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Proposal for a macroscopic test of local realism with phase-space measurements

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We propose a test of local realism based on correlation measurements of continuum valued functions of positions and momenta, known as modular variables. The Wigner representations of these observables are bounded in phase space and, therefore, the associated inequality holds for any state described by a non-negative Wigner function. This agrees with Bell's remark that positive Wigner functions, serving as a valid probability distribution over local (hidden) phase-space coordinates, do not reveal nonlocality. We construct a class of entangled states resulting in a violation of the inequality and thus truly demonstrate nonlocality in phase space. The states can be realized through grating techniques in spacelike separated interferometric setups. The nonlocality is verified from the spatial correlation data that is collected from the screens.

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I. INTRODUCTION

In 1935, Einstein, Podolsky, and Rosen (EPR) argued that the quantum-mechanical description of physical reality is not complete, and thus may be superseded by a more complete realistic theory which reproduces the quantum-mechanical predictions and, at the same time, obeys the locality condition [1]. Bell derived an experimentally testable inequality in his seminal 1964 paper [2], which bounds the correlations between bipartite measurements for any such local hidden-variable theory, but is violated by quantum mechanics. This was a major breakthrough towards empirical tests of quantum mechanics against theories conforming to common sense. Since then, the results constraining the permissible types of hidden-variable models of quantum mechanics have attracted much attention and have been reformulated as the problem of contextual measurements by Kochen and Specker [3] and in terms of temporal correlations by Leggett and Garg [4]. Today, these concepts have mainly been formulated for intrinsic quantum degrees of freedom of microscopic particles such as spins, and tested in various experiments with photons [5], impurity spins [6,7], or superconducting qubits [8]. All experimental observations confirmed the validity of quantum mechanics at this level.

The outstanding challenge, however, is to formulate similar tests with true phase-space measurements, where nonlocality is inferred directly from observing the spatial degree of freedom, for example. This can be viewed as a natural extension of the macroscopic limit of local realism tests [9]. This is closer in spirit to the original EPR argument which uses a phase-space description, a natural concept in the classical world, to better address the reality and locality problems of quantum mechanics. Notably, the Wigner function associated with the entangled state used in the EPR argument, the so-called EPR state, is non-negative everywhere [10]. That is why Bell argued that EPR states do not lead to a violation of the inequalities derived from locality and hidden-variable assumptions [11]. This was because non-negative Wigner functions serve as valid joint probability distributions over local *hidden* positions and momenta. Thereby, in principle, a model of a local hidden variable can be attributed to such states. Banaszek and Wodkiewicz [12], however, showed that by using particular measurements, namely parity, EPR states can reveal nonlocal features, indicating that not only the state itself but the type of correlation measurements is also important in any local realism test. This opens the discussion as to which measurements are good candidates for appropriately testing local hidden-variable models in phase space [13,14]. The problem of constructing a "macroscopic" test of local hidden-variable models depends on choosing proper observables whose Wigner representations satisfy the constraint imposed by the algebraic Clauser-Horne-Shimony-Holt (CHSH) [15] expression. The term "macroscopic" henceforth will be used to refer to the measurement of a particle's phase-space coordinates. This class of measurements has a clear description in the classical limit and its evaluation does not involve sharp measurements that betray "quantum degrees of freedom."

The observables used here are the so-called modular variables [16]. Recently, such variables have found applications in detecting certain continuous-variable (CV) entangled states [17–19] and quantum information [20,21]. Furthermore, they have been used for fundamental tests such as macroscopic realism [22], contextuality [23,24], and even the GHZ test [25]. This strongly suggests that modular variables can be used for Bell inequality tests of local hidden-variable theories as well. Recently, a Bell test with discretized modular variables was proposed [26]. In the present work, we put forward a Bell test with "continuous" modular variables which requires phase-space measurements.

The paper is organized as follows. In Sec. II, we introduce our framework for the Bell test, aiming to use the most "classical-like" variables and measurements. This motivates a macroscopic test of local realism [9]. In Sec. III, we construct a Bell operator from modular variables, for which the violation is achieved only if the state is described by a negative Wigner function. We then proceed with identification of the relevant entangled state, explicitly showing the violation. Finally, in Secs. IV and V, we show how the entire test can be implemented in a double (multislit) grating setup [10]. Grating techniques have been used to experimentally demonstrate quantum matter waves [27]. We summarize by briefly discussing the outlook in Sec. VI and conclude in Sec. VII.

II. FRAMEWORK FOR A MACROSCOPIC BELL TEST

In what follows, we develop a test of local realism which complies with Bell's aforesaid argument. The central problem here is to construct a Bell operator of the CHSH form,

$$\hat{\mathcal{B}} \equiv \hat{A}_1 \otimes (\hat{A}_2 + \hat{A}_2') + \hat{A}_1' \otimes (\hat{A}_2 - \hat{A}_2'), \quad (1)$$

expressed in terms of suitable local CV observables $\hat{A}_i$ which must be restricted to a limited range of values to impose a well-defined classical bound. We therefore require the observables to satisfy the following properties:

(a) Eigenvalues of $\hat{A}_i$, $|a_i| \leqslant 1$, which for bounded observables can be achieved trivially by rescaling. An example of this is the parity operator.

While this condition is enough to obtain a classical bound, we demand an extra constraint which is necessary for probing nonlocality in phase space.

(b) The observable $\hat{A}$ corresponds to a *bounded* c-number function in phase space obtained from the Wigner-Weyl correspondence ($q \leftrightarrow \hat{q}$, $p \leftrightarrow \hat{p}$),

$$|\mathcal{W}_{\hat{A}}(q,p)| \equiv \left| \int dq' e^{ipq'} \left\langle q - \frac{q'}{2} \right| \hat{A} \left| q + \frac{q'}{2} \right\rangle \right| \leqslant 1. \quad (2)$$

This entails

$$\begin{aligned} |\mathcal{W}_{\hat{\mathcal{B}}}(\boldsymbol{q},\boldsymbol{p})| = \big| &\mathcal{W}_{\hat{A}_1}(q_1,p_1)[\mathcal{W}_{\hat{A}_2}(q_2,p_2) + \mathcal{W}_{\hat{A}_2'}(q_2,p_2)] \\ &+ \mathcal{W}_{\hat{A}_1'}(q_1,p_1)[\mathcal{W}_{\hat{A}_2}(q_2,p_2) - \mathcal{W}_{\hat{A}_2'}(q_2,p_2)]\big| \\ \leqslant\ & 2 \end{aligned}$$

for the Wigner representation of the Bell operator where $\boldsymbol{q} \equiv (q_1,q_2)$ and $\boldsymbol{p} \equiv (p_1,p_2)$. Accordingly, for any state, including the EPR state, described by a valid (non-negative) probability distribution over phase space, the following inequality holds:

$$|\langle \hat{\mathcal{B}} \rangle| = \left| \int W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p}) \mathcal{W}_{\hat{\mathcal{B}}}(\boldsymbol{q},\boldsymbol{p}) d\boldsymbol{q} d\boldsymbol{p} \right| \leqslant 2, \quad (3)$$

where $W_{\hat{\rho}}$ is the Wigner quasiprobability distribution corresponding to $\hat{\rho}$ given by $W_{\hat{\rho}} = \mathcal{W}_{\hat{\rho}}/2\pi\hbar$. A violation therefore must necessarily arise from the negativity of the Wigner function describing the state.

Although formally valid, the Bell inequality expressed in terms of displaced parity operators used in Refs. [12,28] voids the second condition; their Wigner representations are given by δ functions which are unbounded in phase space.

The measurement scheme used for evaluating the correlations must have a clear classical limit for any reasonable "macroscopic test." This measurement strategy is in marked contrast to other approaches that use parity measurements [12]. Parity measurements, unlike phase-space measurements, require resolving intrinsic quantum degrees of freedom and thus have no classical analog. It has been shown that for sufficiently sharp measurements, the system inevitably enters a quantum regime and no classical description is possible [29]. Thus such measurements remotely resemble "classical-like" measurements, if at all.

The binary binning of quadrature measurements has also been shown to be a possible scheme [30,31] where entangled Schrödinger Cat states (and their appropriate generalizations) are used. Here, however, to preserve features characteristic of classical dynamics, we aim to adopt a different measurement strategy which retains the continuous spectra and uses phase space exclusively.

Phase-space translation and modular variables

One particular class of bounded observables can be constructed from the quantum-mechanical space translation operator $e^{-i\hat{p}L/\hbar}$. As its name suggests, this operator displaces a particle by a finite distance L, which in our case will be the distance between two adjacent slits. This operator is not an observable, therefore we define a symmetric combination,

$$\hat{X} \equiv \frac{e^{-i\hat{p}L/\hbar} + e^{i\hat{p}L/\hbar}}{2} = \cos(\hat{p}L/\hbar), \quad (4)$$

which is explicitly Hermitian and bounded by ± 1. In fact, the corresponding function $|\mathcal{W}_{\hat{X}}| = |\cos(pL/\hbar)|$ is also manifestly $\leqslant 1$. Further, when $\hat{X}$ is operated on $|p\rangle$, only the modular part of p is relevant to the value of the operator. Thus, we may define

$$\hat{p}_{\text{mod}\frac{h}{L}} \equiv (\hat{p}L/h - \lfloor \hat{p}L/h \rfloor)\frac{h}{L}, \quad (5)$$

and note that measuring $\hat{p}_{\text{mod}\frac{h}{L}}$ is sufficient for obtaining the value of $\hat{X} \equiv X(\hat{p}_{\text{mod}\frac{h}{L}})$. Conversely, measuring $\hat{X}$ only yields $\hat{p}_{\text{mod}\frac{h}{L}}$, not $\hat{p}$. The idea is to construct a Bell operator [see Eq. (1)] from $\hat{X}$ in which the different measurement settings are chosen by transforming it using suitable unitary operators.

III. THE CONSTRUCTION

Consider a localized state $\varphi(q) = \langle q|\varphi\rangle$ symmetric about the position $q = L/2$, where $L \equiv$ length scale and $\varphi_n(q) \equiv \varphi(q - nL)$. We define

$$|\psi_0\rangle \equiv \frac{1}{\sqrt{M}} \sum_{n=-\lfloor \frac{M}{2} \rfloor}^{\lfloor \frac{M-1}{2} \rfloor} |\varphi_{2n+1}\rangle, \quad |\psi_1\rangle \equiv \frac{1}{\sqrt{M}} \sum_{n=-\lfloor \frac{M}{2} \rfloor}^{\lfloor \frac{M-1}{2} \rfloor} |\varphi_{2n}\rangle.$$

Using these states, as illustrated in Fig. 1, we construct the states

$$|\psi_+\rangle \equiv \frac{|\psi_0\rangle + |\psi_1\rangle}{\sqrt{2}}, \quad |\psi_-\rangle \equiv \frac{|\psi_0\rangle - |\psi_1\rangle}{\sqrt{2}}. \quad (6)$$

These states were constructed with a partial translational symmetry which is appropriate to the bounded Hermitian operator $\hat{X}$ discussed earlier. These N-component superposition states can represent a delocalized particle after an N-slit grating. It follows that

$$\langle \psi_+ | \hat{X} | \psi_+ \rangle = \frac{N-1}{N},$$
$$\langle \psi_- | \hat{X} | \psi_- \rangle = -\frac{N-1}{N},$$

where $N \equiv 2M$ is the number of "slits." Before proceeding further, we introduce a unitary operator $\hat{U}$ to implement different measurement settings. Motivated by the spins, we define $\hat{U}$ by its action

$$\hat{U}(\phi)|\psi_0\rangle = e^{i\phi/2}|\psi_0\rangle, \quad \hat{U}(\phi)|\psi_1\rangle = e^{-i\phi/2}|\psi_1\rangle. \quad (7)$$

More explicitly,

$$\hat{U}(\phi) \equiv e^{i\hat{Z}\phi/2},$$

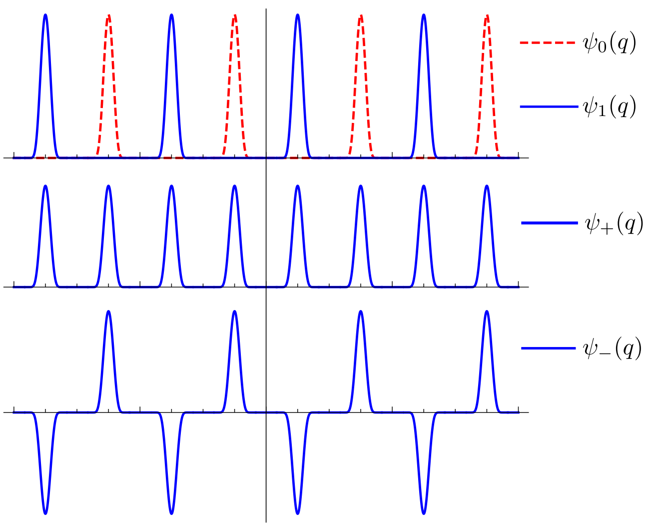

FIG. 1. (Color online) Illustration of multicomponent superposition states $|\psi_\pm\rangle$ and $|\psi_0\rangle$, $|\psi_1\rangle$ for $N = 8$.

where $\hat{Z}$ is such that $\hat{Z}|\psi_0\rangle = |\psi_0\rangle$ and $\hat{Z}|\psi_1\rangle = -|\psi_1\rangle$. We note that $\hat{Z}$ must differentiate between spatial wave functions $\langle q|\varphi\rangle$ and $\langle q - L|\varphi\rangle$. It is thus natural to expect $\hat{Z}$ to be a function of $\hat{q}_{\text{mod}2L}$, i.e., $\hat{Z} \equiv Z(\hat{q}_{\text{mod}2L})$. For consistency then, we conclude that Z must have the form of a square wave and define

$$\hat{Z} \equiv \text{sgn}\left(\sin \frac{\hat{\text{q}}\pi}{\text{L}}\right).$$

The test is performed by considering two particles and their observers, Alice and Bob; they apply the aforesaid local unitaries to define the setting and then measure $\hat{X}$. We claim that the suitable entangled state which will yield a violation, given this scheme, is

$$|\Psi\rangle \equiv \frac{|\psi_+\rangle_1 |\psi_-\rangle_2 - |\psi_-\rangle_1 |\psi_+\rangle_2}{\sqrt{2}}. \tag{8}$$

We now evaluate $\langle\hat{\mathcal{B}}\rangle$. This essentially requires terms such as $\langle\hat{X}(\phi) \otimes \hat{X}(\theta)\rangle$, where $\hat{X}(\theta) \equiv \hat{U}^\dagger(\theta)\hat{X}\hat{U}(\theta)$. It can be shown that (see the Appendix)

$$\langle\hat{X}(\phi) \otimes \hat{X}(\theta)\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi - \theta). \tag{9}$$

Thus, for particular angles, i.e., θ', ϕ, θ, and ϕ' successively separated by $\pi/4$, we get

$$|\langle\hat{\mathcal{B}}\rangle| = \left(\frac{N-1}{N}\right)^2 2\sqrt{2}. \tag{10}$$

The violation, i.e., $|\langle\hat{\mathcal{B}}\rangle| > 2$, requires $N > 6$; see Fig. 2. To interpret this, we must ensure that the assumptions of the framework are satisfied, viz. $|\mathcal{W}_{\hat{X}(\phi)}| \leqslant 1$. To that end, we note that

$$|\mathcal{W}_{\hat{X}(\phi)}(q,p)| = \left|\frac{1}{2}\int dq' e^{ipq'/\hbar}\left\langle q - \frac{q'}{2}\right| \left(e^{-i\hat{Z}\phi/2} \times e^{i\hat{p}\frac{L}{\hbar}} e^{i\hat{Z}\phi/2} + \text{H.c.}\right)\left|q + \frac{q'}{2}\right\rangle\right|$$

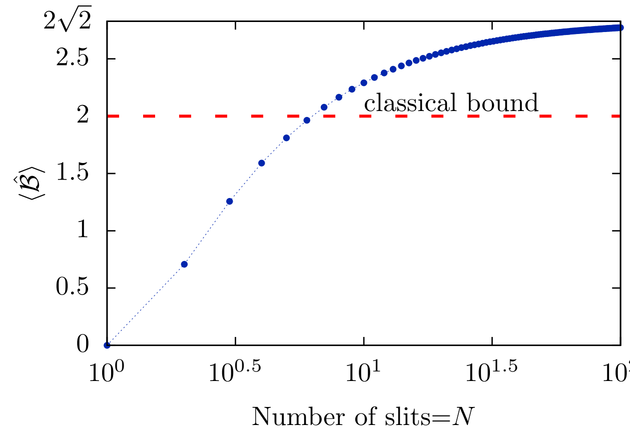

FIG. 2. (Color online) Practically, the number of slits N will be finite. The plot shows $\langle\hat{\mathcal{B}}\rangle$ as a function of N. To get a violation, we need a mere 8 slits; with 50 we almost saturate.

$$= \left|\text{Re}\left(e^{-iZ_-(q)\phi/2} e^{ip\frac{L}{\hbar}} e^{iZ_+(q)\phi/2}\right)\right|$$
$$= |\cos(pL/\hbar \pm Z_\pm(q)\phi)| \leqslant 1, \tag{11}$$

where $Z_\pm(q) \equiv Z[(q \pm \frac{L}{2})\text{mod}2L]$, and we used the fact that $Z(q) = -Z(q + L)$ (omitting the mod$2L$).

Passing remarks

(1) *Pauli-like commutation*. While at first the definition of $\hat{Z}$ might appear arbitrary, we show that it naturally yields Pauli-like algebra. We start with $[\hat{Z}, e^{i\hat{p}L/\hbar}]$. To evaluate it, we multiply the second term with $\int dq|q\rangle\langle q|$ and obtain $\hat{Z}e^{i\hat{p}L/\hbar} + \hat{Z}e^{i\hat{p}L/\hbar}$, where we have used $Z(\hat{q}_{\text{mod}2L}) = -Z((\hat{q} \pm L)_{\text{mod}2L})$,

$$[\hat{Z}, \hat{X}] = 2\hat{Z}\hat{X} = -2i\hat{Y},$$

where $\hat{Y} \equiv i\hat{Z}\hat{X}$. Here, i was introduced to ensure $\hat{Y}^\dagger = \hat{Y}$, since $\hat{X}^\dagger = \hat{X}$ and $\hat{Z}^\dagger = \hat{Z}$. Similarly, $\{\hat{Z}, \hat{X}\} = 0$. From the definition of Y and the anticommutation, $\{\hat{Y}, \hat{X}\} = 0$ and $\{\hat{Y}, \hat{Z}\} = 0$ also follow trivially. We may point out that while $\hat{Z}^2 = 1$, it is not a sum of a two-state projector, and $\hat{X}^2 \neq 1$ in general. This manifests in the following relations:

$$[\hat{X}, \hat{Y}] = -2i\hat{Z}\hat{X}^2 = -2i\hat{X}^2\hat{Z},$$
$$[\hat{Y}, \hat{Z}] = -2i\hat{X}.$$

It is apparent that the exact SU(2) algebra is not necessary to arrive at a violation.

(2) *Asymmetry in Z and X*. Using an analogous momentum translation operator, the following can be derived from the definition of $\hat{p}$:

$$e^{i\hat{p}u}e^{i\hat{q}v} = e^{i\hbar uv}e^{i\hat{q}v}e^{i\hat{p}u}.$$

For appropriate choices of u,v, the translation operators can be made to commute or anticommute. In the former case, it means that one can simultaneously measure modular position and momentum (which is in stark contrast to $\hat{x}$ and $\hat{p}$ measurements), and in the latter case, one can define Pauli-matrix-like commutation. Considering the operator (non-Hermitian for simplicity) $\hat{X} = e^{i\hat{p}L/\hbar}$, defining $\hat{Z} = e^{i\hat{q}2\pi/2L}$ is more natural. They also follow the desired anticommutation

$\{\hat{X},\hat{Z}\}=0$ and we could define $\hat{Y}=i\hat{Z}\hat{X}$ to get a more natural generalization. The question is why did $\hat{Z}=Z(\hat{q}_{\text{mod}2L})$ appear in the analysis. The cause of this asymmetry hinges on the preferential treatment of position space. We could have constructed states of the form $|\psi_0\rangle=\sum_n|q+nd\rangle$ and used the natural definition of $\hat{Z}$ to obtain the violation. The issue is that this forces us to choose a countable superposition of position eigenkets as our desired state.[1] If we start with a better defined and broader class of relevant states, $Z(\hat{q}_{\text{mod}2L})$ appears naturally.

(3) *Commutation and classical limit*. It is well recognized and can be shown that there is a tight relation between nonlocality and noncommutativity of operators. The violation occurs for choices of settings whose corresponding observables do not commute. In our construction, we can demonstrate that the source of violation can be clearly attributed to the noncommutativity between position and momentum, $[\hat{q},\hat{p}]=i\hbar$. This would be regarded as a further illustration that our approach provides a relevant test in phase space. We show that in our case, $[\hat{X}(\theta),\hat{X}(\theta')]\neq 0$. To prove that, we use $\hat{X}(\theta)=\hat{X}e^{i\hat{Z}\theta}$, $e^{i\hat{Z}\theta}=\cos\theta+i\hat{Z}\sin\theta$ and the previous results to arrive at

$$\begin{aligned}[\hat{X}(\theta),\hat{X}(\theta')]&=2i\sin(\theta'-\theta)\hat{Z}\hat{X}^2\\&=2i\hat{Z}\hat{X}^2\neq 0,\end{aligned}$$

where the last equality holds when the angles are as defined earlier. Classically this term not only vanishes, but the different measurement settings also become identical. The Heisenberg equation of motion for the displacement operator,

$$\begin{aligned}\frac{d\hat{X}}{dt}&=i\hbar^{-1}[\hat{Z},\hat{X}]\\&=i\hbar^{-1}[Z(\hat{q}_{\text{mod}2L})-Z(\hat{q}_{\text{mod}2L}\pm L)]\hat{X}\\&=i\hbar^{-1}2\hat{Z}\hat{X},\end{aligned}\tag{12}$$

where $\hat{Z}$ is the potential, shows that $\hat{X}(\theta)$ is essentially $\hat{X}$ at some later time. However, classically, since the particle experiences no force (constant potential), $X(t)=X(t_0)$. This peculiarity is the same as that of the scalar Aharonov-Bohm effect, which is exploited here for realizing different measurement settings. Manifestly then, the noncommutativity of $\hat{q}$ and $\hat{p}$ results in $\hat{X}(t)\neq\hat{X}(t_0)$ (as it follows a nonlocal equation of motion [32]), which is pivotal for the violation.

IV. MEASUREMENT SCHEMES

The scheme requires us to evaluate the correlation functions such as $\langle\hat{X}(\theta)\otimes\hat{X}(\phi)\rangle$. Equivalently, the measurement settings can be chosen by applying the corresponding local unitaries on the entangled state, that is, $|\Psi_{\theta\phi}\rangle=\hat{U}(\theta)\otimes\hat{U}(\phi)|\Psi\rangle$. Therefore, obtaining $|\langle p_1,p_2|\Psi_{\theta\phi}\rangle|^2$ is sufficient for evaluating $\langle\hat{X}(\theta)\otimes\hat{X}(\phi)\rangle=\int dp_1dp_2\cos(p_1L/\hbar)\cos(p_2L/\hbar)|\langle p_1,p_2|\Psi_{\theta\phi}\rangle|^2$.

It is known that in the far-field approximation [10],

$$\left|\left\langle p_1=\frac{p_zq_1}{D},p_2=\frac{p_zq_2}{D}\middle|\Psi_{\theta\phi}\right\rangle\right|^2=\frac{D^2}{p_z^2}\left|\left\langle q_1,q_2\middle|\Psi^{\text{screen}}_{\theta\phi}\right\rangle\right|^2,\tag{13}$$

where $|\Psi^{\text{screen}}_{\theta\phi}\rangle$ is the state of the system at the screen, D is the distance between the gratings and the screens, and p_z is the z component of momentum of the particle. For a photon, $p_z=h/\lambda$, while for a massive particle with mass m, $p_z=mD/T$, where T is the time taken to arrive at the screen from the grating (see Fig. 3). The idea is simply that the momentum distribution at the grating can be recovered by observing the spatial distribution at the screen, sufficiently far away.

V. PHYSICAL IMPLEMENTATION

The test can be implemented in a quantum interferometric setup, using grating techniques to create multicomponent superposition states, as is done in matter-wave experiments for instance. We show that this scheme can be implemented using photons. We harness the two degrees of freedom of a photon, its polarization, and its spatial degree of freedom to construct the required state. With a slightly modified setup, it is possible to do the same with spin and position for matter waves (see the Appendix, Sec. 3). The final setup is given in Fig. 3. We need only consider the quantum-mechanical description along the x axis.

A. Creation of the entangled state

The desired entangled state is $|\Psi\rangle$, as stated in Eq. (8). We start with noting the triviality of constructing a $|\psi_+\rangle$ state [see Eq. (6)]. Consider a source that produces a state $|\gamma\rangle$ at the grating. $\langle q|\gamma\rangle$ is assumed to be a real Gaussian with $\sigma\gg 2NL$. The grating has N slits of width $a\ll L$, separated by a distance L (center to center). After the grating, we obtain $|\psi_+\rangle=\hat{G}|\gamma\rangle$, where $\hat{G}$ may be formally defined accordingly. Similarly, the $|\psi_-\rangle$ state can be constructed by using glass slabs at alternate slits, such that the phase introduced is π. In Fig. 3, if you consider only one particle and disregard everything after the grating, then the setup is expected to produce a $|\psi_+\rangle$ state right after it. To produce the desired entangled state, we start with two entangled photons, such that their polarization state can be expressed as $|\chi\rangle\equiv\frac{|H\rangle_1|V\rangle_2-|V\rangle_1|H\rangle_2}{\sqrt{2}}$. Their spatial description (along the x axis) is initially assumed to be $|\gamma\rangle_1|\gamma\rangle_2$ so that the postgrating state is

$$\frac{|H\rangle_1|V\rangle_2-|V\rangle_1|H\rangle_2}{\sqrt{2}}|\psi_+\rangle_1|\psi_+\rangle_2.$$

If we had glass slabs, whose refractive index (given some orientation) was, say, $\eta_H=1$ for a horizontally polarized beam and $\eta_V=\eta\neq 1$ for vertical polarization, then we could harness the entangled polarization state to create the required spatially entangled state. Birefringent crystals have such polarization-dependent refractive indices. Assume that alternating birefringent crystals have been placed after both the gratings with appropriate thickness so that the subsequent

[1] Such a state is strictly not even in the Hilbert space.

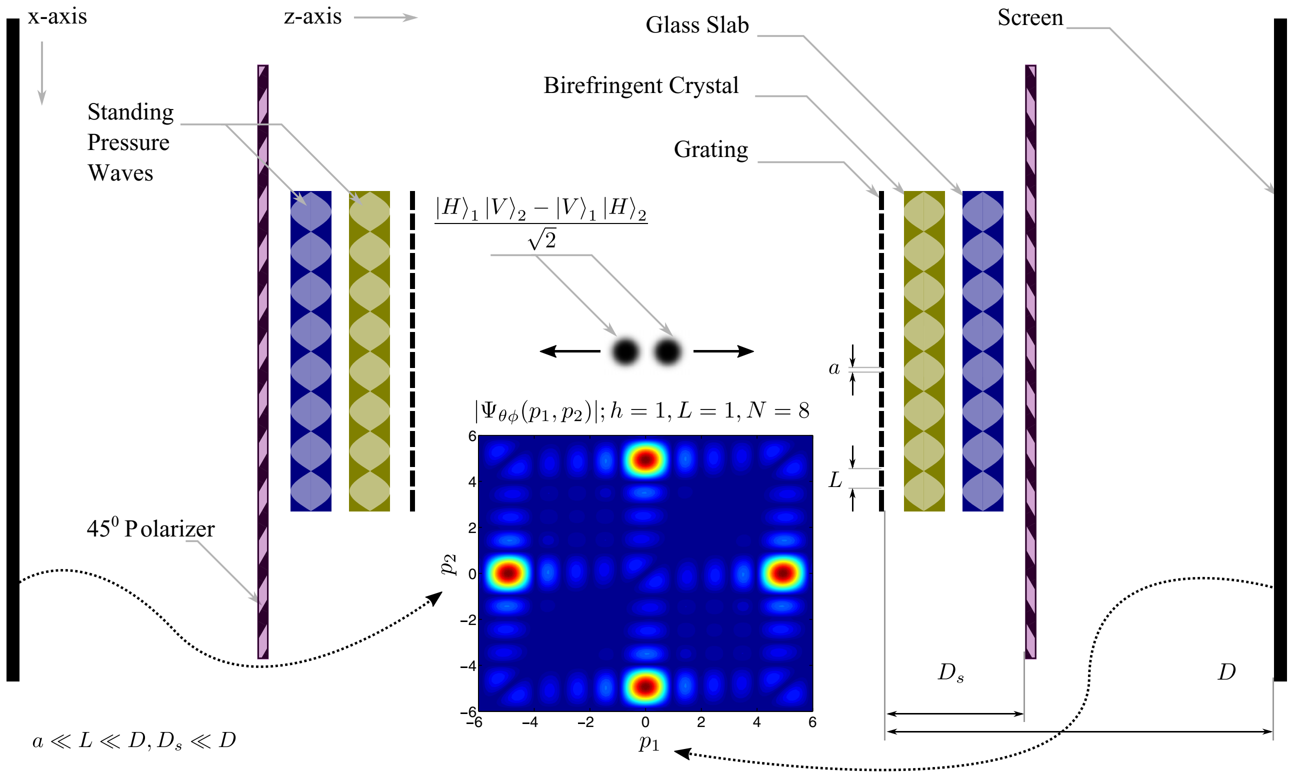

FIG. 3. (Color online) The experimental setup for implementing the test. It includes the scheme for creating the necessary entangled state. See the text for further details.

state is

$$\frac{|H\rangle_1 |V\rangle_2 |\psi_+\rangle_1 |\psi_-\rangle_2 - |V\rangle_1 |H\rangle_2 |\psi_-\rangle_1 |\psi_+\rangle_2}{\sqrt{2}}.$$

If the polarization state is traced out, the resultant state will be mixed, and hence useless. Instead, a 45° polarizer is introduced after which (see the Appendix, Sec. 4) the target entangled state,

$$|\chi_{45}\rangle |\Psi\rangle = |\nearrow\rangle_1 |\nearrow\rangle_2 \frac{|\psi_+\rangle_1 |\psi_-\rangle_2 - |\psi_-\rangle_1 |\psi_+\rangle_2}{\sqrt{2}},$$

is obtained, where $|\nearrow\rangle \equiv (|H\rangle + |V\rangle)/\sqrt{2}$. As a remark, it may be stated that although to arrive at this result we assumed that $\eta_H = 1$, which is physically unreasonable, we can compensate for $\eta_H \neq 1$ by putting appropriate glass slabs at the alternate empty slits to produce zero relative phase when the polarization is horizontal.

B. Measurement settings

The measurement setting is applied by local unitaries such as $\hat{U}(\theta) \otimes \hat{U}(\phi)$. A local unitary can be performed by placing alternating glass slabs of widths such that Eq. (7) holds. These slabs may be placed right after the birefringent crystals, before the polarizer. The final state just after the polarizer is given by $|\Psi_{\theta\phi}\rangle = \hat{U}(\theta) \otimes \hat{U}(\phi)|\Psi\rangle$, where $\theta\phi$ is one of the four possible measurement settings.

C. Effective practical setup

Placing glass slabs may not be suitable for fine gratings, although a similar setup may be possible [33]. Practically we can implement the same scheme using the setup shown in Fig. 3. The first large slab is a birefringent crystal (η_H,η_V), while the adjacent slab is plain glass (η). We generate longitudinal standing pressure waves so that the effective thickness at alternate grating sites is given by d_0,d_1 and l_0,l_1 for the crystal and slab, respectively. The phase difference between a horizontal $|\psi_0\rangle$ and $|\psi_1\rangle$ will be given by $\eta_H(d_0 - d_1) \equiv \phi_A$; note that physically only phase differences are essential. For the vertical component, it will be $\eta_V(d_0 - d_1)$. If we impose $\eta_V(d_0 - d_1) = \pi + \phi_A$, then we would have created[2] the state

$$e^{i\hat{Z}\frac{\phi_A}{2}} \frac{|\psi_+\rangle + |\psi_-\rangle}{\sqrt{2}}$$

for an incident $\frac{|H\rangle+|V\rangle}{\sqrt{2}}$ polarization state. d_0 and d_1 will be constrained by some relation depending on physical properties of the crystal; they will also depend on the amplitude of the longitudinal wave. From this and the imposed constraint, d_0,d_1 and the corresponding amplitude can be determined. However, we do not have the freedom to change ϕ_A. To remedy this, we use the glass slab. It will introduce an additional relative phase, $\eta(l_0 - l_1) \equiv \phi_B$. Here, again l_0,l_1 may satisfy some constraint, but will depend on the amplitude which is adjustable. Thus, by changing this amplitude, we can set the relative phase $\phi = \phi_A + \phi_B$ arbitrarily.

Spatial light modulators may be used to more conveniently implement the aforesaid action of the glass slab and birefringent crystal.

[2](up to an overall phase)

Effectively therefore, this scheme allows for both the creation of the entangled state and changing the measurement settings in a practical way.

VI. DISCUSSION

It is worth adding that one can use an alternative measurement strategy giving the same violation of the inequality; that is, to measure the modular variable with two-valued positive operator-valued measure (POVM) elements $\hat{E}_{\pm}$, given by

$$\hat{E}_{\pm} = \tfrac{1}{2}(\hat{\mathbb{I}} \pm \hat{X}), \tag{14}$$

satisfying $E_+ + E_- = \hat{\mathbb{I}}$. It follows that $\langle \hat{X} \rangle = p_+ - p_-$, where the probabilities of getting $\pm$ outcomes, $p_{\pm} = \langle \hat{E}_{\pm} \rangle$, can be determined from the observed binary statistics read out from an ancillary two-level system [34].

An interesting problem is to develop our approach to finite-dimensional systems, i.e., qudits. A class of Bell inequalities was proposed by Collins *et al.* [35], which is useful for demonstrating nonlocality in high-dimensional entangled states. Our version of the Bell inequality generalized to d-dimensional systems can be achieved by using the discrete translation operators known as Heisenberg-Weyl or generalized Pauli operators, i.e., $e^{-i2\pi \hat{P}l/d}$, whose action is $\hat{e}^{-i2\pi \hat{P}l/d}\,|n\rangle = |n+l\rangle$, where l describes the steps translated in discrete position space with periodic boundary conditions and $\hat{P} = \sum_{k=0}^{d-1} k\,|k\rangle\langle k|$ is the discrete momentum operator. From this, we obtain the relevant discrete modular variable $\hat{X}_d^l = \cos(2\pi \hat{P}l/d)$. We expect that the class of d-dimensional entangled states which demonstrate nonlocality here will be different from those considered by Collins *et al.* [35] and Lee *et al.* [36].

It is obvious from the properties of the modular variables we use that the violation is more pronounced for higher number of slits. One can, however, imagine that those entangled states created with slits which are fewer than the minimum number needed for obtaining a violation must also hold nonlocal properties. To reveal the nonlocality in this range, one may need a more optimal set of observables, which involve a suitable combination of different modular variables, as opposed to the set considered here.

VII. CONCLUSION

In the present work, we constructed a different Bell operator in terms of phase-space measurements via modular variables. In this scheme, there is no possibility for a bipartite system with a positive-definite Wigner function, formally entangled or not, to yield a violation of the inequality. Therefore, a violation of the inequality truly contradicts local (hidden) phase-space models. From this perspective, our scheme is strongly different from the other approaches reported in Refs. [12,37–39] where sharp quantum measurements with no classical analog have been used. The measurement observables in our scheme instead are very simple with a clear classical limit. The relevant entangled states used for achieving a violation of the inequality, however, required the creation of multicomponent superposition states characterized by a negative Wigner function. Interestingly, our scheme also involves the scalar Aharonov-Bohm effect, manifesting another type of nonlocality [32].

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APPENDIX: CLAIMS

Here, we provide more detailed derivations of the results.

1. Useful expectation values

$$\begin{aligned}
\langle \psi_+|\hat{X}|\psi_+\rangle &= \frac{N-1}{N}, \quad \langle \psi_-|\hat{X}|\psi_-\rangle = -\frac{N-1}{N},\\
\langle \psi_0|\hat{X}|\psi_0\rangle &= 0, \quad \langle \psi_1|\hat{X}|\psi_1\rangle = 0,\\
\langle \psi_1|\hat{X}|\psi_0\rangle &= \frac{\frac{N-1}{N} + \frac{N}{N}}{2} = \frac{2N-1}{2N} = \langle \psi_0|\hat{X}|\psi_1\rangle,\\
\langle \psi_-|\hat{X}|\psi_+\rangle &= \frac{-\langle \psi_1|\hat{X}|\psi_0\rangle + \langle \psi_0|\hat{X}|\psi_1\rangle}{2}\\
&= 0 = \langle \psi_+|\hat{X}|\psi_-\rangle,\\
\langle \Psi|\hat{X}\otimes\hat{X}|\Psi\rangle &= \frac{1}{2}(\langle \psi_-|\hat{X}|\psi_-\rangle\langle \psi_+|\hat{X}|\psi_+\rangle\\
&\quad + \langle \psi_+|\hat{X}|\psi_+\rangle\langle \psi_-|\hat{X}|\psi_-\rangle)\\
&= -\left(\frac{N-1}{N}\right)^2.
\end{aligned}$$

2. For arbitrary θ_i and ϕ_i $\langle \hat{U}^\dagger(\phi_i)\hat{X}\hat{U}(\phi_i) \otimes \hat{U}^\dagger(\theta_i)\hat{X}\hat{U}(\theta_i)\rangle = -(\frac{N-1}{N})^2\cos(\phi_i - \theta_i)$

Proof. We start with defining $\phi \equiv \phi_i$, $\theta \equiv \theta_i$, $\delta \equiv \phi - \theta$, $\delta' \equiv \delta/2$. Next, we note that LHS $= \langle \Psi'|\hat{X}\otimes\hat{X}|\Psi'\rangle$ where $|\Psi'\rangle = \hat{U}(\phi_i)\otimes\hat{U}(\theta_i)|\Psi\rangle$,

$$\begin{aligned}
|\Psi'\rangle &= \frac{e^{i\delta'}}{\sqrt{2}}\left(\frac{|\psi_+\rangle - |\psi_-\rangle}{\sqrt{2}}\right)\left(\frac{|\psi_+\rangle + |\psi_-\rangle}{\sqrt{2}}\right)\\
&\quad - \frac{e^{-i\delta'}}{\sqrt{2}}\left(\frac{|\psi_+\rangle + |\psi_-\rangle}{\sqrt{2}}\right)\left(\frac{|\psi_+\rangle - |\psi_-\rangle}{\sqrt{2}}\right)\\
&= \frac{e^{i\delta'}}{2\sqrt{2}}(|\psi_+\psi_+\rangle + |\psi_+\psi_-\rangle - |\psi_-\psi_+\rangle - |\psi_-\psi_-\rangle)\\
&\quad - \frac{e^{-i\delta'}}{2\sqrt{2}}(|\psi_+\psi_+\rangle - |\psi_+\psi_-\rangle + |\psi_-\psi_+\rangle - |\psi_-\psi_-\rangle)\\
&= \frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}|\psi_+\psi_+\rangle + \frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}|\psi_+\psi_-\rangle\\
&\quad - \left(\frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}\right)|\psi_-\psi_+\rangle - \left(\frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}\right)|\psi_-\psi_-\rangle.
\end{aligned}$$

Now using section 1 of this Appendix, we have

$$\begin{aligned}\text{LHS} &= \langle\Psi'|\hat{X}\otimes\hat{X}|\Psi'\rangle\\ &= \frac{1}{2}\left(\frac{N-1}{N}\right)^2\left[\left|\frac{e^{i\delta'}-e^{-i\delta'}}{2}\right|^2\right.\\ &\quad\left.-\left|\frac{e^{i\delta'}+e^{-i\delta'}}{2}\right|^2-\left|\frac{e^{i\delta'}+e^{-i\delta'}}{2}\right|^2+\left|\frac{e^{i\delta'}-e^{-i\delta'}}{2}\right|^2\right]\\ &= -\left(\frac{N-1}{N}\right)^2\frac{1}{2}[2(\cos^2\delta/2-\sin^2\delta/2)]\\ &= -\left(\frac{N-1}{N}\right)^2\cos(\delta).\end{aligned}$$

3. Physical implementation with electrons is also possible

If we can show that the basic components used to describe the photon setup can be translated to the electron setup, then, in principle, we are through. (a) Glass slab: The equivalent is the electric *AB* effect. We need to simply put a capacitor after the slit and the two components will pick up a phase difference. (b) Polarizer: The Stern-Gerlach setup is the classic analog. We simply block the orthogonal component. (c) Birefringent crystal: This is slightly tricky. It can be modeled by using a combination of gradient of magnetic field (as in Stern-Gerlach) and a capacitor. We start with an equivalent superposition of spin states, $\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}|\psi_+\psi_+\rangle$. To construct the spin-dependent $|\psi_-\rangle$ state, we use the magnetic field gradient to spatially separate the $|\uparrow\rangle$ and $|\downarrow\rangle$ states. We place capacitors as described at the spatial position corresponding to, say, $|\downarrow\rangle$. Thereafter, we remove the magnetic field gradient and allow the beams to meet again. This will effectively act as a birefringent crystal, since the phase difference is spin dependent.

4. Action of a polarizer

If we define $|\nearrow\rangle\equiv\frac{|H\rangle+|V\rangle}{\sqrt{2}},|\nwarrow\rangle\equiv-\frac{|H\rangle-|V\rangle}{\sqrt{2}}$ and the 45° projector as $|\nearrow\rangle\langle\nearrow|$, then both $|H\rangle\to|\nearrow\rangle$ and $|V\rangle\to|\nearrow\rangle$ where, of course, with a probability 1/2, the photon will be lost.

5. More on measurement

It is essential to know what ballpark resolution is required for detecting the violation from the screen. We note

$$|\langle p_1,p_2|\Psi_{\theta\phi}\rangle| = |\tilde{\varphi}(p_1)\tilde{\varphi}(p_2)F_{\theta\phi}(p_1,p_2)|,$$

where

$$\begin{aligned}F_{\theta\phi}(p_1,p_2) &= \frac{1}{\sqrt{2}}\sum_{n,m=-\lfloor\frac{M}{2}\rfloor}^{\lfloor\frac{M-1}{2}\rfloor} e^{i(np_1+mp_2)L/\hbar}\{-\cos(\delta')\\ &\quad\times[(-1)^m-(-1)^n]+i\sin(\delta')[1+(-1)^{n+m}]\}.\end{aligned}$$

Here, $\tilde{\varphi}(p)\equiv\langle p|\varphi\rangle$ and $\delta'=(\phi-\theta)/2$. Since the wave function $\varphi(q)$ was assumed sharp with respect to L, $|\tilde{\varphi}(p)|$ will only correspond to a broad envelope, over the range $(-Nh/2L,Nh/2L)$. Thus the main feature of $|\langle p_1,p_2,\Psi_{\theta\phi}\rangle|^2$ will be given by $|F_{\theta\phi}|$, as shown in Fig. 3. Graphically, it is clear that resolving at the scale $p_{\text{typ}}=\frac{h}{L}$ should be sufficient to capture the relevant features. On the screen, this translates to a typical length, $q_{\text{typ}}=\lambda D/L$, which follows from Eq. (13) and $p_z=h/\lambda$ for a photon. This is reminiscent of typical diffraction experiments and is in units that are readily measurable.

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