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Optimal synthesis of multivalued quantum circuits

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Although much work has been devoted to multivalued quantum logic synthesis, the question of whether multivalued quantum circuits are more efficient than conventional binary quantum circuits is still open. This article is devoted to the optimization of generic multivalued quantum circuits. Multivalued quantum Shannon decompositions are improved so that the circuits obtained are asymptotically optimal for all dimensionality d. The syntheses of uniformly multifold controlled R_y rotations are also optimized to make the circuits further simplified. Moreover, the theoretical lower bound of complexity for multivalued quantum circuits is investigated, and a quantity known as the efficiency index is proposed to evaluate the efficiency of the synthesis of various quantum circuits. The algorithm for qudit circuits given here is an efficient synthesis routine which produces improved results for all dimensionality d, whether the number of qudits n is small or large. The two facts, the leading factor of the lower bound of complexity for qudit circuits is small by a factor of $d-1$ in comparison to that for qubit circuits and the asymptotic efficiency index is increased with an increase of dimensionality d, reveal the potential advantage of qudit circuits over generic qubit circuits. Generic n-qudit circuits with $d \geqslant 5$ and generic two-ququart circuits synthesized by the algorithm given here are practical circuits which are more efficient than the most efficient qubit circuits.

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I. INTRODUCTION

Enormous progress has been made in the field of quantum information science over the past two and a half decades. Most approaches to quantum information processing use two-level quantum systems (qubits). However, there is increasing interest in exploiting multilevel quantum systems (qudits) [1–9]. The simplest multilevel system, the three-level quantum system, is called a qutrit, and the four-level quantum system is called a ququart. Qudits are exciting because quantum systems usually have multilevels, and they enable the full use of various resources.

In quantum computing, the algorithms are commonly described by the quantum circuit model. The process of constructing quantum circuits by some elementary components is called synthesis. The complexity of a quantum circuit can be measured in terms of the number of elementary gates required. A large amount of work in these areas has been done for binary quantum computing [10–21]. The controlled-NOT (CNOT) gate is one of most widely used two-qubit elementary gates. It has been shown that the CNOT gate with a one-qubit gates is universal for qubit quantum circuits [10,11]. The best result so far for the synthesis of generic qubit quantum circuits has been given by Shende *et al.*, which was based on quantum Shannon decomposition (QSD) [19].

Many works also have been devoted to multivalued quantum logic synthesis [22–31], but the results in this area are still not satisfying. The results obtained cannot show the advantage of qudits in the complexity of quantum logic synthesis. The gate chosen as the two-qudit elementary gate of the qudit quantum circuit is a crucial issue for multivalued quantum computing, and there have been many proposals. In our recent previous work, the generalized controlled-X (GCX) gate has been proposed as a two-qudit elementary gate for qudit circuits [32,33]. We generalized QSD, the most powerful technique for the synthesis of generic qubit circuits, to the multivalued case. Based on the GCX gate, using multivalued QSD, we improved the results of the synthesis of qudit circuits [33], but there are still some problems. One is that the quantum circuits built by the multivalued QSD algorithm are not asymptotically optimal except that the dimensionality of qudit d is a power of 2. Here, the term "asymptotically optimal" is a specific concept which was first introduced by Bullock *et al.* [23]. A qudit quantum circuit is asymptotically optimal, which means that it can be synthesized asymptotically by $O(\alpha d^{2n})$ two-qudit elementary gates, where α is a constant. The other is that the multivalued quantum circuits in Ref. [33] do not show obvious advantages over the circuits for binary systems. The problem whether multivalued quantum circuits can be more efficient than binary circuits is still open.

This article is devoted to optimizing multivalued quantum circuits and to solving the problems stated above. The multivalued QSD for the qudit d is not a power of 2—it is optimized so that the synthesis of quantum circuits for these qudits also is asymptotically optimal. The synthesis of uniformly multifold controlled R_y rotations is also optimized to make the circuits further simplified. The theoretical lower bound of complexity for qudit quantum circuits is investigated. A quantity known as the efficiency index is proposed to evaluate the efficiency of the synthesis of generic n-qudit circuits. Results and comparisons show that the algorithm given here is an efficient qudit synthesis routine which produces improved results in all respects. The multivalued quantum circuits are indeed more efficient than the binary quantum circuits.

The article is organized as follows: The lower bound of complexity for qudit circuits is investigated in Sec. II. The leading factor of the lower bound of complexity for qudit circuits is small by a factor of $d-1$ in comparison to that for qubit circuits. The optimization of the multivalued QSD and

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uniformly multifold controlled R_y rotations, and the structure and the GCX gate count of optimal qudit circuits, are given in Sec. III. The efficiency of synthesis of quantum circuits is discussed in Sec. IV. The quantity in terms of the efficiency index is proposed in this section. The asymptotic efficiency index is increased with an increase of dimensionality d for these circuits. The efficiency indices of generic n-qudit circuits with $d \geqslant 5$ and generic two-ququart circuits given here are higher than those of the most efficient qubit circuits. Finally, a brief conclusion and a discussion future work are given in Sec. V.

II. LOWER BOUNDS OF GCX GATES

The GCX gate [denoted as GCX$(m \to X^{(ij)})$] is a controlled-U two-qudit gate which implements the $X^{(ij)}$ operation on the target qudit if and only if the control qudit is in the state $|m\rangle$, where $X^{(ij)} = |i\rangle\langle j| + |j\rangle\langle i| + \sum_{k\neq i,j} |k\rangle\langle k|$. The GCX gate essentially is a CNOT gate. For a multilevel quantum system which forms a qudit, two levels in the system form a qubit. If a two-qubit CNOT gate is realized in two such systems, a GCX gate is naturally obtained. The number of the GCX gates required can be used as a unified measure for the complexity of various quantum circuits [33]. It also offers the possibility to compare the efficiency for the synthesis of various quantum circuits.

In quantum computing, the quantum circuit is a unitary transformation on the quantum states. A generic n-qudit quantum circuit is fully determined by $d^{2n} - 1$ real parameters (up to a phase factor). Here, the qudit circuits are constructed by using GCX gates and arbitrary one-qudit gates. The GCX gates do not introduce any parameters, but they provide a kind of barrier that separates one-qudit gates on the same qudit so that they cannot merge into a resulting one-qudit gate for each qudit gate. Intuitively, every GCX gate can be accompanied two one-qudit gates, one for the control qudit G_1 and the other for the target qudit G_2, applied after every GCX gate.

The one-qudit gate corresponds to a $SU(d)$ group, and the $d^2 - 1$ parameters correspond to the bases of $su(d)$ algebra. Without loss of generality, we consider the GCX$(d-1 \to X^{(d-2,d-1)})$ gate and use the natural bases of $u(d)$ algebra $|i\rangle\langle j|$, where $i,j \in 0,1,\ldots,d-1$. There are $2(d-1)$ bases which do not commute with the GCX gate in G_1—they are $|d-1\rangle\langle i|$ and $|i\rangle\langle d-1|$, where $i \in 0,1,\ldots,d-2$, so the gate can separate $2(d-1)$ parameters in G_1. There are $4(d-1)$ such bases in G_2—they are $|d-1\rangle\langle i|$, $|d-2\rangle\langle i|$, where $i \in 0,1,\ldots,d-1$ and $|j\rangle\langle d-1|$, $|j\rangle\langle d-2|$, where $i \in 0,1,\ldots,d-3$. But there are $2(d-1)$ linear combinations of them which commute with the GCX gate—they are $|d-2\rangle\langle i| + |d-1\rangle\langle i|$, where $i \in 0,1,\ldots,d-1$, and $|j\rangle\langle d-2| + |j\rangle\langle d-1|$, where $j \in 0,1,\ldots,d-3$. The GCX gate also can separate $2(d-1)$ independent parameters in G_2. Each GCX gate can bring at most $4(d-1)$ parameters. For generic n-qudit quantum circuits, there are $d^{2n} - n(d^2-1) - 1$ parameters which need to be brought by the GCX gates. The theoretical lower bound of complexity for generic qudit circuits is $[d^{2n} - n(d^2-1) - 1]/[4(d-1)]$. The lower bound of complexity for generic qubit circuits is $(4^n - 3n - 1)/4$ [16]. The leading factor of the lower bound of complexity for qudit circuits is small by a factor of $d-1$ in comparison to that for qubit circuits, revealing the potential advantage of qudit circuits over qubit circuits.

III. OPTIMAL SYNTHESIS OF MULTIVALUED QUANTUM CIRCUITS

A. Synthesis based on multivalued QSD [33]

The first phase of multivalued QSD is to use the cosine-sine decomposition (CSD) [34]. Let the $m \times m$ unitary matrix W be partitioned in 2×2 block form as

$$W = \begin{array}{c|cc} & r & m-r \\ r & & \\ m-r & & \end{array} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{1}$$

with $2r \leqslant m$. Here, r is called the partition size. Let $W = U\Gamma V$ be the CSD of the matrix, then

$$U = \begin{array}{c} \\ r \\ m-r \end{array} \begin{array}{c} \begin{array}{cc} r & m-r \end{array} \\ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \end{array}, \tag{2}$$

$$\Gamma = \begin{array}{c} \\ r \\ r \\ m-2r \end{array} \begin{array}{c} \begin{array}{ccc} r & r & m-2r \end{array} \\ \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{pmatrix} \end{array}, \tag{3}$$

$$V = \begin{array}{c} \\ r \\ m-r \end{array} \begin{array}{c} \begin{array}{cc} r & m-r \end{array} \\ \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \end{array}, \tag{4}$$

where C and S are diagonal matrices of the forms $C = \mathrm{diag}\{\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_r\}$ and $S = \mathrm{diag}\{\sin\theta_1, \sin\theta_2, \ldots, \sin\theta_r\}$, I is the $(m-2r)\times(m-2r)$ identity matrix, and Γ is called the cosine-sine matrix. An n-qudit gate corresponds to a $d^n \times d^n$ unitary matrix. The synthesis of qudit quantum circuits based on CSD was first proposed by Khan *et al.* [27,28]. There, they chose a partition size $r = d^{n-1}$ at each recursion level. Different from Khan *et al.*'s method, we choose a partition size $r = [d/2]d^{n-1}$ for the first level decomposition, then $r = [d/4]d^{n-1}$ for the second level decomposition, and $r = [d/2^k]d^{n-1}$ for the kth level decomposition, where $[a]$ denotes the integer part of a. After κ levels ($\log_2 d \leqslant \kappa < \log_2 d + 1$) of decomposition, $d^{n-1} \times d^{n-1}$ block-diagonal matrices are obtained. The block-diagonal matrices correspond to uniformly controlled $(n-1)$-qudit $[u\Lambda_1(U^{n-1})]$ gates; the cosine-sine matrices correspond to uniformly $(n-1)$-fold controlled $R_y[u\Lambda_{n-1}(R_y)]$ rotations.

The second phase of multivalued QSDs is the further decomposition for the uniformly controlled $(n-1)$-qudit gate. It can be decomposed into d copies of $(n-1)$-qudit gates and $d-1$ copies of controlled $(n-1)$-qudit diagonal $[\Lambda_1(\Delta^{n-1})]$ gates. In the $d=2$ case, the uniformly controlled $(n-1)$-qudit gate is decomposed into a pair of $(n-1)$-qudit gates and a $\Lambda_1(\Delta^{n-1})$ gate, and it is equivalent to the decomposition of the block-diagonal matrix in the original QSD. So the decomposition given here is a generalization of the QSD for qubits.

The synthesis of a generic n-qudit gate involves three kinds of components: $(n-1)$-qudit gates, $\Lambda_1(\Delta^{n-1})$ gates, and $u\Lambda_{n-1}(R_y)$ rotations. The $(n-1)$-qudit gates can be further decomposed in similar ways. So we can construct a generic n-qudit quantum circuit by a recursive way. All the component elements required can be efficiently synthesized based on GCX gates.

B. Optimization of the synthesis stated above

Optimizing the decomposition of matrices. When d is not a power of 2, there are identity submatrices in the cosine-sine matrices of CSD. We can rearrange and change the block-diagonal matrices of CSD to reduce the numbers of two components, the $(n-1)$-qudit gate and the $\Lambda_1(\Delta^{n-1})$ gate. The number of $(n-1)$-qudit gates can be reduced to a minimum d^2.

For example, an n-qutrit gate, corresponding to a $3^n \times 3^n$ unitary matrix, can be decomposed as follows,

$$W_1 = A\Gamma_1 B\Gamma_0 C\Gamma_2 D, \tag{5}$$

with

$$\Gamma_0 = \begin{pmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & C_1 & -S_1 \\ 0 & S_1 & C_1 \end{pmatrix},$$
$$\Gamma_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & C_2 & -S_2 \\ 0 & S_2 & C_2 \end{pmatrix}, \tag{6}$$

$$A = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix},$$
$$C = \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix}, \quad D = \begin{pmatrix} I & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{pmatrix}, \tag{7}$$

where each block matrix in the decomposition above is of size $3^{n-1} \times 3^{n-1}$. We can rewrite it as

$$W_1 = A'\Gamma_1 B'\Gamma_0 C'\Gamma_2 D', \tag{8}$$

with

$$A' = (I \otimes U_2)\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_3' \end{pmatrix},$$
$$B' = (I \otimes X_2)\begin{pmatrix} X_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$
$$C' = (I \otimes V_2)\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & V_3' \end{pmatrix}, \quad D' = \begin{pmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{pmatrix}. \tag{9}$$

where $U_3' = U_2^{-1}U_3$, $X_1 = X_2^{-1}U_2^{-1}U_1$, $V_3' = V_2^{-1}X_2^{-1}X_3V_3$, and $Y_1 = V_2^{-1}X_1$. For matrices A, B, C, D, and D', each corresponds to two $\Lambda_1(\Delta^{n-1})$ gates and three $(n-1)$-qutrit gates; for A', B', and C', each matrix corresponds to one $\Lambda_1(\Delta^{n-1})$ gate and two $(n-1)$-qutrit gates. The optimal synthesis of a generic n-qutrit circuit gate involves nine $(n-1)$-qutrit gates and five $\Lambda_1(\Delta^{n-1})$ gates, three $(n-1)$-qutrit gates and three $\Lambda(\Delta^{n-1})$ gates less than that in the original synthesis.

TABLE I. Numbers of three components for optimal n-qudit quantum circuits.

	d					
Component	3	4	5	6	7	8
$(n-1)$-qudit gate	9	16	25	36	49	64
$\Lambda_1(\Delta^{n-1})$	5	12	17	28	41	56
$u\Lambda_{n-1}(R_y)$	3	6	10	15	21	28

For example again, taking $d = 6$, the n-qudit circuit, corresponding to a $6^n \times 6^n$ unitary matrix, can be decomposed as follows,

$$W_2 = A\Gamma_1 B\Gamma_2 C\Gamma_3 D\Gamma_0 E\Gamma_4 F\Gamma_5 G\Gamma_6 H, \tag{10}$$

with

$$\Gamma_0 = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \tag{11}$$

where each block matrix in Eq. (11) is of size $\frac{6^n}{2} \times \frac{6^n}{2}$, and

$$\Gamma_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & -S_1 & 0 & 0 & 0 \\ 0 & S_1 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C_2 & -S_2 \\ 0 & 0 & 0 & 0 & S_2 & C_2 \end{pmatrix},$$
$$\Gamma_2 = \begin{pmatrix} C_1' & -S_1' & 0 & 0 & 0 & 0 \\ S_1' & C_1' & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2' & -S_2' & 0 \\ 0 & 0 & 0 & S_2' & C_2' & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}, \tag{12}$$
$$\Gamma_3 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1'' & -S_1'' & 0 & 0 & 0 \\ 0 & S_1'' & C_1'' & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & C_2'' & -S_2'' \\ 0 & 0 & 0 & 0 & S_2'' & C_2'' \end{pmatrix},$$

$$\begin{aligned} A &= \mathrm{diag}\{A_1, A_2, A_3, A_4, A_5, A_6\}, \\ B &= \mathrm{diag}\{I, B_2, B_3, I, B_5, B_6\}, \\ C &= \mathrm{diag}\{C_1C_2, C_3, C_4, C_5, C_6\}, \\ D &= \mathrm{diag}\{I, D_2, D_3, I, D_5, D_6\}, \end{aligned} \tag{13}$$

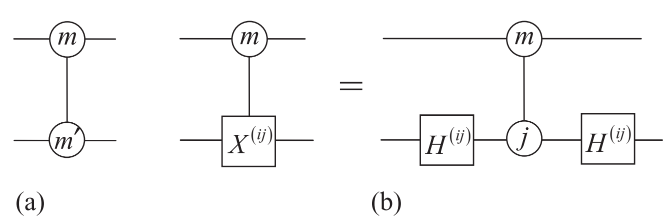

FIG. 1. (a) GCZ gate and (b) transformation between the GCX gate and GCZ gate.

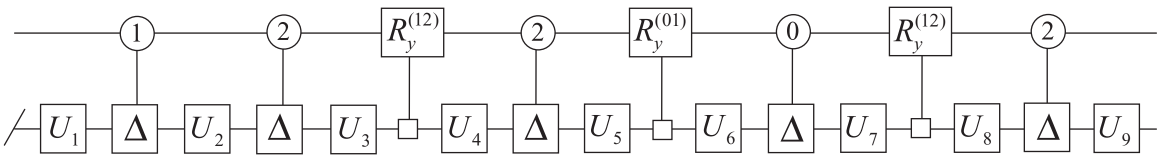

FIG. 2. Structure of a generic n-qutrit circuit. Here, the small square (□) denotes uniform control, and the slash (/) represents multiple qutrits on the line.

where each block matrix in Eqs. (12) and (13) is of size $6^{n-1} \times 6^{n-1}$. The second half of the expression in Eq. (10), $E\Gamma_4 F\Gamma_5 G\Gamma_6 H$, has the same form as the first half, $A\Gamma_1 B\Gamma_2 C\Gamma_3 D$. The first half can be rewritten as

$$A\Gamma_1 B\Gamma_2 C\Gamma_3 D = A'\Gamma_1 B'\Gamma_2 C'\Gamma_3 D', \tag{14}$$

with

$$\begin{aligned} A' &= (I \otimes A_2)\mathrm{diag}\{I, I, A_3', I, A_5', A_6'\},\\ B' &= (I \otimes B_2)\mathrm{diag}\{B_1', I, I, B_4', B_5', I\},\\ C' &= (I \otimes C_2)\mathrm{diag}\{I, I, C_3', I, C_5', C_6'\},\\ D' &= \mathrm{diag}\{D_1', D_2, D_3, D_4', D_5, D_6\}. \end{aligned} \tag{15}$$

Here, $A_3' = A_2^{-1}A_3$, $A_5' = A_2^{-1}A_5$, $A_6' = A_2^{-1}A_6$, $B_1' = B_2^{-1}A_2^{-1}A_1$, $B_4' = B_2^{-1}A_2^{-1}A_4$, $B_5' = B_2^{-1}B_5$, $C_3' = C_2^{-1}B_2^{-1}B_3C_3$, $C_5' = C_2^{-1}C_5$, $C_6' = C_2^{-1}B_2^{-1}B_6C_6$, $D_1' = C_2^{-1}C_1$, and $D_4' = C_2^{-1}C_4$. The second half of the expression can be processed in the same way. The optimal synthesis of a generic n-qudit circuit gate with $d = 6$ involves 36 $(n-1)$-qudit gates and 28 $\Lambda_1(\Delta^{n-1})$ gates, eight $(n-1)$-qudit gates and eight $\Lambda_1(\Delta^{n-1})$ gates less than that in the original synthesis.

The numbers of the three components needed to construct optimal generic multivalued quantum circuits are listed in Table I. A cosine-sine matrix for a qudit system can involve several sets of $u\Lambda_k(R_y)$ rotation. To reduce the number of $(n-1)$-qudit gates to its minimum d^2 is essential for the asymptotic optimality of the synthesis. The number of GCX gates in these d^2 $(n-1)$-qudit gate components account for the vast majority of GCX gate counts of an n-qudit gate if n is large. For example, the number of GCX gates in nine 2-qutrit gate components account for 68.02% of the GCX gate count of a generic 3-qutrit circuit, whereas the number of GCX gates in nine 7-qutrit gate components account for 99.87% of the count of a generic 8-qutrit circuit. The numbers of two other components may be neglected if n is enough large. Hence, the synthesis obtained here is asymptotically optimal, which means that the generic n-qudit circuit can be synthesized asymptotically by $O(\alpha d^{2n})$ GCX gates.

Optimizing the uniformly multifold controlled R_y rotations. The optimization of the qubit $u\Lambda_{n-1}(R_y)$ rotations has been given in Ref. [19] by using controlled-Z (CZ) gates. To optimize the qudit $u\Lambda_{n-1}(R_y)$ rotations, we need a high-dimensional counterpart to the CZ gate. The multivalued extension of the Z operation is a one-qudit operation $Z^{[m]}$ which is specified by $Z^{[m]} = \sum_{k\neq m} |k\rangle\langle k| - |m\rangle\langle m|$. There are only d different forms of the $Z^{[m]}$ operation for a qudit ($d = 0, 1, \ldots, d-1$, respectively), whereas there are $d(d-1)/2$ forms of the $X^{(ij)}$ operation for the qudit. As the GCX gate, the generalized controlled-Z (GCZ) gate [denoted as GCZ$(m - m')$] is defined as a controlled two-qudit gate which implements the $Z^{[m']}$ operation on the target qudit if and only if the control qudit is in the state $|m\rangle$. It is specified by GCZ$(m - m') = \sum_{ij\neq mm'} |ij\rangle\langle ij| - |mm'\rangle\langle mm'|$. As the qubit case, the control qudit and target qudit of the GCZ gate are changeable, and using two generalized Hadamard gates, the GCX and GCZ gates can be transformed into each other. The generalized Hadamard gate $H^{(ij)}$ is a one-qudit gate specified by $H^{(ij)} = \sum_{k\neq i,j} |k\rangle\langle k| +$

TABLE II. The GCX gate count for the synthesis of qudit quantum circuits obtained using the multivalued QSD. In each cell, the upper line denotes the count before optimization [33], and the bottom line denotes the optimized count.

	d					
n	3	4	5	6	7	8
2	44	108	272	510	828	1176
	26	90	176	355	618	980
3	692	2232	10256	25860	52740	85456
	344	1926	5216	15565	53856	72716
4	6860	37800	336144	1158720	2965788	5551504
	3458	32886	136576	577705	1797210	4735948
5	83924	613248	10796560	51109320	166400964	355955600
	32028	534582	3445576	20902225	88346400	303759820
6	1011932	9854970	345689872	2.25×10^9	9.32×10^9	2.28×10^{10}
	291638	8587062	86295576	752674425	4.33×10^9	1.95×10^{10}
7	12157748	1.58×10^8	1.11×10^{10}	9.90×10^{10}	5.22×10^{11}	1.46×10^{12}
	2634932	1.35×10^8	2.16×10^9	2.71×10^{10}	2.12×10^{11}	1.25×10^{12}
8	1.46×10^8	2.52×10^9	3.55×10^{11}	4.36×10^{12}	2.92×10^{13}	9.34×10^{13}
	23744984	2.20×10^9	5.39×10^{10}	9.75×10^{11}	1.04×10^{13}	8.00×10^{13}

TABLE III. The asymptotic efficiency indices for the optimal synthesis of multivalued quantum circuits.

d	3	4	5	6	7	8
$\mathscr{R}^{\text{asy}}$	1.81	1.95	2.48	2.89	3.19	3.53

TABLE V. The CDNOT gate count for the synthesis of ququart quantum circuits [29].

n	2	3	4	5	6	n
Gate count	60	1200	20160	326400	5.2×10^6	$(5/16)4^{2n} - (5/4)4^n$

$(|i\rangle\langle i| + |i\rangle\langle j| + |j\rangle\langle i| - |j\rangle\langle j|)/\sqrt{2}$. The circuit representation of a GCZ gate and its transformation relation with the GCX gate are shown in Fig. 1.

The statements and Fig. 14 in Appendix C of Ref. [33] still hold for qudit $u\Lambda_{n-1}(R_y)$ rotations if all GCX($m \rightarrow X^{(ij)}$) gates are replaced with GCZ($m - j$) gates. Thus a set of qudit $u\Lambda_{n-1}(R_y)$ rotations may be implemented with $2d^{n-2}(d-1)$ GCZ gates, of which $d-1$ GCZ($m-j$) gates ($m = 1,2,\ldots,d-1$, respectively) may be moved furthest to the right (or the left). The rightmost $d-1$ GCZ gates produce a diagonal gate which may be absorbed into the neighboring uniformly controlled $(n-1)$-qudit gate. The reason why the $d-1$ GCZ gates are able to be moved furthest to the right whereas only one CZ gate is able to move as that in the qubit case is that the controlled gates with different control basis states can be exchanged with one another. This saves $d-1$ two-qudit elementary gates for each set of qudit $u\Lambda_k(R_y)$ rotations, totally saving the $(d^{2(n-1)} - 1)n_{uR}/(d+1)$ GCX gates for a generic n-qudit circuit, where n_{uR} is the number of $u\Lambda_{n-1}(R_y)$ components in the circuit and is given in Table I. In the practical process of optimization, it should optimize $u\Lambda_{n-1}(R_y)$ rotations first, then optimize the decomposition of the matrices.

C. Structure and GCX gate count of optimal circuits

The optimal quantum circuit of generic n-qudit circuits involves d^2 $(n-1)$-qudit gates, which are separated by $\Lambda_1(\Delta^{n-1})$ gates or circuits for a cosine-sine matrix. It involves $d^2 - 2^\kappa$ $\Lambda_1(\Delta^{n-1})$ gates and $2^\kappa - 1$ circuits for a cosine-sine matrix. In a multivalued case, a circuit for the cosine-sine matrix usually involves several sets of $u\Lambda_{n-1}(R_y)$ rotations. The structure of a generic n-qutrit circuit is illustrated in Fig. 2.

The GCX gate count of the optimal multivalued quantum circuits given here is tabulated in Table II. For comparison, the count before optimization is also given. The results are improved after optimization, especially for the case d, which is not a power of 2. The optimized circuits have asymptotic optimal features for all dimensionality d, whereas the circuits before optimization are not asymptotically optimal, except for d, which is a power of 2.

IV. EFFICIENCY OF SYNTHESIS OF QUANTUM CIRCUITS

To evaluate the efficiency of the synthesis of generic n-qudit circuits based on GCX gates, we propose a quantity known as the efficiency index ($\mathscr{R}$), which is defined by $\mathscr{R} = d^{2n}/N_n$, where N_n is the number of GCX gates required to synthesize the n-qudit circuit. The quantity $\mathscr{R}$ is the average number of parameters carried by each GCX gate. The larger the quantity $\mathscr{R}$, the more efficient is the synthesis of the circuit. For quantum circuit synthesis which is asymptotically optimal, there is an asymptotic efficiency index $\mathscr{R}^{\text{asy}}$ which is the efficiency index when n is enough large. The asymptotic efficiency indices for optimal synthesis given here are listed in Table III. From the table, it can be seen that the $\mathscr{R}^{\text{asy}}$ is increased with an increase of dimensionality d.

There are several previous works on the synthesis of multivalued quantum circuits based on elementary gates. Based on the controlled-increment (CINC) gate, the synthesis by using the spectrum decomposition algorithm is investigated in Refs. [23,25]. It is asymptotically optimal, which has a leading factor of 2 for the CINC account of the synthesis. Using the GCX gate as the two-qudit elementary gate instead of the CINC gate, the synthesis is greatly simplified [33]. The simplified synthesis still has a leading factor of 2, but for the GCX account. So its $\mathscr{R}^{\text{asy}}$ is equal to 0.5 for all dimensionality d, which less than all values of $\mathscr{R}^{\text{asy}}$ in Table III.

Based on the CINC gate, the syntheses by using the CSD with balanced partitions are investigated in Ref. [28]. The circuits synthesized by this method are simpler than those synthesized by using the spectrum decomposition if n is small, but they are not asymptotically optimal, except that d is a power of 2. The results of this work are given in Table IV. It needs $d-1$ GCX gates to synthesize a CINC gate [33]. Comparing the data in Table II and those in Table IV, and considering that the CINC gate itself has a complex structure, it can be seen that

TABLE IV. The CINC gate count for the synthesis of qudit quantum circuits obtained by using CSD with balanced partitions [28].

	d					
n	3	4	5	6	7	8
2	36	72	280	420	588	784
3	3360	4464	69720	60960	381780	142240
4	20088	40824	670320	1563660	4928616	4750256
5	1382952	685440	252347440	38074200	1.0×10^{10}	3.1×10^8
6	8254764	22254984	2.5×10^9	3.7×10^9	1.4×10^{11}	3.9×10^{10}
7	127837404	357389712	1.4×10^{11}	1.8×10^{11}	1.1×10^{13}	2.5×10^{12}
8	465572880	2.9×10^9	8.8×10^{11}	4.2×10^{12}	9.8×10^{13}	8.0×10^{13}

TABLE VI. The CNOT counts of n-qubit quantum circuits based on QSD.

	2	3	4	5	6	n	$\mathscr{R}^{\text{asy}}$
$l = 1$ [19]	6	36	168	720	2976	$(3/4) \times 4^{2n}–(3/2) \times 4^n$	1.33
$l = 1$,optimal	5	31	147	635	2635	$(2/3) \times 4^{2n}–(3/2) \times 4^n + 1/3$	1.50
$l = 2$ [19]	3	24	120	528	2208	$(9/16) \times 4^{2n}–(3/2) \times 4^n$	1.78
$l = 2$,optimal [19]	3	20	100	444	1868	$(23/48) \times 4^{2n}–(3/2) \times 4^n + 4/3$	2.09

the synthesis of quantum circuits given in this article are much more efficient than those in Ref. [28], even if the n is very small.

Li *et al.* proposed a two-ququart gate, termed the controlled-double-NOT (CDNOT) gate, for four-level quantum systems. Based on the CDNOT gate, they investigated the synthesis of ququart quantum circuits by using the QSD method [29], and the results are tabulated in Table V. A CDNOT gate is a two-ququart controlled gate which implements the $\sigma_x \otimes I_2$ operation on the target ququart if and only if the control ququart is in the state $|m\rangle$, $m \in 2,3$, where σ_x is a Pauli matrix. The $\sigma_x \otimes I_2$ operation is equivalent to two X operations, $X^{(02)}$ and $X^{(13)}$, so a CDNOT gate is equivalent to two GCX gates. The $\mathscr{R}^{\text{asy}}$ of this synthesis is 1.60, still less than that in Table III for ququart 1.95. From the discussion above, it can be seen that our algorithm given here is an efficient multivalued synthesis routine which produces improved results for all dimensionalities d, and for both the small n case and the asymptotic case.

The syntheses of generic n-qubit circuits based on QSD and their asymptotic efficiency indices are listed in Table VI. The qubit counterpart of the optimal synthesis for the generic n-qudit circuits given here are the qubit circuits based on QSD with a recursion bottom out at the one-qubit circuit ($l = 1$) and the optimization for $u\Lambda_{n-1}(R_y)$ rotations (the second line of Table VI), where its $\mathscr{R}^{\text{asy}}$ is 1.50, less than all asymptotic efficiency indices in Table III. For the qubit case, there is a most efficient synthesis for generic two-qubit circuits which reaches its theoretical lower bound of complexity with three CNOT gates. The best result for the synthesis of generic n-qubit circuits is based on QSD with a recursion bottom out at the two-qubit circuit ($l = 2$) and two additional optimizations (the fourth line of Table VI), where its $\mathscr{R}^{\text{asy}}$ is 2.09. Now, the asymptotic efficiency indices of the generic n-qudit circuits with $d \geqslant 5$ are greater than this value. Moreover, the generic two-ququart circuit is more efficient than the most efficient generic four-qubit circuit.

V. CONCLUSION AND FUTURE WORK

We have optimized the synthesis of generic multivalued quantum circuits. The optimal circuits are asymptotically optimal for all dimensionality d, so that we can build efficient quantum circuits when the qudit d is not a power of 2 as well as when d is a power of 2. It is of great significance to make full use of various resources. The algorithm given here is an efficient qudit synthesis routine which produces improved results in all respects.

Multivalued quantum circuits do have advantages over binary quantum circuits. The generic n-qudit circuits with $d \geqslant 5$ and generic two-ququart circuits given here are practical circuits which are more efficient than the most efficient qubit circuits. The leading factor of the lower bound of complexity for qudit circuits is small by a factor of $d - 1$ in comparison to that for qubit circuits, and the asymptotic efficiency index is increased with an increase of dimensionality d, further revealing the advantages and benefits of qudit circuits over generic qubit circuits.

There is still plenty of room for improvement in the synthesis of multivalued quantum circuits. One of most important improvements is to optimize the two-qudit quantum circuits. Since our algorithm given here is recursive, more efficient generic qudit circuits can be obtained from more efficient two-qudit circuits. Qudit systems also can play an important role in duality computers [35,36].

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