Quantum adiabatic evolution with energy degeneracy levels

Qi Zhang

Wilczek Quantum Center and College of Science, Zhejiang University of Technology, Hangzhou 310023, People's Republic of China

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A classical-kind phase-space formalism is developed to address the tiny intrinsic dynamical deviation from what is predicted by Wilczek–Zee theorem during quantum adiabatic evolution on degeneracy levels. In this formalism, the Hilbert space and the aggregate of degenerate eigenstates become the classical-kind phase space and a high-dimensional subspace in the phase space, respectively. Compared with the previous analogous study by a different method, the current result is qualitatively different in that the first-order deviation derived here is always perpendicular to the degeneracy subspace. A tripod-scheme Hamiltonian with two degenerate dark states is employed to illustrate the adiabatic deviation with degeneracy levels.

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I. INTRODUCTION

Quantum adiabatic evolution is always of fundamental interests to physicists since the discovery of quantum adiabatic theorem [1]. It predicts the general and fundamental behavior of a quantum system under slow external driving. Due to the approximated nature of the adiabatic theorem, intrinsic deviation from what is predicted by the adiabatic theorem inevitably arises [2]. Although most of attention has been paid to the adiabatic deviations and conditions in the case of nondegeneracy energy spectra, which seems theoretically more fundamental and simple [3], the study of adiabatic evolution for degeneracy spectrum may be more important in a practical sense, which is more related to holonomic quantum computation and detection of fractional statistics. The study of degenerate adiabatic deviation may be closely associated with assessing the feasibility of topological gates using the concept of Majorana non-Abelian braiding [4].

Because the analytical formulas about the adiabatic deviation in the case of nondegeneracy energy spectra are already very complicated, few people have ever touched upon that for the formidable degeneracy-spectrum case [5–7]. The difficulty mostly comes from the abstract nature of Hilbert space and the quantum kind of formulas. In this paper, we focus on the adiabatic deviations during which the state under study is always degenerate with other orthogonal states by projecting the Hilbert space onto a phase-space of classical form and mapping the eigenstates onto fixed points in the phase-space thus defined, which simplifies greatly the analytical expressions and enables a visual comprehension of the deviations.

In the current theory the aggregate of eigenstates in the degeneracy subspace forms a patch in phase space rather than an isolated point in the nondegenerate case. The patch can be high dimensional according to the degree of degeneracy. With the overall phase of the wave function omitted, each point on the degeneracy subspace is a fixed point of classical form of Hamiltonian. In the first-order theory with respect to slow adiabatic speed, which is of overwhelming importance, the difference between the real state and what is predicted by the Wilczek–Zee theory [8] is always perpendicular to the degeneracy patch (see Fig. 1), with the distance between the average of the oscillating deviation and the Wilczek–Zee point being proportional to the adiabatic speed [see Fig. 2(b)].

In higher-order formulation, the deviation may have components in the degeneracy subspace if, more intuitively and physically, the degeneracy subspace in the whole phase space changes its normal direction during the adiabatic manipulation. We use the example of the tripod scheme, where three laser beams are interacting with a free Rubidium atom, to verify our theory. Theoretically, the tripod scheme has been introduced to implement the non-Abelian vector potential and spin-orbit coupling on neutral atoms [9].

Technically we take advantage of the classical Hamiltonian formulation of the Schrödinger equation [10–12]. Note that this classical formulation is purely mathematical and is *not* the traditional semiclassical limit $\hbar \to 0$.

II. GENERAL QUANTUM DYNAMICS IN THE FORM OF CLASSICAL DYNAMICS

We consider a quantum system described by the Hamiltonian $\hat{H}(R)$, where $R = R(t)$ represents time-dependent parameters in an adiabatic protocol. Different from the ordinary systems, here $\hat{H}(R)$ has a discrete degeneracy spectrum during the entire control protocol. In the case that the system is initially prepared on the degeneracy levels, the adiabatic evolution (geometric phase) can be described by the Wilczek–Zee phase [8]. However, it has been proven that, so long as the protocol is not executed in the mathematical limit $\dot{R} \to 0$ [5–7], the deviation from what is predicted by Wilczek–Zee theory should be expected.

For simplicity and concrete discussion, we assume $\hat{H}(R)$ lives in a finite n-dimensional Hilbert space with two eigenstates being degenerate (the generalization to higher degree of degeneracy is straightforward). Suppose $|D_1\rangle$ and $|D_2\rangle$ are the two degenerate states of $\hat{H}(R)$, any state of the form $c_1|D_1\rangle + c_2|D_2\rangle$ ($|c_1|^2 + |c_2|^2 = 1$) would be an eigenstate. In the energy representation, a state can be expressed as $c_1|D_1\rangle + c_2|D_2\rangle + \sum_{i=3}^{n} c_i|S_i\rangle$, with $|S_i\rangle (i = 3, \ldots, n)$ being nondegeneracy levels.

Due to the abstractness of c numbers in Hilbert space, we alternatively employ real quantities and the corresponding classical form of phase space to address the quantum state in Hilbert space. In particular, in energy representation, we can define the classical phase-space point

$$P_i = \arg(c_{i+1}) - \arg(c_1), \quad Q_i = |c_{i+1}|^2, \tag{1}$$

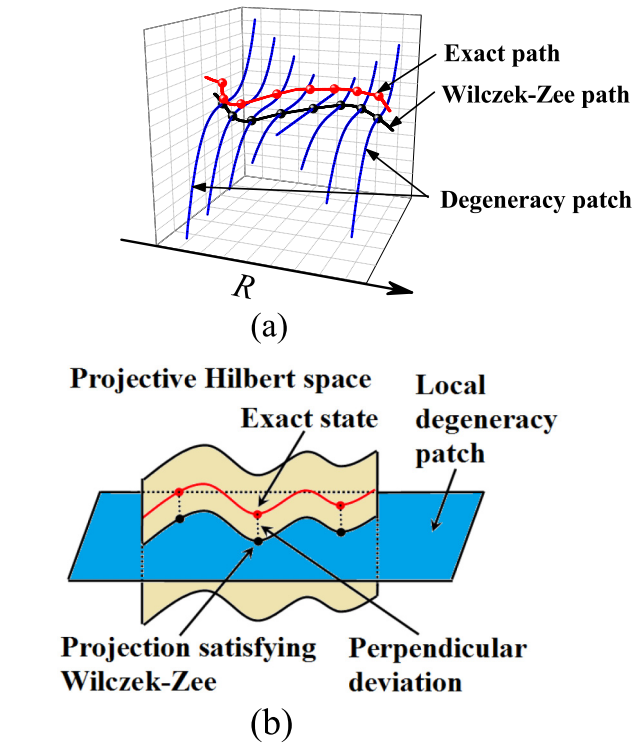

FIG. 1. Illustration of the exact state from first-order calculation and Wilczek–Zee state on the degeneracy subspace. (a) For convenience, the parameter-dependent degeneracy patch is demonstrated by a one-dimensional curve, although it is in fact a $2(m-1)$-dimensional curved patch. The red curve out of the patch and the black curve on the patch are the trajectory of the exact state and that of the Wilczek–Zee state during the degenerate adiabatic evolution. (b) From the perspective of an observer who moves with the patch, a local regime of the patch which seems fixed is illustrated by a two-dimensional patch. According to the first-order perturbation theory, the projection of the exact state on the degeneracy subspace is just the Wilczek–Zee state, which means that the nonadiabatic deviation has a zero first-order component in the degeneracy subspace.

with $i = 1,2,\ldots,n-1$, to describe the wave function and the phase space without any approximation. With the phase space thus defined, the degeneracy region is the patch spanned by (P_1, Q_1) in the phase-space.

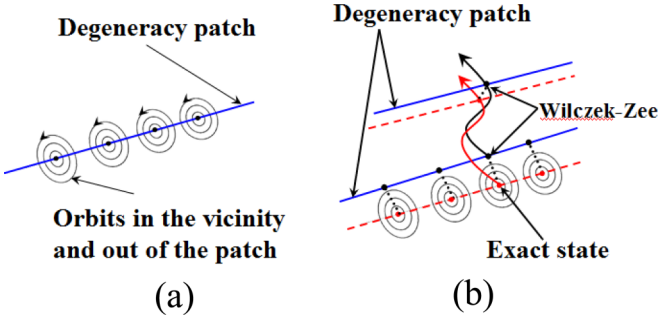

FIG. 2. (a) Exact-state orbits for degeneracy quantum system with fixed parameters. (b) Exact-state orbits and evolving degeneracy subspace for degeneracy quantum system during adiabatic evolution. To illustrate the orbits and degeneracy subspace via a three-dimensional imagination, here the one-dimensional blue lines stand for the instantaneous degeneracy subspace.

Next, let us turn to the evolution of the wave function, which definitely satisfies the Schrödinger equation defined by $\hat{H}(R)$. In order to gain an insight into the dynamics in the perspective of classical form phase space, we should express the Schrödinger equation via the variables (P_i, Q_i) instead of the initial form of the wave function. Generally, as the unitary matrix to diagonalize $\hat{H}(R)$ is R (time) dependent, it is more convenient to take advantage of a general but fixed representation, which may be more physically relevant, rather than the energy representation. In the fixed representation, a wave function can be expressed as

$$|\psi\rangle = \sum_{i=1}^{n} a_i |B_i\rangle, \tag{2}$$

with $|B_i\rangle$ being the orthogonal bases in Hilbert space. The classical phase-space and phase-space point is defined by

$$p_i = \arg(a_{i+1}) - \arg(a_1), \quad q_i = |a_{i+1}|^2, \tag{3}$$

with $i = 1,2,\ldots,n-1$. (P_i, Q_i) and (p_i, q_i) are related via an R-dependent canonical transformation. By construction, the Schrödinger equation then yields the following Hamilton's equations of motion without any approximation [10–12]:

$$\frac{dp_i}{dt} = -\frac{\partial H(R)}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H(R)}{\partial p_i}, \tag{4}$$

where the classical Hamiltonian $H(R)$ is obtained from the quantum Hamiltonian $\hat{H}(R)$ via

$$H(R) = \langle\psi|\hat{H}(R)|\psi\rangle. \tag{5}$$

With the overall phase removed, the phase space in this classical formalism is just the projective Hilbert space, which provides a clear perspective of the wave function and its adiabatic evolution.

One final technical comment is in order. The mapping from the wave-function components a_i to phase-space variables (p_i, q_i) [see Eq. (3)] becomes ambiguous when any one of the wave-function components a_i becomes zero. Fortunately, this ambiguity can be easily overcome by adopting a different representation to reexpress the wave function. For example, a_1 in Eq. (3) is used to remove the overall wave-function phase. If $a_1 = 0$, one can always select another nonzero a_i to carry out a similar mapping.

III. DYNAMICS FOR THE DEGENERATE ADIABATIC DEVIATION: FIRST-ORDER THEORY

During the adiabatic evolution, the initial wave function on the degeneracy patch should always be on the instantaneous degeneracy patch and follow the Wilczek–Zee theory. However, as in the case of nondegenerate adiabatic following [2], the deviation from what is predicted by Wilczek–Zee theory has been proven to arise [5,7]. In the language of phase-space point, the deviation should be expressed as

$$p_i(t) = \bar{p}_i[R(t)] + \delta p_i, \quad q_i(t) = \bar{q}_i[R(t)] + \delta q_i, \tag{6}$$

with $(\delta p_i, \delta q_i)$ being time-dependent deviations from the ideal adiabatic trajectory $[\bar{p}_i(R), \bar{q}_i(R)]$ predicted by Wilczek–Zee theory. In what follows we develop the dynamics for $(\delta p_i, \delta q_i)$ to first order in $\dot{R}$.

Employing the dynamics (4) and the expression (6), the dynamics for $(\delta p_i, \delta q_i)$ associated with the fixed representation $|B_i\rangle$ defined in Eq. (2) can be written as (keeping only the linear term of deviation, i.e., in the first-order approximation)

$$\begin{pmatrix} \frac{d\bar{p}(R)}{dR}\dot{R} + \frac{d\delta p}{dt} \\ \frac{d\bar{q}(R)}{dR}\dot{R} + \frac{d\delta q}{dt} \end{pmatrix} = \Gamma \begin{pmatrix} \delta p \\ \delta q \end{pmatrix}, \tag{7}$$

where

$$\Gamma = \begin{pmatrix} -\frac{\partial^2 H(R)}{\partial q \partial p} & -\frac{\partial^2 H(R)}{\partial q \partial q} \\ \frac{\partial^2 H(R)}{\partial p \partial p} & \frac{\partial^2 H(R)}{\partial p \partial q} \end{pmatrix}_{p=\bar{p}, q=\bar{q}} \tag{8}$$

is an R-dependent $2(n-1) \times 2(n-1)$ matrix obtained from the second-order derivatives of $H(R)$ defined in Eq. (5). In fact, Eq. (7) describes the dynamics in the vicinity of the Wilczek–Zee solution. The expression p (q) without subscript stands for the matrix stack of the whole set of p_i (q_i), with $i = 1,2,\ldots,n-1$, e.g.,

$$\begin{aligned} (\delta p \quad \delta q)^T &\equiv (\delta p_1 \quad \cdots \quad \delta p_{n-1} \quad \delta q_1 \quad \cdots \quad \delta q_{n-1})^T, \\ (\bar{p} \quad \bar{q})^T &\equiv (\bar{p}_1 \quad \cdots \quad \bar{p}_{n-1} \quad \bar{q}_1 \quad \cdots \quad \bar{q}_{n-1})^T. \end{aligned} \tag{9}$$

Because $\hat{H}(R)$ is degenerate, which gives rise to the null dynamics $\dot{p}_i = \dot{q}_i = 0$ as long as the deviation $(\delta p_i, \delta q_i)$ is on the degeneracy patch, the matrix Γ must be of linear dependence and the determinant $|\Gamma| = 0$.

In order to associate the dynamics with the Wilczek–Zee phase, we here carry out a canonical R-dependent transformation from (p_i, q_i) to (P_i, Q_i) defined in Eq. (1), i.e.,

$$\Lambda \equiv (P_1 \quad Q_1 \quad P_2 \quad Q_2 \quad \cdots \quad P_{n-1} \quad Q_{n-1})^T, \tag{10}$$

and

$$\Lambda = U \begin{pmatrix} p \\ q \end{pmatrix}, \quad \bar{\Lambda} = U \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix}, \quad \delta\Lambda = U \begin{pmatrix} \delta p \\ \delta q \end{pmatrix}, \tag{11}$$

where U is a $2(n-1) \times 2(n-1)$ matrix diagonalizing Γ,

$$\Gamma_{\rm dia} = U \Gamma U^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_{2(n-2)} \end{pmatrix}; \tag{12}$$

$\delta\Lambda$ $(\bar{\Lambda})$ is a column vector with each component being the linear combination of $\delta p_i, \delta q_i$ $(\bar{p}_i, \bar{q}_i)$ $(i = 1, \ldots, n-1)$. The first two columns (lines) of $\Gamma_{\rm dia}$ with null diagonal elements stand for the direction along P_1 and Q_1 defined in Eq. (1), i.e., on the degeneracy patch. Even though the nonzero diagonal elements d_i $[i = 1, \ldots, 2(n-2)]$ may generally be complex numbers, the dynamics induced by $\Gamma_{\rm dia}$ and the corresponding vector Λ is totally equivalent to the initial dynamics. The relation between Λ and $(p,q)^T$ is embedded in the R-dependent matrix $U(R)$.

The transformation defined in Eq. (11) is equivalent to the unitary transformation from the original fixed representation $|B_i\rangle$ defined in Eq. (2) to the energy representation. Disregarding the overall phase of wave functions as in the mappings (1) and (3), each energy eigenstate becomes a fixed point in phase space. Since any state in the superposition of two degenerate eigenstates $c_1|D_1\rangle + c_2|D_2\rangle$ is still an eigenstate, arbitrary deviation only in the directions along $P_1 = \arg(c_2) - \arg(c_1)$, $Q_1 = |c_2|^2$ (without component in P_i, Q_i for $i > 1$) will give rise to a fixed point on the degeneracy patch. Clearly then, the diagonal elements of dynamical matrix $\Gamma_{\rm dia}$, corresponding to the deviation in P_1 and Q_1, i.e., along the degeneracy patch, must be zero. When the deviation is on the patch, the temporal evolution of the deviation vanishes.

One can readily generalize that, for m-folder degeneracy, any state in the form $\sum_{j=1}^{m} c_j|D_j\rangle$ is still an eigenstate such that any deviation along the directions $P_1, P_2, \ldots, P_{m-1}$, $Q_1, Q_2, \ldots, Q_{m-1}$ gives rise to a fixed point. Clearly then, the degeneracy regime is $2(m-1)$ dimensional, with the dynamical matrix in the form

$$\Gamma_{\rm dia} = {\rm dia}\left(\overbrace{\underbrace{0,0,\ldots,0}_{2(m-1)\ \text{zeros}}, d_1, d_2, \ldots d_{2(n-m)}}^{2(n-1)\ \text{dimensional}} \right). \tag{13}$$

Thus the dynamics for the deviation reads

$$U \begin{pmatrix} \frac{d\delta p}{dt} \\ \frac{d\delta q}{dt} \end{pmatrix} = \Gamma_{\rm dia}\delta\Lambda - U \begin{pmatrix} \frac{d\bar{p}(R)}{dR} \\ \frac{d\bar{q}(R)}{dR} \end{pmatrix} \dot{R}. \tag{14}$$

Expressed all by new variables, the above equation becomes

$$\frac{d\delta\Lambda}{dt} = \Gamma_{\rm dia}\delta\Lambda - \left(\frac{d\bar{\Lambda}}{dR} - \frac{dU}{dR}U^{-1}\bar{\Lambda} \right)\dot{R} + \frac{dU}{dR}U^{-1}\delta\Lambda\dot{R}. \tag{15}$$

In Eq. (15), the last term on the right-hand side is at least second order with respect to $\dot{R}$ which is negligible in the first-order theory; the second term on the right-hand side is tightly associated with the Wilczek–Zee phase since $\bar{p}$ $(\bar{q})$ is defined as the variable predicted by Wilczek–Zee theory. At this stage, one may naturally ask what this term would be like in the current framework of phase space of the classical kind. To answer this question, employing the original Wilczek–Zee formula becomes compulsory. For doublet degeneracy, the Wilczek–Zee theory predicts the wave function during the adiabatic evolution as

$$|\psi\rangle = c_1(R)|D_1(R)\rangle + c_2(R)|D_2(R)\rangle, \tag{16}$$

with $c_1(R)$ and $c_2(R)$ satisfying

$$\frac{d}{dR}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} \langle D_1|\frac{\partial}{\partial R}|D_1\rangle & \langle D_1|\frac{\partial}{\partial R}|D_2\rangle \\ \langle D_2|\frac{\partial}{\partial R}|D_1\rangle & \langle D_2|\frac{\partial}{\partial R}|D_2\rangle \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{17}$$

Consider now an infinitesimal segment of the adiabatic process dR. After each dR, the difference of the final wave function and the initial one according to Eq. (17) reads

$$\begin{aligned} &|\psi(R+dR)\rangle - |\psi(R)\rangle \\ &= c_1\frac{\partial|D_1\rangle}{\partial R}dR + c_2\frac{\partial|D_2\rangle}{\partial R}dR - c_1\langle D_1|\frac{\partial}{\partial R}|D_1\rangle dR|D_1\rangle \\ &\quad - c_2\langle D_1|\frac{\partial}{\partial R}|D_2\rangle dR|D_1\rangle - c_1\langle D_2|\frac{\partial}{\partial R}|D_1\rangle dR|D_2\rangle \\ &\quad - c_2\langle D_2|\frac{\partial}{\partial R}|D_2\rangle dR|D_2\rangle, \end{aligned} \tag{18}$$

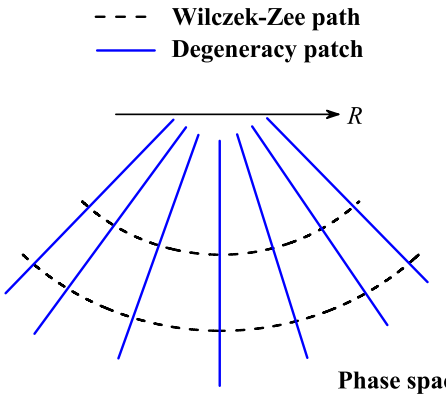

FIG. 3. Illustration of the relationship between Wilczek–Zee path and the evolution of the degeneracy subspace in phase space, defined by Eq. (3), as the parameter R is scanned. For convenience, the parameter-dependent degeneracy patch is demonstrated by a one-dimensional straight line, although it is in fact a $2(m-1)$-dimensional curved patch. According to the Wilczek–Zee formula, the path determined by the Wilczek–Zee solution is always locally perpendicular to the degeneracy patch.

which can be easily proven to be orthogonal with any differential of the wave function on the degeneracy patch, $d|\psi\rangle = dc_1|D_1\rangle + dc_2|D_2\rangle$,

$$[\langle\psi(R+dR)| - \langle\psi(R)|]d|\psi\rangle = 0. \tag{19}$$

Equation (19) implies that, after an infinitesimal time interval, the change of Wilczek–Zee state is orthogonal with any infinitesimal state defined on the degeneracy patch, which is illustrated in Fig. 3. In the language of phase space, the projection of $(\frac{d\bar{p}}{dR}, \frac{d\bar{q}}{dR})$ onto the degeneracy patch vanishes. Employing the same gauge as in Eq. (12), this term assumes the form

$$U\begin{pmatrix}\frac{d\bar{p}(R)}{dR}\\ \frac{d\bar{q}(R)}{dR}\end{pmatrix} = \begin{pmatrix}0\\0\\A_1\\A_2\\\vdots\\A_{2(n-2)}\end{pmatrix}. \tag{20}$$

When generalized to the case of m degree of degeneracy, there will be $2(m-1)$ zeros in the vector,

$$U\begin{pmatrix}\frac{d\bar{p}(R)}{dR}\\ \frac{d\bar{q}(R)}{dR}\end{pmatrix} = \left(\overbrace{\underbrace{0,0,\ldots,0}_{2(m-1)\text{ zeros}},A_1,A_2,\ldots,A_{2(n-m)}}^{2(n-1)\text{ dimensional}}\right)^T. \tag{21}$$

Combining Eqs. (12) and (20) and neglecting the third term on the right-hand side of Eq. (15), one can finally derive the dynamics for the deviation within the first-order approximation. Specifically, the deviations in the degeneracy patch satisfy

$$\frac{d\delta P_1}{dt} = \frac{d\delta Q_1}{dt} = 0, \tag{22}$$

which reveals the fact that, during degenerate adiabatic evolution, the deviation in the first-order approximation is always perpendicular to the degeneracy patch, with the projection of first-order state onto the degeneracy patch satisfying the Wilczek–Zee theory [as shown in Fig. 1(b)]. This property can be qualitatively explained as follows: to fulfill the adiabatic following dictated by adiabatic theorem, a general driving force impelling the system along the adiabatic path must be present. Suppose during the adiabatic process there is no deviation from fixed point (eigenstate), the dynamics for the quantum state, described by p and q defined in Eq. (3), will always be

$$\frac{dp_i}{dt} = -\frac{\partial H(R)}{\partial q_i} \equiv 0, \quad \frac{dq_i}{dt} = \frac{\partial H(R)}{\partial p_i} \equiv 0, \tag{23}$$

giving rise to a constant p and q (quantum state). On the other hand, according to the adiabatic theorem, the state should not be a constant but follow the Wilczek–Zee path during the adiabatic evolution. This paradox legalizes the emergence of the intrinsic adiabatic deviation. However, in the case of degeneracy levels, the deviation in the degeneracy subspace cannot induce a force (all the states on the subspace are dynamical fixed point), i.e., Eq. (23) still holds. Thus it is natural that the deviation prefers to be perpendicular to the degeneracy patch.

This result is in sharp contrast to the previous result of first-order deviation [5,7] obtained by quantum adiabatic perturbation theory [13], where the projection of first-order deviation onto the degeneracy subspace is not zero. This contradiction might arise from the fact that the ansatz of the evolving wave function taken in Ref. [5] has reduced the Hilbert space and is thereby insufficient to describe all possible states during degenerate adiabatic evolution, i.e., the first-order state derived here falls out of the ansatz taken in Ref. [5].

Figure 2(a) shows that the deviation vertical to the patch will generally oscillate. To evaluate the deviation, it is convenient to take the nonzero part of $\Gamma_{\rm dia}$ and the right-hand side of Eq. (20) associated with the dynamics perpendicular to the degeneracy subspace,

$$\Gamma^{\rm NZ}_{\rm dia} = \begin{pmatrix}d_1 & 0 & \cdots & 0\\ 0 & d_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & d_{2(n-2)}\end{pmatrix}, \tag{24}$$

$$M = \begin{pmatrix}A_1\\A_2\\\vdots\\A_{2(n-2)}\end{pmatrix}. \tag{25}$$

The dynamics for the deviation then reads

$$\frac{d\delta\Lambda^{\rm NZ}}{dt} = \Gamma^{\rm NZ}_{\rm dia}\left[\delta\Lambda^{\rm NZ} - \Gamma^{\rm NZ,-1}_{\rm dia} M\dot{R}\right], \tag{26}$$

which clearly shows that the deviation orthogonal to the degeneracy region behaves like a multidimensional harmonic oscillator. The averaged deviation, proportional to the adiabatic speed $\dot{R}$ as shown in Fig. 2(b), reads

$$\delta\Lambda^{\rm NZ} = \Gamma^{\rm NZ,-1}_{\rm dia} M\dot{R}, \tag{27}$$

and the first-order deviation in the representation of Λ, according to our convention (10), should be written as

$$\delta^1\Lambda = (0 \quad 0 \quad (\delta\Lambda^{\rm NZ})^T)^T, \tag{28}$$

which shows clearly again that the first-order deviation vanishes in the degeneracy subspace.

As mentioned above, the relation between vector Λ and the initial variables (p_i, q_i) is embedded in U, i.e.,

$$\begin{pmatrix}\delta^1 p\\ \delta^1 q\end{pmatrix} = U^{-1}\delta^1\Lambda, \tag{29}$$

from which the first-order wave function can be obtained in the representation defined by Eq. (3).

Because the first-order adiabatic deviation behaves like a harmonic oscillator, it forms another Hamiltonian dynamics like that in the nondegenerate case [14]. As the center of the oscillator depends on $\dot{R}$, the first-order deviation will undergo a tiny adiabatic evolution as both R and $\dot{R}$ evolve slowly, which is identical to the situation in the nondegenerate adiabatic process.

The above treatment can be naturally generalized to the cases of higher-degeneracy degrees. The only difference is that there are $2\times(m-1)$ zero diagonal elements of $\Gamma_{\rm dia}$ shown in Eq. (13) and the same number of zero-elements of vector shown in Eq. (21), with m being the degree of degeneracy.

IV. DYNAMICS FOR THE DEGENERATE ADIABATIC DEVIATION: HIGH-ORDER THEORY

In the last section, the first-order deviation is shown to be orthogonal to the degeneracy subspace. In this section, we give the general formula for the high-order deviation.

Returning back to Eq. (15), the deviation $\delta\Lambda$ is in fact the sum of all orders of the deviations,

$$\delta\Lambda = \delta^1\Lambda + \delta^2\Lambda + \cdots. \tag{30}$$

The dynamics of $\delta\Lambda \equiv \delta P_i, \delta Q_i$ is in fact the Taylor expansion instead of the first-order approximation associated with $\Gamma_{\rm dia}$. However, the higher-order terms in the expansion is only associated with $\Gamma_{\rm dia}^{\rm NZ}$ since the deviation on the degeneracy patch can never generate any driving force.

First, let us consider the second-order term in Eq. (15). The dynamics for the second-order deviation then reads

$$\frac{d\delta^2\Lambda}{dt} = \frac{1}{2}\delta\Gamma_{\rm dia}\delta^1\Lambda + \frac{dU}{dR}U^{-1}\delta^1\Lambda\dot{R}, \tag{31}$$

where $\delta\Gamma_{\rm dia}$ is defined as

$$\begin{aligned}\delta\Gamma_{\rm dia} &= \sum_i \left(\frac{\partial\Gamma_{\rm dia}}{\partial P_i}\right)_{\bar{p},\bar{q}}\delta P_i + \sum_i \left(\frac{\partial\Gamma_{\rm dia}}{\partial Q_i}\right)_{\bar{p},\bar{q}}\delta Q_i\\ &\equiv \left(\left(\frac{\partial\Gamma_{\rm dia}}{\partial P}\right)_{\bar{p},\bar{q}}, \left(\frac{\partial\Gamma_{\rm dia}}{\partial Q}\right)_{\bar{p},\bar{q}}\right)\delta^1\Lambda,\end{aligned} \tag{32}$$

which has the same matrix dimension as $\Gamma_{\rm dia}$.

Next, according to Eq. (15), the dynamics for the kth-order deviation can be iteratively obtained as

$$\frac{d\delta^k\Lambda}{dt} = \sum_{j=1}^{k-1}(\Delta^j\Gamma_{\rm dia})\delta^{k-j}\Lambda + \frac{dU}{dR}U^{-1}\delta^{k-1}\Lambda\dot{R}, \tag{33}$$

with

$$\Delta^j\Gamma_{\rm dia} = \mathcal{T}^j\left\{\sum_{i=1}^{j}\frac{1}{(i+1)!}\left[\left(\frac{\partial}{\partial P},\frac{\partial}{\partial Q}\right)\sum_{r=1}^{j}\delta^r\Lambda\right]^i\Gamma\right\} \tag{34}$$

The function $\mathcal{T}^j(\cdots)$ in Eq. (34) is to take the jth-order terms in $(\cdots)$.

Because dU/dR is generally very different from U itself, the deviation with the order higher than one will not be zero on the degeneracy subspace. However, if the normal direction of the degeneracy subspace in the whole Hilbert phase space keeps fixed as parameter R changes, dU/dR vanishes and the last term in Eq. (15) is always zero, which means that, in this case, deviations of all orders are vertical to the degeneracy subspace and the formulation reduces to that for the nondegenerate case [14]. This sheds more light on the difference between the degenerate and nondegenerate adiabatic evolutions.

As seen iteratively from Eqs. (31) and (33), arbitrary-order deviation evolves dynamically like a harmonic oscillator, with the center of the kth order depending on the temporal derivatives of R up to the kth order. This situation is identical with that in the nondegenerate case.

V. NUMERICAL SIMULATIONS

To verify our results, we employ the tripod-scheme Hamiltonian implemented by three laser beams interacting with a rubidium atom. For convenience we adopt the same configuration as in Ref. [15], where two laser beams are counterpropagating along the x axis and the third laser beam is along the z axis. The associated Hamiltonian under the rotating wave approximation (RWA) is given by

$$H_4 = \sum_{n=1}^{3}\Omega_n|0\rangle\langle n| + \text{H.c.}, \tag{35}$$

with

$$\Omega_1 = \frac{\Omega_0\sin(\xi)}{\sqrt{2}}e^{-ik_l x}, \tag{36}$$

$$\Omega_2 = \frac{\Omega_0\sin(\xi)}{\sqrt{2}}e^{ik_l x}, \tag{37}$$

$$\Omega_3 = \Omega_0\cos(\xi)e^{ik_l z}, \tag{38}$$

where the parameter ξ is set to satisfy $\cos(\xi) = \sqrt{2}-1$, as in Ref. [15], and k_l is the wave vector of the laser fields.

The Hamiltonian H_4 has two degenerate states with null eigenvalue. We denote these two degenerate states as $|D_{1(2)}\rangle$, and it is straightforward to find their spatial dependence as follows [15]:

$$\begin{aligned}|D_1\rangle &= (|\tilde{1}\rangle - |\tilde{2}\rangle)e^{-i\kappa' z}/\sqrt{2}\\ |D_2\rangle &= [\cos(\xi)(|\tilde{1}\rangle + |\tilde{2}\rangle)/\sqrt{2} - \sin(\xi)|3\rangle]e^{-i\kappa' z},\end{aligned} \tag{39}$$

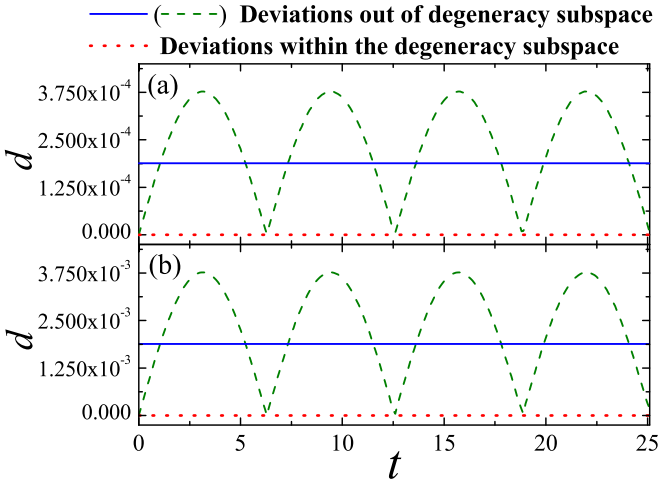

FIG. 4. Numerical adiabatic deviations from the Wilczek–Zee theory measured by the dimensionless quantity d for the tripod-scheme Hamiltonian as z is scanned with $x = 1$. The dashed green curve and dotted red curve are the results of deviations in the perpendicular direction and in the degeneracy subspace, respectively, when the initial state is set on $|D_2\rangle$; with the initial state set according to Eq. (27), the blue solid line is for the perpendicular deviation. (a) z is scanned with velocity $v = 0.0005$ and (b) z is scanned with velocity $v = 0.005$. Throughout, x and z are in units of $1/k_l$, v is in units of $\Omega_0/\hbar k_l$, and t is in units of $\hbar/\Omega_0$.

where

$$\kappa' \equiv k_l[1 - \cos(\xi)], \tag{40}$$

$$|\tilde{1}\rangle \equiv |1\rangle e^{ik_l(x+z)}, \tag{41}$$

$$|\tilde{2}\rangle \equiv |2\rangle e^{-ik_l(x-z)}. \tag{42}$$

In the numerical simulation, we consider two scenarios: the quantum state emanates from (i) degeneracy subspace and (ii) the state predicted by Eq. (27). Then the parameter x or z is scanned as in the tripod scheme and we calculate (1) the distance between the real state derived by numerically integrating the Schrödinger equation and its projection state in degeneracy subspace and (2) the distance between the projection state and the state obtained by Wilczek–Zee theory. According to our results, the former, which stands for the deviation in the vertical direction, should be a first-order quantity in $\dot{z}(\dot{x})$ while the latter, which stands for the deviation in the degeneracy subspace, should be at least second order. The typical results depicted in Fig. 4 as well as other numerical results clearly demonstrate this property, which verifies our theory numerically. Here the distance between two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is defined as $d = \sqrt{(\langle\psi_1| - \langle\psi_2|)(|\psi_1\rangle - |\psi_2\rangle)}$.

VI. CONCLUSION

In summary, the deviation during quantum adiabatic evolution for degeneracy energy levels is studied both analytically and numerically. In the first-order formulation with respect to adiabatic speed, the deviation between exact state and Wilczek–Zee state will always be in the direction perpendicular to the degeneracy subspace. Thus, the deviation in the degeneracy subspace will be at least second order. Our findings are of fundamental interest to non-Abelian quantum computation and topological braiding. The implications of this work for designing optimal protocols of degenerate adiabatic quantum gates should be a fascinating topic in our future studies.

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[1] M. Born and V. A. Fock, Eur. Phys. J. A **51**, 165 (1928).
[2] T. T. Nguyen-Dang, E. Sinelnikov, A. Keller, and O. Atabek, Phys. Rev. A **76**, 052118 (2007); C. P. Sun, Phys. Rev. D **41**, 1318 (1990); Z. Wu, Phys. Rev. A **40**, 6852 (1989); M. Maamache and Y. Saadi, *ibid.* **78**, 052109 (2008); Phys. Rev. Lett. **101**, 150407 (2008); D. M. Tong, K. Singh, L. C. Kwek, and C. H. Oh, *ibid.* **95**, 110407 (2005); V. I. Yukalov, Phys. Rev. A **79**, 052117 (2009); C. DeGrandi, V. Gritsev, and A. Polkovnikov, Phys. Rev. B **81**, 012303 (2010); C. DeGrandi, A. Polkovnikov, and A. W. Sandvik, *ibid.* **84**, 224303 (2011); A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011); D. Cheung, P. Hoyer, and N. Wiebe, J. Phys. A: Math. Theor. **44**, 415302 (2011); S. Jansen, M. B. Ruskai, and R. Seiler, J. Math. Phys. **48**, 102111 (2007).
[3] K. P. Marzlin and B. C. Sanders, Phys. Rev. Lett. **93**, 160408 (2004); J. Ortigoso, Phys. Rev. A **86**, 032121 (2012); M. H. S. Amin, Phys. Rev. Lett. **102**, 220401 (2009); R. MacKenzie, A. Morin-Duchesne, H. Paquette, and J. Pinel, Phys. Rev. A **76**, 044102 (2007); D. Comparat, *ibid.* **80**, 012106 (2009); Y. Zhao, *ibid.* **77**, 032109 (2008); J. Ma, Y. Zhang, E. Wang, and B. Wu, Phys. Rev. Lett. **97**, 128902 (2006); S. Duki, H. Mathur, and O. Narayan, *ibid.* **97**, 128901 (2006); Z. Wu and H. Yang, Phys. Rev. A **72**, 012114 (2005); J. Du, L. Hu, Y. Wang, J. Wu, M. Zhao, and D. Suter, Phys. Rev. Lett. **101**, 060403 (2008); D. M. Tong, *ibid.* **104**, 120401 (2010); A Ambainis and O. Regev, arXiv:quant-ph/0411152.
[4] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001); E. Cobanera and G. Ortiz, Phys. Rev. A **89**, 012328 (2014).
[5] G. Rigolin and G. Ortiz, Phys. Rev. Lett. **104**, 170406 (2010).
[6] G. Rigolin and G. Ortiz, Phys. Rev. A **85**, 062111 (2012).
[7] G. Rigolin and G. Ortiz, Phys. Rev. A **90**, 022104 (2014).
[8] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
[9] J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, Phys. Rev. Lett. **95**, 010404 (2005); J. Y. Vaishnav and C. W. Clark, *ibid.* **100**, 153002 (2008).
[10] S. Weinberg, Ann. Phys. (NY) **194**, 336 (1989).
[11] A. Heslot, Phys. Rev. D **31**, 1341 (1985).
[12] J. Liu, B. Wu, and Q. Niu, Phys. Rev. Lett. **90**, 170404 (2003).
[13] G. Rigolin, G. Ortiz, and V. H. Ponce, Phys. Rev. A **78**, 052508 (2008).
[14] Q. Zhang, J. Gong, and B. Wu, New J. Phys. **16**, 123024 (2014).
[15] G. Juzeliūnas, J. Ruseckas, A. Jacob, L. Santos, and P. Öhberg, Phys. Rev. Lett. **100**, 200405 (2008).