_{+}1_{+}}}{\left[\omega_{k} - \omega_{0} - \frac{\sqrt{2} - 1}{2}\Gamma s\right]^{2} + \gamma_{2_{+}1_{+}}^{2}},$$
(43a)

$$I(2_{+} \to 1_{-}) = \frac{S_{2_{+},1_{-}}}{\pi} \frac{\gamma_{2_{+}1_{-}}}{\left[\omega_{k} - \omega_{0} - \frac{\sqrt{2} + 1}{2}\Gamma s\right]^{2} + \gamma_{2_{+}1_{-}}^{2}},$$
(43b)

$$I(2_{-} \to 1_{+}) = \frac{S_{2_{-},1_{+}}}{\pi} \frac{\gamma_{2_{-}1_{+}}}{\left[\omega_{k} - \omega_{0} + \frac{\sqrt{2} + 1}{2}\Gamma s\right]^{2} + \gamma_{2_{-}1_{+}}^{2}},$$
(43c)

$$I(2_{-} \to 1_{-}) = \frac{S_{2_{-},1_{+}}}{\pi} \frac{\gamma_{2_{-}1_{-}}}{\left[\omega_{k} - \omega_{0} + \frac{\sqrt{2} - 1}{2}\Gamma s\right]^{2} + \gamma_{2_{-}1_{-}}^{2}},$$
(43d)

$$I(1_+ \to 0) = \frac{2+r}{4} \frac{1}{\pi} \frac{\gamma_{1+0}}{\left[\omega_k - \omega_0 - \frac{\Gamma_s}{2}\right]^2 + \gamma_{1+0}^2}, \quad (43e)$$

$$I(1_{-} \to 0) = \frac{2-r}{4} \frac{1}{\pi} \frac{\gamma_{1_{-}0}}{\left[\omega_k - \omega_0 + \frac{\Gamma_s}{2}\right]^2 + \gamma_{1_{-}0}^2}, \quad (43f)$$

where the branching ratios (relative to the total decay rate of the n = 2 manifold) are given by

$$S_{i,j} = \Gamma_{i,j}/4\Gamma, \tag{44}$$

with $\Gamma_{i,j}$ given by Eqs. (37). The corresponding dressed states and spectra for two atoms and two oscillators are given in the Appendix.

In all the cases considered, we have not included the nonresonant van der Waals shifts in the definition of the dressed states based on the assumption that inequality (4) is valid.

IV. SUMMARY

We have derived equations that characterize the dynamics of a system consisting of a two-level atom and a quantum oscillator coupled by the vacuum radiation field. In contrast to the two-atom or two-oscillator case, the atom-oscillator system leads to an *unclosed* hierarchy of equations for the operators that determine the radiation pattern emitted by the atomoscillator system. Instead of using the operator equations, we formulated the problem in terms of the density-matrix elements associated with the problem. These density-matrix equations can be solved once the initial conditions are specified. As an example, we looked at the radiation pattern, the decay dynamics, and the spectrum for an initial condition in which the oscillator is in its n = 1 state and the atom in its excited state. Dressed states (dressed by the static interaction between the atom and oscillator) proved to be especially useful for interpreting the results. Although we looked at the limit in which $\gamma_o = \gamma_a \equiv \gamma$, another interesting limit is $\gamma_o \gg \gamma_a$. This limit could correspond to a nanoantenna coupled to a quantum dot in which the antenna is the oscillator and the quantum dot is the "two-level atom." For $\gamma_o \gg \gamma_a$, the antenna results in an

enhanced radiation rate for the quantum dot. We have limited the discussion to spontaneous emission from the coupled atom-oscillator system, but the results can be generalized to allow for driving of either the atom or the oscillator (or both) by a cw or pulsed optical field.

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APPENDIX

In this Appendix, we summarize the results for the twoatom and two-oscillator cases.

1. Two atoms

The Hamiltonian is

$$H_{a} = \hbar \omega_{0} \sum_{j=1,2} \sigma_{ee}^{(j)} + \hbar \sum_{\mathbf{k},\lambda} \omega_{k} a_{\mathbf{k}\lambda}^{\dagger} a_{\mathbf{k}\lambda} + \hbar \sum_{\mathbf{k},\lambda} \sum_{j=1,2} (g_{\mathbf{k}} \sigma_{+}^{(j)} a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}_{j}} + g_{\mathbf{k}}^{*} a_{\mathbf{k}\lambda}^{\dagger} \sigma_{-}^{(j)} e^{-i\mathbf{k}\cdot\mathbf{R}_{j}}),$$
(A1)

and the differential time-averaged power is

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}_a \rangle \cdot \mathbf{u}_R$$

$$= 2\gamma_a \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta [\langle \sigma_+^{(1)}(\tau) \sigma_-^{(1)}(\tau) \rangle$$

$$+ \langle \sigma_+^{(2)}(\tau) \sigma_-^{(2)}(\tau) \rangle$$

$$+ 2 \operatorname{Re}(\langle \sigma_+^{(1)}(\tau) \sigma_-^{(2)}(\tau) \rangle e^{-i\alpha})]. \quad (A2)$$

In obtaining the Heisenberg equations of motion for the operators needed in this equation, we find that an additional operator, $\sigma_{+}^{(1)}(\tau)\sigma_{-}^{(1)}(\tau)\sigma_{+}^{(2)}(\tau)\sigma_{-}^{(2)}(\tau)$, is needed. It is convenient to define

$$M(\tau) = \langle \sigma_{+}^{(1)}(\tau)\sigma_{-}^{(1)}(\tau)\rangle + \langle \sigma_{+}^{(2)}(\tau)\sigma_{-}^{(2)}(\tau)\rangle$$
$$= \langle \sigma_{ee}^{(1)}(\tau)\rangle + \langle \sigma_{ee}^{(2)}(\tau)\rangle, \qquad (A3a)$$

$$P(\tau) = \langle \sigma_{+}^{(1)}(\tau) \sigma_{-}^{(2)}(\tau) \rangle + \langle \sigma_{+}^{(2)}(\tau) \sigma_{-}^{(1)}(\tau) \rangle,$$
(A3b)

$$D(\tau) = \langle \sigma_{+}^{(1)}(\tau) \sigma_{-}^{(1)}(\tau) \rangle - \langle \sigma_{+}^{(2)}(\tau) \sigma_{-}^{(2)}(\tau) \rangle$$
$$= \langle \sigma_{ee}^{(1)}(\tau) \rangle - \langle \sigma_{ee}^{(2)}(\tau) \rangle, \qquad (A3c)$$

$$Y(\tau) = i(\langle \sigma_{+}^{(2)}(\tau)\sigma_{-}^{(1)}(\tau)\rangle - \langle \sigma_{+}^{(1)}(\tau)\sigma_{-}^{(2)}(\tau)\rangle),$$
(A3d)

$$T(\tau) = \langle \sigma_{+}^{(1)}(\tau) \sigma_{-}^{(1)}(\tau) \sigma_{+}^{(2)}(\tau) \sigma_{-}^{(2)}(\tau) \rangle = \left\langle \sigma_{ee}^{(1)}(\tau) \sigma_{ee}^{(2)}(\tau) \right\rangle.$$

Using the Hamiltonian (A1), it is then possible to obtain the Heisenberg equations of motion for the operators appearing in Eqs. (A3), as well as for the field annihilation and creation operators. Eliminating the field states, we can arrive at the closed equations:

$$\frac{dM}{d\tau} = -2\gamma_a M - 2r\gamma_a P, \qquad (A4a)$$

$$\frac{dP}{d\tau} = -2\gamma_a P - 2r\gamma_a M + 8r\gamma_a T, \qquad (A4b)$$

$$\frac{dT}{d\tau} = -4\gamma_a T,\tag{A4c}$$

$$\frac{dD}{d\tau} = -2\gamma_a D + 2s\gamma_a Y,\tag{A4d}$$

$$\frac{dY}{d\tau} = -2\gamma_a Y - 2s\gamma_a D. \tag{A4e}$$

These equations can be solved easily for arbitrary initial conditions to get the radiation pattern,

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}_a \rangle \cdot \mathbf{u}_R = 2\gamma_a \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta$$
$$\times [M(\tau) + P(\tau) \cos \alpha + Y(\tau) \sin \alpha].$$
(A5)

The general solution of Eqs. (A4) for M, P, and Y is

$$M(\tau) = -4e^{-4\gamma_a \tau} T_0 \frac{r^2}{1 - r^2} + e^{-2\gamma_+ \tau} \left[\frac{M_0}{2} + \frac{P_0}{2} + T_0 \frac{2r}{1 - r^2} \right] + e^{-2\gamma_- \tau} \left[\frac{M_0}{2} - \frac{P_0}{2} - T_0 \frac{2r}{1 - r^2} \right],$$
(A6a)

$$P(\tau) = -4e^{-4\gamma_a \tau} T_0 \frac{r}{1-r^2} + e^{-2\gamma_+ \tau} \left[\frac{M_0}{2} + \frac{P_0}{2} + T_0 \frac{2r}{1-r^2} \right] - e^{-2\gamma_- \tau} \left[\frac{M_0}{2} - \frac{P_0}{2} - T_0 \frac{2r}{1-r^2} \right],$$
(A6b)

$$Y(\tau) = e^{-2\gamma_a \tau} [Y_0 \cos(2\gamma_a s \tau) - D_0 \sin(2\gamma_a s \tau)], \qquad (A6c)$$

with

$$\gamma_{\pm} = \gamma_a (1 \pm r). \tag{A7}$$

The time-integrated values of these variables are

$$^{\infty} d\tau M(\tau) = \frac{1}{2\gamma_a} \frac{M_0 - rP_0 - 2r^2 T_0}{1 - r^2}, \quad (A8a)$$

$$\int_{0}^{\infty} d\tau P(\tau) = \frac{1}{2\gamma_a} \frac{P_0 - rM_0 + 2rT_0}{1 - r^2}, \quad (A8b)$$

$$\int_0^\infty d\tau Y(\tau) = \frac{1}{2\gamma_a} \frac{-sD_0 + Y_0}{1 + s^2}.$$
 (A8c)

Note that if the initial condition corresponds to both atoms being inverted, then $M_0 = 2$, $T_0 = 1$, $D_0 = Y_0 = P_0 = 0$. In

(A3e)

this limit, the time-integrated signal is

$$\frac{dW}{d\Omega} = W_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta, \tag{A9}$$

and there is no contribution from the interference term.

For dressed states and dressed-state energies defined by

$$|0\rangle = |gg\rangle, \quad E_0^a = 0, \tag{A10a}$$

$$|\pm\rangle = \frac{|eg\rangle \pm |ge\rangle}{\sqrt{2}}, \quad E^a_{\pm} = \hbar(\omega_0 \pm \gamma_a s), \quad (A10b)$$

$$|2\rangle = |ee\rangle, \quad E_2^a = 2\hbar\omega_0, \tag{A10c}$$

the decay rates appearing in Fig. 5(b) have been calculated by Lehmberg [3]. The spectrum consists of a pair of Lorentzians centered at $\omega_k = \omega_0 - \gamma_a s$ and a pair of Lorentzians centered at $\omega_k = \omega_0 + \gamma_a s$. Explicitly, the spectrum is given by

$$I(2 \to +) = \frac{1+r}{2} \frac{1}{\pi} \frac{\gamma_{2+}}{[\omega_k - \omega_0 + \gamma_a s]^2 + \gamma_{2+}^2}, \quad (A11a)$$

$$I(2 \to -) = \frac{1-r}{2} \frac{1}{\pi} \frac{\gamma_{2-}}{[\omega_k - \omega_0 - \gamma_a s]^2 + \gamma_{2-}^2}, \quad (A11b)$$

$$I(+ \to 0) = \frac{1+r}{2} \frac{1}{\pi} \frac{\gamma_{+}}{[\omega_{k} - \omega_{0} - \gamma_{a}s]^{2} + \gamma_{+}^{2}}, \quad (A11c)$$

$$I(- \to 0) = \frac{1 - r}{2} \frac{1}{\pi} \frac{\gamma_{-}}{[\omega_k - \omega_0 + \gamma_a s]^2 + \gamma_{-}^2}, \quad (A11d)$$

where

$$\gamma_{2\pm} = 2\gamma_{\pm}.\tag{A12}$$

2. Two oscillators

In this case,

$$H_{o} = \hbar \omega_{0} \sum_{j=1,2} b^{\dagger(j)} b^{(j)} + \hbar \sum_{\mathbf{k},\lambda} \omega_{k} a^{\dagger}_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \hbar \sum_{\mathbf{k},\lambda} \sum_{j=1,2} (g'_{\mathbf{k}} b^{\dagger(j)} a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}_{j}} + g'^{*}_{\mathbf{k}} a^{\dagger}_{\mathbf{k}\lambda} b^{(j)} e^{-i\mathbf{k}\cdot\mathbf{R}_{j}}),$$
(A13)

and

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}_o \rangle \cdot \mathbf{u}_R = 2\gamma_o \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta$$
$$\times [\langle b^{\dagger(1)}(\tau) b^{(1)}(\tau) \rangle + \langle b^{\dagger(2)}(\tau) b^{(2)}(\tau) \rangle$$
$$+ 2 \operatorname{Re}(\langle b^{\dagger(1)}(\tau) b^{(2)}(\tau) \rangle e^{-i\alpha})]. \quad (A14)$$

Defining

$$M(\tau) = \langle n^{(1)}(\tau) \rangle + \langle n^{(2)}(\tau) \rangle, \qquad (A15a)$$

$$P(\tau) = \langle b^{\dagger(1)}(\tau)b^{(2)}(\tau) \rangle + \langle b^{\dagger(2)}(\tau)b^{(1)}(\tau) \rangle, \quad (A15b)$$

$$D(\tau) = \langle n^{(1)}(\tau) \rangle - \langle n^{(2)}(\tau) \rangle, \qquad (A15c)$$

$$Y(\tau) = i(\langle b^{\dagger(2)}(\tau)b^{(1)}(\tau)\rangle - \langle b^{\dagger(1)}(\tau)b^{(2)}(\tau)\rangle), \quad (A15d)$$

where

$$n^{(j)}(\tau) = b^{\dagger(j)}(\tau)b^{(j)}(\tau),$$
 (A16)

we can arrive at the closed equations:

$$\frac{dM}{d\tau} = -2\gamma_o M - 2r\gamma_o P, \qquad (A17a)$$

$$\frac{dP}{d\tau} = -2\gamma_o P - 2r\gamma_o M, \qquad (A17b)$$

$$\frac{dD}{d\tau} = -2\gamma_o D + 2s\gamma_o Y, \qquad (A17c)$$

$$\frac{dY}{d\tau} = -2\gamma_o Y - 2s\gamma_o D.$$
(A17d)

Note that the function T that was present in Eqs. (A3) for the two-atom case is absent from these equations, a result related to the linearity of the oscillator dynamics. These equations can be solved easily for arbitrary initial conditions to get the radiation pattern,

$$\frac{d^2 W(\Omega, \tau)}{d\tau d\Omega} = R^2 \langle \mathbf{S}_a \rangle \cdot \mathbf{u}_R = 2\gamma_o \hbar \omega_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta \\ \times [M(\tau) + P(\tau) \cos \alpha + Y(\tau) \sin \alpha].$$
(A18)

The general solution of Eqs. (A17) for M, P, and Y is

$$M(\tau) = e^{-2\gamma_o \tau} [M_0 \cosh(2\gamma_o r \tau) - P_0 \sinh(2\gamma_o r \tau)], \quad (A19a)$$

$$P(\tau) = e^{-2\gamma_o \tau} [P_0 \cosh(2\gamma_o r \tau) - M_0 \sinh(2\gamma_o r \tau)], \quad (A19b)$$

$$Y(\tau) = e^{-2\gamma_o \tau} [Y_0 \cos(2\gamma_o s \tau) - D_0 \sin(2\gamma_o s \tau)].$$
(A19c)

The time-integrated values of these variables are

$$\int_{0}^{\infty} d\tau M(\tau) = \frac{1}{2\gamma_o} \frac{M_0 - rP_0}{1 - r^2},$$
 (A20a)

$$\int_{0}^{\infty} d\tau P(\tau) = \frac{1}{2\gamma_{o}} \frac{P_{0} - rM_{0}}{1 - r^{2}},$$
 (A20b)

$$\int_{0}^{\infty} d\tau Y(\tau) = \frac{1}{2\gamma_{o}} \frac{-sD_{0} + Y_{0}}{1 + s^{2}}.$$
 (A20c)

Note that if the initial condition corresponds to both oscillators in their first excited states, then $M_0 = 2$, $D_0 = Y_0 = P_0 = 0$, and the time-integrated radiation pattern is

$$\frac{dW}{d\Omega} = W_0 \left(\frac{3}{8\pi}\right) \sin^2 \theta \left(\frac{1 - r \cos \alpha}{1 - r^2}\right).$$
(A21)

Now there is a contribution from the interference term.

The amplitude equations for the two oscillators, derived using the same method as that used for the atom-oscillator system, are

$$\dot{c}_{m,m'} = -(\sqrt{m} + \sqrt{m'})\gamma_o c_{m,m'} - (r+is)\gamma_o [\sqrt{m(m'+1)}c_{m-1,m'+1} - \sqrt{(m+1)m'}c_{m+1,m'-1}], \qquad (A22)$$

where the unprimed index is for oscillator 1 and the primed one is for oscillator 2. Only states having a fixed value of m + m' = n are coupled in the RWA. In other words, for a given value of *n*, there are *n* coupled equations. The corresponding "in terms" for the density matrix equations are

$$\dot{\rho}_{m,m';m'',m'''}^{ln} = 2\gamma_o\sqrt{(m+1)(m''+1)\rho_{m+1,m';m''+1,m'''}} + 2\gamma_o\sqrt{(m'+1)(m'''+1)}\rho_{m,m'+1;m'',m'''+1} + 2r\gamma_o\sqrt{(m+1)(m'''+1)}\rho_{m+1,m';m'',m'''+1} + 2r\gamma_o\sqrt{(m'+1)(m''+1)}\rho_{m,m'+1;m''+1,m'''}.$$
(A23)

The dressed states and dressed-state eigenfrequencies are obtained by finding the normal modes of Eqs. (A22), keeping only terms on the right-hand side of the equation proportional to *s*. The calculation can be done analytically [13]. For a given m + m' = n there are *n* eigenfrequencies equally spaced with spacing $2\gamma_o s$, symmetrically located about the unperturbed energy. The corresponding eigenkets $|n,q\rangle^o$ are given by

$$|n,q\rangle^o = \sum_{n_1=0}^n A(n_1,n-n_1,q)|n_1,n-n_1\rangle,$$
 (A24)

where $-n \leq q \leq n$, $|n_1, n_2\rangle = |n_1\rangle |n_2\rangle$ are the eigenkets associated with the *uncoupled* oscillators, and

$$\begin{aligned} A(n_1, n - n_1, q) \\ &= \frac{1}{2^{n/2}} \frac{1}{\left(n_1 - \frac{n-q}{2}\right)!} \sqrt{\frac{n_1! \left(\frac{n+q}{2}\right)!}{(n - n_1)! \left(\frac{n-q}{2}\right)!}} \\ &\times F\left(\frac{q - n}{2}, -(n - n_1); 1 + n_1 + \frac{q - n}{2}; -1\right), \end{aligned}$$
(A25)

where F(a,b;c;z) is a hypergeometric function.

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(A26b)

For n = 2, the appropriate dressed states and dressed-state energies are

$$|0\rangle^{o} = |0,0\rangle, \quad E_{0}^{o} = 0,$$
 (A26a)

$$|1_{\pm}\rangle^{o} \equiv |1, \pm 1\rangle^{o} = \frac{|1, 0\rangle \pm |0, 1\rangle}{\sqrt{2}}, \quad E_{1_{\pm}}^{o} = \hbar(\omega_{0} \pm \gamma_{o}s),$$

$$\begin{aligned} |2_{\pm}\rangle^{o} &\equiv |2, \pm 2\rangle^{o} = \frac{|2, 0\rangle + |0, 2\rangle \pm \sqrt{2} |1, 1\rangle}{\sqrt{2}}, \\ E^{o}_{2_{\pm}} &= \hbar(2\omega_{0} \pm 2\gamma_{o}s), \end{aligned}$$
(A26c)

$$|2_0\rangle^o \equiv |2,1\rangle^o = \frac{|2,0\rangle - |0,2\rangle}{\sqrt{2}}, \quad E_{2_0}^o = 2\hbar\omega_0.$$
 (A26d)

The decay rates shown in Fig. 5(c) can be obtained from Eqs. (A23) and (A26). The spectrum consists of a pair of Lorentzians centered at $\omega_k = \omega_0 - \gamma_o s$ and a pair of Lorentzians centered at $\omega_k = \omega_0 + \gamma_o s$. Explicitly, the spectrum is given by

$$I(2_{+} \to 1_{+}) = \frac{1+r}{2} \frac{1}{\pi} \frac{\gamma_{2_{+}1_{+}}}{[\omega_{k} - \omega_{0} - \gamma_{o}s]^{2} + \gamma_{2_{+}1_{+}}^{2}}, \quad (A27)$$

$$I(2_{-} \to 1_{-}) = \frac{1-r}{2} \frac{1}{\pi} \frac{\gamma_{2_{-}1_{-}}}{[\omega_{k} - \omega_{0} + \gamma_{o}s]^{2} + \gamma_{2_{-}1_{-}}^{2}}, \quad (A28)$$

$$I(1_{+} \to 0) = \frac{1+r}{2} \frac{1}{\pi} \frac{\gamma_{1+0}}{[\omega_{k} - \omega_{0} - \gamma_{o}s]^{2} + \gamma_{1+0}^{2}}, \quad (A29)$$

$$I(1_{-} \to 0) = \frac{1-r}{2} \frac{1}{\pi} \frac{\gamma_{1_{-}0}}{[\omega_k - \omega_0 + \gamma_o s]^2 + \gamma_{1_{-}0}^2}, \quad (A30)$$

where

$$\gamma_{2_{\pm}1_{\pm}} = 3\gamma_o(1\pm r),$$
 (A31)

$$\gamma_{1\pm 0} = \gamma_o (1\pm r).$$
 (A32)

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which the atom is in its excited state and the oscillator is in its ground state or one in which the atom is in its ground state and the oscillator in its n = 1 state), T(t) = 0 at all times, and we recover the two-atom or two-oscillator limit for the same initial conditions.

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