Quantifying coherence in infinite-dimensional systems

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We investigate the quantification of coherence in the infinite-dimensional systems, and especially, we focus on the infinite-dimensional bosonic systems in the Fock space. We find that given the average energy constraints, the relative entropy of coherence serves as a well-defined quantification of coherence in the infinite-dimensional systems, however, the l_1 norm of coherence fails. Via using the relative entropy of coherence as the quantification of coherence, we generalize the case to multimode Fock spaces, and some special examples are considered. It is shown that with a finite average particle number, increasing the number of modes of light can enhance the relative entropy of coherence. With the mean energy constraint, our results can also be extended to other infinite-dimensional systems.

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I. INTRODUCTION

Quantum coherence arising from the quantum superposition principle is a fundamental aspect of quantum physics [1]. The laser [2] and superfluidity [3] are two famous examples of quantum coherence, whose effects are evident at the macroscopic scale. The quantum tomography and Leggett-Garg inequalities that test the correlations of a single system at different times [4] are shown useful for the test for the existence of quantum coherence [5]. However, the framework of the quantification of coherence has only been methodically investigated recently. The first attempt to address the classification of quantum coherence as physical resources is given by Baumgratz *et al.*, who have established a rigorous framework for the quantification of coherence based on distance measures in the finite-dimensional setting [6]. With such a foundational framework for coherence, one can find the appropriate distance measures to quantify the quantum coherence in a fixed basis by measuring the distance between the quantum state $\hat{\rho}$ and its nearest incoherent state. After the framework was proposed, it received increasing attention. Streltsov *et al.* used the entanglement to provide an operational quantification of coherence [7]. Du *et al.* focused on the interconversion of the coherent states by means of incoherent operations using the concept of majorization relations [8]. Xi *et al.* gave a clear quantitative analysis on the connections between relative entropy of coherence, quantum discord, and one-way quantum deficit in the bipartite quantum system [9]. Bromley *et al.* found the freezing conditions in which the coherence remains unchanged during the nonunitary dynamics [10]. Up to now, all the results for quantifying the quantum coherence assumed the finite-dimensional setting, which is neither necessary nor desirable [6]. In consideration of the relevant physical situations such as quantum optics states of light, further investigations are required on quantifying the quantum coherence in the infinite-dimensional systems.

In this paper, we aim to investigate the quantification of coherence in the infinite-dimensional systems. Specifically, we focus on the infinite-dimensional bosonic systems in the Fock space [11] which are used to describe the most notable quantum optics states of light [12] and Gaussian states [13–15]. We show that when considering the average energy constraints, the relative entropy of coherence serves as a well-defined quantification of coherence in the infinite-dimensional systems, but the l_1 norm of coherence fails. Via using the relative entropy of coherence, we generalize the results to the multimode Fock spaces, and special examples are considered. It is shown that with a finite average particle number, increasing the number of modes of light can enhance the relative entropy of coherence, which clearly shows the advantage of multimode quantum optics. Our results can also be extended to other infinite-dimensional systems with energy constraints. Our work investigates the experimentally relevant cases and the most easy-to-use quantifiers, which is significant and essential in quantum physics as well as quantum optics.

II. CONDITIONS FOR QUANTIFICATION OF COHERENCE IN FOCK SPACE

We consider a Hilbert space $\mathcal{H}$ with a finite dimension $D = \dim(\mathcal{H})$. For a fixed reference basis $\{|i\rangle\}$ of $\mathcal{H}$, we define $\hat{\delta} = \sum_i \delta_i |i\rangle\langle i|$ with an arbitrary set of nonnegative probabilities $\{\delta_i\}$ as an incoherent state and $\mathcal{I} \subset \mathcal{H}$ as a set of the incoherent states. As presented in Ref. [6], any proper measure of the coherence $C(\hat{\rho})$ must satisfy the following postulates:

(C1) $C(\hat{\rho}) \geqslant 0$ for $\forall$ density operator $\hat{\rho}$ defined in the Hilbert space $\mathcal{H}$ and $C(\hat{\delta}) = 0$ iff $\hat{\delta} \in \mathcal{I}$.

(C2a) Monotonicity under all the incoherent completely positive and trace-preserving (ICPTP) maps $\Phi_{\text{ICPTP}}(\circ)$: $C(\hat{\rho}) \geqslant C(\Phi_{\text{ICPTP}}(\hat{\rho}))$, where $\Phi_{\text{ICPTP}}(\circ) \equiv \sum_n \hat{K}_n \circ \hat{K}_n^\dagger$. Here $\{\hat{K}_n\}$ is a set of Kraus operators that satisfies $\sum_n \hat{K}_n^\dagger \hat{K}_n = \mathbb{I}$ and $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subset \mathcal{I}$.

(C2b) Monotonicity for the average coherence under the subselection based on measurement outcomes: $C(\hat{\rho}) \geqslant \sum_n p_n C(\hat{\rho}_n)$, where $\hat{\rho}_n = \hat{K}_n \hat{\rho} \hat{K}_n^\dagger / p_n$ and $p_n = \text{Tr}(\hat{K}_n \hat{\rho} K_n^\dagger)$ for all $\{\hat{K}_n\}$ with $\sum_n \hat{K}_n^\dagger \hat{K}_n = \mathbb{I}$ and $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subset \mathcal{I}$.

(C3) Nonincreasing under the mixing of quantum states: $\sum_n p_n C(\hat{\rho}_n) \geqslant C(\sum_n p_n \hat{\rho}_n)$.

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We note that conditions (C2b) and (C3) automatically imply condition (C2a) [6].

Two kinds of measures for coherence in the finite-dimensional systems [6] satisfy all the conditions mentioned above including the relative entropy of coherence defined as

$$C_{\text{rel.ent.}}(\hat{\rho}) = S(\hat{\rho}_{\text{diag}}) - S(\hat{\rho}), \tag{1}$$

and the l_1 norm of coherence defined as

$$C_{l_1}(\hat{\rho}) = \sum_{i \neq j} |\rho_{ij}|, \tag{2}$$

where $\hat{\rho} = \sum_{ij} \rho_{ij} |i\rangle\langle j|$, $\hat{\rho}_{\text{diag}} = \sum_i \rho_{ii} |i\rangle\langle i|$, and $S(\hat{\rho}) = -\text{Tr}(\hat{\rho}\log\hat{\rho})$ is the von-Neumann entropy. It has been shown that the promising fidelity of coherence does not in general satisfy (C2b) under the subselection of the measurement condition [16].

We then generalize the problem to the infinite-dimensional systems, furthermore, we consider the Fock space. Generally, the bosonic single mode Hilbert space $\mathcal{H}$ is spanned by the basis $\{|n\rangle\}_{n=0}^{\infty}$ called the Fock (number state) basis where n is an integer. Fock states are the eigenstates of the number operator $\hat{n} := \hat{a}^\dagger\hat{a}$ where we have $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. Referring to the development of the entanglement theory in infinite-dimensional systems, the problem of quantification of coherence can be addressed by requiring energy constraints [17], which is experimentally relevant. The quantification of the quantum coherence may be divergent if no energy constraints are assumed, so we require a new condition for this case:

(C4) If the first-order moment, the average particle number, is finite $\langle\hat{n}\rangle < \infty$, it should fulfill $C(\hat{\rho}) < \infty$.

Given the proper definition of incoherent states, in the infinite-dimensional systems, incoherent operations and maximal coherent states, the proofs of (C1)–(C3) of these two definitions (1) and (2) do not require the finite-dimensional setting. It is because there are very relevant physical situations that require infinite-dimensional systems for their description. The incoherent states and incoherent operations defined in Ref. [6] can be easily generalized to the case in infinite-dimensional systems. In the Fock space, the set of incoherent states can be defined as $\mathcal{I} \subset \mathcal{H}$ and all density operators $\hat{\delta} \in \mathcal{I}$ are of the form $\hat{\delta} = \sum_{n=0}^{\infty} \delta_n |n\rangle\langle n|$. For (C2), Kraus operators $\{\hat{K}_n\}$ satisfy $\sum_{n=0}^{\infty} \hat{K}_n^\dagger \hat{K}_n = \mathbb{I}$ and $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subset \mathcal{I}$ are $d_n \times d_{\text{in}}$ matrices where $d_{\text{in}} \to \infty$. Given these premises, our problem turns to verifying condition (C4): whether these quantifications of coherence fulfilling (C1)–(C3) can serve as a unit for coherence or be finite $C(\rho) < \infty$, when the energy constraint is taken into consideration. That is, incoherent states, maximal coherent states, and the maximum quantification of coherence need to be carefully tested.

III. RELATIVE ENTROPY OF COHERENCE

In this section, we show that the relative entropy of coherence $C_{\text{rel.ent.}}$ fulfills the requirements of the quantification of coherence for the states in the infinite-dimensional Hilbert space. At the beginning, it is easy to find that the diagonal mixed states such as thermal states have zero coherence $C_{\text{rel.ent.}} = 0$. When the mean particle number is finite $\bar{n} := \langle\hat{n}\rangle < \infty$, we can figure out the maximal coherent state as

$$|\psi_m\rangle = \sum_{n=0}^{\infty} \frac{\bar{n}^{n/2}}{(\bar{n}+1)^{(n+1)/2}} e^{i\varphi_n} |n\rangle, \tag{3}$$

which makes (C4) saturated:

$$C_{\text{rel.ent.}}^{\max} = (\bar{n}+1)\log(\bar{n}+1) - \bar{n}\log\bar{n} < \infty. \tag{4}$$

This result can be directly obtained from the fact that the thermal state as $\hat{\rho}^{\text{th}}(\bar{n}) = \sum_{n=0}^{\infty} (\bar{n}^n/(\bar{n}+1)^{n+1})|n\rangle\langle n|$ reaches the maximum von Neumann entropy. Therefore, we can conclude that relative entropy of coherence is an appropriate quantification of the coherence in the infinite-dimensional systems. ■

Hereafter, the relative entropy shown in Figs. 1 and 2 is calculated by using the natural logarithm function ln. The normalized second-order correlation function of the maximal coherent state (3) can be calculated as $g^2(0) = \langle\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\rangle/\langle\hat{n}\rangle^2 = 2$ which is the same as the thermal state. Given a linear phase generation $\varphi_n = n\varphi$, the state (3), a pure state with a thermal distribution (PSTD), has been shown to be the eigenstate of the SG-phase operator $\sum_{n=0}^{\infty} |n\rangle\langle n+1|$ with eigenvalue $\sqrt{\bar{n}/(\bar{n}+1)}e^{i\varphi}$ [18]. A proposal of the generation of PSTD in the "photon box" [19] has been also presented in Ref. [18]. Compared with two well-known Gaussian states, the coherent state $|\alpha\rangle := \hat{D}(\alpha)|0\rangle$ and the squeezed vacuum state $|0,\xi\rangle = \hat{S}(\xi)|0\rangle$, the particle number distributions and the coherence quantifications of relative entropy are shown in Figs. 1(a) and 1(b), respectively. The PSTD has the geometric distribution, the coherent state has the Poisson distribution, and the squeezed vacuum state has zero distributions for odd photon numbers. Given different average photon numbers, the

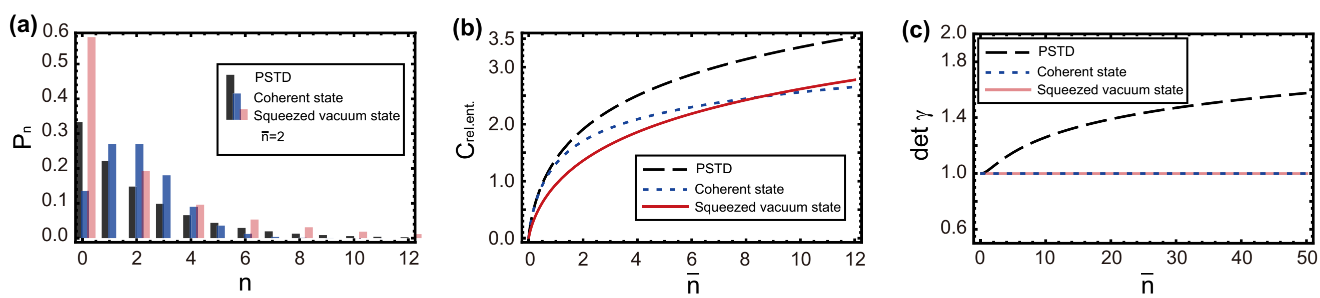

FIG. 1. (a) Photon number distributions of the PSTD, the coherent state, and the squeezed vacuum state against average particle number. (b) Relative entropies of coherence of the PSTD, the coherent state, and the squeezed vacuum state against average particle number. (c) Determinants of the covariance matrices γ of these three states against the mean particle number.

PSTD always has the largest relative entropy of coherence. The squeezed state has a larger relative entropy of coherence that the coherent state for the larger average photon number. In Fig. 1(c), the determinants of the coherence variance matrices γ of these three states against the mean particle number are given. Since a Gaussian state is pure iff $\det\gamma = 1$ [13–15], we conclude that the PSTD with form (3) is a non-Gaussian state, except for $\bar{n} \to 0$. Therefore, the PSTD cannot be easily constructed by acting the squeezing and displacement operators on the vacuum state. For details, please see the appendix.

Furthermore, in the infinite-dimensional systems, we consider the interconversion between relative entropy of coherence $C_{\text{rel.ent.}}$ and relative entropy of entanglement $E_{\text{rel.ent.}}$ [20]. It is shown in Ref. [7] that the nonzero coherence of a system can be converted to the entanglement between the system (S) and an initially incoherent ancilla (A). For the contractive distance to be the relative entropy and $\dim(\text{A}) \geqslant \dim(\text{S})$, there exits an incoherent operation $\Phi^{\text{SA}}_{\text{ICPTP}}(\circ)$ on the combined system such that

$$E_{\text{rel.ent.}}\left[\Phi^{\text{SA}}_{\text{ICPTP}}(\hat{\rho} \otimes |0\rangle_{\text{A}}\langle 0|)\right] = C_{\text{rel.ent.}}(\hat{\rho}). \tag{5}$$

To generalize this result to the infinite-dimensional system, we choose $\dim(\text{A}) \to \infty$ and consider the incoherent operation as $\Phi^{\text{SA}}_{\text{ICPTP}}(\circ) = U \circ U^\dagger$ with the unitary operation,

$$U = \sum_{i,j=0}^{\infty} |i\rangle\langle i| \otimes |i+j\rangle_{\text{A}}\langle j|. \tag{6}$$

Therefore, for the infinite-dimensional systems, we find the connection between the relative entropy of coherence and the relative entropy of entanglement. For instance, we consider the maximal coherent state (3) and obtain

$$\Phi^{\text{SA}}_{\text{ICPTP}}(|\psi_m\rangle\langle\psi_m| \otimes |0\rangle_{\text{A}}\langle 0|) = |\text{TMSV}\rangle\langle\text{TMSV}|, \tag{7}$$

where $|\text{TMSV}\rangle \equiv \sum_{n=0}^{\infty} \frac{\bar{n}^{n/2}}{(\bar{n}+1)^{(n+1)/2}} e^{i\varphi_n}|n\rangle|n\rangle_{\text{A}}$ is the two-mode squeezed vacuum (TMSV) state and maximizes the relative entropy of entanglement, given a finite average particle number.

IV. l_1 NORM OF COHERENCE

Next, we show that the l_1 norm of coherence does not satisfy (C4) in the infinite-dimensional systems. With a set of the particle number distributions $\{P_n \in [0,1]\}$ of a pure state, the identity condition, $\sum_{n=0}^{\infty} P_n = 1$, and the finite energy constraint (C4), $\sum_{n=0}^{\infty} nP_n = \bar{n} < \infty$, are two constraint conditions. Here, we only consider the pure states, because for any mixed state, we can find a pure state with the same particle number distribution that achieves a larger l_1 norm of coherence. That is, given the finite energy constraint, the possible maximal coherent state for the maximal l_1 norm of coherence should be a pure state.

The l_1 norm of coherence of a pure state with $\{P_n\}$ can be written as

$$C_{l_1}(\hat{\rho}) = \sum_{m,n=0}^{\infty} \sqrt{P_m P_n} - 1. \tag{8}$$

The maximum of l_1 norm of coherence should occur as the first variation is zero,

$$\delta C_{l_1} = \sum_{m,n=0}^{\infty} \sqrt{P_m/P_n}\,\delta P_n = 0, \tag{9}$$

with two constraints,

$$\sum_{n=0}^{\infty} \delta P_n = 0, \text{ and } \sum_{n=0}^{\infty} n\delta P_n = 0. \tag{10}$$

Using the method of Lagrange multipliers with two Lagrange multipliers λ_1 and λ_2 to combine Eqs. (9) and (10), a series of equations can be obtained for n being the non-negative integers:

$$\frac{\sum_{m=0}^{\infty} \sqrt{P_m}}{\sqrt{P_n}} + \lambda_1 n + \lambda_2 = 0, \tag{11}$$

the solutions of which, also called Karush-Kuhn-Tucker (KKT) conditions [21], are written as $\{P_n = (\frac{\sum_{m=0}^{\infty}\sqrt{P_m}}{\lambda_1 n + \lambda_2})^2\}$. If we assume that $\mathcal{S} \equiv \sum_{m=0}^{\infty}\sqrt{P_m}$ is finite, we can obtain that $\mathcal{S} = \sum_{m=0}^{\infty}[\mathcal{S}/(\lambda_1 n + \lambda_2)]$ relates to the Riemann Zeta function [22] and is infinite. Therefore, the solutions that fulfill Eq. (11) obviously are not a set of particle number distributions. Moreover, mathematically, l_1 norm of coherence (8) is a concave function in probability space which makes the KKT conditions also sufficient for the optimality [21]. Then, we can conclude that analytically for the optimal problem of the l_1 norm of coherence, no maximal coherent state that satisfies C(4) can be derived. Therefore, the l_1 norm of coherence does not seem to be a well-defined quantification of coherence in Fock space, because it does not have a well-defined maximal coherent state, which completes the proof. ■

We here note that with a stronger condition (C4′), in which the second-order moment is finite $\langle\hat{n}^2\rangle < \infty$, we can find a well-defined maximal coherent state for the l_1 norm of coherence and $C(\hat{\rho}) < \infty$ could be met.

V. RELATIVE ENTROPY OF COHERENCE IN MULTIMODE FOCK SPACE

We have shown that the relative entropy of coherence $C_{\text{rel.ent.}}$ fulfills the requirements of the quantification of coherence even for the states in the single-mode Fock space. Then, we generalize this result to the d-mode Fock space $\mathcal{H} = \otimes_{i=1}^{d}\mathcal{H}_i$. It has the basis $\{|\boldsymbol{n}\rangle = \otimes_{i=1}^{d}|n_i\rangle_i\}$ consisting of the multimode Fock states and the probability distributions $\{P_{\boldsymbol{n}}\}$, where the vector is defined as $\boldsymbol{n} = (n_1, n_2 \ldots, n_d)$ and we define $|\boldsymbol{n}|_1 = \sum_{i=1}^{d} n_i$. After simple calculations, the maximal coherent state should have a distribution as $P_{\boldsymbol{n}}^{\max} = \bar{n}_{\text{t}}^{|\boldsymbol{n}|_1}/[(\bar{n}_{\text{t}}+1)^{|\boldsymbol{n}|_1+1}\mathbb{C}^{d-1}_{|\boldsymbol{n}|_1+d-1}]$ with the finite average total particle number written as $\bar{n}_{\text{t}} \equiv \sum_{\boldsymbol{n}} P_{\boldsymbol{n}}|\boldsymbol{n}|_1$ and can be written as

$$\left|\psi_m^d\right\rangle = \sum_{\boldsymbol{n}} \frac{\bar{n}_{\text{t}}^{|\boldsymbol{n}|_1/2}}{\left[(\bar{n}_{\text{t}}+1)^{|\boldsymbol{n}|_1+1}\mathbb{C}^{d-1}_{|\boldsymbol{n}|_1+d-1}\right]^{1/2}}|\boldsymbol{n}\rangle. \tag{12}$$

The maximum relative entropy of coherence for d-mode Fock space can be calculated as

$$C^{\max,d}_{\text{rel.ent.}} = C^{\max,d=1}_{\text{rel.ent.}} + S_d(\bar{n}_{\text{t}}), \tag{13}$$

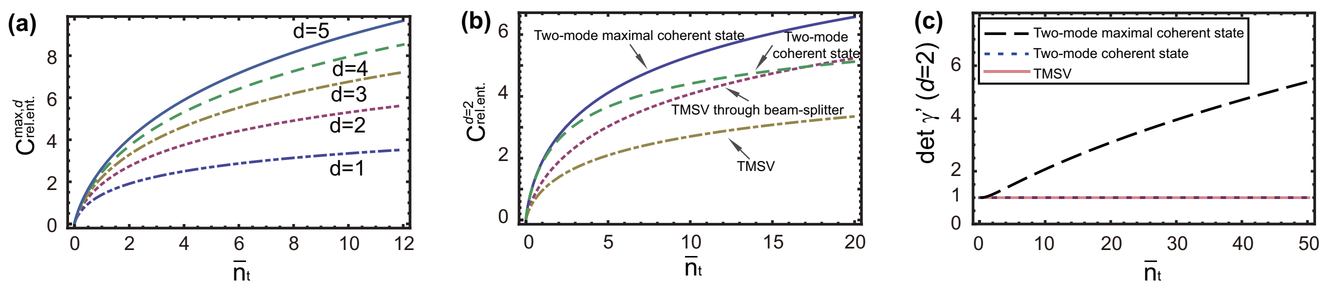

FIG. 2. Relative entropies of coherence of multimode states. (a) Relative entropies of coherence of maximal coherent states with $d = 1,2\ldots,5$. (b) For two-mode states $d = 2$, relative entropies of coherence of the maximal coherent state, two-mode coherent state, TMSV, and TMSV through a 50:50 beam splitter. (c) For two-mode states $d = 2$, determinants of the covariance matrices γ' of these three states against the total mean particle number.

where $S_d(\bar{n}_t) \equiv \sum_{n=0}^{\infty}[\bar{n}_t^n/(\bar{n}_t+1)^{n+1}]\log(\mathbb{C}_{n+d-1}^{d-1})$ is a convergent series, and $C_{\text{rel.ent.}}^{\max,d=1}$ is given in Eq. (4). Since $S_d(\bar{n}_t) > S_{d'}(\bar{n}_t)$ if $d > d'$, we show in Fig. 2(a) that given a fixed average total particle number $\bar{n}_t$, the relative entropy of coherence increases as the number of modes d increases. This result is significant because with a finite average particle number, increasing the number of modes of light can enhance the coherence as a resource in quantum information processing. Also the advantages of multimode quantum optics have been recently interpreted in the quantum metrology [23], optical quantum computation [24], and other photonic technologies [25,26].

For comparison, we then consider the two-mode coherent state, the two-mode squeezed vacuum (TMSV) state, and the TMSV passing a 50:50 beam splitter as special examples. The probability of the last one has been shown to be efficient to beat the shot noise limit (SNL) in the quantum metrology [27]. The TMSV can be written as $|\text{TMSV}\rangle = \sum_{n=0}^{\infty}(\bar{n}_t/2)^{\frac{n}{2}}/(\bar{n}_t/2+1)^{\frac{n+1}{2}}|n\rangle_1|n\rangle_2$, and the TMSV through a 50:50 beam splitter is written as [22,27]

$$\hat{U}_{\text{BS}}|\text{TMSV}\rangle = \sum_{n=0}^{\infty}\frac{\left(\frac{\bar{n}_t}{2}\right)^{\frac{n}{2}}}{\left(\frac{\bar{n}_t}{2}+1\right)^{\frac{n+1}{2}}}\sum_{k=0}^{n}(-1)^k \times \frac{\mathbb{C}_n^k[(2n-2k)!(2k)!]^{\frac{1}{2}}}{2^n n!}|2n-2k\rangle_1|2k\rangle_2, \tag{14}$$

where $\hat{U}_{\text{BS}} := \exp[i\pi(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)/2]$ is the unitary transformation of a 50:50 beam splitter with $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) the annihilation (creation) operators for two modes, respectively. Compared with the maximal coherent state for $d = 2$, we show in Fig. 2(b) the relative entropies of coherence of these three Gaussian two-mode states against the total average particle number $\bar{n}_t$. It is obviously shown that the TMSV through a 50:50 beam splitter has a larger coherence than TMSV. The reason may be that in each subspace with a definite total photon number, the state after the beam splitter is spanned by more bases. Moreover, we show in the appendix and in Fig. 2(c) that the two-mode maximal coherent state is not a Gaussian state.

VI. CONCLUSIONS

In conclusion, we have investigated the quantifications of coherence in the infinite-dimensional systems, as there are very relevant physical situations that require the infinite-dimensional systems for their descriptions. A new constraint condition (C4) has been suggested for this problem, with which the relative entropy of coherence has been shown to be a well-defined quantification of coherence in infinite-dimensional systems, but the l_1 norm of coherence fails. We have also considered quantifying the coherence in the multimode Fock space. Given a fixed average total particle number, the relative entropy of coherence increases as the number of modes increases, which is significant because the coherence as a resource in quantum information processing is larger when increasing the number of modes. This work investigates experimentally relevant infinite-dimensional systems and the most general and easy to use quantifiers, which is important for experimental and theoretic applications in quantum physics as well as quantum optics. Moreover, our results can be easily extended to other infinite-dimensional systems.

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APPENDIX A: RELATIVE ENTROPY OF COHERENCE OF THE COHERENT STATE AND THE SQUEEZED VACUUM STATE

The well-known coherent state can be written as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\alpha^n/\sqrt{n!}|n\rangle, \tag{A1}$$

with the particle number distribution $P_n^{\text{cs}} = e^{-\bar{n}}\bar{n}^n/n!$ and $\bar{n} = |\alpha|^2$. The relative entropy as a quantification of coherence can be calculated as

$$C_{\text{rel.ent.}}^{\text{cs}} = e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n\log n!}{n!} - \bar{n}\log\frac{\bar{n}}{e}, \tag{A2}$$

which is shown in Fig. 1(b). A squeezed state $|\alpha,\xi\rangle$ may be generated by first acting with the squeeze operator $\hat{S}(\xi)$ on the vacuum followed by the displacement operator $\hat{D}(\alpha)$ with particle number distribution ($\xi = re^{i2\phi}$) [28],

$$P_n^{\rm ss} = \frac{\exp\left[-|\alpha|^2 - \frac{1}{2}\tanh r(\alpha^{*2}e^{i\phi} + \alpha^2 e^{-i\phi})\right]}{2^n n!\cosh r}\tanh^n r\left|H_n\left(\frac{\alpha + \alpha^* e^{i\phi}\tanh r}{\sqrt{2e^{i\phi}\tanh r}}\right)\right|^2, \tag{A3}$$

where $H_n(z)$ is the nth Hermite polynomial. For the squeezed vacuum state, $\alpha = 0$ and $|H_n(0)| = 2^{n/2}(n-1)!!$ when n is even, we obtain

$$P_n^{\rm sv} = \frac{\tanh^n r[(n-1)!!]^2}{n!\cosh r}, \tag{A4}$$

where $\bar{n} = \sinh^2 r$. Then we can calculate the relative entropy using $C_{\rm rel.ent.}^{\rm sv} = \sum_{n=0}^{\infty} P_n^{\rm sv}\log P_n^{\rm sv}$ in Fig. 1(b).

APPENDIX B: COVARIANCE MATRIX OF PSTD AND THE TWO-MODE MAXIMAL COHERENT STATE

Canonical variables can be written in terms of creation and annihilation operators as

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger),\ \hat{p} = \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger),\ \hbar = 1. \tag{B1}$$

For the state in the one-mode Fock space, by defining a vectorial operator $\boldsymbol{R} = (\hat{x},\hat{p})^T$, we can calculate the covariance matrix as follows:

$$\gamma = 2\begin{pmatrix}{\rm Cov}_\rho(\hat{x},\hat{x}) & {\rm Cov}_\rho(\hat{x},\hat{p})\\ {\rm Cov}_\rho(\hat{p},\hat{x}) & {\rm Cov}_\rho(\hat{p},\hat{p})\end{pmatrix} - iJ_1 \tag{B2}$$

$$= 2\begin{pmatrix}\bar{n} + \frac{1}{2} + \langle\hat{a}^2\rangle - 2\langle\hat{a}\rangle^2 & 0\\ 0 & \bar{n} + \frac{1}{2} - \langle\hat{a}^2\rangle\end{pmatrix}, \tag{B3}$$

where $J_1 = \binom{0\ \ 1}{-1\ 0}$, the covariance of two operators is defined as ${\rm Cov}_\rho(\hat{A},\hat{B}) \equiv {\rm Tr}(\rho\hat{A}\hat{B}) - {\rm Tr}(\rho\hat{A}){\rm Tr}(\rho\hat{B})$, and

$$\langle\hat{a}^2\rangle = \frac{\bar{n}}{(\bar{n}+1)^2}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n\sqrt{n+2}\sqrt{n+1}, \tag{B4}$$

$$\langle\hat{a}\rangle = {\rm Li}_{-\frac{1}{2}}\left(\frac{\bar{n}}{\bar{n}+1}\right)\Big/\sqrt{\bar{n}(\bar{n}+1)}, \tag{B5}$$

with ${\rm Li}_k(z) = \sum_{n=1}^{\infty} z^n/n^k$ the polylogarithm function. The determinant of the covariance matrix in Eq. (B3) is calculated numerically and shown in Fig. 1(c) against the mean particle number $\bar{n}$. Then, we conclude the PSTD is non-Gaussian for $\bar{n} > 0$.

Similarly, for the two-mode maximal coherent state,

$$\left|\psi_m^{d=2}\right\rangle = \sum_{n=0}^{\infty}\left[\frac{\bar{n}_{\rm t}^n}{(\bar{n}_{\rm t}+1)^{n+1}(n+1)}\right]^{\frac{1}{2}}\sum_{k=0}^{n}|k\rangle_1|n-k\rangle_2, \tag{B6}$$

the covariance matrix, with $\hat{b}^\dagger$ and $\hat{b}$ the creation and annihilation operators of the second mode, can be written as

$$\gamma' = 2\begin{pmatrix}{\rm Cov}_\rho(\hat{x}_1,\hat{x}_1) & {\rm Cov}_\rho(\hat{x}_1,\hat{p}_1) & {\rm Cov}_\rho(\hat{x}_1,\hat{x}_2) & {\rm Cov}_\rho(\hat{x}_1,\hat{p}_2)\\ {\rm Cov}_\rho(\hat{p}_1,\hat{x}_1) & {\rm Cov}_\rho(\hat{p}_1,\hat{p}_1) & {\rm Cov}_\rho(\hat{p}_1,\hat{x}_2) & {\rm Cov}_\rho(\hat{p}_1,\hat{p}_2)\\ {\rm Cov}_\rho(\hat{x}_2,\hat{x}_1) & {\rm Cov}_\rho(\hat{x}_2,\hat{p}_1) & {\rm Cov}_\rho(\hat{x}_2,\hat{x}_2) & {\rm Cov}_\rho(\hat{x}_2,\hat{p}_2)\\ {\rm Cov}_\rho(\hat{p}_2,\hat{x}_1) & {\rm Cov}_\rho(\hat{p}_2,\hat{p}_1) & {\rm Cov}_\rho(\hat{p}_2,\hat{x}_2) & {\rm Cov}_\rho(\hat{p}_2,\hat{p}_2)\end{pmatrix} - iJ_1 \oplus J_1 \tag{B7}$$

$$= 2\begin{pmatrix}\bar{n}' + \frac{1}{2} + \langle\hat{a}^2\rangle - 2\langle\hat{a}\rangle^2 & 0 & \langle\hat{a}\hat{b}\rangle + \langle\hat{a}\hat{b}^\dagger\rangle - 2\langle\hat{a}\rangle^2 & 0\\ 0 & \bar{n}' + \frac{1}{2} - \langle\hat{a}^2\rangle & 0 & \langle\hat{a}\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle\\ \langle\hat{a}\hat{b}\rangle + \langle\hat{a}\hat{b}^\dagger\rangle - 2\langle\hat{a}\rangle^2 & 0 & \bar{n}' + \frac{1}{2} + \langle\hat{b}^2\rangle - 2\langle\hat{b}\rangle^2 & 0\\ 0 & \langle\hat{a}\hat{b}^\dagger\rangle - \langle\hat{a}\hat{b}\rangle & 0 & \bar{n}' + \frac{1}{2} - \langle\hat{b}^2\rangle\end{pmatrix}, \tag{B8}$$

where $\bar{n}' = \bar{n}_{\rm t}/2$. Here, we have used the fact that

$$\langle\hat{a}^2\rangle = \sum_{n=0}^{\infty}\frac{\bar{n}_{\rm t}^{n+1}}{(\bar{n}_{\rm t}+1)^{n+2}}\sum_{k=0}^{n}\sqrt{\frac{(k+1)(k+2)}{(n+1)(n+3)}}, \tag{B9}$$

$$\langle\hat{a}\rangle = \sum_{n=0}^{\infty}\frac{\bar{n}_{\rm t}^{n+\frac{1}{2}}}{(\bar{n}_{\rm t}+1)^{n+\frac{3}{2}}}\sum_{k=0}^{n}\sqrt{\frac{(k+1)}{(n+1)(n+2)}}, \tag{B10}$$

$$\langle \hat{a}\hat{b} \rangle = \sum_{n=0}^{\infty} \frac{\bar{n}_t^{n+1}}{(\bar{n}_t+1)^{n+2}} \sum_{k=0}^{n} \sqrt{\frac{(k+1)(n-k+1)}{(n+1)(n+3)}}, \tag{B11}$$

and

$$\langle \hat{a}\hat{b}^\dagger \rangle = \sum_{n=0}^{\infty} \frac{\bar{n}_t^{n}}{(\bar{n}_t+1)^{n+1}} \sum_{k=0}^{n} \frac{\sqrt{(k+1)(n-k)}}{n+1}. \tag{B12}$$

The determinant of the covariance matrix in Eq. (B8) is calculated numerically and shown in Fig. 2(c) against the total mean particle number $\bar{n}_t$.

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