Relativistic ponderomotive Hamiltonian of a Dirac particle in a vacuum laser field

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We report a point-particle ponderomotive model of a Dirac electron oscillating in a high-frequency field. Starting from the Dirac Lagrangian density, we derive a reduced phase-space Lagrangian that describes the relativistic time-averaged dynamics of such a particle in a geometrical-optics laser pulse propagating in vacuum. The pulse is allowed to have an arbitrarily large amplitude provided that radiation damping and pair production are negligible. The model captures the Bargmann-Michel-Telegdi (BMT) spin dynamics, the Stern-Gerlach spin-orbital coupling, the conventional ponderomotive forces, and the interaction with large-scale background fields (if any). Agreement with the BMT spin precession equation is shown numerically. The commonly known theory in which ponderomotive effects are incorporated in the particle effective mass is reproduced as a special case when the spin-orbital coupling is negligible. This model could be useful for studying laser-plasma interactions in relativistic spin-1/2 plasmas.

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I. INTRODUCTION

In recent years, many works have been focused on incorporating quantum effects into classical plasma physics [1,2]. In particular, various models have been proposed to marry spin equations with classical equations of plasma dynamics. This includes the early works by Takabayasi [3,4] as well as the most recent works presented in Refs. [5–13]. Of particular interest in this regard is the regime wherein particles interact with high-frequency electromagnetic (EM) radiation. In this regime, it is possible to introduce a simpler time-averaged description, in which particles experience effective time-averaged, or "ponderomotive," forces [14–16]. It was shown recently that the inclusion of spin effects yields intriguing corrections to this time-averaged dynamics [17,18]. However, current "spin-ponderomotive" theories remain limited to regimes where (i) the particle de Broglie wavelength is much less than the radiation wavelength and (ii) the radiation amplitude is small enough so that it can be treated as a perturbation. These conditions are far more restrictive than those of spinless particle theories, where nonperturbative, relativistic ponderomotive effects can be accommodated within the effectively modified particle mass [19–23]. One may wonder then: is it possible to derive a fully relativistic, and yet transparent, theory accounting also for the spin dynamics and the Stern-Gerlach-type spin-orbital coupling?

Excitingly, the answer is yes, and the purpose of this paper is to propose such a description. More specifically, what we report here is a point-particle ponderomotive model of a Dirac electron [24]. Starting from the Dirac Lagrangian density, we derive a phase-space Lagrangian (75) in canonical coordinates with a Hamiltonian (76) that describes the relativistic time-averaged dynamics of such particle in a geometrical-optics (GO) laser pulse propagating in vacuum [25]. The pulse is allowed to have an arbitrarily large amplitude (as long as radiation damping and pair production are negligible) and, in case of nonrelativistic interactions, a wavelength comparable to the electron de Broglie wavelength. The model captures the spin dynamics, the spin-orbital coupling, the conventional ponderomotive forces, and the interaction with large-scale background fields (if any). Agreement with the Bargmann-Michel-Telegdi (BMT) spin precession equation [26] is shown numerically. The aforementioned "effective-mass" theory for spinless particles is reproduced as a special case when the spin-orbital coupling is negligible. Also notably, the point-particle Lagrangian that we derive has a canonical structure, which could be helpful in simulating the corresponding dynamics using symplectic methods [27–29].

This work is organized as follows. In Sec. II the basic notation is defined. In Sec. III the main assumptions used throughout the work are presented. To arrive at the point-particle ponderomotive model, Secs. IV–VII apply successive approximations and reparametrizations to the Dirac Lagrangian density. Specifically, in Sec. IV we derive a ponderomotive Lagrangian density that captures the average dynamics of a Dirac particle. In Sec. V we obtain a reduced Lagrangian model that explicitly shows orbital-spin-coupling effects. In Sec. VI we deduce a "fluid" Lagrangian model that describes the particle wave-packet dynamics. In Sec. VII we calculate the point-particle limit of such "fluid" model. In Sec. VIII the ponderomotive model is numerically compared to a generalized nonaveraged BMT model. In Sec. IX the main results are summarized.

II. NOTATION

The following notation is used throughout the paper. The symbol "$\doteq$" denotes definitions, "H.c." denotes "Hermitian conjugate," and "c.c." denotes "complex conjugate." Unless indicated otherwise, we use natural units so that the speed of light and the Plank constant equal unity ($c = \hbar = 1$). The identity $N \times N$ matrix is denoted by $\mathbb{I}_N$. The Minkowski metric is adopted with signature $(+, -, -, -)$. Greek indices span from 0 to 3 and refer to space-time coordinates $x^\mu = (x^0, \mathbf{x})$ with x^0 corresponding to the time variable t. Also, $\partial_\mu \equiv \partial/\partial x^\mu = (\partial_t, \boldsymbol{\nabla})$ and $d^4x \equiv dt\, d^3x$. Latin indices span from 1 to 3 and denote the spatial variables, i.e., $\mathbf{x} = (x^1, x^2, x^3)$ and $\partial_i \equiv \partial/\partial x^i$. Summation over repeated indexes is assumed. In particular, for arbitrary four-vectors a and b, we have $\mathsf{a} \cdot \mathsf{b} \equiv a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$ and $\mathsf{a}^2 \equiv \mathsf{a} \cdot \mathsf{a}$. The Feynman slash notation is used: $\not{a} \doteq a_\mu \gamma^\mu$, where $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ are the Dirac matrices (see below). The average of an arbitary complex-valued function $f(\mathsf{x},\Theta)$ with respect to a phase Θ is denoted by $\langle f \rangle$. In Euler-Lagrange equations (ELEs), the notation "δa :" means that the corresponding equation is obtained by extremizing the action integral with respect to a.

III. BASIC FORMALISM

As for any quantum particle or nondissipative wave [30], the dynamics of a Dirac electron [24] is governed by the least action principle $\delta\mathcal{A} = 0$, where $\mathcal{A}$ is the action integral

$$\mathcal{A} = \int \mathfrak{L}\, d^4x, \tag{1}$$

and $\mathfrak{L}$ is the Lagrangian density given by [31]

$$\mathfrak{L} = \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - \bar{\psi}q\not{A}\psi - \bar{\psi}m\psi. \tag{2}$$

Here q and m are the particle charge and mass, ψ is a complex four-component wave function, and $\bar{\psi} \doteq \psi^\dagger\gamma^0$ is its Dirac conjugate. The Dirac matrices γ^μ satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}_4, \tag{3}$$

where $g^{\mu\nu}$ is the Minkowski metric tensor. Hence,

$$\not{a}\not{b} + \not{b}\not{a} = 2(\mathsf{a}\cdot\mathsf{b})\mathbb{I}_4, \tag{4}$$

$$\not{a}\not{a} = \mathsf{a}^2\mathbb{I}_4, \tag{5}$$

for any pair of four-vectors a and b. In this work, the standard representation of the Dirac matrices is used:

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{6}$$

where $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are the 2×2 Pauli matrices. Note that these matrices satisfy

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{7}$$

We consider the interaction of an electron with an EM field such that the four-vector potential A has the form

$$\mathsf{A}(\epsilon\mathsf{x},\Theta) = \mathsf{A}_{\text{bg}}(\epsilon\mathsf{x}) + \mathsf{A}_{\text{osc}}(\epsilon\mathsf{x},\Theta). \tag{8}$$

Here $\mathsf{A}_{\text{bg}}(\epsilon\mathsf{x})$ describes a background field (if any) that is slow, as determined by a small dimensionless parameter ϵ, which is yet to be specified. The other part of the vector potential,

$$\mathsf{A}_{\text{osc}}(\epsilon\mathsf{x},\Theta) = \text{Re}[\mathsf{A}_{\text{osc,c}}(\epsilon\mathsf{x})e^{i\Theta}], \tag{9}$$

describes a rapidly oscillating EM wave field, e.g., a laser pulse. Here $\mathsf{A}_{\text{osc,c}}(\epsilon\mathsf{x})$ is a complex four-vector describing the laser envelope with a slow space-time dependence, and Θ is a rapid phase. The EM wave frequency is defined by $\omega(\epsilon\mathsf{x}) \doteq -\partial_t\Theta$, and the wave vector is $\mathbf{k}(\epsilon\mathsf{x}) \doteq \nabla\Theta$. Accordingly, $k^\mu \doteq -\partial^\mu\Theta = (\omega,\mathbf{k})$. We describe A_{osc} within the GO approximation [32] and assume that the interaction takes place in vacuum. Then, the four-wavevector k satisfies the vacuum dispersion relation

$$\mathsf{k}^2 = \omega^2 - \mathbf{k}^2 = 0, \tag{10}$$

which can also be expressed as

$$\not{k}\not{k} = \mathsf{k}^2\mathbb{I}_4 = 0. \tag{11}$$

Furthermore, a Lorentz gauge condition is chosen for the oscillatory field such that

$$\partial_\mu A^\mu_{\text{osc}} = 0. \tag{12}$$

In this work, we neglect radiation damping and assume

$$\omega'/\omega_c \ll 1, \tag{13}$$

where $\omega_c \doteq m$ is the Compton frequency and ω' is the frequency in the electron rest frame. Then, pair production (and annihilation) can be neglected. We also assume

$$\epsilon \doteq \max\left(\frac{1}{\omega\tau},\frac{1}{|\mathbf{k}|\ell}\right) \ll 1, \tag{14}$$

where τ and ℓ are the characteristic temporal and spatial scales of k, A_{bg}, and $\mathsf{A}_{\text{osc,c}}$. Using this ordering and the Lagrangian density (2), we aim to derive a reduced Lagrangian density that describes the ponderomotive (Θ-averaged) dynamics of an electron accurately enough to capture the spin-orbital coupling effects to the leading order in ϵ. As shown in Refs. [33,34], this requires that $O(\epsilon)$ terms be retained when approximating the Lagrangian density (2). Such reduced Lagrangian density is derived as follows.

IV. PONDEROMOTIVE MODEL

In this section, we derive a ponderomotive Lagrangian density for the four-component Dirac wave function.

A. Wave function parametrization

Consider the following representation for the four-component wave function:

$$\psi(\mathsf{x}) = \xi e^{i\theta}. \tag{15}$$

Here $\theta(\mathsf{x})$ is a fast real phase, and $\xi(\epsilon\mathsf{x},\Theta)$ is a complex four-component vector slow compared to $\theta(\mathsf{x})$. In these variables, the Lagrangian density (2) is expressed as

$$\mathfrak{L} = \frac{i}{2}[\bar{\xi}\gamma^\mu(\partial_\mu\xi) - (\partial_\mu\bar{\xi})\gamma^\mu\xi] + \bar{\xi}(\not{\pi} - q\not{A}_{\text{osc}} - m\mathbb{I}_4)\xi, \tag{16}$$

where

$$\pi^\mu(\epsilon\mathsf{x}) \doteq p^\mu - qA^\mu_{\text{bg}}, \tag{17}$$

$$p_\mu(\epsilon\mathsf{x}) \doteq -\partial_\mu\theta. \tag{18}$$

It is convenient to parametrize ξ in terms of the "semiclassical" Volkov solution (Appendix A) since the latter becomes the exact solution of the Dirac equation in the limit of vanishing ϵ. Specifically, we write

$$\xi(\epsilon\mathsf{x},\Theta) = \Xi e^{i\tilde{\theta}}\varphi. \tag{19}$$

Here $\varphi(\epsilon\mathsf{x},\Theta)$ is a near-constant function with an asymptotic representation of the form

$$\varphi(\epsilon\mathsf{x},\Theta) = \sum_{n=-\infty}^{\infty} \epsilon^{|n|}\varphi_n(\epsilon\mathsf{x})e^{in\Theta} \tag{20}$$

(so that $\varphi \to \varphi_0 \to$ const at $\epsilon \to 0$), the real phase $\tilde{\theta}(\epsilon\mathsf{x},\Theta)$ is given by

$$\tilde{\theta}(\epsilon\mathsf{x},\Theta) \doteq \frac{q}{\pi\cdot\mathsf{k}}\int^{\Theta}\pi\cdot\mathsf{A}_{\rm osc}\,d\Theta' - \frac{q^2}{2(\pi\cdot\mathsf{k})}\int^{\Theta}\left(\mathsf{A}_{\rm osc}^2 - \langle\mathsf{A}_{\rm osc}^2\rangle\right)d\Theta' \tag{21}$$

and has the property $\langle\tilde{\theta}\rangle = 0$, and $\Xi(\epsilon\mathsf{x},\Theta)$ is a matrix defined as follows:

$$\Xi(\epsilon\mathsf{x},\Theta) \doteq \mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{k}\not{A}_{\rm osc}. \tag{22}$$

Notice also that the Dirac conjugate of ξ is given by

$$\bar{\xi} = (\Xi e^{i\tilde{\theta}}\varphi)^\dagger\gamma^0 = e^{-i\tilde{\theta}}\varphi^\dagger\Xi^\dagger\gamma^0 = e^{-i\tilde{\theta}}\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0, \tag{23}$$

where

$$\begin{aligned}\gamma^0\Xi^\dagger\gamma^0 &= \gamma^0\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{A}_{\rm osc}^\dagger\gamma^0\gamma^0\not{k}^\dagger\right]\gamma^0\\ &= \mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{A}_{\rm osc}\not{k}.\end{aligned} \tag{24}$$

Here we used Eqs. (3), (7), and (22).

B. Lagrangian density in the new variables

Inserting Eqs. (19) and (23) into Eq. (16) leads to

$$\begin{aligned}\mathfrak{L} = {}& \underbrace{\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\not{\pi}\,\Xi\varphi}_{=\mathfrak{L}_1}\underbrace{-\bar{\varphi}m\gamma^0\Xi^\dagger\gamma^0\Xi\varphi}_{=\mathfrak{L}_2}\underbrace{-\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0 q\not{A}_{\rm osc}\Xi\varphi}_{=\mathfrak{L}_3}\\ &+\underbrace{\frac{i}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi(\partial_\mu\varphi) - \text{c.c.}]}_{=\mathfrak{L}_4}\underbrace{-\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0(\not{\partial}\tilde{\theta})\Xi\varphi}_{=\mathfrak{L}_5}\\ &+\underbrace{\frac{i}{2}\bar{\varphi}[\gamma^0\Xi^\dagger\gamma^0\gamma^\mu(\partial_\mu\Xi) - \text{H.c.}]\varphi}_{=\mathfrak{L}_6}.\end{aligned} \tag{25}$$

Let us explicitly calculate each term in Eq. (25). Substituting Eqs. (22) and (24) into $\mathfrak{L}_1$ leads to

$$\begin{aligned}\mathfrak{L}_1 &= \bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\not{\pi}\,\Xi\varphi\\ &= \bar{\varphi}\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{A}_{\rm osc}\not{k}\right]\not{\pi}\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{k}\not{A}_{\rm osc}\right]\varphi\\ &= \bar{\varphi}\left[\not{\pi} + \frac{q}{2(\pi\cdot\mathsf{k})}(\not{A}_{\rm osc}\not{k}\not{\pi} + \not{\pi}\not{k}\not{A}_{\rm osc})\right.\\ &\quad\left. + \frac{q^2}{4(\pi\cdot\mathsf{k})^2}\not{A}_{\rm osc}\not{k}\not{\pi}\not{k}\not{A}_{\rm osc}\right]\varphi\\ &= \bar{\varphi}\left[\not{\pi} + q\not{A}_{\rm osc} + \frac{q\mathsf{A}_{\rm osc}\cdot\mathsf{k}}{\pi\cdot\mathsf{k}}\not{\pi} - \frac{q\mathsf{A}_{\rm osc}\cdot\pi}{\pi\cdot\mathsf{k}}\not{k}\right.\\ &\quad\left. + \frac{q^2\mathsf{A}_{\rm osc}\cdot\mathsf{k}}{\pi\cdot\mathsf{k}}\not{A}_{\rm osc} - \frac{q^2\mathsf{A}_{\rm osc}^2}{2(\pi\cdot\mathsf{k})}\not{k}\right]\varphi,\end{aligned} \tag{26}$$

where we used Eqs. (4), (5), and (11). Similarly,

$$\begin{aligned}\mathfrak{L}_2 &= -\bar{\varphi}m\gamma^0\Xi^\dagger\gamma^0\Xi\varphi\\ &= -\bar{\varphi}m\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{A}_{\rm osc}\not{k}\right]\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{k}\not{A}_{\rm osc}\right]\varphi\\ &= -\bar{\varphi}m\left[\mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}(\not{A}_{\rm osc}\not{k} + \not{k}\not{A}_{\rm osc})\right]\varphi\\ &= -\bar{\varphi}m\left(1 + \frac{q\mathsf{A}_{\rm osc}\cdot\mathsf{k}}{\pi\cdot\mathsf{k}}\right)\varphi,\end{aligned} \tag{27}$$

where we used Eq. (11) to obtain the third line and Eq. (4) to get the last line. For $\mathfrak{L}_3$, one obtains

$$\begin{aligned}\mathfrak{L}_3 &= -\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0 q\not{A}_{\rm osc}\Xi\varphi\\ &= -\bar{\varphi}\left[\mathbb{I}_4 + \frac{q\not{A}_{\rm osc}\not{k}}{2(\pi\cdot\mathsf{k})}\right]q\not{A}_{\rm osc}\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\rm osc}}{2(\pi\cdot\mathsf{k})}\right]\varphi\\ &= -\bar{\varphi}\left[q\not{A}_{\rm osc} + \frac{q^2}{\pi\cdot\mathsf{k}}\not{A}_{\rm osc}\not{k}\not{A}_{\rm osc}\right.\\ &\quad\left. + \frac{q^3}{4(\pi\cdot\mathsf{k})^2}\not{A}_{\rm osc}\not{k}\not{A}_{\rm osc}\not{k}\not{A}_{\rm osc}\right]\varphi\\ &= -\bar{\varphi}\left[q\not{A}_{\rm osc} + \frac{2q^2\mathsf{A}_{\rm osc}\cdot\mathsf{k}}{\pi\cdot\mathsf{k}}\not{A}_{\rm osc} - \frac{q^2\mathsf{A}_{\rm osc}^2}{\pi\cdot\mathsf{k}}\not{k}\right.\\ &\quad\left. + \frac{q^3\mathsf{A}_{\rm osc}\cdot\mathsf{k}}{2(\pi\cdot\mathsf{k})^2}\not{A}_{\rm osc}\not{k}\not{A}_{\rm osc}\right]\varphi.\end{aligned} \tag{28}$$

The terms $\mathfrak{L}_4$, $\mathfrak{L}_5$, and $\mathfrak{L}_6$ in Eq. (25) involve space-time derivatives of $(\tilde{\theta},\Xi,\varphi)$, which have slow space-time and rapid Θ dependencies. For convenience, let us write the derivative operator ∂_μ as follows:

$$\partial_\mu f(\epsilon\mathsf{x},\Theta) = \epsilon d_\mu f(\epsilon\mathsf{x},\Theta) - k_\mu\partial_\Theta f(\epsilon\mathsf{x},\Theta), \tag{29}$$

where f is an arbitrary function and d_μ indicates a derivation with respect to the first argument of f. In this notation, $\mathfrak{L}_4$ can be written as follows:

$$\begin{aligned}\mathfrak{L}_4 &= \frac{i}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi(\partial_\mu\varphi) - \text{c.c.}]\\ &= -\frac{i}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\not{k}\,\Xi(\partial_\Theta\varphi) - \text{c.c.}]\\ &\quad + \frac{i\epsilon}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi(d_\mu\varphi) - \text{c.c.}]\\ &= -\frac{i}{2}[\bar{\varphi}\not{k}(\partial_\Theta\varphi) - (\partial_\Theta\bar{\varphi})\not{k}\varphi]\\ &\quad + \frac{i\epsilon}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi(d_\mu\varphi) - \text{c.c.}],\end{aligned} \tag{30}$$

where in the third line, we used Eqs. (11), (22), and (24). Similarly, substituting Eq. (21) into $\mathfrak{L}_5$ leads to

$$\begin{aligned}\mathfrak{L}_5 &= -\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0(\not{\partial}\tilde{\theta})\Xi\varphi\\ &= \bar{\varphi}\left[\mathbb{I}_4 + \frac{q\not{A}_{\rm osc}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\not{k}(\partial_\Theta\tilde{\theta})\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\rm osc}}{2(\pi\cdot\mathsf{k})}\right]\varphi\\ &\quad - \epsilon\bar{\varphi}\left[\mathbb{I}_4 + \frac{q\not{A}_{\rm osc}\not{k}}{2(\pi\cdot\mathsf{k})}\right](\not{d}\tilde{\theta})\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\rm osc}}{2(\pi\cdot\mathsf{k})}\right]\varphi\\ &= \bar{\varphi}\not{k}\varphi\left[\frac{q\mathsf{A}_{\rm osc}\cdot\pi}{\pi\cdot\mathsf{k}} - \frac{q^2\mathsf{A}_{\rm osc}^2}{2(\pi\cdot\mathsf{k})} + \frac{q^2\langle\mathsf{A}_{\rm osc}^2\rangle}{2(\pi\cdot\mathsf{k})}\right]\\ &\quad - \epsilon\bar{\varphi}\left[\mathbb{I}_4 + \frac{q\not{A}_{\rm osc}\not{k}}{2(\pi\cdot\mathsf{k})}\right](\not{d}\tilde{\theta})\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\rm osc}}{2(\pi\cdot\mathsf{k})}\right]\varphi.\end{aligned} \tag{31}$$

Finally, the last term $\mathfrak{L}_6$ gives

$$\begin{aligned}\mathfrak{L}_6 &= \frac{i}{2}\bar{\varphi}[\gamma^0\Xi^\dagger\gamma^0\gamma^\mu(\partial_\mu\Xi) - \text{H.c.}]\varphi \\ &= \frac{i\epsilon}{2}\bar{\varphi}\left\{\left[\mathbb{I}_4 + \frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\not{d}\left[\frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\right. \\ &\quad\left. - d_\mu\left[\frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\gamma^\mu\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\right\}\varphi. \end{aligned} \tag{32}$$

Substituting Eqs. (26)–(32) into Eq. (25) leads to

$$\begin{aligned}\mathfrak{L} &= -\frac{i}{2}[\bar{\varphi}\not{k}(\partial_\Theta\varphi) - (\partial_\Theta\bar{\varphi})\not{k}\varphi] \\ &\quad + \bar{\varphi}\left[\not{\pi} + \not{k}\frac{q^2\langle\mathsf{A}^2_{\text{osc}}\rangle}{2(\pi\cdot\mathsf{k})} - m\mathbb{I}_4\right]\varphi + \mathcal{F} + \mathcal{G},\end{aligned} \tag{33}$$

where

$$\mathcal{F} \doteq \frac{i\epsilon}{2}[\bar{\varphi}\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi(d_\mu\varphi) - \text{c.c.}] \tag{34}$$

and

$$\begin{aligned}\mathcal{G} &\doteq \bar{\varphi}\left[\frac{q\chi}{\pi\cdot\mathsf{k}}(\not{\pi} - m\mathbb{I}_4) - \frac{q^2\chi}{\pi\cdot\mathsf{k}}\not{A}_{\text{osc}} - \frac{q^3\chi}{2(\pi\cdot\mathsf{k})^2}\not{A}_{\text{osc}}\not{k}\not{A}_{\text{osc}}\right]\varphi \\ &\quad - \epsilon\bar{\varphi}\left[\mathbb{I}_4 + \frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right](\not{d}\tilde{\theta})\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\varphi \\ &\quad + \frac{i\epsilon}{2}\bar{\varphi}\left\{\left[\mathbb{I}_4 + \frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\not{d}\left[\frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\right. \\ &\quad \left. - d_\mu\left[\frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\gamma^\mu\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\right\}\varphi.\end{aligned} \tag{35}$$

Here we introduced $\chi(\epsilon\mathsf{x},\Theta) \doteq \mathsf{k}(\epsilon\mathsf{x})\cdot\mathsf{A}_{\text{osc}}(\epsilon\mathsf{x},\Theta)$. From Eqs. (12) and (29), one has $k_\mu\partial_\Theta A^\mu_{\text{osc}} = \epsilon d_\mu A^\mu_{\text{osc}}$, so

$$\chi = \epsilon\int^\Theta d_\mu A^\mu_{\text{osc}}(\epsilon\mathsf{x},\Theta')\,d\Theta'. \tag{36}$$

Hence, it is seen that $\chi = O(\epsilon)$, so $\mathcal{G} = O(\epsilon)$.

C. Approximate Lagrangian density

The reduced Lagrangian density $\mathcal{L}$ that governs the time-averaged, or ponderomotive, dynamics can be derived as the time average of $\mathfrak{L}$, as usual [35,36]. In our case, the time average coincides with the Θ average, so

$$\mathcal{L} \doteq \langle\mathfrak{L}\rangle. \tag{37}$$

Remember that we are interested in calculating $\mathcal{L}$ with accuracy up to $O(\epsilon)$. Using Eqs. (11) and (20) and also the fact that χ is shifted in phase from A_{osc} by $\pi/2$ [cf. Eq. (36)], it can be shown that $\langle\mathcal{G}\rangle = O(\epsilon^2)$. Therefore, the contribution of $\mathcal{G}$ to $\mathcal{L}$ can be neglected. Similarly, we can also neglect the first term in Eq. (33) since

$$-\frac{i}{2}\langle\bar{\varphi}\not{k}(\partial_\Theta\varphi) - (\partial_\Theta\bar{\varphi})\not{k}\varphi\rangle = \sum_{n=-\infty}^{\infty} n\epsilon^{|2n|}\bar{\phi}_n\not{k}\phi_n = O(\epsilon^2), \tag{38}$$

where we substituted the asymptotic expansion (20). The second term in Eq. (33) gives

$$\begin{aligned}&\left\langle\bar{\varphi}\left[\not{\pi} + \not{k}\frac{q^2\langle\mathsf{A}^2_{\text{osc}}\rangle}{2(\pi\cdot\mathsf{k})} - m\mathbb{I}_4\right]\varphi\right\rangle \\ &\quad = \bar{\varphi}_0\left[\not{\pi} + \not{k}\frac{q^2\langle\mathsf{A}^2_{\text{osc}}\rangle}{2(\pi\cdot\mathsf{k})} - m\mathbb{I}_4\right]\varphi_0 + O(\epsilon^2).\end{aligned} \tag{39}$$

By following similar considerations, we also calculate $\langle\mathcal{F}\rangle$, namely, as follows. Averaging the first term in $\mathcal{F}$ gives

$$\begin{aligned}&\langle\bar{\varphi}_0\gamma^0\Xi^\dagger\gamma^0\gamma^\mu\Xi d_\mu\varphi_0\rangle \\ &= \bar{\varphi}_0\left\langle\left[\mathbb{I}_4 + \frac{q\not{A}_{\text{osc}}\not{k}}{2(\pi\cdot\mathsf{k})}\right]\gamma^\mu\left[\mathbb{I}_4 + \frac{q\not{k}\not{A}_{\text{osc}}}{2(\pi\cdot\mathsf{k})}\right]\right\rangle d_\mu\varphi_0 \\ &= \bar{\varphi}_0\left[\gamma^\mu + \frac{q^2}{4(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}\not{k}\gamma^\mu\gamma^\nu k_\nu\not{A}_{\text{osc}}\rangle\right]d_\mu\varphi_0 \\ &= \bar{\varphi}_0\left[\gamma^\mu + \frac{q^2}{4(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}\not{k}(2k^\mu - \not{k}\gamma^\mu)\not{A}_{\text{osc}}\rangle\right]d_\mu\varphi_0 \\ &= \bar{\varphi}_0\left[\gamma^\mu + k^\mu\frac{q^2}{2(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}\not{k}\not{A}_{\text{osc}}\rangle\right]d_\mu\varphi_0 \\ &= \bar{\varphi}_0\Gamma^\mu d_\mu\varphi_0,\end{aligned} \tag{40}$$

where we used Eqs. (3) and (11). We also introduced the modified Dirac matrices

$$\begin{aligned}\Gamma^\mu(\epsilon\mathsf{x}) &\doteq \gamma^\mu + k^\mu\frac{q^2}{2(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}\not{k}\not{A}_{\text{osc}}\rangle \\ &= \gamma^\mu + k^\mu\frac{q^2}{2(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}(2\chi - \not{A}_{\text{osc}}\not{k})\rangle \\ &= \gamma^\mu - k^\mu\frac{q^2}{2(\pi\cdot\mathsf{k})^2}\langle\not{A}_{\text{osc}}\not{A}_{\text{osc}}\rangle\not{k} \\ &= \gamma^\mu - k^\mu\frac{q^2\langle\mathsf{A}^2_{\text{osc}}\rangle}{2(\pi\cdot\mathsf{k})^2}\not{k}.\end{aligned} \tag{41}$$

Gathering the previous results, we obtain the following reduced Lagrangian density:

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}[\bar{\phi}\Gamma^\mu(\partial_\mu\phi) - (\partial_\mu\bar{\phi})\Gamma^\mu\phi] \\ &\quad + \bar{\phi}\left[\not{\pi} + \not{k}\frac{q^2\langle\mathsf{A}^2_{\text{osc}}\rangle}{2(\pi\cdot\mathsf{k})} - m\mathbb{I}_4\right]\phi + O(\epsilon^2),\end{aligned} \tag{42}$$

where $\phi \doteq \varphi_0$. Since only slow space-time dependencies appear in Eq. (42), we dropped the "ϵd_μ" notation for slow space-time derivatives and returned to the "∂_μ" notation.

V. REDUCED MODEL

In this section, the Lagrangian density (42) is further simplified by considering only positive-kinetic-energy particle states. The resulting model describes two-component wave functions instead of four-component wave functions, which leads to explicit identification of the spin-coupling term.

A. Particle and antiparticle states

First let us briefly review the case when ϵ is vanishingly small. In this case, $\partial_\mu\phi$ can be neglected, so Eq. (42) can be approximated as

$$\mathcal{L}_0[\theta,\phi,\bar{\phi}] = \bar{\phi}\left[\not{\pi} + \not{k}\frac{q^2\langle \mathsf{A}_{\rm osc}^2\rangle}{2(\pi\cdot \mathsf{k})} - m\mathbb{I}_4\right]\phi, \tag{43}$$

where ϕ, $\bar{\phi}$, and θ can be treated as independent variables. [The Lagrangian density $\mathcal{L}_0$ depends on θ in the sense that it depends on π_μ, which is defined through $\partial_\mu\theta$ (Sec. IV A).] When varying the action with respect to $\bar{\phi}$, the corresponding ELE is

$$\delta\bar{\phi}:\quad (\not{\lambda} - m\mathbb{I}_4)\phi = 0, \tag{44}$$

where

$$\lambda^\mu(\epsilon \mathsf{x}) \doteq \pi^\mu + \alpha k^\mu \tag{45}$$

is a quasi four-momentum [37] and

$$\alpha(\epsilon \mathsf{x}) \doteq \frac{q^2\langle \mathsf{A}_{\rm osc}^2\rangle}{2(\pi\cdot \mathsf{k})}. \tag{46}$$

The local eigenvalues are obtained by solving

$$\det(\not{\lambda} - m\mathbb{I}_4) = 0. \tag{47}$$

Since the local dispersion relation (47) has the same form as that of the free-streaming Dirac particle [38], one has

$$\lambda\cdot\lambda = \pi\cdot\pi + q^2\langle \mathsf{A}_{\rm osc}^2\rangle = m^2, \tag{48}$$

where we used Eq. (10). Solving for π^0 leads to

$$\pi^0 = -\partial_t\theta - qV_{\rm bg} = \pm\sqrt{(\nabla\theta - q\mathbf{A}_{\rm bg})^2 + m_{\rm eff}^2}. \tag{49}$$

Here $m_{\rm eff}$ is the "effective mass" [19–21] given by

$$m_{\rm eff}^2(\epsilon \mathsf{x}) \doteq m^2 - q^2\langle \mathsf{A}_{\rm osc}^2\rangle(\epsilon \mathsf{x}). \tag{50}$$

(The minus sign is due to the chosen metric signature.) Equation (49) is the well known Hamilton-Jacobi equation that governs the ponderomotive dynamics of a relativistic spinless particle interacting with an oscillating EM vacuum field and a slowly varying background EM field [39–42]. The two roots in Eq. (49) represent solutions for the particle and the antiparticle states.

B. Eigenmode decomposition

Corresponding to the eigenvalues given by Eq. (49), there exists four orthonormal eigenvectors h_q which are obtained from Eq. (44) and represent the particle and the antiparticle states. Since h_q form a complete basis, one can write $\phi = h_q\eta^q$, where η^q are scalar functions. Recall also that pair production is neglected in our model due to the assumption (13). Let us hence focus on particle states, merely for clarity, which correspond to positive kinetic energies

$$\varepsilon_{\rm eff} = \sqrt{\boldsymbol{\pi}^2 + m_{\rm eff}^2} \tag{51}$$

in the limit of vanishing ϵ. We will assume that only such states are actually excited (we call these eigenmodes "active"), whereas the antiparticle states acquire nonzero amplitudes only through the medium inhomogeneities (we call these eigenmodes "passive"). When designating the active mode eigenvectors by $h_{1,2}$ and the passive mode eigenvectors by $h_{3,4}$, we have

$$\eta^q = \begin{cases} O(\epsilon^0), & q = 1,2 \\ O(\epsilon^1), & q = 3,4 \end{cases} \tag{52}$$

As shown in Ref. [33], due to the mutual orthogonality of all h_q, the contribution of passive modes to $\mathcal{L}$ is $o(\epsilon)$, so it can be neglected entirely. In other words, for the purpose of calculating $\mathcal{L}$, it is sufficient to adopt $\phi \approx h_1\eta^1 + h_2\eta^2$. It is convenient to write this active eigenmode decomposition in a matrix form

$$\phi(\epsilon \mathsf{x}) = \Psi\eta, \tag{53}$$

where

$$\Psi(\epsilon \mathsf{x}) = \sqrt{\frac{m+\lambda^0}{2\varepsilon_{\rm eff}}}\begin{pmatrix} \mathbb{I}_2 \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix} \tag{54}$$

is a 4×2 matrix having h_1 and h_2 as its columns and

$$\eta(\epsilon \mathsf{x}) \doteq \begin{pmatrix}\eta^1 \\ \eta^2\end{pmatrix}. \tag{55}$$

It is to be noted that $\eta^1(\epsilon \mathsf{x})$ and $\eta^2(\epsilon \mathsf{x})$ describe wave envelopes corresponding to the spin-up and spin-down states.

When inserting the eigenmode representation (53) into Eq. (42), one obtains [33]

$$\mathcal{L} = \mathcal{K} - \eta^\dagger(\mathcal{E} - \mathcal{U})\eta + o(\epsilon), \tag{56}$$

where

$$\mathcal{K} \doteq \frac{i}{2}[\eta^\dagger\Psi^\dagger\gamma^0\Gamma^\mu\Psi(\partial_\mu\eta) - \text{c.c.}], \tag{57}$$

$$\mathcal{E} \doteq \partial_t\theta + \varepsilon_{\rm eff} + qV_{\rm bg}, \tag{58}$$

$$\mathcal{U} \doteq \frac{i}{2}[\Psi^\dagger\gamma^0\Gamma^\mu(\partial_\mu\Psi) - \text{H.c.}]. \tag{59}$$

The terms $\mathcal{K}$ and $\mathcal{U}$, which are of order ϵ, represent corrections to the lowest-order (in ϵ) Lagrangian density. Specifically, for $\mathcal{K}$ one obtains (Appendix B 1)

$$\mathcal{K} = \frac{i}{2}[\eta^\dagger(d_t\eta) - (d_t\eta^\dagger)\eta], \tag{60}$$

where $d_t \doteq \partial_t + \mathbf{v}_0\cdot\nabla$ is a convective derivative associated to the zeroth-order velocity field

$$\mathbf{v}_0(\epsilon \mathsf{x}) \doteq \frac{\partial\varepsilon_{\rm eff}}{\partial\mathbf{p}} = \frac{\boldsymbol{\pi}}{\varepsilon_{\rm eff}}. \tag{61}$$

Regarding $\mathcal{U}$, one obtains the ponderomotive spin-orbit-coupling Hamiltonian (Appendix B 2)

$$\mathcal{U} = \tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_{\rm eff}, \tag{62}$$

where

$$\begin{aligned}\boldsymbol{\Omega}_{\text{eff}}(\epsilon \mathsf{x}) = \frac{q}{\varepsilon_{\text{eff}}}\left(\mathbf{B}_{\text{bg}} - \frac{\boldsymbol{\lambda} \times \mathbf{E}_{\text{bg}}}{m + \lambda^0}\right) + \frac{q^2}{2\varepsilon_{\text{eff}}(\pi \cdot \mathsf{k})}\left[\mathbf{k} \times \nabla\langle \mathsf{A}_{\text{osc}}^2\rangle - \frac{(\boldsymbol{\lambda} \times \mathbf{k})\partial_t\langle \mathsf{A}_{\text{osc}}^2\rangle}{m + \lambda^0} - \frac{k^0\boldsymbol{\lambda} \times \nabla\langle \mathsf{A}_{\text{osc}}^2\rangle}{m + \lambda^0}\right] \\ + \frac{q^2\langle \mathsf{A}_{\text{osc}}^2\rangle}{2\varepsilon_{\text{eff}}(\pi \cdot \mathsf{k})^2}\left\{\left(\frac{k^0\boldsymbol{\lambda}}{m + \lambda^0} - \mathbf{k}\right) \times [k^0 q\mathbf{E}_{\text{bg}} + \mathbf{k} \times q\mathbf{B}_{\text{bg}} - (\pi^\mu \partial_\mu)\mathbf{k}] - \frac{\boldsymbol{\lambda} \times \mathbf{k}}{m + \lambda^0}[\mathbf{k} \cdot q\mathbf{E}_{\text{bg}} - (\pi^\mu\partial_\mu)k^0]\right\} \quad (63)\end{aligned}$$

and $\pi \cdot \mathsf{k} = \varepsilon_{\text{eff}}(\omega - \mathbf{k} \cdot \mathbf{v}_0)$.

When substituting Eqs. (51), (60), and (62) into Eq. (56), one obtains the following effective Lagrangian density:

$$\begin{aligned}\mathcal{L} = -\eta^\dagger\left(\partial_t\theta + \sqrt{\boldsymbol{\pi}^2 + m_{\text{eff}}^2} + qV_{\text{bg}}\right)\eta \\ + \frac{i}{2}[\eta^\dagger(d_t\eta) - (d_t\eta^\dagger)\eta] + \frac{1}{2}\eta^\dagger\boldsymbol{\sigma} \cdot \boldsymbol{\Omega}_{\text{eff}}\eta. \quad (64)\end{aligned}$$

The first line of Eq. (64) represents the zeroth-order Lagrangian density that would describe a spinless relativistic electron. The second line, which is of order ϵ, introduces spin-orbit-coupling effects. Also note that the Lagrangian density (64) is analogous to that describing circularly-polarized EM waves in isotropic dielectric media when polarization effects are included [43].

VI. CONTINUOUS-WAVE MODEL

Here we construct a "fluid" description of the Dirac electron described by Eq. (64). Let us adopt the representation $\eta = z\sqrt{\mathcal{I}}$, where $\mathcal{I}(\mathsf{x}) \doteq \eta^\dagger\eta$ is a real function (called the action density) and $z(\mathsf{x})$ is a unit vector such that $z^\dagger z \equiv 1$. (From now on, we drop ϵ in the function arguments to simplify the notation, but we will continue to assume that the corresponding functions are slow.) Since the common phase of the two components of z can be attributed to θ, we parametrize z in terms of just two real functions $\zeta(\mathsf{x})$ and $\vartheta(\mathsf{x})$:

$$z(\vartheta,\zeta) = \begin{pmatrix} e^{-i\vartheta/2}\cos(\zeta/2) \\ e^{i\vartheta/2}\sin(\zeta/2)\end{pmatrix}. \quad (65)$$

As in the case of the Pauli particle [34], ζ determines the relative fraction of "spin-up" and "spin-down" quanta. Note that, under this reparametrization, the spin vector $\mathbf{S}(\mathsf{x})$ is

$$\mathbf{S} \doteq \frac{1}{2}z^\dagger\boldsymbol{\sigma}z = \begin{pmatrix}\sin\zeta\cos\vartheta \\ \sin\zeta\sin\vartheta \\ \cos\zeta\end{pmatrix}, \quad (66)$$

and $S \doteq |\mathbf{S}| = 1/2$.

Expressing Eq. (64) in terms of the four independent variables $(\theta,\mathcal{I},\zeta,\vartheta)$ leads to

$$\begin{aligned}\mathcal{L}[\theta,\mathcal{I},\zeta,\vartheta] = -\mathcal{I}\Big[\partial_t\theta + \sqrt{\boldsymbol{\pi}^2 + m_{\text{eff}}^2} + qV_{\text{bg}} \\ - \tfrac{1}{2}(d_t\vartheta)\cos\zeta - \mathbf{S}(\zeta,\vartheta) \cdot \boldsymbol{\Omega}_{\text{eff}}\Big], \quad (67)\end{aligned}$$

where one can immediately recognize the first line of Eq. (67) as Hayes' representation of the Lagrangian density of a GO wave [44]. Four ELEs are yielded. The first one is the action conservation theorem

$$\delta\theta: \quad \partial_t\mathcal{I} + \nabla \cdot (\mathcal{I}\mathbf{V}) = 0. \quad (68)$$

The flow velocity is given by $\mathbf{V} = \mathbf{v}_0 + \mathbf{u}$, where

$$\mathbf{u} \doteq -\frac{\partial}{\partial\mathbf{p}}\left[\frac{1}{2}(\mathbf{v}_0 \cdot \nabla\vartheta)\cos\zeta + \mathbf{S}(\zeta,\vartheta) \cdot \boldsymbol{\Omega}_{\text{eff}}\right] \quad (69)$$

is the spin-driven deflection of the electron's center of mass. The second ELE is a Hamilton-Jacobi equation

$$\begin{aligned}\delta\mathcal{I}: \quad \partial_t\theta + \sqrt{\boldsymbol{\pi}^2 + m_{\text{eff}}^2} + qV_{\text{bg}} \\ - \tfrac{1}{2}(d_t\vartheta)\cos\zeta - \mathbf{S}(\zeta,\vartheta) \cdot \boldsymbol{\Omega}_{\text{eff}} = 0, \quad (70)\end{aligned}$$

whose gradient yields the momentum equation

$$\begin{aligned}\partial_t\boldsymbol{\pi} + (\mathbf{v}_0 \cdot \nabla)\boldsymbol{\pi} = q\mathbf{E}_{\text{bg}} + q\mathbf{v}_0 \times \mathbf{B}_{\text{bg}} \\ + \frac{\nabla\langle \mathsf{A}_{\text{osc}}^2\rangle}{2\varepsilon_{\text{eff}}} + \nabla\left[\frac{1}{2}(d_t\vartheta)\cos\zeta + \mathbf{S} \cdot \boldsymbol{\Omega}_{\text{eff}}\right]. \\ (71)\end{aligned}$$

Note that the first line is the well known relativistic momentum equation. The first term in the second line represents the well known nonlinear ponderomotive force due to the oscillating EM field [41], and the last two terms represent the ponderomotive Stern-Gerlach spin force. Finally, the remaining two ELEs are

$$\delta\zeta: \quad (d_t\vartheta)\sin\zeta = 2(\partial_\zeta\mathbf{S}) \cdot \boldsymbol{\Omega}_{\text{eff}}, \quad (72)$$

$$\delta\vartheta: \quad \partial_t(\mathcal{I}\cos\zeta) + \nabla \cdot (\mathbf{v}_0\mathcal{I}\cos\zeta) = 2(\partial_\vartheta\mathbf{S}) \cdot \boldsymbol{\Omega}_{\text{eff}}. \quad (73)$$

These equations describe the phase-averaged electron spin precession. Together, Eqs. (68)–(73) provide a complete "fluid" description of the ponderomotive dynamics of a Dirac electron.

VII. POINT-PARTICLE MODEL

A. Ponderomotive model

The ray equations corresponding to the above field equations can be obtained as a point-particle limit. In this limit, $\mathcal{I}$ can be approximated with a delta function,

$$\mathcal{I}(t,\mathbf{x}) = \delta[\mathbf{x} - \mathbf{X}(t)], \quad (74)$$

where $\mathbf{X}(t)$ is the location of the center of the wave packet. As in Refs. [33,34], the Lagrangian density (67) can be replaced by a point-particle Lagrangian $L_{\text{eff}} \doteq \int \mathcal{L}\, d^3x$, namely,

$$\begin{aligned}L_{\text{eff}}[\mathbf{X},\mathbf{P},Z,Z^\dagger] \\ = \mathbf{P} \cdot \dot{\mathbf{X}} + \frac{i\hbar}{2}(Z^\dagger\dot{Z} - \dot{Z}^\dagger Z) - H_{\text{eff}}(t,\mathbf{X},\mathbf{P},Z,Z^\dagger), \quad (75)\end{aligned}$$

where the effective Hamiltonian is given by

$$H_{\text{eff}}(t,\mathbf{X},\mathbf{P},Z,Z^\dagger) \doteq \gamma_{\text{eff}}mc^2 + q\mathbf{V}_{\text{bg}} - \frac{\hbar}{2}Z^\dagger\boldsymbol{\sigma} \cdot \boldsymbol{\Omega}_{\text{eff}}Z. \quad (76)$$

Here $\mathbf{P}(t) \doteq \hbar\nabla\theta(t,\mathbf{X}(t))$ is the canonical momentum, and $Z(t) \doteq z(t,\mathbf{X}(t))$ is a complex two-component vector. For clarity, we have reintroduced c and $\hbar$. The effective Lorentz factor associated with the particle oscillation-center motion is

$$\gamma_{\text{eff}}(t,\mathbf{X},\mathbf{P}) \doteq \sqrt{1 + a_0^2 + \left(\frac{\mathbf{P}}{mc} - \frac{q\mathbf{A}_{\text{bg}}}{mc^2}\right)^2}, \qquad (77)$$

where

$$a_0^2(t,\mathbf{X}) \doteq -\frac{q^2\langle \mathsf{A}_{\text{osc}}^2\rangle}{m^2c^4} \qquad (78)$$

is positive under the assumed metric. For example, suppose a standard representation of the laser vector potential is $\mathbf{A}_{\text{osc}} = \text{Re}[\mathbf{A}_\perp(\mathbf{x})e^{i\Theta}]$, where $\mathbf{A}_\perp \cdot \mathbf{k} = 0$ [39,45,46]. Then, the Lorentz condition (12) determines the scalar potential envelope $V_{\text{osc,c}} = i(\nabla \cdot \mathbf{A}_\perp)c^2/\omega = O(\epsilon)$. Hence, Eq. (78) yields

$$a_0^2 \approx \frac{q^2|\mathbf{A}_\perp|^2}{2m^2c^4}, \qquad (79)$$

where we neglected a term $O(\epsilon^2)$. Note also that, loosely speaking, a_0^2 is the measure of the particle quiver energy in units mc^2. Accordingly, nonrelativistic interactions correspond to $a_0 \ll 1$.

The effective precession frequency $\mathbf{\Omega}_{\text{eff}}$ is given by

$$\mathbf{\Omega}_{\text{eff}}(t,\mathbf{X},\mathbf{P}) = \mathbf{\Omega}_1 + \mathbf{\Omega}_2 + \mathbf{\Omega}_3 + O(\epsilon^2), \qquad (80)$$

where

$$\mathbf{\Omega}_1(t,\mathbf{X},\mathbf{P}) \doteq \frac{q}{\gamma_{\text{eff}}mc}\left(\mathbf{B}_{\text{bg}} - \frac{\mathbf{\Lambda} \times \mathbf{E}_{\text{bg}}}{mc + \Lambda^0}\right), \qquad (81)$$

$$\mathbf{\Omega}_2(t,\mathbf{X},\mathbf{P}) \doteq -\frac{mc^2}{2\gamma_{\text{eff}}(\Pi\cdot\mathsf{k})}\left[\mathbf{k}\times\nabla a_0^2 - \frac{(\mathbf{\Lambda}\times\mathbf{k})\partial_t a_0^2}{mc^2 + \Lambda^0 c} - \frac{\omega\mathbf{\Lambda}\times\nabla a_0^2}{mc^2+\Lambda^0 c}\right], \qquad (82)$$

$$\begin{aligned}\mathbf{\Omega}_3(t,\mathbf{X},\mathbf{P}) \doteq &-\frac{mc^2a_0^2}{2\gamma_{\text{eff}}(\Pi\cdot\mathsf{k})^2}\left(\frac{\omega\mathbf{\Lambda}}{mc^2+\Lambda^0 c} - \mathbf{k}\right)\\ &\times\left[\frac{\omega q\mathbf{E}_{\text{bg}}}{c^2} + \frac{\mathbf{k}\times q\mathbf{B}_{\text{bg}}}{c} - (\Pi^\mu\partial_\mu)\mathbf{k}\right]\\ &+\frac{mca_0^2}{2\gamma_{\text{eff}}(\Pi\cdot\mathsf{k})^2}\frac{\mathbf{\Pi}\times\mathbf{k}}{mc+\Lambda^0}[\mathbf{k}\cdot q\mathbf{E}_{\text{bg}} - (\Pi^\mu\partial_\mu)\omega].\end{aligned} \qquad (83)$$

Here $\Pi^\mu = (mc\gamma_{\text{eff}},\mathbf{P} - q\mathbf{A}_{\text{bg}}/c)$, $k^\mu = (\omega/c,\mathbf{k})$, $\partial_\mu = (c^{-1}\partial_t,\nabla)$, and

$$\Lambda^\mu(t,\mathbf{X},\mathbf{P}) = \Pi^\mu - k^\mu\frac{m^2c^2a_0^2}{2(\Pi\cdot\mathsf{k})}. \qquad (84)$$

Notably, $\Lambda^\mu \to \Pi^\mu$ at $a_0 \to 0$ and $\Lambda^\mu \to \Pi^\mu - k^\mu mc^2a_0/(2\omega)$ at $a_0 \to +\infty$. Also, if the spin-orbital interaction is neglected, the present model yields the spinless ponderomotive model that was developed in Ref. [42] for a particle interacting with a laser pulse and a slow background field simultaneously.

Treating $\mathbf{X}(t)$, $\mathbf{P}(t)$, $Z(t)$, and $Z^\dagger(t)$ as independent variables leads to the following ELEs:

$$\delta\mathbf{P}:\quad \dot{\mathbf{X}} = mc^2\partial_{\mathbf{P}}\gamma_{\text{eff}} - \partial_{\mathbf{P}}(\mathbf{S}\cdot\mathbf{\Omega}_{\text{eff}}), \qquad (85)$$

$$\delta\mathbf{X}:\quad \dot{\mathbf{P}} = -\partial_{\mathbf{X}}(mc^2\gamma_{\text{eff}} + qV_{\text{bg}}) + \partial_{\mathbf{X}}(\mathbf{S}\cdot\mathbf{\Omega}_{\text{eff}}), \qquad (86)$$

$$\delta Z^\dagger:\quad \dot{Z} = \frac{i}{2}\mathbf{\Omega}_{\text{eff}}\cdot\boldsymbol{\sigma}Z, \qquad (87)$$

$$\delta Z:\quad \dot{Z}^\dagger = -\frac{i}{2}Z^\dagger\mathbf{\Omega}_{\text{eff}}\cdot\boldsymbol{\sigma}, \qquad (88)$$

where $\mathbf{S}(t)$ is the particle spin vector,

$$\mathbf{S}(t) \doteq \frac{\hbar}{2}Z^\dagger(t)\boldsymbol{\sigma}Z(t), \qquad (89)$$

and $S = \hbar/2$. Equations (77)–(89) form a complete set of equations. The first terms on the right-hand side of Eqs. (85) and (86) describe the dynamics of a relativistic spinless particle in agreement with earlier theories [39–42]. The second terms describe the ponderomotive spin-orbit coupling. Equations (87) and (88) also yield the following ponderomotive equation for spin precession,

$$\dot{\mathbf{S}} = \mathbf{S}\times\mathbf{\Omega}_{\text{eff}}, \qquad (90)$$

which can be checked by direct substitution. Equations (75)–(90) are the main result of this work.

B. Extended BMT model

Let us compare our ponderomotive point-particle Lagrangian (75) with the complete point-particle Lagrangian of a Dirac electron [33]

$$\begin{aligned}&L_{\text{XBMT}}[\mathbf{X},\mathbf{P},Z,Z^\dagger]\\ &= \mathbf{P}\cdot\dot{\mathbf{X}} + \frac{i\hbar}{2}(Z^\dagger\dot{Z} - \dot{Z}^\dagger Z) - H_{\text{XBMT}}(t,\mathbf{X},\mathbf{P},Z,Z^\dagger),\end{aligned} \qquad (91)$$

where the Hamiltonian is given by

$$H_{\text{XBMT}}(t,\mathbf{X},\mathbf{P},Z,Z^\dagger) \doteq \gamma mc^2 + qV - \frac{\hbar}{2}Z^\dagger\boldsymbol{\sigma}\cdot\mathbf{\Omega}_{\text{BMT}}Z \qquad (92)$$

and the BMT precession frequency [26] is

$$\mathbf{\Omega}_{\text{BMT}}(t,\mathbf{X},\mathbf{P}) = \frac{q}{mc}\left[\frac{\mathbf{B}}{\gamma} - \frac{(\mathbf{v}_0/c)\times\mathbf{E}}{1+\gamma}\right]. \qquad (93)$$

Here $\mathbf{v}_0 \doteq \mathbf{\Pi}/(\gamma m)$, and

$$\gamma(t,\mathbf{X},\mathbf{P}) \doteq \sqrt{1 + \left(\frac{\mathbf{P}}{mc} - \frac{q\mathbf{A}}{mc^2}\right)^2}. \qquad (94)$$

Note that the EM potentials and fields in Eqs. (91)–(94) include both the rapidly oscillating EM wave and the background EM field (if any). Obviously, $L_{\text{eff}} \to L_{\text{XBMT}}$ when $\mathsf{A}_{\text{osc}} \to 0$; i.e., the two Lagrangians are equivalent in the absence of the EM wave.

The corresponding ELEs are

$$\delta\mathbf{P}:\quad \dot{\mathbf{X}} = mc^2\partial_{\mathbf{P}}\gamma - \partial_{\mathbf{P}}(\mathbf{S}\cdot\mathbf{\Omega}_{\text{BMT}}), \qquad (95)$$

$$\delta\mathbf{X}:\quad \dot{\mathbf{P}} = -\partial_{\mathbf{X}}(mc^2\gamma + qV) + \partial_{\mathbf{X}}(\mathbf{S}\cdot\mathbf{\Omega}_{\text{BMT}}), \qquad (96)$$

$$\delta Z^\dagger:\quad \dot{Z} = \frac{i}{2}\mathbf{\Omega}_{\text{BMT}}\cdot\boldsymbol{\sigma}Z, \qquad (97)$$

$$\delta Z:\quad \dot{Z}^\dagger = -\frac{i}{2}Z^\dagger\mathbf{\Omega}_{\text{BMT}}\cdot\boldsymbol{\sigma}. \qquad (98)$$

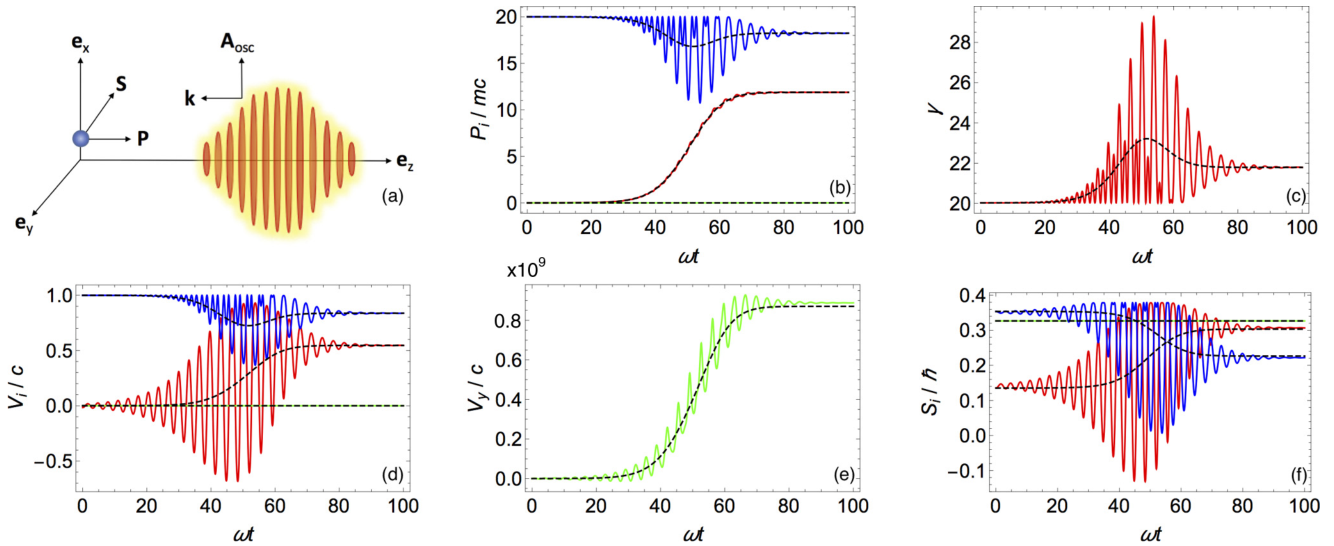

FIG. 1. (Color online) Motion of a single Dirac electron under the action of a relativistically intense laser pulse (numerical simulation): the black dashed curves correspond to the ponderomotive model described by the Lagrangian (75); the colored curves correspond to the XBMT model described by the Lagrangian (91). (a) Schematic of the interaction; yellow and red is the laser field, blue is the particle; arrows denote the direction of the laser wave vector $\mathbf{k}$, the oscillating vector potential $\mathbf{A}_{\rm osc}$, the particle canonical momentum $\mathbf{P}$, and the particle spin $\mathbf{S}$. The unit vectors along the reference axes are denoted by $\mathbf{e}_i$. (b)–(f) show the components of the particle canonical momentum $\mathbf{P}$, Lorentz factor γ, velocity $\mathbf{V}$, and spin $\mathbf{S}$. The red, green, and blue lines correspond to projections on the x, y, and z axes, respectively. We consider an electron initially traveling along the z axis and colliding with a counterpropagating laser pulse. The initial position of the particle is $\mathbf{X}_0 = (\ell/2)\mathbf{e}_x$, the initial momentum is $\mathbf{P}_0/(mc) = 20\mathbf{e}_z$, and the normalized initial spin vector is $\mathbf{S}_0/\hbar = 0.14\mathbf{e}_x + 0.33\mathbf{e}_y + 0.35\mathbf{e}_z$. The envelope of the vector potential of the laser pulse is $q\mathbf{A}_{\rm osc}/(mc^2) = 30\,{\rm sech}[(z - 5\ell + ct)/\ell]\exp[-(x^2 + y^2)/\ell^2]\mathbf{e}_x$, where $\ell = 20|\mathbf{k}|^{-1}$. These parameters correspond to a maximum intensity $I_{\max} \simeq 1.23 \times 10^{21}$ W/cm^2 for a 1 μm laser.

These equations also yield the BMT spin precession equation, similar to Eq. (90), with $\boldsymbol{\Omega}_{\rm eff}$ replaced by $\boldsymbol{\Omega}_{\rm BMT}$. However, as opposed to the original BMT model [26,31], Eqs. (95)–(98) also capture the spin-orbital coupling. Because of that, they represent a generalization of the BMT model, which we call "extended BMT" (XBMT).

The XBMT model applies, in principle, to arbitrary fields, provided that (i) the spin-orbital coupling is weak and (ii) the particle de Broglie wavelength $\lambda\!\!\!^{-}$ remains much shorter than the smallest spatial scale of the EM fields. In application to the particle motion in a laser field, it can describe details that the ponderomotive model misses due to phase averaging. In *this* sense, the XBMT model is more precise than the ponderomotive model above. However, the XBMT (and, similarly, BMT) model is also more complicated for the same reason and, in application to laser fields, requires $\lambda\!\!\!^{-}/\lambda_L \ll 1$, where λ_L is the laser wavelength. No such assumption was made to derive the ponderomotive model above. Instead, Eq. (13) was assumed, which implies

$$\lambda\!\!\!^{-}/\lambda_L \ll c/v_0, \tag{99}$$

where v_0 is the particle speed. For nonrelativistic particles ($v_0 \ll c$), this can be satisfied even at $\lambda_L \lesssim \lambda\!\!\!^{-}$. In *that* sense, the ponderomotive model is, perhaps surprisingly, more general than XBMT.

VIII. NUMERICAL SIMULATIONS

To test our ponderomotive model, we applied it to simulate the single-particle motion and compare the results with the XBMT model in two cases. In the first test case, we consider the dynamics of a Dirac electron colliding with a counterpropagating relativistically strong ($a_0 \gg 1$) laser pulse. The simulation parameters are given in the caption of Fig. 1, and a schematic of the interaction is presented in Fig. 1(a). From Figs. 1(b)–1(e), it is seen that the ponderomotive model accurately describes the mean evolution of the particle momentum, kinetic energy, and velocity. The main contribution to the variations in V_x and V_z is the ponderomotive force caused by spatial gradient of the effective mass. However, the acceleration on the xz plane is caused by the Stern-Gerlach force, as shown in Fig. 1(e). Also notice that the ponderomotive model is extremely accurate in describing the particle spin precession, as can be seen in Fig. 1(f).

In the second test case, we consider a Dirac electron immersed in a background magnetic field along the z axis and interacting with a laser plane wave traveling along the z axis. The simulation parameters are given in Fig. 2. As can be seen in Figs. 2(a)–2(f), the ponderomotive model accurately describes the particle position, momentum, velocity, and spin. Notably, these simulations also support the spinless model developed in Ref. [42] for a particle interacting with a relativistic laser field and a large-scale background field simultaneously.

IX. CONCLUSIONS

In this paper, we report a point-particle ponderomotive model of a Dirac electron oscillating in a high-frequency field. Starting from the first-principle Dirac Lagrangian density, we derived a reduced phase-space Lagrangian that describes the relativistic time-averaged dynamics of such particle in a geometrical-optics laser pulse in vacuum. The pulse is allowed

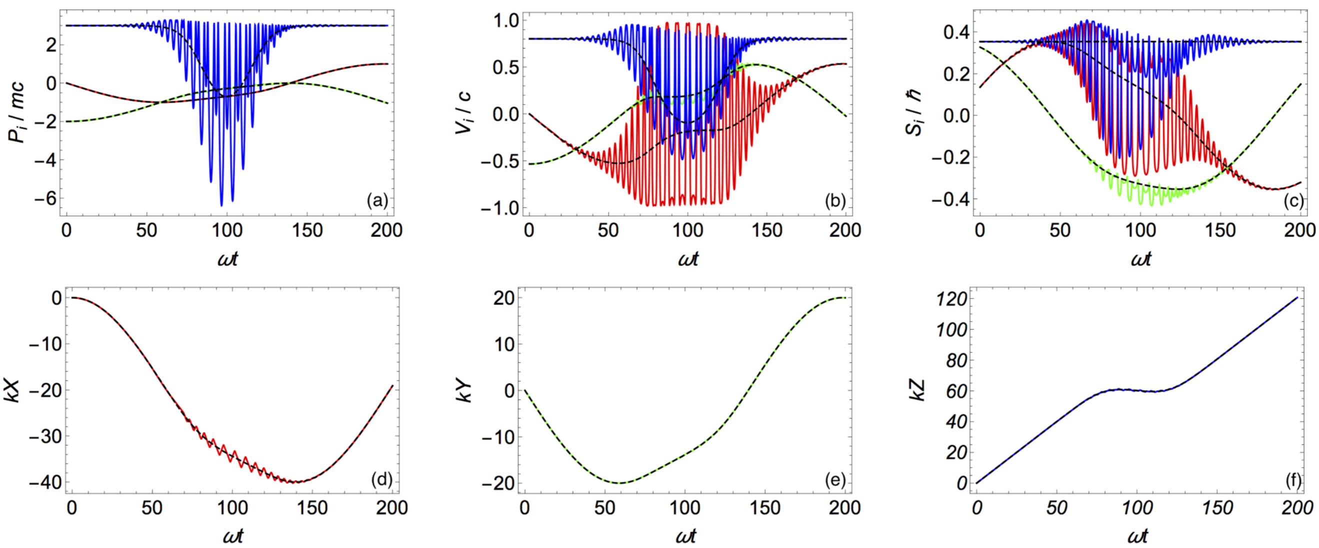

FIG. 2. (Color online) Motion of a Dirac electron under the action of an external background field and a relativistically intense laser pulse (numerical simulation): the black dashed curves correspond to the ponderomotive model described by the Lagrangian (75); the colored curves correspond to the XBMT model described by the Lagrangian (91). (a)–(c) show the components of the particle canonical momentum **P**, velocity **V**, and spin **S**. The red, green, and blue lines correspond to projections on the x, y, and z axes, respectively. We consider an electron initially traveling along the z axis and colliding with a counterpropagating laser pulse. (d)–(f) show the components of the particle position **X**. The initial position of the particle is $\mathbf{X}_0 = \mathbf{0}$, the initial momentum is $\mathbf{P}_0/(mc) = (-2\mathbf{e}_y + 3\mathbf{e}_z)$, and the normalized initial spin vector is $\mathbf{S}_0/\hbar = 0.14\mathbf{e}_x + 0.33\mathbf{e}_y + 0.35\mathbf{e}_z$. A background magnetic field is added such that $q\mathbf{A}_{\text{bg}}/(mc^2) = 0.1(-y\mathbf{e}_x + x\mathbf{e}_y)/2$, which corresponds to a static homogeneous magnetic field $B_{\text{bg}} \simeq 10.7$ MG aligned towards the z axis. The envelope of the vector potential of the laser pulse is assumed to have the form $q\mathbf{A}_{\text{osc}}/(mc^2) = 10\,\text{sech}[(z - 8\ell + ct)/\ell]\mathbf{e}_x$, where $\ell = 20|\mathbf{k}|^{-1}$. These parameters correspond to a maximum laser intensity $I_{\max} \simeq 1.37 \times 10^{20}$ W/cm^2 for a 1 μm laser.

to have an arbitrarily large amplitude (as long as radiation damping and pair production are negligible) and, in case of nonrelativistic interactions, a wavelength comparable to the electron de Broglie wavelength. The model captures the BMT spin dynamics, the Stern-Gerlach spin-orbital coupling, the conventional ponderomotive forces, and the interaction with large-scale background fields (if any). Agreement with the BMT spin precession equation is shown numerically. Also, the well known theory in which ponderomotive effects are incorporated in the particle effective mass is reproduced as a special case when the spin-orbital coupling is negligible.

As a final note, the underlying essence of this paper is to illustrate the convenience of using the Lagrangian wave formalism for deriving reduced point-particle models. To derive the ponderomotive model above by using the point-particle equations of motion and spin would have been a torturous task. However, the bilinear structure of the wave Lagrangian enabled a straightforward deduction of the reduced model. Following this reasoning, we believe that the ability to treat particles and waves on the same footing, i.e. as fields, may have far-reaching implications, e.g., for plasma theory. This will be discussed in future publications.

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APPENDIX A: SEMICLASSICAL VOLKOV STATES

Volkov states are eigenstates of the Dirac equation with a homogeneous EM vacuum field [47–49]. Here we present a derivation of these states. Consider the second-order Dirac equation,

$$\left(D_\mu D^\mu + m^2 + \tfrac{1}{2} q \sigma_{\mu\nu} F^{\mu\nu}\right)\psi = 0, \tag{A1}$$

where $iD_\mu \doteq i\partial_\mu - qA_\mu$ is the covariant derivative, $\sigma_{\mu\nu} \doteq i[\gamma_\mu, \gamma_\nu]/2$ is twice the (relativistic) spin operator, and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the EM tensor. We start with the case where A_{bg} is constant and $\mathsf{A}_{\text{osc}}(\Theta)$ is strictly periodic. Since Eq. (A1) is linear, we search for ψ in the Floquet-Bloch form. Specifically, we consider $\psi = ue^{i\theta}$, where u is a periodic four-component function of Θ and $p_\mu \doteq -\partial_\mu \theta$ is constant. It is also convenient to rewrite u in the form $u = e^{i\tilde{\theta}}\Xi\varphi$, where $\Xi(\Theta)$ is a matrix operator, $\tilde{\theta}(\Theta)$ is a real scalar function, and φ is a constant four-component spinor. This leads to

$$\begin{aligned}&\left[\pi^2 - m^2 + 2(\pi\cdot\mathsf{k})\partial_\Theta\tilde{\theta} - 2q(\pi\cdot\mathsf{A}_{\text{osc}}) + q^2\mathsf{A}_{\text{osc}}^2\right]\Xi\varphi\\&\quad - 2i(\pi\cdot\mathsf{k})(\partial_\Theta\Xi)\varphi - \tfrac{1}{2}q\sigma_{\mu\nu}F^{\mu\nu}\Xi\varphi = 0,\end{aligned} \tag{A2}$$

where $\pi^\mu \doteq p^\mu - qA^\mu_{\text{bg}}$.

Equation (A2) can be satisfied identically if we require that $\tilde{\theta}$ and Ξ satisfy the following equations:

$$\pi^2 - m^2 + 2(\pi\cdot\mathsf{k})\partial_\Theta\tilde{\theta} - 2q(\pi\cdot\mathsf{A}_{\text{osc}}) + q^2\mathsf{A}_{\text{osc}}^2 = 0, \tag{A3}$$

$$-2i(\pi\cdot\mathsf{k})(\partial_\Theta\Xi) - \tfrac{1}{2}q\sigma_{\mu\nu}F^{\mu\nu}\Xi = 0. \tag{A4}$$

The integration constants can be chosen arbitrarily since they merely redefine φ. We hence require $\Xi \to \mathbb{I}_4$ at vanishing

$\mathsf{A}_{\rm osc}$ and $\langle\tilde{\theta}\rangle = 0$ (so that $\tilde{\theta}$ represents a phase shift due to the oscillating EM field). For Ξ, this gives

$$\begin{aligned}\Xi(\epsilon\mathsf{x},\Theta) &= \mathcal{T}\exp\left[\frac{iq}{4(\pi\cdot\mathsf{k})}\int^{\Theta}\sigma_{\mu\nu}F^{\mu\nu}(\Theta')\,d\Theta'\right]\\ &= \mathbb{I}_4 + \frac{q}{2(\pi\cdot\mathsf{k})}\not{k}\not{A}_{\rm osc}(\Theta),\end{aligned}\tag{A5}$$

where we used

$$\begin{aligned}\sigma_{\mu\nu}F^{\mu\nu} &= \sigma_{\mu\nu}(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu})\\ &= -\sigma_{\mu\nu}\left(k^{\mu}\partial_{\Theta}A^{\nu}_{\rm osc} - k^{\nu}\partial_{\Theta}A^{\mu}_{\rm osc}\right)\\ &= -i(\not{k}\partial_{\Theta}\not{A}_{\rm osc} - \partial_{\Theta}\not{A}_{\rm osc}\not{k})\\ &= -2i\not{k}\partial_{\Theta}\not{A}_{\rm osc}.\end{aligned}\tag{A6}$$

Here $\not{k}\not{A}_{\rm osc} + \not{A}_{\rm osc}\not{k} = 0$ since $\mathsf{k}\cdot\mathsf{A}_{\rm osc} = 0$ [see Eq. (12)]. We note that the ordered exponential [denoted by $\mathcal{T}\exp(\ldots)$] becomes an ordinary exponential due to

$$\begin{aligned}&\sigma_{\mu\nu}F^{\mu\nu}(\Theta_1)\sigma_{\alpha\xi}F^{\alpha\xi}(\Theta_2)\\ &\quad = -4\not{k}[\partial_{\Theta}\not{A}_{\rm osc}(\Theta_1)]\not{k}[\partial_{\Theta}\not{A}_{\rm osc}(\Theta_2)]\\ &\quad = 4\not{k}\not{k}\partial_{\Theta}\not{A}_{\rm osc}(\Theta_1)\not{A}_{\rm osc}(\Theta_2)\\ &\quad = 0,\end{aligned}\tag{A7}$$

where we substituted Eq. (11). To obtain $\tilde{\theta}$, we average Eq. (A3). This leads to Eq. (47), which serves as a dispersion relation for π_{μ}. Subtracting Eq. (47) from Eq. (A3) and solving for $\tilde{\theta}$ leads to Eq. (21). Finally, if one substitutes $\psi = \Xi e^{i\theta+\tilde{\theta}}\varphi$ into the first-order Dirac equation, one finds that constant φ indeed satisfies that equation.

The above solution can be extended also to a wave with a slowly inhomogeneous amplitude; i.e., when the vector potential has the form $\mathsf{A}(\epsilon\mathsf{x},\Theta)$. This can be done by substituting the ansatz $\psi = \Xi e^{i\theta+\tilde{\theta}}\varphi$ into the Dirac equation with the same Ξ and $\tilde{\theta}$, as before, and *requiring* that p_{μ} is slow. This will lead to an equation for φ with a perturbation linear in ϵ. Hence, one can construct a solution for φ as an asymptotic power series in ϵ. The general form of such series is given by Eq. (20), and finding the coefficients φ_n explicitly is not needed here.

APPENDIX B: AUXILIARY FORMULAS

1. Kinetic term $\mathcal{K}$

Let us reexpress Eq. (57) as

$$\mathcal{K} = \frac{i}{2}[\eta^{\dagger}\Psi^{\dagger}\gamma^0\Gamma^0\Psi(\partial_t\eta) + \eta^{\dagger}\Psi^{\dagger}\gamma^0\boldsymbol{\Gamma}\cdot\Psi(\nabla\eta) - \text{c.c.}].\tag{B1}$$

Substituting Eqs. (41), (45), (46), and (54) into $\Psi^{\dagger}\gamma^0\Gamma^0\Psi$ leads to

$$\begin{aligned}\Psi^{\dagger}\gamma^0\Gamma^0\Psi &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left(\gamma^0\gamma^0 - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\gamma^0\not{k}\right)\begin{pmatrix}\mathbb{I}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\mathbb{I}_4 - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\begin{pmatrix}\mathbb{I}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\begin{pmatrix}\mathbb{I}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0\mathbb{I}_2 - \frac{(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} + k^0\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\right]\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\left\{1 + \frac{\boldsymbol{\lambda}^2}{(m+\lambda^0)^2} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\left[k^0 - \frac{2\mathbf{k}\cdot\boldsymbol{\lambda}}{m+\lambda^0} + k^0\frac{\boldsymbol{\lambda}^2}{(m+\lambda^0)^2}\right]\right\}\mathbb{I}_2\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\left\{1 + \frac{(\lambda^0)^2-m^2}{(m+\lambda^0)^2} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\left[k^0 - \frac{2\mathbf{k}\cdot\boldsymbol{\lambda}}{m+\lambda^0} + k^0\frac{(\lambda^0)^2-m^2}{(m+\lambda^0)^2}\right]\right\}\mathbb{I}_2,\\ &= \mathbb{I}_2,\end{aligned}\tag{B2}$$

where $\lambda\cdot\lambda = m^2$ from Eq. (48). Also notice that $\lambda\cdot\mathsf{k} = \pi\cdot\mathsf{k}$ from Eqs. (10) and (45). Similarly,

$$\begin{aligned}\Psi^{\dagger}\gamma^0\boldsymbol{\Gamma}\Psi &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left(\gamma^0\boldsymbol{\gamma} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\gamma^0\not{k}\right)\begin{pmatrix}\mathbb{I}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\begin{pmatrix}\mathbb{I}_2\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\begin{pmatrix}\frac{\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\\ \boldsymbol{\sigma}\end{pmatrix} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0\mathbb{I}_2 - \frac{(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} + k^0\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\right]\\ &= \frac{m+\lambda^0}{2\varepsilon_{\rm eff}}\left\{\frac{\boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) + (\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\boldsymbol{\sigma}}{m+\lambda^0} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\left[k^0 - \frac{2\mathbf{k}\cdot\boldsymbol{\lambda}}{m+\lambda^0} + k^0\frac{\boldsymbol{\lambda}^2}{(m+\lambda^0)^2}\right]\right\}\mathbb{I}_2\\ &= \frac{\boldsymbol{\lambda}-\mathbf{k}\alpha}{\varepsilon_{\rm eff}}\mathbb{I}_2 = \frac{\boldsymbol{\pi}}{\varepsilon_{\rm eff}}\mathbb{I}_2.\end{aligned}\tag{B3}$$

Hence, notice the following corollary of Eqs. (B2) and (B3) that we will use below:

$$(\Psi^\dagger\gamma^0\Gamma^\mu\Psi)\partial_\mu = \mathbb{I}_2\left(\partial_t + \frac{\boldsymbol{\pi}}{\varepsilon_{\text{eff}}}\cdot\nabla\right) = \mathbb{I}_2 d_t, \tag{B4}$$

where d_t is the same as defined in Sec. V B. Substituting Eq. (B4) into Eq. (57) leads to Eq. (60).

2. Expression for $\mathcal{U}$

An alternative representation of $\mathcal{U}$ in Eq. (59) is

$$\mathcal{U} = -\text{Im}[\Psi^\dagger\gamma^0\Gamma^\mu(\partial_\mu\Psi)], \tag{B5}$$

where "Im" is short for the "anti-Hermitian part of." To calculate $\partial_\mu\Psi$, let us consider Ψ as a function

$$\Psi(t,\mathbf{x}) = \Psi(\varepsilon_{\text{eff}}(t,\mathbf{x}),\lambda^0(t,\mathbf{x}),\boldsymbol{\lambda}(t,\mathbf{x})). \tag{B6}$$

Notice that the contribution to Eq. (B5) from the partial derivative with respect to ε_{eff} is zero. This is shown by using Eqs. (54) and (B4):

$$\begin{aligned}
&\text{Im}\left[\Psi^\dagger\gamma^0\Gamma^\mu\left(\partial_{\varepsilon_{\text{eff}}}\Psi\right)\partial_\mu\varepsilon_{\text{eff}}\right]\\
&\quad= -\tfrac{1}{2}\text{Im}[(\Psi^\dagger\gamma^0\Gamma^\mu\Psi)\partial_\mu\ln\varepsilon_{\text{eff}}]\\
&\quad= -\tfrac{1}{2}\text{Im}(\mathrm{d}_t\ln\varepsilon_{\text{eff}})\\
&\quad= 0
\end{aligned} \tag{B7}$$

since ε_{eff} is real. Then, $\mathcal{U} = -\mathcal{P}_t - \mathcal{P}_x - \mathcal{Q}_t - \mathcal{Q}_x$, where

$$\mathcal{P}_t \doteq \text{Im}[\Psi^\dagger\gamma^0\Gamma^0(\partial_{\lambda_i}\Psi)(\partial_t\lambda_i)], \tag{B8}$$

$$\mathcal{P}_x \doteq \text{Im}[\Psi^\dagger\gamma^0\boldsymbol{\Gamma}(\partial_{\lambda_i}\Psi)\cdot(\nabla\lambda_i)], \tag{B9}$$

$$\mathcal{Q}_t \doteq \text{Im}[\Psi^\dagger\gamma^0\Gamma^0(\partial_{\lambda^0}\Psi)(\partial_t\lambda^0)], \tag{B10}$$

$$\mathcal{Q}_x \doteq \text{Im}[\Psi^\dagger\gamma^0\boldsymbol{\Gamma}(\partial_{\lambda^0}\Psi)\cdot(\nabla\lambda^0)]. \tag{B11}$$

When substituting Eqs. (41), (45), (46), and (54) into $\mathcal{P}_t$, we obtain

$$\begin{aligned}
\mathcal{P}_t &= \text{Im}\left[\Psi^\dagger\gamma^0\Gamma^0\left(\partial_{\lambda_i}\Psi\right)(\partial_t\lambda^i)\right] = \frac{m+\lambda^0}{2\varepsilon_{\text{eff}}}\text{Im}\left[\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\gamma^0\Gamma^0\begin{pmatrix}0\\ \frac{\boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\right]\\
&= \frac{m+\lambda^0}{2\varepsilon_{\text{eff}}}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\mathbb{I}_4 - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\begin{pmatrix}0\\ \frac{\boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\begin{pmatrix}0\\ \boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda}\end{pmatrix} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}-(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda})\\ \boldsymbol{\sigma}\cdot k^0\partial_t\boldsymbol{\lambda}\end{pmatrix}\right]\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\text{Im}\left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda})}{m+\lambda^0} + \frac{k^0\alpha}{\pi\cdot\mathsf{k}}(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\partial_t\boldsymbol{\lambda}) - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\boldsymbol{\sigma}\cdot k^0\partial_t\boldsymbol{\lambda})}{m+\lambda^0}\right]\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\boldsymbol{\sigma}\cdot\left[\frac{\boldsymbol{\lambda}\times\partial_t\boldsymbol{\lambda}}{m+\lambda^0} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\frac{\boldsymbol{\lambda}\times k^0\partial_t\boldsymbol{\lambda}}{m+\lambda^0} + \frac{\alpha}{\pi\cdot\mathsf{k}}\mathbf{k}\times k^0\partial_t\boldsymbol{\lambda}\right].
\end{aligned} \tag{B12}$$

The next term, $\mathcal{P}_x$, is calculated similarly:

$$\begin{aligned}
\mathcal{P}_x &= \text{Im}\left[\Psi^\dagger\gamma^0\boldsymbol{\Gamma}\left(\partial_{\lambda_i}\Psi\right)\cdot(\nabla\lambda^i)\right] = \frac{m+\lambda^0}{2\varepsilon_{\text{eff}}}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\cdot\begin{pmatrix}0\\ \frac{\nabla(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\end{pmatrix}\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\begin{pmatrix}(\boldsymbol{\sigma}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\\ 0\end{pmatrix} - \frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}-(\boldsymbol{\sigma}\cdot\mathbf{k})(\mathbf{k}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\\ k^0(\mathbf{k}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\end{pmatrix}\right]\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\text{Im}\left[(\boldsymbol{\sigma}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) + \frac{\alpha}{\pi\cdot\mathsf{k}}(\boldsymbol{\sigma}\cdot\mathbf{k})(\mathbf{k}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\mathbf{k}\cdot\nabla)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\right]\\
&= \frac{1}{2\varepsilon_{\text{eff}}}\boldsymbol{\sigma}\cdot\left[\nabla\times\boldsymbol{\lambda} + \frac{\alpha}{\pi\cdot\mathsf{k}}\mathbf{k}\times(\mathbf{k}\cdot\nabla)\boldsymbol{\lambda} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\frac{\boldsymbol{\lambda}\times(\mathbf{k}\cdot\nabla)\boldsymbol{\lambda}}{m+\lambda^0}\right].
\end{aligned} \tag{B13}$$

Furthermore, the expressions for $\mathcal{Q}_t$ and $\mathcal{Q}_x$ are given by

$$\begin{aligned}
\mathcal{Q}_t &= \text{Im}[\Psi^\dagger\gamma^0\Gamma^0(\partial_{\lambda^0}\Psi)(\partial_t\lambda^0)] = \frac{m+\lambda^0}{2\varepsilon_{\text{eff}}}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\mathbb{I}_4 - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\begin{pmatrix}0\\ -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{(m+\lambda^0)^2}\partial_t\lambda^0\end{pmatrix}\right\}\\
&= \frac{\partial_t\lambda^0}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\text{Im}\left\{\left(\mathbb{I}_2 \quad \tfrac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\right)\left[\begin{pmatrix}0\\ -\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\end{pmatrix} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\\ -k^0\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\end{pmatrix}\right]\right\}\\
&= \frac{\partial_t\lambda^0}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\text{Im}\left[-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0} - \frac{k^0\alpha}{\pi\cdot\mathsf{k}}(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) + \frac{(k^0)^2\alpha}{\pi\cdot\mathsf{k}}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\right] = -\frac{k^0\partial_t\lambda^0}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\frac{\alpha}{\pi\cdot\mathsf{k}}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\lambda}),
\end{aligned} \tag{B14}$$

$$
\begin{aligned}
\mathcal{Q}_x &= \mathrm{Im}[\Psi^\dagger\gamma^0\boldsymbol{\Gamma}(\partial_{\lambda^0}\Psi)\cdot(\nabla\lambda^0)]\\
&= \frac{m+\lambda^0}{2\varepsilon_{\text{eff}}}\mathrm{Im}\left\{\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} - \mathbf{k}\frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}k^0 & -\boldsymbol{\sigma}\cdot\mathbf{k}\\ -\boldsymbol{\sigma}\cdot\mathbf{k} & k^0\end{pmatrix}\right]\cdot\begin{pmatrix}0\\ -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{(m+\lambda^0)^2}\nabla\lambda^0\end{pmatrix}\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\mathrm{Im}\left\{\begin{pmatrix}\mathbb{I}_2 & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}}{m+\lambda^0}\end{pmatrix}\left[\begin{pmatrix}-(\boldsymbol{\sigma}\cdot\nabla\lambda^0)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\\ 0\end{pmatrix} - (\mathbf{k}\cdot\nabla\lambda^0)\frac{\alpha}{\pi\cdot\mathsf{k}}\begin{pmatrix}(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\\ -k^0(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\end{pmatrix}\right]\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\mathrm{Im}\left\{-(\boldsymbol{\sigma}\cdot\nabla\lambda^0)(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) - (\mathbf{k}\cdot\nabla\lambda^0)\frac{\alpha}{\pi\cdot\mathsf{k}}\left[(\boldsymbol{\sigma}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}) - k^0\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})}{m+\lambda^0}\right]\right\}\\
&= \frac{1}{2\varepsilon_{\text{eff}}(m+\lambda^0)}\boldsymbol{\sigma}\cdot\left[\boldsymbol{\lambda}\times\nabla\lambda^0 - \frac{\alpha}{\pi\cdot\mathsf{k}}(\mathbf{k}\times\boldsymbol{\lambda})(\mathbf{k}\cdot\nabla\lambda^0)\right].
\end{aligned}\tag{B15}
$$

Substituting Eqs. (B12)–(B15) leads to

$$
\mathcal{U} = -\frac{1}{2\varepsilon_{\text{eff}}}\boldsymbol{\sigma}\cdot\left[\nabla\times\boldsymbol{\lambda} + \frac{\boldsymbol{\lambda}\times(\nabla\lambda^0+\partial_t\boldsymbol{\lambda})}{m+\lambda^0} + \frac{\alpha}{\pi\cdot\mathsf{k}}\mathbf{k}\times(k^\mu\partial_\mu)\boldsymbol{\lambda} - \frac{\alpha}{\pi\cdot\mathsf{k}}\frac{k^0\boldsymbol{\lambda}\times(k^\mu\partial_\mu)\boldsymbol{\lambda} + (\mathbf{k}\times\boldsymbol{\lambda})(k^\mu\partial_\mu)\lambda^0}{m+\lambda^0}\right].\tag{B16}
$$

Equation (B16) can be simplified as follows. The first term can be rewritten as

$$
\nabla\times\boldsymbol{\lambda} = \nabla\times(\boldsymbol{\pi}+\mathbf{k}\alpha) = \nabla\times(\nabla\theta - q\mathbf{A}_{\text{bg}} + \mathbf{k}\alpha) = -q\mathbf{B}_{\text{bg}} - \mathbf{k}\times\nabla\alpha\tag{B17}
$$

since $\nabla\times\mathbf{k} = \nabla\times\nabla\Theta = 0$. Moreover, we note that $\partial^2_{\mu\nu}\theta = \partial^2_{\nu\mu}\theta$. Hence,

$$
\begin{aligned}
\nabla\lambda^0 &= \nabla\pi^0 + \alpha\nabla k^0 + k^0\nabla\alpha \simeq -\nabla(\partial_t\theta + qV_{\text{bg}}) - \alpha\partial_t\mathbf{k} + k^0\nabla\alpha\\
&= -\partial_t(\nabla\theta - q\mathbf{A}_{\text{bg}}) - q(\nabla V_{\text{bg}} + \partial_t\mathbf{A}_{\text{bg}}) - \alpha\partial_t\mathbf{k} + k^0\nabla\alpha = -\partial_t\boldsymbol{\lambda} + q\mathbf{E}_{\text{bg}} + k^0\nabla\alpha + \mathbf{k}\partial_t\alpha.
\end{aligned}\tag{B18}
$$

Similarly, the numerator of the last term simplifies to

$$
k^0\boldsymbol{\lambda}\times(k^\mu\partial_\mu)\boldsymbol{\lambda} + (\mathbf{k}\times\boldsymbol{\lambda})(k^\mu\partial_\mu)\lambda^0 = \boldsymbol{\lambda}\times[k^0(k^\mu\partial_\mu)\boldsymbol{\pi} - \mathbf{k}(k^\mu\partial_\mu)\pi^0],\tag{B19}
$$

where

$$
(k^\mu\partial_\mu)\mathbf{k} = k^0\partial_t\mathbf{k} + (\mathbf{k}\cdot\nabla)\mathbf{k} = k^0\nabla\partial_t\Theta + k^i\nabla\partial_i\Theta = -k^0\nabla k^0 + k^i\nabla k_i = -\nabla[(k^0)^2 - \mathbf{k}^2]/2 = 0\tag{B20}
$$

and $(k^\mu\partial_\mu)k^0 = 0$. Here we used Eq. (10). By substituting Eqs. (B17)–(B19) and explicitly showing the derivatives of α, we obtain

$$
\begin{aligned}
\mathcal{U} &= \frac{q}{2\varepsilon_{\text{eff}}}\boldsymbol{\sigma}\cdot\left(\mathbf{B}_{\text{bg}} - \frac{\boldsymbol{\lambda}\times\mathbf{E}_{\text{bg}}}{m+\lambda^0}\right) + \frac{q^2}{4\varepsilon_{\text{eff}}(\pi\cdot\mathsf{k})}\boldsymbol{\sigma}\cdot\left[\mathbf{k}\times\nabla\langle\mathsf{A}^2_{\text{osc}}\rangle - \frac{(\boldsymbol{\lambda}\times\mathbf{k})\partial_t\langle\mathsf{A}^2_{\text{osc}}\rangle}{m+\lambda^0} - \frac{k^0\boldsymbol{\lambda}\times\nabla\langle\mathsf{A}^2_{\text{osc}}\rangle}{m+\lambda^0}\right]\\
&\quad + \frac{\alpha}{2\varepsilon_{\text{eff}}(\pi\cdot\mathsf{k})}\boldsymbol{\sigma}\cdot\left(\frac{k^0\boldsymbol{\lambda}}{m+\lambda^0} - \mathbf{k}\right)\times[\nabla(\pi\cdot\mathsf{k}) + (k^\mu\partial_\mu)\boldsymbol{\pi}] - \frac{\alpha}{2\varepsilon_{\text{eff}}(\pi\cdot\mathsf{k})}\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{\lambda}\times\mathbf{k})}{m+\lambda^0}[(k^\mu\partial_\mu)\pi^0 - \partial_t(\pi\cdot\mathsf{k})].
\end{aligned}\tag{B21}
$$

We can simplify the last two lines of Eq. (B21) with

$$
\begin{aligned}
\nabla(\pi\cdot\mathsf{k}) + (k^\mu\partial_\mu)\boldsymbol{\pi} &= \nabla(\pi^0k^0 - \mathbf{k}\cdot\boldsymbol{\pi}) + k^0\partial_t\boldsymbol{\pi} + (\mathbf{k}\cdot\nabla)\boldsymbol{\pi} \simeq k^0q\mathbf{E}_{\text{bg}} + \pi^0\nabla k^0 + (\mathbf{k}\cdot\nabla)\boldsymbol{\pi} - \nabla(\mathbf{k}\cdot\boldsymbol{\pi})\\
&= k^0q\mathbf{E}_{\text{bg}} - \pi^0\nabla\partial_t\Theta + (\mathbf{k}\cdot\nabla)\boldsymbol{\pi} - \pi^i\nabla\partial_i\Theta - k_i\nabla\pi^i\\
&= k^0q\mathbf{E}_{\text{bg}} - \pi^0\partial_t\mathbf{k} - (\boldsymbol{\pi}\cdot\nabla)\mathbf{k} + (\mathbf{k}\cdot\nabla)\boldsymbol{\pi} - k_i\nabla\pi^i = k^0q\mathbf{E}_{\text{bg}} - (\pi^\mu\partial_\mu)\mathbf{k} - \mathbf{k}\times(\nabla\times\boldsymbol{\pi})\\
&= k^0q\mathbf{E}_{\text{bg}} - (\pi^\mu\partial_\mu)\mathbf{k} - \mathbf{k}\times[\nabla\times(\nabla\theta - q\mathbf{A}_{\text{bg}})] = k^0q\mathbf{E}_{\text{bg}} + \mathbf{k}\times q\mathbf{B}_{\text{bg}} - (\pi^\mu\partial_\mu)\mathbf{k},
\end{aligned}\tag{B22}
$$

$$
\begin{aligned}
(k^\mu\partial_\mu)\pi^0 - \partial_t(\pi\cdot\mathsf{k}) &= k^0\partial_t\pi^0 + \mathbf{k}\cdot\nabla\pi^0 - \partial_t(\pi^0k^0 - \boldsymbol{\pi}\cdot\mathbf{k}) = \mathbf{k}\cdot\nabla\pi^0 + \partial_t(\boldsymbol{\pi}\cdot\mathbf{k}) - \pi^0\partial_tk^0\\
&\simeq \mathbf{k}\cdot(-\partial_t\boldsymbol{\pi} + q\mathbf{E}_{\text{bg}}) + \partial_t(\boldsymbol{\pi}\cdot\mathbf{k}) - \pi^0\partial_tk^0 = q\mathbf{k}\cdot\mathbf{E}_{\text{bg}} - \pi^0\partial_tk^0 + \boldsymbol{\pi}\cdot\partial_t\mathbf{k}\\
&= q\mathbf{k}\cdot\mathbf{E}_{\text{bg}} - \pi^0\partial_tk^0 + \boldsymbol{\pi}\cdot\partial_t\nabla\Theta = q\mathbf{k}\cdot\mathbf{E}_{\text{bg}} - \pi^0\partial_tk^0 - (\boldsymbol{\pi}\cdot\nabla)k^0 = q\mathbf{k}\cdot\mathbf{E}_{\text{bg}} - (\pi^\mu\partial_\mu)k^0.
\end{aligned}\tag{B23}
$$

Hence, we obtain Eqs. (62) and (63).

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