Non-Markovianity measure using two-time correlation functions

Md. Manirul Ali,* Ping-Yuan Lo, Matisse Wei-Yuan Tu, and Wei-Min Zhang†
Department of Physics, National Cheng Kung University, Tainan 70101, Taiwan
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We investigate non-Markovianity measure using two-time correlation functions for open quantum systems. We define non-Markovianity measure as the difference between the exact two-time correlation function and the one obtained from quantum regression theorem in the Born-Markov approximation. Such non-Markovianity can easily be measured in experiments. We found that the non-Markovianity dynamics in different time scale crucially depends on the system-environment coupling strength and other physical parameters such as the initial temperature of the environment and the initial state of the system. In particular, we obtain the short-time and long-time behaviors of non-Markovianity for different spectral densities. We find that the thermal fluctuation always reduce the non-Markovian memory effect. Also, the non-Markovianity measure shows nontrivial initial state dependence in different time scales.

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I. INTRODUCTION

Born-Markov approximation [1–4], which ignores the memory effects between the system and its environments, is widely used in the study of open quantum systems. However, in quantum information processing, many quantum devices exhibit *non-Markovian* (memory) behaviors which prohibits the use of the Born-Markov approximation. In particular, non-Markovian memory effects are typically significant for structured environments at low temperature in the strong system-environment coupling regime. Various concepts of non-Markovianity were introduced recently to define the border between Markovian and non-Markovian quantum evolution, although the very definition of non-Markovianity is still a debated issue. Many different measures of *non-Markovianity* have been proposed in the literature to quantify memory effects in open systems, based on, for example, the divisibility of the dynamical map [5–8], the distinguishability of states [9–12], quantum entanglement [13–17], quantum Fisher information [18–21], mutual information [22–25], geometrical characterization [8,26], and the decay rate of the master equation itself [27,28]. Predominately, almost all these measures of non-Markovianity are introduced in terms of mathematical quantities. Most of them do not directly relate to physically measurable observables, although some of them are experimentally measured through reconstruction of the output state in terms of density matrix, using quantum state tomography [11,29]. Also, some of these measures even quantitatively diverge from each other [25,30,31].

Recently, a general theory of non-Markovian dynamics is developed for open quantum systems of bosons (fermions) interacting with a general bosonic (fermionic) environment through particle-tunneling processes [32], where exact dynamics of open quantum systems is explored through the nonequilibrium Green's functions which account for all the information of non-Markovian back-action memory effects. The relation between the exact master equation and the nonequilibrium Green's functions manifests the fundamental importance of correlation functions in the description of the non-Markov dynamics for open systems. Therefore, non-Markovianity measure extracted from correlation functions in time domain has a special significance. In particular, two-time correlation functions are experimentally measurable. For example, the two-time correlation functions of the electromagnetic field emitted by an atom can be measured through fluorescence spectrum [33]; the two-time correlation functions of the number of emitted photons give the information about the photon statistics and describe the behavior of photon bunching and antibunching [34,35]; the two-time correlation functions have also been measured in optical measurements [36–38] and light-harvesting photosynthesis for testing the non-Markovian memory effects [39]; and the two-time correlation functions of the electric current through nanostructure devices are used in quantum transport to study the current fluctuations and noise spectrum [40–44]. Intuitively, two-time correlation functions correlating a past event with its future provide useful information about the system-environment back-action processes revealing the non-Markovian memory effects.

In this paper we introduce a non-Markovianity measure using two-time correlation functions. In open quantum systems, when the Born-Markov approximation [1–4] is valid, two-time correlation functions can be calculated using quantum regression theorem where the memory effect is totally ignored. Our non-Markovianity measure in terms of two-time correlation functions is defined as the departure of the exact dynamics of the two-time correlation function from the one obtained through quantum regression theorem within the Born-Markov approximations. Our definition of non-Markovianity measure is valid for arbitrary finite temperature of the environment. We also show that the non-Markovianity defined in this way is a function of time and therefore the non-Markovianity measure is a dynamical quantity, which is physically expected. Such a non-Markovianity definition also depends crucially on various systems and environment parameters and on the nature of system-environment coupling. As a specific example, we take an open system described by the famous Fano-Anderson model [45,46]. Then the two-time correlation function we used to define the non-Markovianity measure is just the first-order coherence function in quantum optics, which can be easily measured in experiments [1–4].

*mani@mail.ncku.edu.tw
†wzhang@mail.ncku.edu.tw

The rest of the paper is organized as follows. In Sec. II A we consider the open system described by the Fano-Anderson model, where we show that the exact two-time correlation functions of system operators can be obtained through the exact master equation associated with the nonequilibrium Green's functions [32]. In Sec. II B we calculate the two-time correlation function under Born-Markov approximation using the quantum regression theorem. We define our non-Markovianity measure in Sec. II C using two-time correlation functions, where we also include a relative discussion of our definition with other relevant non-Markovianity measure. In Sec. II D we present the numerical results of the dynamics of the non-Markovianity measure for different system-environment coupling, different initial temperature of the environment, and different initial states of the system. Our results also show nontrivial interesting non-Markovian features on the dependence of the system and environmental parameters, in particular, the existence of both the short-time and long-time behavior of the non-Markovianity. In Sec. III we extend our definition of non-Markovianity measure to general open quantum systems. A comparative study of the non-Markovianity measure with other approaches is given in Sec. III B. Finally, a conclusion is presented in Sec. IV.

II. THE NON-MARKOVIANITY MEASURE BASED ON TWO-TIME CORRELATION FUNCTION

A. Two-time correlation function obtained from the exact master equation

We consider an open system described by the famous Fano-Anderson Hamiltonian [45,46]

$$H = \hbar\omega_0 a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \sum_k \hbar V_k (a^\dagger b_k + a b_k^\dagger), \quad (1)$$

where $a^\dagger$ and a are the creation and annihilation operators of a single frequency mode. A general non-Markovian environment is modeled as a collection of infinite modes, and $b_k^\dagger$ and b_k are the corresponding creation and annihilation operators of the kth mode with frequency ω_k. The parameter V_k is the tunneling amplitude between the system and its environment. The model Hamiltonian (1) has wide applications in atomic, photonic, and condensed matter physics [47,48].

We consider initially the environment be at thermal equilibrium. Tracing over all the environmental degrees of freedom using the influence functional approach [49,50] in the coherent state representation [51], the exact master equation of such open systems has been derived [52–54]:

$$\begin{aligned}\frac{d}{dt}\rho(t) =& -i\omega_0'(t)[a^\dagger a, \rho(t)] \\ &+ \gamma(t)[2a\rho(t)a^\dagger - a^\dagger a\rho(t) - \rho(t)a^\dagger a] \\ &+ \widetilde{\gamma}(t)[a\rho(t)a^\dagger + a^\dagger\rho(t)a - a^\dagger a\rho(t) - \rho(t)aa^\dagger].\end{aligned} \quad (2)$$

The time-dependent coefficients in the master equation are fully determined by the nonequilibrium Green's functions $u(t,t_0)$ and $v(t,t)$, which satisfy the following equation of motion and the nonequilibrium fluctuation-dissipation theorem [32], respectively:

$$\dot{u}(t,t_0) + i\omega_0 u(t,t_0) + \int_{t_0}^{t} d\tau\, g(t,\tau)u(\tau,t_0) = 0, \quad (3)$$

$$v(t,t+\tau) = \int_{t_0}^{t} d\tau_1 \int_0^{t+\tau} d\tau_2 u(t,\tau_1)\tilde{g}(\tau_1,\tau_2)u^*(t+\tau,\tau_2). \quad (4)$$

The integral kernels in Eqs. (3) and (4) characterize all the non-Markovian back-action between the system and the reservoir, and can be determined by the spectral density $J(\omega)$ of the reservoir through the relations $g(t,\tau) = \int_0^\infty d\omega J(\omega)e^{-i\omega(t-\tau)}$ and $\tilde{g}(\tau_1,\tau_2) = \int_0^\infty d\omega J(\omega)\bar{n}(\omega,T)e^{-i\omega(\tau_1-\tau_2)}$, where the spectral density is defined by $J(\omega) = \sum_k |V_k|^2\delta(\omega-\omega_k)$, and $\bar{n}(\omega,T) = \frac{1}{e^{\hbar\omega/k_BT}-1}$ is the particle number distribution of the bosonic reservoir.

The functions $u(t,t_0)$ and $v(t,t+\tau)$ are indeed related to the two-time correlation functions $\langle[a(t),a^\dagger(t_0)]\rangle$ and $\langle a^\dagger(t+\tau)a(t)\rangle$, respectively, which are the two basic nonequilibrium Green's functions in the Schwinger-Keldysh theory [32]. These two-time correlation functions can also be calculated directly from the equation of motion approach [55,56]. In fact, all two-time correlation functions of system operators can be constructed in terms of the above nonequilibrium Green's functions. For example, as we have shown in our previous works [53,57], the exact two-time correlation function $\langle a^\dagger(t)a(t+\tau)\rangle$ can be expressed in terms of $u(t,t_0)$ and $v(t,t+\tau)$ as follows:

$$\langle a^\dagger(t)a(t+\tau)\rangle = u^*(t,t_0)n(t_0)u(t+\tau,t_0) + v^*(t,t+\tau). \quad (5)$$

The exact analytic solution of the integrodifferential equation (3) is recently given in [32],

$$u(t,t_0) = \mathcal{Z}e^{-i\omega_b(t-t_0)} + \int_0^\infty d\omega \frac{J(\omega)e^{-i\omega(t-t_0)}}{[\omega-\omega_0-\Delta(\omega)]^2+\gamma^2(\omega)}, \quad (6)$$

where $\Delta(\omega) = \mathcal{P}\int_0^\infty d\omega' \frac{J(\omega')}{\omega-\omega'}$ is a principal-value integral and $\gamma(\omega) = \pi J(\omega)$, which are the real and imaginary parts of the self-energy correction to the system induced by the system-environment coupling,

$$\Sigma(\omega \pm i0^+) = \int_0^\infty d\omega' \frac{J(\omega')}{\omega-\omega'\pm i0^+} = \Delta(\omega) \mp i\gamma(\omega). \quad (7)$$

The first term in Eq. (6) is the contribution of the dissipationless localized mode. The localized mode frequency ω_b is determined by the pole condition $\omega_b - \omega_0 - \Delta(\omega_b) = 0$, and $\mathcal{Z} = [1-\Sigma'(\omega_b)]^{-1}$ corresponds to the residue at the pole, which gives the amplitude of the localized mode. Thus, the two-time correlation functions can be exactly calculated from the exact solution of Eq. (6) through Eq. (4).

B. Two-time correlation functions in Born-Markov approximation using quantum regression theorem

Two-time correlation functions can also be calculated through the *quantum regression theorem* [2] in the Born-Markov approximation, if one ignores the back-action or

memory effect. Quantum regression theorem states that evolution equations of the two-time correlation functions of system observables is the same as the evolution equation for single-time expectation values of the observables, which is usually valid within the Born-Markov approximation [2]. The single time expectation values of a complete set of system operators $\hat{O}_i$ can be calculated using the Born-Markovian (BM) approximated master equation

$$\frac{\partial}{\partial t}\langle \hat{O}_i(t)\rangle = \mathrm{Tr}_S\left[\hat{O}_i(0)\frac{\partial}{\partial t}\rho(t)\right] = \sum_j M_{ij}(t)\langle \hat{O}_j(t)\rangle, \quad (8)$$

where the evolution matrix elements M_{ij} are determined from the BM approximated master equation (for more detailed derivation, see Appendix A or Ref. [2]).

According to quantum regression theorem, the evolution equation (8) for single-time expectation values is also valid for the two-time correlation functions

$$\frac{\partial}{\partial \tau}\langle \hat{O}_1(t)\hat{O}_i(t+\tau)\rangle = \sum_j M_{ij}(\tau)\langle \hat{O}_1(t)\hat{O}_j(t+\tau)\rangle. \quad (9)$$

The operator $\hat{O}_1$ can be any system operator, not necessarily one of the $\{\hat{O}_i\}$, and $M_{ij}(\tau)$ is the same matrix given in Eq. (8).

The BM approximated master equation for the open system of Eq. (1) is well known in the literature [2] and it has the same form as the exact master equation (2) for the system considered in Sec. II A. Thus, the coefficients in the BM master equation can also be derived from the exact master equation (2) using a perturbation expansion up to the second order [52]. The result is

$$\gamma(t) \to \gamma(t) \simeq \int_0^t d\tau \int_0^\infty d\omega J(\omega)\cos[(\omega-\omega_0)(t-\tau)], \quad (10)$$

$$\tilde{\gamma}(t) \to \tilde{\gamma}(t) \simeq 2\int_0^t d\tau \int_0^\infty d\omega J(\omega)\bar{n}(\omega,T)\cos[(\omega-\omega_0)(t-\tau)], \quad (11)$$

$$\omega_0'(t) \to \omega_0'(t) \simeq \omega_0 - \int_0^t d\tau \int_0^\infty d\omega J(\omega)\sin[(\omega-\omega_0)(t-\tau)], \quad (12)$$

which is consistent with the results in Ref. [2]. Using the BM approximated master equation, one can obtain the evolution matrix $M_{ij}(\tau)$ in the complete set of system operators $(a, a^\dagger, a^\dagger a, \mathbb{1})$:

$$\mathbf{M}(\tau) = \begin{pmatrix} \beta(\tau) & 0 & 0 & 0 \\ 0 & \beta^*(\tau) & 0 & 0 \\ 0 & 0 & -2\gamma(\tau) & \tilde{\gamma}(\tau) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (13)$$

where $\beta(\tau) = -[\gamma(\tau) + i\omega_0'(\tau)]$. Hence, the particle number $\langle a^\dagger(t)a(t)\rangle$ calculated from the BM master equation satisfies the following equation:

$$\frac{d}{dt}\langle a^\dagger(t)a(t)\rangle_{\mathrm{BM}} = -2\gamma(t)\langle a^\dagger(t)a(t)\rangle_{\mathrm{BM}} + \tilde{\gamma}(t). \quad (14)$$

By choosing the operator $\hat{O}_1 = a^\dagger$ in Eq. (9), the evolution equation for the two-time correlation function $\langle a^\dagger(t)a(t+\tau)\rangle$ based on quantum regression theorem is given by

$$\frac{\partial}{\partial \tau}\langle a^\dagger(t)a(t+\tau)\rangle_{\mathrm{BM}} = -[\gamma(\tau) + i\omega_0'(\tau)]\langle a^\dagger(t)a(t+\tau)\rangle_{\mathrm{BM}}, \quad (15)$$

which can be solved explicitly with the solution

$$\langle a^\dagger(t)a(t+\tau)\rangle_{\mathrm{BM}} = \langle a^\dagger(t)a(t)\rangle_{\mathrm{BM}} e^{-\int_0^\tau [\gamma(\tau')+i\omega_0'(\tau')]d\tau'}. \quad (16)$$

In the next subsection we will use the exact two-time correlation function and the one obtained in the BM approximation to define the non-Markovianity measure.

C. Definition of non-Markovianity and its measure

Our definition of the non-Markovianity measure quantifies the departure of the exact dynamics of two-time correlation function from the one obtained in the BM approximation. Thus, non-Markovianity measure in terms of two-time correlation functions is given by

$$\mathcal{N}(t,\tau) = |C(t,\tau) - C_{\mathrm{BM}}(t,\tau)|, \quad (17)$$

where $C(t,\tau)$ and $C_{\mathrm{BM}}(t,\tau)$ are the first order coherence functions in terms of two-time correlation functions for the *exact* and BM dynamics, respectively,

$$C(t,\tau) = \frac{\langle a^\dagger(t)a(t+\tau)\rangle}{\sqrt{\langle a^\dagger(t)a(t)\rangle\langle a^\dagger(t+\tau)a(t+\tau)\rangle}} \quad (18)$$

and

$$C_{\mathrm{BM}}(t,\tau) = \frac{\langle a^\dagger(t)a(t+\tau)\rangle_{\mathrm{BM}}}{\sqrt{\langle a^\dagger(t)a(t)\rangle_{\mathrm{BM}}\langle a^\dagger(t+\tau)a(t+\tau)\rangle_{\mathrm{BM}}}}. \quad (19)$$

Here $\langle a^\dagger(t)a(t)\rangle$ and $\langle a^\dagger(t)a(t)\rangle_{\mathrm{BM}}$ can be calculated simply from Eqs. (5) and (16) by setting $\tau = 0$. In other words, we take $C_{\mathrm{BM}}(t,\tau)$ as a reference point to account for the non-Markovianity of the open system dynamics.

Based on this definition (17), it is clear that such a non-Markovianity can be directly measured in experiments through the measurement of the exact two-time correlation function or more precisely the first order coherence function $C(t,\tau)$. Meantime, the two-time correlation functions are given for arbitrary finite temperature of the environment. Hence the non-Markovianity measure defined in Eq. (17) takes into account the proper temperature dependence, which will be explicitly described in Sec. II D. Our non-Markovianity measure is also an explicit function of time. Physically, this must be the case because non-Markovianity characterizes the memory effect which must be time dependent in general. It is also clear from the analytical expression (17) through Eq. (5) that this non-Markovianity measure $\mathcal{N}(t,\tau)$ depends on the initial state of the system. This initial state dependence is indeed expected as a typical non-Markovian memory effect.

Here we should also point out that a relative relation between non-Markovianity and quantum regression theorem has been recently discussed by Guarnieri *et al.* [31], with applications to pure dephasing models. In their work, the possible connection between non-Markovianity and the validity

of quantum regression theorem is explored through a relative error defined by

$$Z = \left|1 - \frac{\langle A(t_2)B(t_1)\rangle_{\rm qrt}}{\langle A(t_2)B(t_1)\rangle_{\rm exc}}\right|, \tag{20}$$

where $\langle A(t_2)B(t_1)\rangle_{\rm qrt}$ is the two-time correlation function calculated from quantum regression theorem using the exact master equation, and $\langle A(t_2)B(t_1)\rangle_{\rm exc}$ is the exact two-time correlation function. Notice that the two-time correlation function $\langle A(t_2)B(t_1)\rangle_{\rm qrt}$ obtained from the exact master equation through the quantum regression theorem is in general inadequate. This is because quantum regression theorem is based on a crucial assumption that the total density operator of the system plus environment must be approximately factorized and the environment must always remain in the initial state at an arbitrary time, namely

$$\rho_{\rm tot}(t) = \rho(t) \otimes \rho_E(t_0). \tag{21}$$

The detail derivation of this condition is given in Appendix A (also see Ref. [2]). For non-Markov evolution based on the exact master equation, the condition of Eq. (21) cannot be satisfied in general and therefore the two-time correlation functions calculated from such a naive quantum regression theorem is often inadequate.

To show this inadequacy explicitly, let us apply the inadequate quantum regression theorem used in Ref. [31] to our exact master equation (2). It is straightforward to show (see Appendix B) that the two-time correlation function based on such a quantum regression theorem is given by

$$\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt} = u^*(t,t_0)n(t_0)u(t,t_0)u(\tau,t_0) + v(t,t)u(\tau,t_0). \tag{22}$$

This result is different from Eq. (5) which is exact, it is also different from Eq. (16) which is the two-time correlation function in the BM approximation. Thus, the two-time correlation function of Eq. (22), if it is applicable, may contain partial non-Markovianity, and therefore cannot be used as a reference point for non-Markovianity measure. We consider the two-time correlation calculated in the BM approximation through the quantum regression theorem because the Born-approximation ensures that quantum regression theorem is valid, while the Markov approximation ensures that the obtained two-time correlation function contains no memory effect.

D. Results and discussion

In order to evaluate explicitly the two-time correlation functions and the non-Markovianity measure $\mathcal{N}(t,\tau)$, we need to specify the spectral density $J(\omega)$ of the environment. In the following we consider the Ohmic-type spectral density which can simulate a large class of thermal bath [58]

$$J(\omega) = \eta\omega\left(\frac{\omega}{\omega_c}\right)^{s-1} e^{-\omega/\omega_c}, \tag{23}$$

where η is the coupling strength between the system and the environment, and ω_c is the frequency cutoff of the environmental spectra. When $s = 1$, <1, and >1, the corresponding environments are called as Ohmic, sub-Ohmic, and super-Ohmic in the literature, respectively. With the spectral density specified, the Green's function $u(t,t_0)$ and $v(t,t+\tau)$ are determined by Eqs. (3) and (4). The non-Markovianity measure can then be calculated from Eq. (17) through the two-time correlation functions of Eqs. (5) and (16).

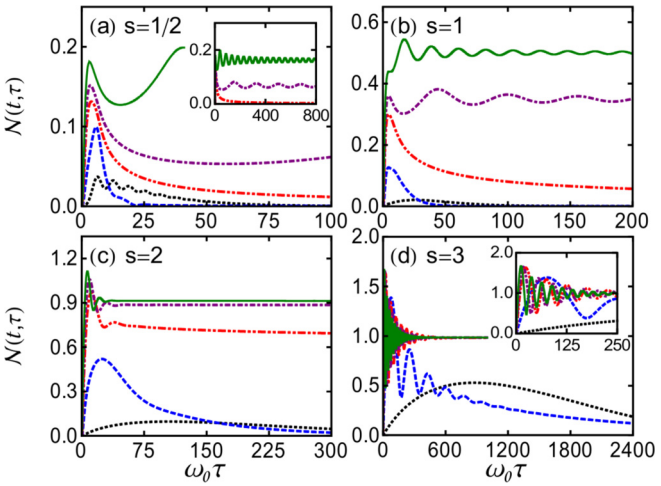

FIG. 1. (Color online) The non-Markovianity measure $\mathcal{N}(t,\tau)$ is plotted as a function of τ with $t = 0$ at a fixed temperature ($T = 1$ K) for different system-environment coupling strengths: $\eta = 0.1\eta_c$ (black dotted line), $\eta = 0.5\eta_c$ (blue dashed line), $\eta = 1.0\eta_c$ (dashed double-dotted red line), $\eta = 1.2\eta_c$ (dashed dotted violet line), and $\eta = 1.5\eta_c$ (solid green line). Four different spectral densities are considered (a) sub-Ohmic ($s = 1/2$) (b) Ohmic ($s = 1$), (c) super-Ohmic ($s = 2$), and (d) super-Ohmic ($s = 3$) for the bosonic reservoir. The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $n(t_0) = 1$.

In Fig. 1 we plot the non-Markovianity measure $\mathcal{N}(t,\tau)$ as a function of time τ for different system-environment coupling strengths η at a fixed initial temperature $T = 1$ K. For simplicity, we also set $t = 0$. The non-Markovianity measure is examined for different spectral densities with $s = 1/2$ [Fig. 1(a) for sub-Ohmic], $s = 1$ [Fig. 1(b) for Ohmic], $s = 2$ [Fig. 1(c) for super-Ohmic], and $s = 3$ [Fig. 1(d) for super-Ohmic]. Different curves represent different coupling strengths, namely $\eta = 0.1\eta_c$ (black dotted line), $\eta = 0.5\eta_c$ (blue dashed line), $\eta = 1.0\eta_c$ (dashed double-dotted red line), $\eta = 1.2\eta_c$ (dashed dotted violet line), and $\eta = 1.5\eta_c$ (solid green line). Here η_c is the critical coupling strength [32] for the occurrence of the localized mode given in Eq. (6), with frequency ω_b located in $\omega < 0$ when the coupling strength exceeds the critical value $\eta_c = \omega_0/[\omega_c\Gamma(s)]$, where $\Gamma(s) = \int_0^\infty dx x^{s-1}e^{-x}$ is the gamma function. Notice that the renormalized frequency ω_b of the open system in strong coupling regime is negative, but the total energy remains positive definite within our exact master equation formalism. We observed that for spectral densities with $s \leqslant 2$ and when the coupling strength is below the critical value [see black-dotted and blue-dashed curves in Figs. 1(a)–1(c) for $\eta < \eta_c$], the non-Markovianity measure increases very fast and approaches a maximum value in a very short time scale, and then decays asymptotically to zero. This indicates that when the system-environment coupling is weak ($\eta < \eta_c$), the non-Markovianity dynamics only shows a short-time behavior, as was pointed out by Caldeira and Leggett in quantum Brownian

motion [50]. For stronger coupling strengths ($\eta > \eta_c$), the non-Markovianity dynamics is significantly different, and we find that the non-Markovianity measure tends to approach a saturated value in the long-time limit after a few slow oscillations [see dashed-dotted-violet and solid-green lines in Figs. 1(a)–1(c)]. This behavior of non-Markovianity dynamics is contributed from the localized mode given by the first term in Eq. (6). This corresponds to the localized-mode-induced dissipationless dynamics discovered recently in [32], as a long-time non-Markovian behavior.

For super-Ohmic case with $s > 2$, the above short-time and long-time behavior of non-Markovianity dynamics is slightly changed. In this case, for weak coupling $\eta < \eta_c$, the decay rate of the non-Markovianity measure becomes very slow [see Fig. 1(d)], where the non-Markovianity measure is calculated up to $\omega_0\tau \sim$ a few thousands. This result is quite different from the case of $s < 2$ where the non-Markovianity measure already decays to zero at $\omega_0\tau \sim$ few hundreds or less. This slow decay for $s > 2$ was discussed long time ago in Ref. [59], but in Ref. [59] such a delayed damping was interpreted as an undamped process and that the system would never go to its equilibrium for $s > 2$. This interpretation is not fully correct. As we can see from the black-dotted curve and the blue-dashed curve in Fig. 1(d), it shows that the non-Markovianity measure damps very very slowly but will eventually decay to zero, so that the system will approach equilibrium after a very long time. However, the system will not approach equilibrium only in the strong-coupling regime ($\eta > \eta_c$) when the localized mode has significant contribution [32,60–62]. In this case, the non-Markovianity dynamics will show the true long-time behavior, namely the memory effect will be kept forever. This is given by the dashed-dotted-violet curve ($\eta = 1.2\eta_c$) and solid-green curve ($\eta = 1.5\eta_c$) of Fig. 1(d), where the non-Markovianity measure reaches a finite saturated value in the long-time limit. To sum up, in Fig. 1 we show the existence of both the short-time and long-time features of non-Markovianity dynamics for different system-environment coupling strength.

In Fig. 2 we discuss the temperature dependence of the non-Markovianity dynamics using two different type of coupling strengths: a weak coupling $\eta = 0.5\eta_c$ [see Figs. 2(a), 2(c), 2(e), and 2(g)] and a strong coupling $\eta = 1.5\eta_c$ [see Figs. 2(b), 2(d), 2(f), and 2(h)]. The temperature dependence of the non-Markovianity originates from the particle number distribution $\bar{n}(\omega, T)$ and nonequilibrium thermal fluctuation $v(t, t+\tau)$ involved in the two-time correlation functions of Eq. (5). We plot again the non-Markovianity measure for different spectral densities with $s = 1/2$ [Figs. 2(a) and 2(b) for sub-Ohmic], $s = 1$ [Figs. 2(c) and 2(d) for Ohmic], $s = 2$ [Figs. 2(e) and 2(f) for super-Ohmic], and $s = 3$ [Figs. 2(g) and 2(h) for super-Ohmic], where the temperature dependence of non-Markovianity is given for $T = 1$ K (solid blue line), $T = 5$ K (red dashed line), and $T = 50$ K (green dotted line). When the coupling strength is weak ($\eta < \eta_c$), non-Markovianity dynamics still shows a short-time behavior as expected but it takes longer time to decay to zero at low temperature [see solid blue curves in Figs. 2(a), 2(c), and 2(e) for $T = 1$ K]. For a higher temperature, the magnitude of non-Markovianity decays much faster [see green-dotted curves in Figs. 2(a), 2(c), and 2(e) for $T = 50$ K]. This indicates that the non-Markovian

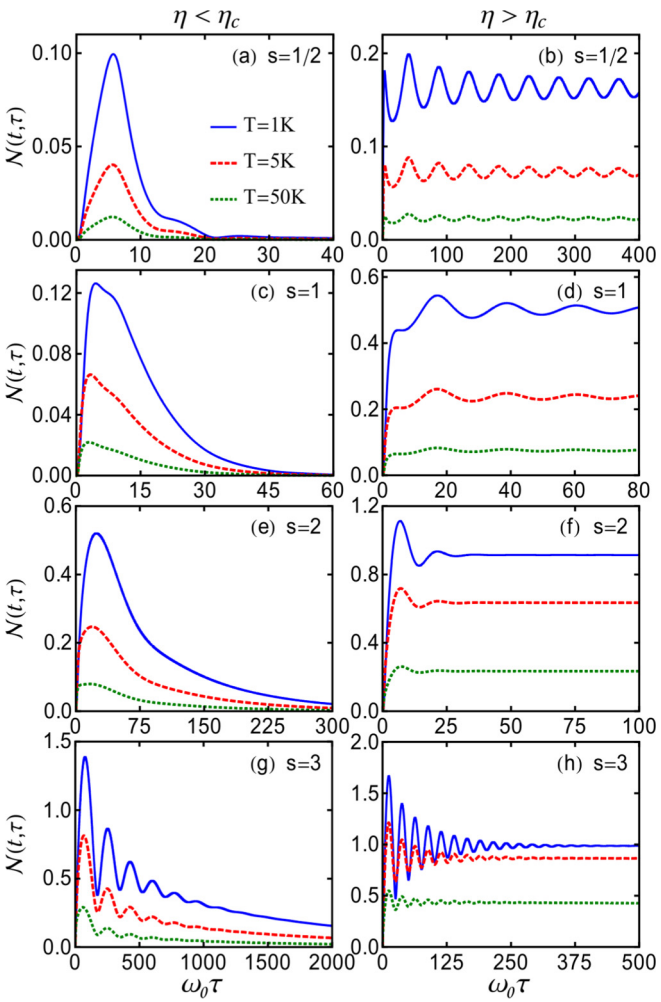

FIG. 2. (Color online) Temperature dependence of non-Markovianity measure is shown as a function of τ at three different temperatures: $T = 1$ K (solid blue line), $T = 5$ K (red dashed line), and $T = 50$ K (dotted green line) with two type of coupling strengths: (a), (c), (e), and (g) weak coupling ($\eta = 0.5\eta_c$) and (b), (d), (f), and (h) strong coupling ($\eta = 1.5\eta_c$). Four different spectral densities for the bosonic environment are considered: (a) and (b) sub-Ohmic ($s = 1/2$), (c) and (d) Ohmic ($s = 1$), (e) and (f) super-Ohmic ($s = 2$), and (g) and (h) super-Ohmic ($s = 3$). The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $n(t_0) = 1$, with a fixed value of $t = 0$.

memory effect is lost faster when temperature is higher. In stronger coupling regime ($\eta > \eta_c$), non-Markovianity measure also tends to approach a steady value in the long-time limit due to the localized mode contribution but the magnitude of this stationary values of non-Markovianity become higher for lower temperature of the environment [see Figs. 2(b) and 2(d)]. This shows that in both the weak and strong coupling regimes, thermal fluctuation always reduce the non-Markovian degree. It may be worth pointing out that in the calculation of the two-time correlation function including the thermal particle fluctuation, one should take a sufficient value of the environment temperature T such that the environment contains at least one more particle. Otherwise, the thermal fluctuation contribution is not so meaningful.

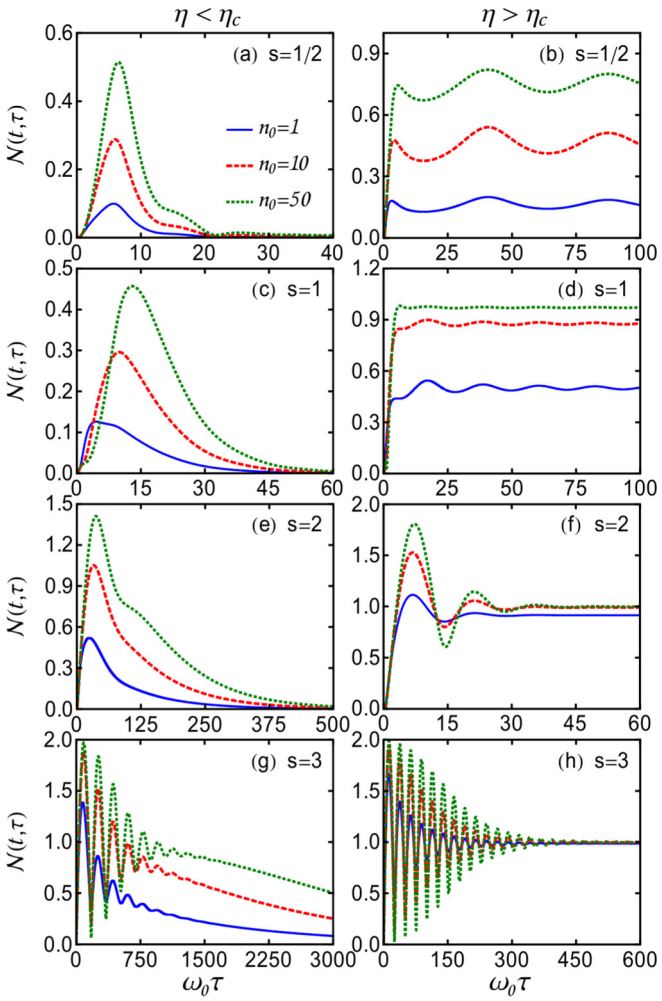

FIG. 3. (Color online) Initial state dependence of non-Markovianity measure is shown as a function of τ for three different $n(t_0)$: $n(t_0) = 1$ (solid blue line), $n(t_0) = 10$ (red dashed line), and $n(t_0) = 50$ (green dotted line) with two type of coupling strengths: (a), (c), (e), and (g) weak coupling ($\eta = 0.5\eta_c$) and (b), (d), (f), and (h) strong coupling ($\eta = 1.5\eta_c$). Four different spectral densities for the bosonic environment is investigated: (a) and (b) sub-Ohmic ($s = 1/2$), (c) and (d) Ohmic ($s = 1$), (e) and (f) super-Ohmic ($s = 2$), and (g) and (h) super-Ohmic ($s = 3$). The other parameters are taken as $\omega_0 = 10$ GHz, $\omega_c = 5\omega_0$, and $T = 1$ K, with a fixed value of $t = 0$.

In Fig. 3 we show the initial state dependence of non-Markovianity measure. Here the different initial state is characterized by different initial particle number $n(t_0)$. We take $n(t_0) = 1$ (solid blue line), $n(t_0) = 10$ (red dashed line), and $n(t_0) = 50$ (green dotted line) in Fig. 3, with a fixed temperature ($T = 1$ K) of the environment for weak coupling $\eta = 0.5\eta_c$ [see Figs. 3(a), 3(c), 3(e), and 3(g)] and strong coupling $\eta = 1.5\eta_c$ [see Figs. 3(b), 3(d), 3(f), and 3(h)]. The non-Markovianity dynamics shows an explicit $n(t_0)$ dependence. We find that the magnitude of non-Markovianity is enhanced as the initial particle number $n(t_0)$ is increased in the weak coupling regime ($\eta < \eta_c$), although the decay time scale of the non-Markovianity looks similar for different $n(t_0)$ [see Figs. 3(a), 3(c), and 3(e) for spectral densities with $s \leqslant 2$]. But this initial state dependence is only a short-time behavior in the weak system-environment coupling regime. However, for strong coupling ($\eta > \eta_c$) with $s < 2$, we find that the non-Markovianity shows an explicit long-time dependence of the initial state, due to the existence of the localized modes in the strong coupling regime [see Figs. 3(b) and 3(d) for $\eta = 1.5\eta_c$]. Thus, the magnitude of non-Markovianity measure increases in general, as the initial particle number $n(t_0)$ increases. On the other hand, for a super-Ohmic case with $s > 2$ in the weak coupling regime $\eta < \eta_c$, a very slow decay of the $n(t_0)$-dependent non-Markovianity degree is observed [see Fig. 3(g)]. In strong coupling regime, short-time dynamics of non-Markovianity shows strong oscillations which is $n(t_0)$ dependent [see Fig. 3(h)]. Whereas the long-time steady value of non-Markovianity does not significantly depend on $n(t_0)$.

III. EXTENSION TO GENERAL OPEN QUANTUM SYSTEMS

A. Non-Markovianity measure for arbitrary open systems

We now extend the above definition of non-Markovianity measure, based on two-time correlation function, to general open quantum systems

$$\mathcal{N}(t_1,t_2) = \left| \frac{\langle A(t_1)B(t_2)\rangle}{\sqrt{\langle A(t_1)B(t_1)\rangle\langle A(t_2)B(t_2)\rangle}} - \frac{\langle A(t_1)B(t_2)\rangle_{\rm BM}}{\sqrt{\langle A(t_1)B(t_1)\rangle_{\rm BM}\langle A(t_2)B(t_2)\rangle_{\rm BM}}} \right|, \quad (24)$$

where A and B are two physical observables of the system. The exact two-time correlation function $\langle A(t_1)B(t_2)\rangle$ can be obtained from the experiment. For example, two-time current-current correlation $\langle i(t_1)i(t_2)\rangle$ in nanoelectronic systems is experimentally measured [44]. The two-time particle number correlation $\langle n(t_1)n(t_2)\rangle$ for photonic systems is also experimentally measured through photon bunching and anti-bunching experiments [34,35]. In addition, two-time intensity correlation functions have been experimentally measured in optical measurements [36–38]. The two-time correlation function $\langle A(t_1)B(t_2)\rangle_{\rm BM}$ can be evaluated through the BM master equation. Explicitly, for general open systems with the system-environment coupling given by

$$H_I = \sum_{i,k}(V_{ik}L_iE_k^\dagger + V_{ik}^*L_i^\dagger E_k), \quad (25)$$

where V_{ik} is the arbitrary system-reservoir coupling, and L_i and E_k are operators of the system and its environment, respectively. Using BM approximation, one can derive the Lindblad form of the memoryless master equation, as given in standard textbook [63]

$$\frac{d}{dt}\rho(t) = -i[H_S',\rho(t)] + \sum_{ij}\frac{\gamma_{ij}(t)}{2}[2L_i\rho(t)L_j^\dagger - L_j^\dagger L_i\rho(t) - \rho(t)L_j^\dagger L_i]. \quad (26)$$

Here H_S' is the modified system Hamiltonian including the Lamb shift, the decay rates $\gamma_{ij}(t)$ are obtained from the Fourier

transform of the reservoir correlation functions:

$$\gamma_{ij}(t) = \int_0^\infty d\omega e^{i\omega t}\langle E_i^\dagger(t)E_j(0)\rangle, \tag{27}$$

where $E_i(t) = \sum_k V_{ik}e^{iH_Bt}E_ke^{-iH_Bt}$. The BM approximation of two-time correlation function $\langle A(t_1)B(t_2)\rangle_{\rm BM}$ can be calculated using the quantum regression theorem based on above master equation (26). Combining the experimental measurement of the two-time correlation functions which is regarded as the exact one with the theoretical calculation of BM approximation of the corresponding two-time correlation functions, one can finally obtain the non-Markovianity measure given by Eq. (24).

B. A comparative study of the non-Markovianity measure

Here we apply our definition of non-Markovianity measure (24) for a two-level system coupled to a non-Markovian dephasing environment, and compare our results with other proposals of the non-Markovianity measure in the literature, in particular, the non-Markovianity measure in terms of the trace distance of a pair of quantum states [9,10] and divisibility of the dynamical map [7,13]. The two-level dephasing model is described by the Hamiltonian [63]

$$H = \frac{1}{2}\omega_0\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_z(g_k b_k + g_k^* b_k^\dagger), \tag{28}$$

where ω_0 is the energy difference between ground state $|0\rangle$ and excited state $|1\rangle$ of the two-level system. The time evolution of the reduced density matrix of the two-level system is given by the exact master equation in the Schrödinger picture [64]

$$\frac{d}{dt}\rho(t) = -\frac{i\omega_0}{2}[\sigma_z,\rho(t)] + \frac{\Gamma(t)}{2}[\sigma_z\rho(t)\sigma_z - \rho(t)], \tag{29}$$

where

$$\Gamma(t) = 4\int_0^\infty d\omega J(\omega)\coth\left(\frac{\beta\omega}{2}\right)\frac{\sin(\omega t)}{\omega}, \tag{30}$$

with $J(\omega) = \sum_k |g_k|^2\delta(\omega-\omega_k)$ and $\beta = 1/k_BT$. We consider the following Ohmic-type spectral density for this model

$$J(\omega) = \eta\omega e^{-\omega/\omega_c}. \tag{31}$$

The non-Markovianity measure based on the trace distance [9–12] is given by $D(\rho_1,\rho_2) = (1/2)\,\mathrm{Tr}\,|\rho_1-\rho_2|$ (distinguishability between a pair of quantum states ρ_1 and ρ_2 of the system). Markov processes try to continuously reduce the distinguishability of physical states indicating a flow of information from the open system to its environment. An increase of $D(\rho_1(t),\rho_2(t))$ under a quantum process is interpreted as a reversed flow of information from the environment back to the open system, owing to the non-Markovian memory effect. The time evolution of the trace distance between two states for the above dephasing model is given by [65,66]

$$D(\rho_1(t),\rho_2(t)) = \sqrt{a^2 + |G(t)|^2|b|^2}, \tag{32}$$

where $a = \rho_1^{11}(0) - \rho_2^{11}(0)$ is the difference of the excited state populations and $b = \rho_1^{10}(0) - \rho_2^{10}(0)$ is the difference of the coherences of the two initial states. The function $G(t)$ is related

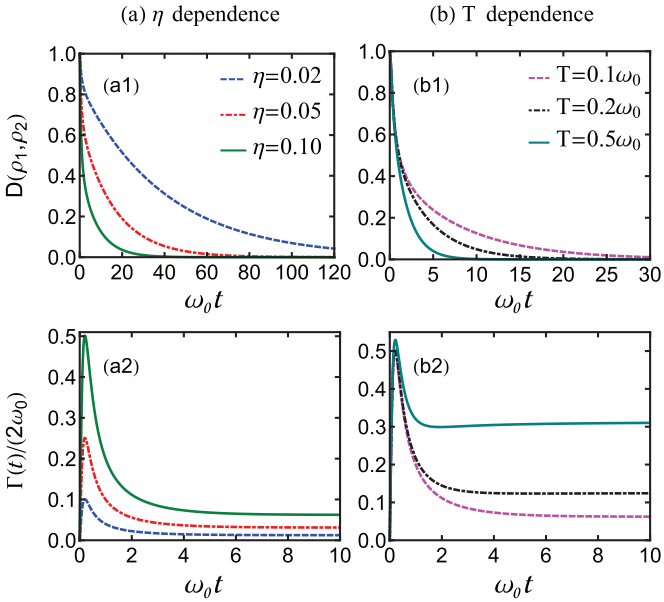

FIG. 4. (Color online) We show η dependence of the trace distance $D(\rho_1(t),\rho_2(t))$ and dimensionless decay rate $\Gamma(t)/(2\omega_0)$ in (a_1) and (a_2), respectively, with a fixed temperature of the reservoir $k_BT = 0.1\omega_0$. Different curves in (a_1) and (a_2) represent different coupling strengths, namely $\eta = 0.02$ (blue dashed line), $\eta = 0.05$ (red dashed dotted line), and $\eta = 0.10$ (solid green line). The temperature dependence of the same quantities $D(\rho_1(t),\rho_2(t))$ and dimensionless $\Gamma(t)/(2\omega_0)$ are shown in (b_1) and (b_2), respectively, with a fixed coupling strength $\eta = 0.1$. Different curves in (b_1) and (b_2) represent different temperatures, namely $k_BT = 0.1\omega_0$ (dashed violet line), $k_BT = 0.2\omega_0$ (dashed dotted black line), and $k_BT = 0.5\omega_0$ (solid cyan line). The cutoff frequency $\omega_c = 5\omega_0$.

to the decoherence through the relation

$$G(t) = \exp\left\{-i\omega_0 t - \int_0^t d\tau\Gamma(\tau)\right\}. \tag{33}$$

One can calculate the trace distance for a pair of initial states with $a = 0$ and $|b| = 1$ for which the growth of trace distance is maximal [65]. In this case, it gives $D(\rho_1(t),\rho_2(t)) = |G(t)|$, as shown from Eq. (32). Hence, the absolute value of the decoherence function is equal to the trace distance $D(\rho_1(t),\rho_2(t))$ for this model. Considering an Ohmic spectral density of Eq. (31) with $s = 1$, the characteristic feature of the trace distance dynamics is illustrated in Figs. 4(a_1) and 4(b_1). In Fig. 4(a_1) we show the dynamics of trace distance $D(\rho_1(t),\rho_2(t))$ with varying coupling strengths. The trace distance (or the absolute value of the decoherence function) decreases monotonically and the rate of decay increases with the increase of coupling strength η for a fixed value of temperature T. The temperature dependence of the trace distance is shown in Fig. 4(b_1), where an enhanced decay rate of the trace distance is observed for increasing temperatures. For this Ohmic spectrum, one never sees an increase of trace distance in any given time interval, even if the coupling strength is increased or in a regime where the reservoir temperature is very low. The non-Markovianity is always zero according to the concept of trace distance [9–12].

Another widely used concept of non-Markovianity is based on the divisibility condition [7,13], according to which

the quantum evolution is Markovian if the corresponding dynamical map is CP divisible (where CP stands for complete positivity). The dynamical map $\Phi(t,0)$ is defined to be divisible if for all $t,\tau \geqslant 0$ the CP map $\Phi(t+\tau,0)$ can be decomposed by the two CP maps $\Phi(t+\tau,t)$ and $\Phi(t,0)$ as follows:

$$\Phi(t+\tau,0) = \Phi(t+\tau,t)\Phi(t,0). \tag{34}$$

Now, for this model, the exact dynamical map Φ given by the time-local master equation (29) is divisible when the decay rate $\Gamma(t)$ is positive definite [9,66]. The negative decay rate at any given time interval will imply a breakdown of the divisibility condition. The quantum evolution will then become non-Markovian according to this criterion. In Fig. 4(a_2) we plot the decay rate $[\Gamma(t)/2]$ for this dephasing model with varying coupling strengths, while Fig. 4(b_2) shows the temperature dependence of the decay rate dynamics. Figures 4(a_2) and 4(b_2) indicate that the decay rate $\Gamma(t)$ remains positive for varying coupling strengths and temperatures. Thus, for this Ohmic spectral density (31), the dynamical map given by the master equation (29) is always completely positive and hence non-Markovian character of the dynamics never shows up according to the divisibility criterion [7,13].

Now, we calculate the non-Markovianity measure using our definition (24) based on two-time correlation function as follows:

$$\mathcal{N}(t,\tau) = \left| \frac{\langle\sigma_+(t)\sigma_-(t+\tau)\rangle}{\sqrt{\langle\sigma_+(t)\sigma_-(t)\rangle\langle\sigma_+(t+\tau)\sigma_-(t+\tau)\rangle}} - \frac{\langle\sigma_+(t)\sigma_-(t+\tau)\rangle_{\rm BM}}{\sqrt{\langle\sigma_+(t)\sigma_-(t)\rangle_{BM}\langle\sigma_+(t+\tau)\sigma_-(t+\tau)\rangle_{\rm BM}}} \right|. \tag{35}$$

For $t=0$, the exact two-time correlation function $\langle\sigma_+(t)\sigma_-(t+\tau)\rangle$ for this dephasing model [64] is given by

$$\langle\sigma_+(t)\sigma_-(t+\tau)\rangle = \exp\left\{-i\omega_0\tau - \int_0^\tau d\tau'\Gamma(\tau')\right\}\langle\sigma_+\sigma_-\rangle. \tag{36}$$

Note that although the physical origin of the two concepts of non-Markovianity (in terms of trace distance and the two-time correlation function) are very different, the mathematical form of trace distance (33) and the two-time correlation function (36) is exactly the same. Interestingly, for this dephasing model, the exact master equation (29) is the same as the BM master equation. Hence, one can apply quantum regression theorem to this master equation (29) to obtain the two-time correlation function $\langle\sigma_+(t)\sigma_-(t+\tau)\rangle_{\rm BM}$ under BM approximation, which comes out to be same as the exact two-time correlation function $\langle\sigma_+(t)\sigma_-(t+\tau)\rangle$ given by Eq. (36). As a result, the non-Markovianity vanishes according to our definition (35). This agrees with the conclusion obtained from Fig. 4 for other two approaches that there is no non-Markovianity for the pure dephasing model in Ohmic environment. In this case, the result also agrees with the result in Ref. [31].

IV. CONCLUSION

To summarize, we have introduced a new definition of non-Markovianity measure using two-time correlation functions. Our non-Markovianity measure quantifies the difference between the exact two-time correlation function and the one obtained in the Born-Markov approximation through the quantum regression theorem. The Born-Markov dynamics of the two-time correlation function completely ignores the memory effect and therefore can be taken as a reference point for measuring non-Markovianity. Such a definition of non-Markovianity also takes into account the proper temperature dependence given explicitly in two-time correlation functions, and can be measured directly in experiments. We also find that the degree of non-Markovianity is reduced in general due to thermal fluctuations for hotter environments. Also, our non-Markovianity measure is an explicit function of time. We discover that, in general, non-Markovianity shows both the short-time and long-time behavior, which crucially depends on the system-environment coupling. The long-time memory effect is the significant contribution of the localized mode induced by the strong system-environment coupling. The non-Markovianity measure also depends on the initial state of the open system in different time scales, as a nontrivial non-Markovian memory effect. Our approach of quantifying non-Markovianity is applicable to the more general class of open quantum systems interacting with a general environment. Further experimental investigations are required to explore these nontrivial dynamics of the non-Markovianity through the measurement of two-time correlation functions.

ACKNOWLEDGMENTS

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APPENDIX A: GENERALIZED QUANTUM REGRESSION THEOREM UNDER BORN APPROXIMATION

Here we present a generalized quantum regression theorem (also see Ref. [2]). The total density operator of the system plus environment is governed by the quantum Liouville equation in the Schrödinger picture

$$\frac{\partial}{\partial\tau}\rho_{\rm tot}(\tau) = \frac{1}{i\hbar}[H,\rho_{\rm tot}(\tau)], \tag{A1}$$

with a formal solution

$$\rho_{\rm tot}(\tau) = e^{-\frac{i}{\hbar}H\tau}\rho_{\rm tot}(0)e^{\frac{i}{\hbar}H\tau}. \tag{A2}$$

Consider an initial state $\rho_{\rm tot}(0)$ of the total system to be uncorrelated and the reservoir to be in a thermal equilibrium state, that is,

$$\rho_{\rm tot}(0) = \rho(0)\otimes\rho_E(0), \quad \rho_E(0) = \frac{e^{-\beta H_E}}{{\rm Tr}\, e^{-\beta H_E}}, \tag{A3}$$

where $H_E = \sum_k \hbar\omega_k b_k^\dagger b_k$ and $\beta = 1/k_B T$ is the initial temperature of the reservoir. In this case, if no nonlinear dynamics is involved, tracing over all the environmental degrees of freedom is easily carried out to obtain a time-local master

equation in terms of the reduced density operator

$$\frac{\partial}{\partial\tau}\rho(\tau) = \mathcal{L}(\tau)\rho(\tau), \tag{A4}$$

where $\rho(\tau) = \mathrm{Tr}_E[\rho_{\mathrm{tot}}(\tau)]$. One can then calculate the mean values of a complete set of system operators $\hat{O}_i$, $i = 1,2,\ldots,$ using Eq. (A4). The expectation value of an operator $\hat{O}_i$ obey the relation

$$\frac{\partial}{\partial\tau}\langle\hat{O}_i(\tau)\rangle = \langle\hat{O}_i(0)\dot{\rho}(\tau)\rangle = \langle\hat{O}_i(0)\mathcal{L}(\tau)\rho(\tau)\rangle. \tag{A5}$$

For each $\hat{O}_i$ one can define

$$\langle\hat{O}_i(0)\mathcal{L}(\tau)\rho(\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j(\tau)\rangle. \tag{A6}$$

Then Eq. (A5) becomes

$$\frac{\partial}{\partial\tau}\langle\hat{O}_i(\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j(\tau)\rangle. \tag{A7}$$

On the other hand, given two system operators $\hat{O}_1$ and $\hat{O}_2$, their two-time correlation function is defined more conveniently in the Heisenberg picture,

$$\left\langle\hat{O}_1^H(t_1)\hat{O}_2^H(t_2)\right\rangle = \mathrm{Tr}_{S\oplus E}\left[\rho_{\mathrm{tot}}^H\hat{O}_1^H(t_1)\hat{O}_2^H(t_2)\right]. \tag{A8}$$

Then going back to the Schrödinger picture through the transformations

$$\hat{O}_1^H(t_1) = e^{\frac{iHt_1}{\hbar}}\hat{O}_1e^{-\frac{iHt_1}{\hbar}}, \tag{A9}$$

$$\hat{O}_2^H(t_2) = e^{\frac{iHt_2}{\hbar}}\hat{O}_2e^{-\frac{iHt_2}{\hbar}}, \tag{A10}$$

$$\rho_{\mathrm{tot}}^H = e^{\frac{iHt}{\hbar}}\rho_{tot}(t)e^{-\frac{iHt}{\hbar}}, \tag{A11}$$

also taking $t_1 = t$ and $t_2 = t+\tau$, and using the cyclic property of trace, we obtain

$$\begin{aligned}\left\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\right\rangle &= \mathrm{Tr}_{S\oplus E}\left[\hat{O}_2e^{-\frac{iH\tau}{\hbar}}\rho_{\mathrm{tot}}(t)\hat{O}_1e^{\frac{iH\tau}{\hbar}}\right]\\ &= \mathrm{Tr}_{S\oplus E}\left[\hat{O}_2\rho_{\mathrm{tot},\hat{O}_1}(t,\tau)\right],\end{aligned} \tag{A12}$$

where we have used the fact that the state of the total system $\rho_{\mathrm{tot}}(t)$ in the Schrödinger picture acted by the operator $\hat{O}_1$ becomes a new state denoted by $\rho_{\mathrm{tot},\hat{O}_1}(t,0) \equiv \rho_{\mathrm{tot}}(t)\hat{O}_1$ such that

$$\rho_{\mathrm{tot},\hat{O}_1}(t,\tau) = e^{-\frac{iH\tau}{\hbar}}\rho_{\mathrm{tot},\hat{O}_1}(t,0)e^{\frac{iH\tau}{\hbar}}. \tag{A13}$$

Now, if $\rho_{\mathrm{tot}}(t)$ is factorized as

$$\rho_{\mathrm{tot}}(t) = \rho(t)\otimes\rho_E(0), \tag{A14}$$

then

$$\rho_{\mathrm{tot},\hat{O}_1}(t,0) = \rho_{\hat{O}_1}(t,0)\otimes\rho_E(0), \tag{A15}$$

where $\rho_{\hat{O}_1}(t,0) \equiv \rho(t)\hat{O}_1$ is the state of the system acted by $\hat{O}_1$ at time t. Thus, Eq. (A12) can be reduced to

$$\begin{aligned}\left\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\right\rangle &= \mathrm{Tr}_S[\hat{O}_2\,\mathrm{Tr}_E\{\rho_{\mathrm{tot},\hat{O}_1}(t,\tau)\}]\\ &= \mathrm{Tr}_S[\hat{O}_2\rho_{\hat{O}_1}(t,\tau)],\end{aligned} \tag{A16}$$

with $\rho_{\hat{O}_1}(t,\tau) = \mathrm{Tr}_E\{\rho_{\mathrm{tot},\hat{O}_1}(t,\tau)\}$.

As one can see, Eq. (A15) for $\rho_{\mathrm{tot},\hat{O}_1}(t,0)$ obeys the same initially factorized condition of Eq. (A3) for $\rho_{\mathrm{tot}}(0)$, and the formal solution (A13) for $\rho_{\mathrm{tot},\hat{O}_1}(t,\tau)$ has the same form of Eq. (A2) for $\rho_{\mathrm{tot}}(\tau)$. Thus, the master equation for $\rho_{\hat{O}_1}(t,\tau)$ with respect to τ must be the same as that for $\rho(\tau)$, namely

$$\frac{\partial}{\partial\tau}\rho_{\hat{O}_1}(t,\tau) = \mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau). \tag{A17}$$

This gives

$$\frac{\partial}{\partial\tau}\left\langle\hat{O}_1^H(t)\hat{O}_2^H(t+\tau)\right\rangle = \langle\hat{O}_2\mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau)\rangle. \tag{A18}$$

For $\hat{O}_2$ we choose a complete set of system operators $\hat{O}_i$ so that for each $\hat{O}_i$,

$$\langle\hat{O}_i(0)\mathcal{L}(\tau)\rho_{\hat{O}_1}(t,\tau)\rangle = \sum_j M_{ij}(\tau)\langle\hat{O}_j\rho_{\hat{O}_1}(t,\tau)\rangle. \tag{A19}$$

Thus we reach

$$\frac{\partial}{\partial\tau}\left\langle\hat{O}_1^H(t)\hat{O}_i^H(t+\tau)\right\rangle = \sum_j M_{ij}(\tau)\left\langle\hat{O}_1^H(t)\hat{O}_j^H(t+\tau)\right\rangle. \tag{A20}$$

This is the generalized quantum regression theorem, namely the form of the evolution equation (A7) for single-time expectation values is the same as that for the two-time correlation functions (A20). Notice that Eq. (A15) is the well-known Born approximation in the derivation of the perturbative master equation up to the second order for system-environment coupling. Thus, the above generalized quantum regression theorem is only valid under Born approximation.

APPENDIX B: CALCULATING $\langle a^\dagger(t)a(t+\tau)\rangle_{\mathrm{qrt}}$

Using the exact master equation (2), one can obtain the evolution matrix $M_{ij}(\tau)$ of Eq. (A7) by calculating the single-time expectation values of a complete set of system operators (a, $a^\dagger$, n, and $\mathbb{1}$), where $n = \langle a^\dagger a\rangle$:

$$\mathbf{M}(\tau) = \begin{pmatrix}\alpha(\tau) & 0 & 0 & 0\\ 0 & \alpha^*(\tau) & 0 & 0\\ 0 & 0 & -2\gamma(\tau) & \tilde{\gamma}(\tau)\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{B1}$$

where $\alpha(\tau) = -[\gamma(\tau) + i\omega_0'(\tau)]$. The time-dependent coefficients in the exact master equation (2) are explicitly given by

$$\omega_0'(\tau) = -\mathrm{Im}\left[\frac{\dot{u}(\tau,t_0)}{u(\tau,t_0)}\right], \tag{B2}$$

$$\gamma(\tau) = -\mathrm{Re}\left[\frac{\dot{u}(\tau,t_0)}{u(\tau,t_0)}\right], \tag{B3}$$

$$\tilde{\gamma}(\tau) = \dot{v}(\tau,\tau) + 2v(\tau,\tau)\gamma(\tau). \tag{B4}$$

Now, naively applying generalized quantum regression theorem as done in Ref. [31] (this is indeed incorrect because the exact master equation does not involve the Born approximation), the evolution equation for the two-time correlation

function $\langle a^\dagger(t)a(t+\tau)\rangle$ is given by

$$\begin{aligned}\frac{d}{d\tau}\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt} &= -[\gamma(\tau)+i\omega_0'(\tau)]\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt}\\ &= \frac{\dot u(\tau,t_0)}{u(\tau,t_0)}\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt}. \end{aligned}\tag{B5}$$

The exact solution of this equation is

$$\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt} = n(t)u(\tau,t_0),\tag{B6}$$

where $n(t)=\langle a^\dagger(t)a(t)\rangle$, which is given by [52]

$$n(t) = v(t,t) + u^*(t,t_0)n(t_0)u(t,t_0).\tag{B7}$$

Thus, the two-time correlation function naively using the generalized quantum regression theorem is given by

$$\begin{aligned}\langle a^\dagger(t)a(t+\tau)\rangle_{\rm qrt} &= u^*(t,t_0)n(t_0)u(t,t_0)u(\tau,t_0)\\ &\quad + v(t,t)u(\tau,t_0).\end{aligned}\tag{B8}$$

This solution is not the exact two-time correlation function given by Eq. (5) nor the result of two-time correlation function obtained under the Born-Markov approximation, i.e., Eq. (16).

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