Nonclassical properties of the q-coherent and q-cat states of the Biedenharn-Macfarlane q oscillator with $q > 1$

H. Fakhri* and A. Hashemi†

Department of Theoretical Physics and Astrophysics, Faculty of Physics, University of Tabriz, P. O. Box 51666-16471, Tabriz, Iran

(Received 10 August 2015; published 4 January 2016)

This paper has been motivated by a recent paper by Dey [Phys. Rev. D **91**, 044024 (2015)] on the known Arik-Coon q oscillator. We construct q coherent, even and odd q-cat states in Fock representation for the Biedenharn-Macfarlane q oscillator with $q > 1$ and study their nonclassical properties. The q-coherent states minimize the Heisenberg uncertainty relation between the generalized position and momentum operators as well as the x and y components of a q-deformed su(1,1) algebra in the Schwinger boson representation. The latter is also minimized by the even and odd q-cat states. We show that, contrary to the undeformed harmonic oscillator, the squeezing effect in both position and momentum operators can be exhibited by odd q-cat states. It is also violated by even q-cat states. Furthermore, it is shown that the antibunching effect and sub-Poissonian or super-Poissonian statistics can simultaneously appear by each of the even or odd q-cat states. Finally, a unitary Fock representation of the q-deformed su(1,1) algebra is obtained by the q-deformed Bargmann-Fock realization.

DOI: 10.1103/PhysRevA.93.013802

I. INTRODUCTION

During recent decades, coherent and cat states have made their appearance in a large number of works in nonlinear and quantum optics, both experimentally and theoretically. For the first time Schrödinger introduced coherent states as nonspreading wave packets that minimize the Heisenberg uncertainty relation for the undeformed harmonic oscillator [1]. They are the closest to classical states in this sense that they remain localized around a classical trajectory and do not change their functional form with time. Afterwards, Glauber showed these states are the eigenstates of the annihilation operator and are obtained by the application of the unitary displacement operator on the ground state [2]. Then, the two latter ideas were generalized separately to the noncompact and arbitrary Lie groups and led to construct coherent states in two distinct classes, the so-called Barut-Girardello and Klauder-Perelomov classes [3–5]. A comprehensive review of this development can be found in Refs. [6–10]. Photon antibunching and squeezing in a quantum system are two of the most remarkable properties of the radiation field which differentiate the quantum correlations from those of the classical ones, and considering and comparing quantum and classical correlations remain the subject of further discussions since the early days of quantum mechanics. Mandel showed that for steady-state resonance fluorescence radiation, antibunching, and sub-Poissonian behavior occur simultaneously when there is a tuning between the exciting field and the atom [11]. He also showed that for resonance fluorescence generated under special conditions, the squeezing effect and sub-Poissonian behavior need not accompany each other [12]. It is necessary to mention the fact that, the cat states—which are also called even and odd coherent states—are defined as the superposition of two coherent states with opposite phase [13,14]. Both the even and odd cat states are eigenstates of the square of the annihilation operator and exhibit the squeezing effect and photon antibunching, respectively [15,16].

On the other hand, in continuation of the efforts to improve some properties of quantum field theory, several authors have proposed some changes in the canonical commutation relations by adding a quantum parameter q, which in turn has led to q deformations of the undeformed oscillator algebra. Field theories based on q deformation of the canonical commutation relations as such possess a small violation of the Pauli exclusion principle and deviations from the Bose statistics [17,18]. For this reason several attempts have been made to introduce the various deformations of the one-dimensional harmonic oscillator algebra in the last decades. The most known q oscillators are Arik-Coon [19], Biedenharn-Macfarlane [20–22], and Chung *et al.* [23]. Also, it must be emphasized that the deformed oscillator algebras generated by the operators $\{1,a,a^\dagger,N\}$ have been unified by introducing a positive analytic function $\Phi(x)$ with $\Phi(0)=0$ in the following form [24–26]:

$$[N,a^\dagger]=a^\dagger, \quad [N,a]=-a,$$
$$aa^\dagger=\Phi(N+1), \quad a^\dagger a=\Phi(N). \tag{1.1}$$

If we choose the structure function as $\Phi(x)=xF^2(x-1)$, then from the general scheme (1.1), we get the bosonization scheme for the deformed oscillator algebra with the bosonic creation and annihilation operators

$$[N,a^\dagger]=a^\dagger, \quad [N,a]=-a, \quad a^\dagger=b^\dagger F(N), \quad a=F(N)b, \tag{1.2}$$

in which $\{1,b,b^\dagger,N\}$ is the undeformed oscillator algebra

$$[N,b^\dagger]=b^\dagger, \quad [N,b]=-b, \quad bb^\dagger=N+1, \quad b^\dagger b=N. \tag{1.3}$$

The bosonization scheme (1.2) was initially proposed by Jannussis *et al.* in Ref. [27] and was afterwards called the f deformation in Ref. [28]. Some q-oscillator algebras that provide

*hfakhri@tabrizu.ac.ir

†a.hashemi@tabrizu.ac.ir

the boson realizations of quantum algebras can be derived from the deformed oscillator algebras given above as special cases. For example, it has already been reported in Refs. [29,30] that the Biedenharn-Macfarlane q-oscillator algebra is directly extracted from (1.1) and (1.2). We will introduce those later in Sec. II. The well known Arik-Coon q-oscillator algebra,

$$[N,a^\dagger] = a^\dagger, \quad [N,a] = -a, \\ aa^\dagger - qa^\dagger a = 1, \quad aa^\dagger - a^\dagger a = q^N, \tag{1.4}$$

can be directly obtained by choosing $\Phi(N) = [[N]]$ and $F(N) = \sqrt{\frac{[[N+1]]}{N+1}}$ where $[[N]] = \frac{1-q^N}{1-q}$. In the past decades, after the paper by Arik and Coon [19], the constructions and properties of the coherent and squeezed states associated with a wide variety of q-oscillator algebras were the object of numerous studies (see, for instance, [19,30–40]). Recently, Dey in Ref. [41] has studied classical-like properties of the nonlinear coherent states as well as nonclassical behaviors of the Schrödinger cat states for the Arik-Coon q-oscillator algebra (1.4) with $0 < q < 1$. So, it will be interesting to follow the studies along the line of Ref. [41] for the Biedenharn-Macfarlane q-oscillator algebra. The purpose of this paper is to use the Fock representation of the Biedenharn-Macfarlane q-oscillator algebra with $q > 1$ to construct q coherent, even and odd q-cat states and consider their nonclassical properties in detail.

The plan of the paper is as follows. Section II contains a brief review of the Biedenharn-Macfarlane q-oscillator algebra and its unitary lowest weight representation, the so-called Fock representation. In Sec. III we associate with the Fock representation a pair of new boson creation and annihilation operators that, in turn, gives a way to introduce a generalized type of Hermitian position and momentum operators. They are also used to make a q-deformed su(1,1) algebra that is separately represented by both even and odd Fock states with positive and negative parities. In Sec. IV we present q-coherent states associated with Fock representation and realize the resolution of the identity condition by an appropriate positive definite measure on the whole complex plane. It is shown that the Heisenberg uncertainty relation between the generalized position and momentum operators as well as the x and y components of the q-deformed su(1,1) is minimized by the q-coherent states. In Sec. V, the even and odd q-cat states associated with two orthogonal subspaces of the Fock space representation together with their positive definite measures are constructed for realizing the resolution of the identity condition over the entire complex plane. We show that these q-cat states minimize the Heisenberg uncertainty relation between the x and y components of the q-deformed su(1,1) algebra and exhibit the squeezing effect in both generalized position and momentum operators. It is also shown that the antibunching effect appears simultaneously with sub-Poissonian or super-Poissonian statistics by each of the even and odd q-cat states in some ranges of the parameter q and the distance from the origin on the complex plane. In Sec. VI we obtain the unitary Fock representation of the q-deformed su(1,1) algebra by the q-deformed Bargmann-Fock realization on two Hilbert spaces of even and odd entire holomorphic functions associated with the even and odd q-cat states, respectively. Finally, in the Appendix we give a q analog of Euler's formula for a factorial function by the symmetric quantum numbers.

II. UNITARY REPRESENTATION OF THE BIEDENHARN-MACFARLANE Q-OSCILLATOR ALGEBRA $\mathcal{A}_q$

The well-known Biedenharn-Macfarlane q-oscillator (unital and associative) algebra $\mathcal{A}_q = \langle a,a^\dagger,q^N,q^{-N}\rangle$ over $\mathbb{C}$ is defined by the commutation relations (with $q \in \mathbb{R} \setminus \{-1,0,1\}$) [20,21,29,42]

$$q^N a = q^{-1} a q^N, \quad q^N a^\dagger = q a^\dagger q^N, \\ aa^\dagger - qa^\dagger a = q^{-N}, \quad aa^\dagger - q^{-1}a^\dagger a = q^N, \tag{2.1}$$

where the first and second relations are adjoint of each other and the third and fourth relations are self-adjoint. It is also called the symmetric q oscillator since the relations (2.1) are invariant under the transformation $q \to q^{-1}$. The commutation relations (2.1) for the bosonic creation and annihilation operators $a^\dagger$ and a can be stated as follows:

$$[N,a^\dagger] = a^\dagger, \quad [N,a] = -a, \quad aa^\dagger = [N+1], \quad a^\dagger a = [N], \tag{2.2}$$

where $[N] \equiv \frac{q^N - q^{-N}}{q-q^{-1}}$. As it was mentioned above, the algebra relations (1.1) and (1.2) simply reduce to (2.2), if we take $\Phi(N) = [N]$ and $F(N) = \sqrt{\frac{[N+1]}{N+1}}$, respectively [29,30]. In what follows we limit ourselves to the case $q > 0$ to deal with the unitary lowest weight representation of the symmetric q-oscillator algebra $\mathcal{A}_q$. It is supposed that the generators of $\mathcal{A}_q$ are the linear operators on the Hilbert space $\mathcal{H} = \text{Lin. Span}\{|n\rangle | n \in \mathbb{N}_0;\ \langle n|m\rangle = \delta_{nm},\ \sum_{n=0}^{\infty} |n\rangle\langle n| = I\}$ with $\langle .|.\rangle$ as a scalar product and I as the identity operator on that space. Then, an irreducible representation of the algebra $\mathcal{A}_q$ with the lowest weight on $\mathcal{H}$ is given by

$$a|n\rangle = \sqrt{[n]}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle, \\ N|n\rangle = n|n\rangle. \tag{2.3}$$

It is now a unitary representation since the dagger plays the role of a star structure and N is a self-adjoint operator and the two operators $a^\dagger$ and a are Hermitian conjugates of each other with respect to the scalar product. The irreducible representation (2.3) is briefly called a Fock representation of the Biedenharn-Macfarlane q-oscillator algebra $\mathcal{A}_q$ with $q > 0$.

III. Q-DEFORMED POSITION-MOMENTUM AND SU(1,1) ALGEBRAS FROM $\mathcal{A}_q$

By defining new creation and annihilation operators as $A = a\, q^{\frac{N}{2}}$ and $A^\dagger = q^{\frac{N}{2}} a^\dagger$ we get a different feature of q-deformed oscillator algebra,

$$AA^\dagger = A^\dagger A + q^{2N+1} = q^2 A^\dagger A + q, \\ [N,A^\dagger] = A^\dagger, \quad [N,A] = -A. \tag{3.1}$$

Their quadratic powers are faithful to the following commutation relations

$$A^2 A^{\dagger 2} = q^8 A^{\dagger 2} A^2 + q^3(q^2+1)^2 A^\dagger A + q^2(q^2+1)$$
$$= A^{\dagger 2}A^2 + \frac{q^2+1}{q^2-1}[(q^4+1)q^{2N} - (q^2+1)]q^{2N},$$
$$q^N A^2 = q^{-2}A^2 q^N, \quad q^N A^{\dagger 2} = q^2 A^{\dagger 2} q^N. \tag{3.2}$$

The generalized position and momentum operators associated with the unitary symmetric q oscillator are given by

$$x \equiv \frac{1}{\sqrt{2}}(A^\dagger + A), \quad p \equiv \frac{i}{\sqrt{2}}(A^\dagger - A), \quad [x,p] = iq^{2N+1}. \tag{3.3}$$

Also, a unitary q-deformed version of su(1,1) in the Schwinger boson representation is introduced by

$$J_x \equiv \frac{1}{4}(A^{\dagger 2} + A^2), \quad J_y \equiv \frac{1}{4i}(A^{\dagger 2} - A^2),$$
$$J_z \equiv \frac{1}{4}(2N+1),$$
$$[J_x, J_y] = -i\frac{q^2+1}{8(q^2-1)}[(q^4+1)q^{4J_z-1} - (q^2+1)]q^{4J_z-1},$$
$$[J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \tag{3.4}$$

The Hilbert space $\mathcal{H}$ can be uniquely decomposed into two orthogonal subspaces $\mathcal{H}^e = \text{Lin. Span}\{|2k\rangle \,|\, k \in \mathbb{N}_0\}$ and $\mathcal{H}^o = \text{Lin. Span}\{|2k+1\rangle \,|\, k \in \mathbb{N}_0\}$ with positive and negative parity states, respectively. Each of the subspaces $\mathcal{H}^e$ and $\mathcal{H}^o$ is separately a Fock representation space of the q-deformed su(1,1) algebra:

$$J_+|n\rangle = \frac{1}{2}q^{n+\frac{3}{2}}\sqrt{[n+1][n+2]}|n+2\rangle,$$
$$J_z|n\rangle = \frac{2n+1}{4}|n\rangle, \tag{3.5}$$
$$J_-|n+2\rangle = \frac{1}{2}q^{n+\frac{3}{2}}\sqrt{[n+1][n+2]}|n\rangle.$$

It is clear that the undeformed harmonic oscillator algebra and the classical su(1,1) Lie algebra are obtained as limiting cases of the algebra relations (3.1) and (3.4) when q tends to 1, respectively. Without loss of generality we shall assume $q > 1$ from now on.

IV. THE MINIMUM-UNCERTAINTY Q-COHERENT STATES

Let w be a complex variable with the polar form as $w = |w|e^{i\varphi}, 0 \leqslant |w| < \infty, 0 \leqslant \varphi < 2\pi$. The q-coherent states $|w\rangle$ of the carrier space of the Fock representation of $\mathcal{A}_q$ are defined as the normalized eigenvectors of the annihilation operator A, i.e., $A|w\rangle = w|w\rangle$, with $w \in \mathbb{C}$ as eigenvalues. They can be written in terms of the Fock bases $\{|n\rangle\}_{n\in\mathbb{N}_0}$ as

$$|w\rangle = \frac{1}{\sqrt{\widetilde{e}_{q^{-1}}(q^{-1}|w|^2)}}\sum_{n=0}^{\infty}\frac{q^{-\frac{n(n+1)}{4}}w^n}{\sqrt{[n]!}}|n\rangle, \tag{4.1}$$

where the tilde notation for the q-exponential function is defined in the Appendix. From the remarks in the Appendix it also follows that the coherent states $|w\rangle$ form an overcomplete basis of $\mathcal{H}$. More precisely:

(i) There exists a resolution of the unity in $\mathcal{H}$,

$$\int_{\mathbb{C}} d\mu_q(w,\bar{w})|w\rangle\langle w| = \sum_{n=0}^{\infty}|n\rangle\langle n|, \tag{4.2}$$

with the positive definite measure as

$$d\mu_q(w,\bar{w}) = \frac{1}{1+(q-q^{-1})|w|^2}\frac{\widetilde{d}_q|w|^2\,d\varphi}{2\pi}. \tag{4.3}$$

For a precise definition of the measure $\widetilde{d}_q|w|^2$, see the Appendix.

(ii) They are not orthogonal, and their overlapping relations have the forms

$$\langle w|w'\rangle = \frac{\widetilde{e}_{q^{-1}}(q^{-1}\bar{w}w')}{\sqrt{\widetilde{e}_{q^{-1}}(q^{-1}|w|^2)\widetilde{e}_{q^{-1}}(q^{-1}|w'|^2)}}. \tag{4.4}$$

The Heisenberg uncertainty relation between x and p as well as J_x and J_y is minimized by q-coherent states $|w\rangle$,

$$\langle w|(\Delta x)^2|w\rangle\langle w|(\Delta p)^2|w\rangle$$
$$= \tfrac{1}{4}|\langle w|[x,p]|w\rangle|^2 = \tfrac{1}{4}[(q^2-1)|w|^2+q]^2, \tag{4.5}$$
$$\langle w|(\Delta J_x)^2|w\rangle\langle w|(\Delta J_y)^2|w\rangle$$
$$= \tfrac{1}{4}|\langle w|[J_x,J_y]|w\rangle|^2$$
$$= \tfrac{1}{256}[(q^8-1)|w|^4 + q^3(q^2+1)^2|w|^2 + q^2(q^2+1)]^2. \tag{4.6}$$

In the limit $q \to 1$, they reduce to the familiar values 1/4 and $(2|w|^2+1)^2/16$ for undeformed harmonic oscillator, given in Refs. [1,43].

V. THE Q-CAT STATES AND THEIR NONCLASSICAL PROPERTIES

Let us briefly present the nonclassical properties we are going to consider in this section. The Glauber-Sudarshan P function which is analogous to the phase-space distributions of statistical mechanics, and is a criterion for nonclassicality based on the photon-number distribution of the field, characterizes the properties of the quantum states [44,45]. States for which P function is negative or more singular than a δ function are nonclassical. The quadrature squeezing, as a nonclassical behavior, is possible only for states for which the P function is negative in some regions of the phase space. With respect to the vacuum, the variance of one of the quadratures is less, while the variance of the other quadrature is more [46,47]. Indeed, according to the uncertainty relation, when the variance of one of the quadratures becomes squeezed the variance of the other quadrature is expanded. Furthermore, photon bunching and photon antibunching are characterized by inequalities $g^{(2)}(0) > 1$ and $g^{(2)}(0) < 1$ for the second-order intensity correlation function, respectively [48]. States for which the antibunching inequality holds are known as nonclassical fields since they can only be interpreted in terms of the quantum mechanical formalism. Therefore, photon antibunching effect reflects the corpuscular nature of light. Also, the deviation of standard photon statistics from the Poisson distribution is characterized by the Mandel's Q parameter [11]. That is, the zero value of the Q parameter corresponds to Poissonian statistics, whereas the negative and positive values indicate sub-Poissonian and super-Poissonian statistics, respectively. Sub-Poissonian radiation is called nonclassical light, since its photocount distribution is narrower than a Poissonian one with the same intensity. The main focus of this section is on the nonclassical properties, squeezing and antibunching effects as well as sub-Poissonian statistics, for the q-cat states of the Biedenharn-Macfarlane q oscillator and a comparison with those of undeformed and Arik-Coon oscillators.

Two even and odd q-cat states as summations over even and odd Fock states, respectively, i.e.,

$$|z\rangle^e \equiv \frac{\sum_{n=0}^{\infty} \frac{q^{-\frac{n(2n+1)}{2}} z^{2n}}{\sqrt{[2n]!}} |2n\rangle}{\sqrt{\cosh_{q^{-1}}(q^{-1}|z|^2)}} = \frac{\cosh_{q^{-1}}\left(q^{-\frac{1}{2}} z a^\dagger q^{\frac{N}{2}}\right)|0\rangle}{\sqrt{\cosh_{q^{-1}}(q^{-1}|z|^2)}}, \tag{5.1}$$

$$|z\rangle^o \equiv \frac{\sum_{n=0}^{\infty} \frac{q^{-\frac{(2n+1)(2n+2)}{4}} z^{2n+1}}{\sqrt{[2n+1]!}} |2n+1\rangle}{\sqrt{\sinh_{q^{-1}}(q^{-1}|z|^2)}} = \frac{\sinh_{q^{-1}}\left(q^{-\frac{1}{2}} z a^\dagger q^{\frac{N}{2}}\right)|0\rangle}{\sqrt{\sinh_{q^{-1}}(q^{-1}|z|^2)}}, \tag{5.2}$$

are the orthonormalized eigenstates of the square of the boson annihilation operator A. Therefore, $A^2|z\rangle^e = z^2|z\rangle^e$, $A^2|z\rangle^o = z^2|z\rangle^o$, ${}^e\langle z|z\rangle^e = {}^o\langle z|z\rangle^o = 1$ and ${}^e\langle z'|z\rangle^o = 0$. Here, z is again a complex variable with the polar representation as $z = |z|e^{i\theta}$, $0 \leqslant |z| < \infty$, $0 \leqslant \theta < 2\pi$. We obtain the following expressions if the cat states are to overlap:

$${}^e\langle z'|z\rangle^e = \frac{\cosh_{q^{-1}}(q^{-1}\bar{z}'z)}{\sqrt{\cosh_{q^{-1}}(q^{-1}|z'|^2)\cosh_{q^{-1}}(q^{-1}|z|^2)}}, \tag{5.3}$$

$${}^o\langle z'|z\rangle^o = \frac{\sinh_{q^{-1}}(q^{-1}\bar{z}'z)}{\sqrt{\sinh_{q^{-1}}(q^{-1}|z'|^2)\sinh_{q^{-1}}(q^{-1}|z|^2)}}. \tag{5.4}$$

By using the q-integral relation given in the Appendix, i.e., (A15), it is easy to show that for the nonnegative definite weight functions

$$\mu_q^e(|z|) \equiv \frac{\cosh_{q^{-1}}(q^{-1}|z|^2)}{\widetilde{e}_{q^{-1}}(q|z|^2)} = \frac{1}{2[1+|z|^2(q-q^{-1})]} + \frac{1}{2}\prod_{n=0}^{\infty}\frac{1-|z|^2q^{-2n-2}(q-q^{-1})}{1+|z|^2q^{-2n}(q-q^{-1})}, \tag{5.5}$$

$$\mu_q^o(|z|) \equiv \frac{\sinh_{q^{-1}}(q^{-1}|z|^2)}{\widetilde{e}_{q^{-1}}(q|z|^2)} = \frac{1}{2[1+|z|^2(q-q^{-1})]} - \frac{1}{2}\prod_{n=0}^{\infty}\frac{1-|z|^2q^{-2n-2}(q-q^{-1})}{1+|z|^2q^{-2n}(q-q^{-1})}, \tag{5.6}$$

resolutions of the identity condition are saturated as below:

$$\int_{\mathbb{C}} \frac{\widetilde{d}_q|z|^2\, d\theta}{2\pi}\mu_q^e(|z|)\,|z\rangle^{e\,e}\langle z| = \sum_{n=0}^{\infty}|2n\rangle\langle 2n|, \tag{5.7}$$

$$\int_{\mathbb{C}} \frac{\widetilde{d}_q|z|^2\, d\theta}{2\pi}\mu_q^o(|z|)\,|z\rangle^{o\,o}\langle z| = \sum_{n=0}^{\infty}|2n+1\rangle\langle 2n+1|. \tag{5.8}$$

One can easily see that the q-coherent states with opposite phases are superpositions of the even and odd q-cat states

$$|\pm z\rangle = \frac{\sqrt{\cosh_{q^{-1}}(q^{-1}|z|^4)}\,|z\rangle^e \pm \sqrt{\sinh_{q^{-1}}(q^{-1}|z|^4)}\,|z\rangle^o}{\sqrt{\widetilde{e}_{q^{-1}}(q^{-1}|z|^4)}}. \tag{5.9}$$

The boson annihilation operator A maps the even and odd q-cat states to each other as below:

$$A|z\rangle^e = \sqrt{z^2\tanh_{q^{-1}}(q^{-1}|z|^2)}\,|z\rangle^o,$$
$$A|z\rangle^o = \sqrt{z^2\coth_{q^{-1}}(q^{-1}|z|^2)}\,|z\rangle^e. \tag{5.10}$$

Furthermore, all the necessary expectation values in the q-cat states are

$$\begin{aligned}
&{}^e\langle z|A|z\rangle^e = {}^e\langle z|A^\dagger|z\rangle^e = {}^o\langle z|A|z\rangle^o = {}^o\langle z|A^\dagger|z\rangle^o = 0,\\
&{}^e\langle z|A^2|z\rangle^e = {}^o\langle z|A^2|z\rangle^o = z^2,\\
&{}^e\langle z|A^{\dagger 2}|z\rangle^e = {}^o\langle z|A^{\dagger 2}|z\rangle^o = \bar{z}^2,\\
&{}^e\langle z|A^\dagger A|z\rangle^e = |z|^2\tanh_{q^{-1}}(q^{-1}|z|^2),\\
&{}^o\langle z|A^\dagger A|z\rangle^o = |z|^2\coth_{q^{-1}}(q^{-1}|z|^2).
\end{aligned} \tag{5.11}$$

It is straightforward to conclude that the covariance of the operators x and p as well as J_x and J_y over the normalized even and odd q-cat states vanishes. Therefore, the minimum uncertainty relations between the x and y components of operator $\mathbf{J}$ in the even and odd q-cat states are calculated in terms of their variances as below:

$$\begin{aligned}
&{}^e\langle z|(\Delta J_x)^2|z\rangle^{e\,e}\langle z|(\Delta J_y)^2|z\rangle^e\\
&= \tfrac{1}{256}[(q^8-1)|z|^4 + q^3(1+q^2)^2|z|^2\\
&\quad\times \tanh_{q-1}(q^{-1}|z|^2) + q^2(1+q^2)]^2,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&{}^o\langle z|(\Delta J_x)^2|z\rangle^2\rangle^{o\,o}\langle z|(\Delta J_y)^2|z\rangle^o\\
&= \tfrac{1}{256}[(q^8-1)|z|^4 + q^3(1+q^2)^2|z|^2\\
&\quad\times \coth_{q-1}(q^{-1}|z|^2) + q^2(1+q^2)]^2.
\end{aligned} \tag{5.13}$$

These results correspond to those of the undeformed harmonic oscillator in the limit $q \to 1$ [16].

Now, it will be interesting to consider the deviation of the minimum uncertainty for the position and momentum operators in the even and odd q-cat states. First, from (5.11) we get the following results for the variance of the position and momentum operators with respect to even and odd q-cat states

$${}^e\langle z|(\Delta x)^2|z\rangle^e = \tfrac{1}{2}[q + 2|z|^2\cos 2\theta + (1+q^2)|z|^2\tanh_{q^{-1}}(q^{-1}|z|^2)], \tag{5.14}$$

$${}^e\langle z|(\Delta p)^2|z\rangle^e = \tfrac{1}{2}[q - 2|z|^2\cos 2\theta + (1+q^2)|z|^2\tanh_{q^{-1}}(q^{-1}|z|^2)], \tag{5.15}$$

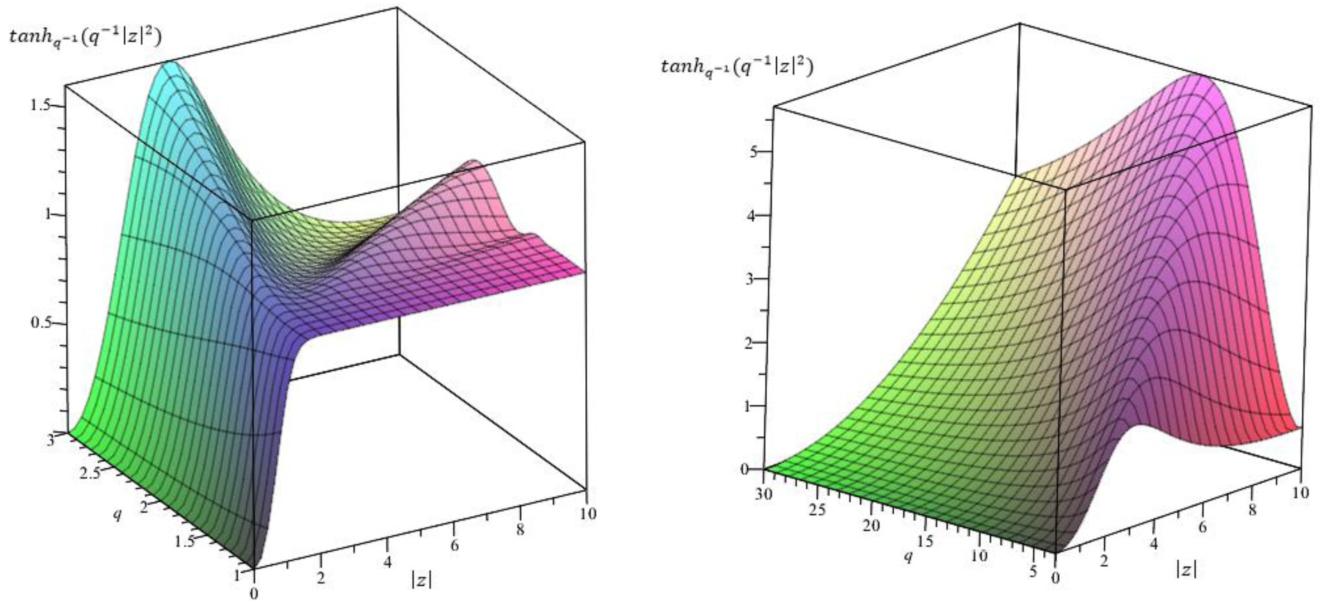

FIG. 1. The plots of the function $\tanh_{q^{-1}}(q^{-1}|z|^2)$ against q and $|z|$. They denote that $\tanh_{q^{-1}}(q^{-1}|z|^2)$ can be both smaller and larger than the unit. In the first case, the squeezing effect is exhibited while in the second case, it is violated by even q-cat states, contrary to what occurs in the limit $q \to 1$.

and

$$^{o}\langle z|(\Delta x)^2|z\rangle^{o} = \frac{1}{2}[q + 2|z|^2\cos 2\theta + (1+q^2)|z|^2\coth_{q^{-1}}(q^{-1}|z|^2)], \quad (5.16)$$

$$^{o}\langle z|(\Delta p)^2|z\rangle^{o} = \frac{1}{2}[q - 2|z|^2\cos 2\theta + (1+q^2)|z|^2\coth_{q^{-1}}(q^{-1}|z|^2)]. \quad (5.17)$$

From (3.1) and (3.3), we note that the Heisenberg uncertainty relation between the position and momentum operators in an arbitrary normalized state can be expressed as below:

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geqslant \frac{1}{4}|\langle q + (q^2-1)A^\dagger A\rangle|^2. \quad (5.18)$$

It is clear that this uncertainty relation is not minimized by the even and odd q-cat states. This allows us to consider a nonclassical behavior, i.e., squeezing effect, in the position and momentum operators by the even and odd q-cat states. If $\pm\cos 2\theta + \tanh_{q^{-1}}(q^{-1}|z|^2) < 0$, then $^{e}\langle z|(\Delta x)^2|z\rangle^{e}$ or $^{e}\langle z|(\Delta p)^2|z\rangle^{e} < \frac{1}{2}\,{}^{e}\langle q + (q^2-1)A^\dagger A\rangle^{e}$, and even q-cat states are "squeezed" in x or p, respectively. If $\pm\cos 2\theta + \coth_{q^{-1}}(q^{-1}|z|^2) < 0$, then $^{o}\langle z|(\Delta x)^2|z\rangle^{o}$ or $^{o}\langle z|(\Delta p)^2|z\rangle^{o} < \frac{1}{2}\,{}^{o}\langle q + (q^2-1)A^\dagger A\rangle^{o}$, and odd q-cat states are squeezed in x or p, respectively. If we fix $\theta = \frac{\pi}{2}$ for squeezing in x and $\theta = 0$ for squeezing in p, then squeezing conditions for even and odd q-cat states appear in the forms $\tanh_{q^{-1}}(q^{-1}|z|^2) < 1$ and $\coth_{q^{-1}}(q^{-1}|z|^2) < 1$, respectively. The first inequality is clearly converted to a familiar one in the limit $q \to 1$ while the second one does not have such a limit. Indeed, it means that for the undeformed harmonic oscillator, only even cat states can exhibit the squeezing effect, which is in agreement with the findings of Refs. [15,16]. We have plotted the changes in the functions $\tanh_{q^{-1}}(q^{-1}|z|^2)$ and $\coth_{q^{-1}}(q^{-1}|z|^2)$ in terms of q and $|z|$, in Figs. 1 and 2, respectively. They show that in some ranges of the independent variables q and $|z|$, both even and odd q-cat states of the Biedenharn-Macfarlane q oscillator exhibit the squeezing effect in both quadrature variances. Also, the inequalities are violated in some other ranges. Consequently, the squeezing effect in both position and momentum operators disappears. While according to Ref. [41], the even q-cat states of the Arik-Coon q oscillator exhibit the squeezing effect but its odd q-cat states do not exhibit this nonclassical phenomena.

It is now easy to show that the second-order intensity q-correlation function (for zero delay time) and q-Mandel parameter associated with an arbitrary normalized state of the model, i.e.,

$$g^{(2)}(0) \equiv \frac{\langle A^{\dagger 2}A^2\rangle}{\langle A^\dagger A\rangle^2}, \quad Q \equiv \frac{\langle (A^\dagger A)^2\rangle - \langle A^\dagger A\rangle^2}{\langle A^\dagger A\rangle} - 1, \quad (5.19)$$

saturate the following equation

$$Q = \langle A^\dagger A\rangle[q^2 g^{(2)}(0) - 1] + q - 1. \quad (5.20)$$

This expression implies the deviation of the photon number probability from the Poissonian distribution, and it is obviously reduced to the standard formula in the limit of $q \to 1$ [49]. According to (5.20), photon coherent state $g_e^{(2)}(0) = 1$, bunching $g_e^{(2)}(0) > 1$, and antibunching $g_e^{(2)}(0) < 1$ correspond, respectively, to Poissonian, super-Poissonian, and sub-Poissonian distributions in the undeformed harmonic oscillator. However, we deal with a different variety of properties for $q > 1$. It is not hard to verify the following results by direct calculations:

$$g_e^{(2)}(0) = \coth^2_{q^{-1}}(q^{-1}|z|^2),$$
$$Q_e = q^2|z|^6\coth_{q^{-1}}(q^{-1}|z|^2) - |z|^2\tanh_{q^{-1}}(q^{-1}|z|^2) + q - 1, \quad (5.21)$$

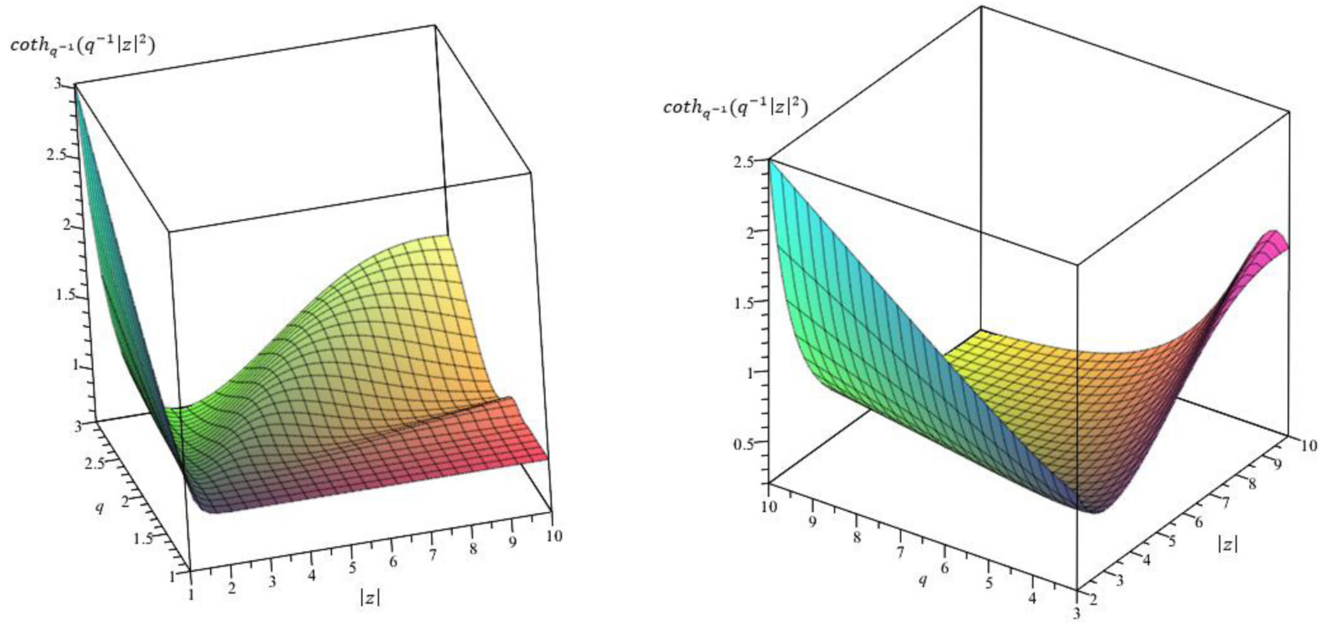

FIG. 2. The plots of the function $\coth_{q^{-1}}(q^{-1}|z|^2)$ against q and $|z|$. They denote that $\coth_{q^{-1}}(q^{-1}|z|^2)$ can be both larger and smaller than the unit. In the first case, the squeezing effect is violated while in the second case, it is exhibited by odd q-cat states, contrary to what occurs in the limit $q \to 1$.

$$g_o^{(2)}(0) = \tanh^2_{q^{-1}}(q^{-1}|z|^2),$$
$$Q_o = q^2|z|^6 \tanh_{q^{-1}}(q^{-1}|z|^2) - |z|^2 \coth_{q^{-1}}(q^{-1}|z|^2) + q - 1. \quad (5.22)$$

It is quite clear from the above results that in the limit $q \to 1$, the light field of even cat states does not exhibit the nonclassical antibunching effect. On the contrary, this effect is exhibited by the light field of odd cat states in all ranges of the variable $|z|$. This is exactly as has been reported in Refs. [15,16]. From Figs. 1 and 2, we immediately notice that the antibunching effect appears for both even and odd q-cat states in some ranges of the parameters q and $|z|$. These findings, together with (5.20), imply that sub-Poissonian statistics and the antibunching effect, sub-Poissonian statistics and the bunching effect, and super-Poissonian statistics and the antibunching effect as well as super-Poissonian statistics and the bunching effect can simultaneously occur by both even and odd q-cat states in some ranges of the parameters q and $|z|$. While according to Ref. [41], the Mandel parameters associated with the odd and even q-cat states of the Arik-Coon q oscillator could take negative (sub-Poissonian distribution) and either positive or negative values (sub- and super-Poissonian distributions), depending on the range of the parameters q and $|z|$, respectively.

VI. THE q-DEFORMED BARGMANN-FOCK REALIZATION OF THE q-DEFORMED su(1,1) ALGEBRA

We are now going to derive the q-deformed Bargmann-Fock realization of the Fock representation of the q-deformed su(1,1) algebra on two Hilbert spaces of even and odd entire holomorphic functions associated with the even and odd q-cat states, respectively. An even (odd) entire holomorphic function is an even (odd) analytic map of the complex plane $\mathbb{C}$ into itself. Each of the spaces of even and odd entire holomorphic functions, which are expressed as even and odd power series of z with complex coefficients, denoted by $\mathbb{C}_e[z]$ and $\mathbb{C}_o[z]$, constitutes a vector space with respect to the operations of pointwise addition and scalar multiplication. It is now straightforward to see that the coefficients in each of the infinite superpositions of the even and odd q-cat states $|z\rangle^e$ and $|z\rangle^o$, i.e.,

$$u^e_{2k}(z) \equiv [\cosh_{q^{-1}}(q^{-1}|z|^2)]^{\frac{1}{2}}\ {}^e\langle\bar{z}|2k\rangle = \frac{q^{\frac{-2k(2k+1)}{4}} z^{2k}}{\sqrt{[2k]!}}, \quad (6.1)$$

$$u^o_{2k+1}(z) \equiv [\sinh_{q^{-1}}(q^{-1}|z|^2)]^{\frac{1}{2}}\ {}^o\langle\bar{z}|2k+1\rangle = \frac{q^{\frac{-(2k+1)(2k+2)}{4}} z^{2k+1}}{\sqrt{[2k+1]!}}, \quad (6.2)$$

constitute separately an orthonormal basis of $\mathbb{C}_e[z]$ and $\mathbb{C}_o[z]$ with respect to the positive definite scalar products as

$$\left(u^e_{2k}, u^e_{2k'}\right)^e \equiv \int_{\mathbb{C}} \overline{u^e_{2k}(z)} u^e_{2k'}(z) \frac{\mu^e_q(|z|)}{\cosh_{q^{-1}}(q^{-1}|z|^2)} \frac{\widetilde{d}_q|z|^2\, d\theta}{2\pi} = \delta_{k\,k'}, \quad (6.3)$$

$$\left(u^o_{2k+1}, u^o_{2k'+1}\right)^o \equiv \int_{\mathbb{C}} \overline{u^o_{2k+1}(z)} u^o_{2k'+1}(z) \frac{\mu^o_q(|z|)}{\sinh_{q^{-1}}(q^{-1}|z|^2)} \frac{\widetilde{d}_q|z|^2\, d\theta}{2\pi} = \delta_{k\,k'}, \quad (6.4)$$

respectively.

Let now $\mathcal{F}^e$ and $\mathcal{F}^o$ denote Hilbert space completions of $(\mathbb{C}_e[z],(.,.)^e)$ and $(\mathbb{C}_o[z],(.,.)^o)$. We can obtain the q-deformed

Bargmann-Fock realization of the Fock representation of the q-deformed su(1,1) algebra by isomorphisms from $\mathcal{H}^e$ and $\mathcal{H}^o$ to $\mathcal{F}^e$ and $\mathcal{F}^o$ which send $|2k\rangle$ and $|2k+1\rangle$ to $u^e_{2k}(z)$ and $u^o_{2k+1}(z)$, respectively. If we denote q differentiation with respect to the variable z by ${}^z\widetilde{D}_q$, then it is easy to see that

$$A^2 u^e_{2k}(z) \equiv \sqrt{\cosh_{q^{-1}}(q^{-1}|z|^2)}\,{}^e\langle \bar{z}|A^2|2k\rangle = {}^z\widetilde{D}_q^2 q^{2z\frac{d}{dz}-1} u^e_{2k}(z), \quad (6.5)$$

$$A^2 u^o_{2k+1}(z) \equiv \sqrt{\sinh_{q^{-1}}(q^{-1}|z|^2)}\,{}^o\langle \bar{z}|A^2|2k+1\rangle = {}^z\widetilde{D}_q^2 q^{2z\frac{d}{dz}-1} u^o_{2k+1}(z). \quad (6.6)$$

Therefore, under the isomorphisms from $\mathcal{H}^e$ and $\mathcal{H}^o$ to $\mathcal{F}^e$ and $\mathcal{F}^o$, the explicit form of operator A^2 in the spaces $\mathcal{F}^e$ and $\mathcal{F}^o$ is obtained as below:

$$A^2 = {}^z\widetilde{D}_q^2 q^{2z\frac{d}{dz}-1}. \quad (6.7)$$

In a similar way, we get

$$A^{\dagger 2} = z^2, \quad N = z\frac{d}{dz}. \quad (6.8)$$

For example, one can easily show that A^2 in Eq. (6.7) is actually the adjoint of $A^{\dagger 2}$ in Eq. (6.8) with respect to the bilinear forms on $\mathcal{F}^e$ and $\mathcal{F}^o$. Therefore, we have here obtained a unitary Fock representation of the q-deformed su(1,1) algebra by the q-deformed Bargmann-Fock realization.

VII. CONCLUDING REMARKS

A class of q-coherent states on the carrier space of the Fock representation corresponding to the Biedenharn-Macfarlane q-oscillator algebra was introduced and shown to minimize an uncertainty relation between the position and momentum operators. The Biedenharn-Macfarlane boson creation and annihilation operators were used to make a q-deformed version of su(1,1) algebra that is represented by both even and odd Fock subspaces. We have shown that the even and odd q-cat states associated with these Fock subspaces, which themselves are superpositions of the two q-coherent states with an opposite phase, have substantially nonclassical different properties with respect to those of the undeformed boson oscillator and the Arik-Coon q oscillator. The even cat states corresponding to the latter two models exhibit the squeezing effect, whereas the odd cat states do not exhibit this type of nonclassical behavior [41,46,47]. Furthermore, in the undeformed boson oscillator, the even and odd cat states display only super-Poissonian and sub-Poissonian photon statistics, respectively [15,16]. Besides, in the Arik-Coon q oscillator, the odd q-cat states exhibit only sub-Poissonian statistics of photons while the even q-cat states exhibit sub-Poissonian or super-Poissonian distributions which depend on the range of the parameters q and $|z|$ [41]. So, for the Arik-Coon q oscillator, the quantum photon antibunching effect is visible not only through the odd q-cat states but also through the even q-cat states. While, according to our findings in this work, the even and odd q-cat states of the Biedenharn-Macfarlane q oscillator can demonstrate the squeezing effect in both position and momentum operators. They also exhibit the photon antibunching and sub-Poissonian or super-Poissonian statistics simultaneously. The magnitudes of squeezing and antibunching can be controlled by enlargement of the parameter q, too. It was shown that, as expected, the even and odd q-cat states minimize the Heisenberg uncertainty relation between the x and y components of the q-deformed su(1,1) algebra. We have also found the q-deformed Bargmann-Fock realization of the q-deformed su(1,1) algebra on two Hilbert spaces of even and odd entire holomorphic functions which in turn can be utilized to generate two different classes of q-special functions. Finally, it would be interesting to consider the nonclassical properties of the q-cat states of the more complicated deformed oscillators such as the Chung-Chung-Nam-Um q oscillator [23].

APPENDIX: q ANALOGUE OF EULER'S FORMULA BASED ON THE SYMMETRIC QUANTUM NUMBERS

For a fixed parameter q, the symmetric q-differentiation operator $\widetilde{D}_q$ and q integration as an inverse operation to this q differentiation are defined as [42]

$$\widetilde{D}_q F(x) := \frac{F(qx)-F(q^{-1}x)}{(q-q^{-1})x} \equiv f(x) \quad (A1)$$

and

$$\int_0^x f(x)\widetilde{d}_q x := F(x) - F(0). \quad (A2)$$

The Leibniz rule for the symmetric q derivative of the two functions' product is [42]

$$\widetilde{D}_q(F(x)G(x)) = F(qx)\widetilde{D}_q G(x) + G(q^{-1}x)\widetilde{D}_q F(x), \quad (A3)$$

$$\widetilde{D}_q(F(x)G(x)) = F(q^{-1}x)\widetilde{D}_q G(x) + G(qx)\widetilde{D}_q F(x). \quad (A4)$$

By symmetry, both relations (A3) and (A4) remain unaltered under the interchange of F and G.

Now, we go on to define a q antiderivative and obtain a q-integral representation of the q factorial in terms of the symmetric quantum numbers. Thus, for $q>1$, the q integrals of a function $f(x)$ on the intervals $[0,a]$ and $[0,\infty)$ are, respectively, calculated as follows

$$\int_0^a f(x)\,\widetilde{d}_q x = a(q-q^{-1})\sum_{j=0}^{\infty} q^{-2j-1} f(q^{-2j-1}a), \quad (A5)$$

$$\int_0^\infty f(x)\,\widetilde{d}_q x = (q-q^{-1})\sum_{j=-\infty}^{\infty} q^{2j+1} f(q^{2j+1}). \quad (A6)$$

In Eqs. (A5) and (A6), the q integration by part for the parameter a, whether finite or infinite, is immediate:

$$\int_0^a F(qx)\widetilde{D}_q G(x)\,\widetilde{d}_q x = F(a)G(a) - F(0)G(0) - \int_0^a G(q^{-1}x)\widetilde{D}_q F(x)\,\widetilde{d}_q x, \quad (A7)$$

$$\int_0^a F(q^{-1}x)\widetilde{D}_q G(x)\,\widetilde{d}_q x = F(a)G(a) - F(0)G(0) - \int_0^a G(qx)\widetilde{D}_q F(x)\,\widetilde{d}_q x. \quad (A8)$$

(A7) and (A8) with each other are symmetric with respect to the interchange of F and G.

Recall that a known q analog of the exponential function with convergence radius equal to infinity for each q is [42,50]

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{(q;q)_n} = (-x;q)_\infty, \tag{A9}$$

where $(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$ and $(a;q)_0 = 1$. In what follows, to abbreviate the formulas, we define the following q-exponential function with a tilde notation:

$$\widetilde{e}_q(x) \equiv \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]!} = E_{q^2}[(1-q^2)x]. \tag{A10}$$

Its associated hyperbolic sine and cosine functions are

$$\sinh_q x \equiv \frac{\widetilde{e}_q(x) - \widetilde{e}_q(-x)}{2} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)} x^{2n+1}}{[2n+1]!},$$
$$\cosh_q x \equiv \frac{\widetilde{e}_q(x) + \widetilde{e}_q(-x)}{2} = \sum_{n=0}^{\infty} \frac{q^{n(2n-1)} x^{2n}}{[2n]!}. \tag{A11}$$

It is now straightforward to show that, for a fixed parameter q, we have

$$\widetilde{D}_q \widetilde{e}_q(x) = \widetilde{e}_q(qx), \tag{A12}$$

$$\widetilde{D}_q \frac{1}{\widetilde{e}_q(x)} = \frac{-1}{\widetilde{e}_q(q^{-1}x)}. \tag{A13}$$

Equation (A12), regarding the explicit definition of $\widetilde{D}_q$ given in Eq. (A1), straightforwardly gives the following expression for $\widetilde{e}_{q^{-1}}(x)$:

$$\widetilde{e}_{q^{-1}}(x) = \prod_{k=0}^{\infty}(1 - xq^{-2k-1}(q^{-1}-q)), \quad \text{for} \quad q > 1. \tag{A14}$$

Finally, one can show that the following q-integral representation formula is held for the q-factorial $[n]! = [n][n-1]\cdots[1]$:

$$\int_0^\infty \frac{x^n}{\widetilde{e}_{q^{-1}}(qx)} \widetilde{d}_q x = q^{\frac{n(n+1)}{2}}[n]!, \quad \text{for} \quad q > 1. \tag{A15}$$

From (A14) we note that the integrand has no singular point in the interval of integration.

[1] E. Schrödinger, Der stetige Übergang von der Mikro-zur Makromechanik, Naturwissenschaften **14**, 664 (1926).
[2] R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. **131**, 2766 (1963).
[3] A. O. Barut and L. Girardello, New coherent states associated with non-compact groups, Commun. Math. Phys. **21**, 41 (1971).
[4] A. M. Perelomov, Coherent states for arbitrary Lie group, Commun. Math. Phys. **26**, 222 (1972).
[5] K. Wodkiewicz and J. H. Eberly, Coherent states, squeezed fluctuations, and the $SU(2)$ and $SU(1,1)$ groups in quantum-optics applications, J. Opt. Soc. Am. B **2**, 458 (1985).
[6] J. R. Klauder and B. S. Skagerstam, *Coherent States - Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[7] A. M. Perelomov, *Generalized Coherent States and their Applications* (Springer, Berlin, 1986).
[8] W. M. Zhang, D. H. Feng, and R. Gilmore, Coherent states: Theory and some applications, Rev. Mod. Phys. **62**, 867 (1990).
[9] D. H. Feng, J. R. Klauder, and M. R. Strayer, *Coherent States: Past, Present, and Future* (World Scientific, Singapore, 1994).
[10] S. T. Ali, J. P. Antoine, and J. P. Gazeau, *Coherent States, Wavelets and Their Generalizations* (Springer-Verlag, Berlin, 2000).
[11] L. Mandel, Sub-Poissonian photon statistics in resonance fluorescence, Opt. Lett. **4**, 205 (1979).
[12] L. Mandel, Squeezed States and Sub-Poissonian Photon Statistics, Phys. Rev. Lett. **49**, 136 (1982).
[13] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, Even and odd coherent states and excitations of a singular oscillator, Physica **72**, 597 (1974).
[14] M. Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Springer-Verlag, Berlin, 2007).
[15] M. Hillery, Amplitude-squared squeezing of the electromagnetic field, Phys. Rev. A **36**, 3796 (1987).
[16] Y. Xia and G. Guo, Nonclassical properties of even and odd coherent states, Phys. Lett. A **136**, 281 (1989).
[17] O. W. Greenberg, Example of Infinite Statistics, Phys. Rev. Lett. **64**, 705 (1990).
[18] R. N. Mahapatra, Infinite statistics and a possible small violation of the Pauli principle, Phys. Lett. B **242**, 407 (1990).
[19] M. Arik and D. D. Coon, Hilbert spaces of analytic functions and generalized coherent states, J. Math. Phys. **17**, 524 (1976).
[20] L. C. Biedenharn, The quantum group $SU_q(2)$ and a q-analogue of the boson operators, J. Phys. A **22**, L873 (1989).
[21] A. J. Macfarlane, On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, J. Phys. A: Math. Gen. **22**, 4581 (1989).
[22] H. Yan, q-deformed oscillator algebra as a quantum group, J. Phys. A: Math. Gen. **23**, L1155 (1990).
[23] W.-S. Chung, K.-S. Chung, S.-T. Nam, and C.-I. Um, Generalized deformed algebra, Phys. Lett. A **183**, 363 (1993).
[24] C. Daskaloyannis, Generalized deformed oscillator and nonlinear algebras, J. Phys. A: Math. Gen. **24**, L789 (1991).
[25] C. Daskaloyannis and K. Ypsilantis, A deformed oscillator with Coulomb energy spectrum, J. Phys. A: Math. Gen. **25**, 4157 (1992).
[26] C. Daskaloyannis, Generalized deformed oscillator corresponding to the modified Pöschl-Teller energy spectrum, J. Phys A: Math. Gen. **25**, 2261 (1992).
[27] A. Jannussis, G. Brodimas, D. Sourlas, and V. Zisis, Lett. Nuovo Cimento **30**, 123 (1981).
[28] V. I. Man'ko and R. V. Mendes, On the nonlinearity interpretation of q- and f-deformation and some applications, J. Phys. A: Math. Gen. **31**, 6037 (1998).

[29] M. Irac-Astaud and G. Rideau, Bargmann representations for deformed harmonic oscillators, Rev. Math. Phys. **10**, 1061 (1998).
[30] A. I. Solomon and J. Katriel, On q-squeezed states, J. Phys. A: Math. Gen. **23**, L1209 (1990).
[31] J. Katriel and A. I. Solomon, Generalized q-bosons and their squeezed states, J. Phys. A: Math. Gen. **24**, 2093 (1991).
[32] L.-M. Kuang and F.-B. Wang, The $\mathrm{su}_q(1,1)q$-coherent states and their nonclassical properties, Phys. Lett. A **173**, 221 (1993).
[33] L. C. Kwek and C. H. Oh, The general q-oscillator algebra and its coherent states, Czech. J. Phys. **48**, 1423 (1998).
[34] J.-S. Wang, B.-Y. Wang, and C.-Y. Sun, Even and odd qs-coherent states and their photon-statistical properties, Phys. Lett. A **246**, 464 (1998).
[35] K. A. Penson and A. I. Solomon, New generalized coherent states, J. Math. Phys. **40**, 2354 (1999).
[36] R. Parthasarathy and R. Sridhar, A diagonal representation of the quantum density matrix using q-boson oscillator coherent states, Phys. Lett. A **305**, 105 (2002).
[37] X.-M. Liu and C. Quesne, Even and odd q-deformed charge coherent states and their nonclassical properties, Phys. Lett. A **317**, 210 (2003).
[38] V. V. Eremin and A. A. Meldianov, The q-harmonic oscillator, coherent states, and the uncertainty relation, Theor. Math. Phys. **147**, 709 (2006).
[39] J. P. Gazeau and M. A. del Olmo, Pisot q-coherent states quantization of the harmonic oscillator, Annal. Phys. (NY) **330**, 220 (2013).
[40] S. Dey, A. Fring, L. Gouba, and P. G. Castro, Time-Dependent q-Deformed Coherent States for Generalized Uncertainty Relations, Phys. Rev. D **87**, 084033 (2013).
[41] S. Dey, q-Deformed Noncommutative Cat States and Their Nonclassical Properties, Phys. Rev. D **91**, 044024 (2015).
[42] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations* (Springer-Verlag, Berlin 1997).
[43] R. R. Puri and G. S. Agarwal, $SU(1,1)$ coherent states defined via a minimum-uncertainty product and an equality of quadrature variances, Phys. Rev. A **53**, 1786 (1996).
[44] R. J. Glauber, Photon Correlations, Phys. Rev. Lett. **10**, 84 (1963).
[45] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. **10**, 277 (1963).
[46] C. C. Gerry and P. L. Knight, Quantum superpositions and Schrödinger cat states in quantum optics, Am. J. Phys. **65**, 964 (1997).
[47] C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005).
[48] D. F. Walls, Squeezed states of light, Nature (London) **306**, 141 (1983).
[49] F.-L. Li, X.-S. Li, D. L. Lin, and T. F. George, Dynamics of an M-level atom interacting with cavity fields. III. Nonclassical behavior of the initially squeezed field, Phys. Rev. A **45**, 3133 (1992).
[50] E. G. Kalnins, W. Miller, and S. Mukherjee, Models of q-algebra representations: matrix elements of $U_q(su_2)$, Contemp. Math. **160**, 191 (1994).