Analytical solution for the Klein-Gordon equation and action function of the solution for the Dirac equation in counterpropagating laser waves

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Nonperturbative calculation of QED processes involving a strong electromagnetic field, especially provided by strong laser facilities at present and in the near future, generally resorts to the Furry picture with the use of analytical solutions of particle dynamical equations, such as the Klein-Gordon equation and Dirac equation. However, only for limited field configurations such as a plane-wave field could the equations be solved analytically. Studies have shown significant interest in QED processes in a strong field composed of two counterpropagating laser waves, but the exact solution in such a field is out of reach. In this paper, inspired by the observation of the structure of the solutions in a plane-wave field, we develop a method and obtain the analytical solution for the Klein-Gordon equation and equivalently the action function of the solution for the Dirac equation in this field, under a largest dynamical parameter condition that there exists an inertial frame in which the particle free momentum is far larger than the other field dynamical parameters. The applicable range of the solution is demonstrated and its validity is proven clearly. The result has the advantages of Lorentz covariance, clear structure, and close similarity to the solution in a plane-wave field, and thus favors convenient application.

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I. INTRODUCTION

QED is the successful theory to describe the interaction between particles and photons. Conventionally the calculation is carried out in a perturbative manner since the interaction is characterized by the fine-structure constant $\ll 1$. However, in an intense laser field with A_μ being the four potential, the laser-particle interaction is characterized by the classical nonlinearity parameter $\xi = \frac{\sqrt{-\langle e^2 A^2\rangle}}{m_e}$, where e is the electron charge, m_e is the electron mass, and $\langle \cdots \rangle$ represents time averaging [1]. If $\xi \gtrsim 1$ the interaction is in the nonperturbative multiphoton regime. For example, the SLAC E-144 experiment with $\xi \approx 0.3$ found an electron-positron-pair-production rate scaling of $R \sim \xi^{10}$ [2] and in the perturbative theory this would be interpreted as the typical ξ^{2N_0} dependence with $N_0 = 5$ photon absorption. However, the calculation [3] reveals that on average more than 6 photons are absorbed in the process, and thus demonstrates the onset of nonperturbative effects on the experiment. It is also found by calculation that if the intensity of the laser field in the above experiment is enhanced such that $\xi \sim 1$, photon orders up to $N \approx 50$ give significant contributions to the total rate and thus the process enters the fully nonperturbative regime. Therefore, special techniques are required to tackle such strong-field nonperturbative problems in order to take the effects of the laser field properly into account.

The general approach is to employ the Furry picture [4], where the laser field is treated as a classical background field and the particle state is represented by the exact solution of the particle dynamical equation in the laser field, the so-called laser-dressed state. For the ideal situation of the laser field being a plane wave or a constant-crossed field (it can be viewed as a plane wave with an infinitely long wavelength), analytical solutions of the Klein-Gordon equation for a neutral scalar particle or $\pi^\pm$ meson and the Dirac equation for a fermion have been obtained exactly [5–9], which are generally named Volkov states after the author of Ref. [5]. By using such laser-dressed state the particle-laser interaction has been taken into account to all orders, and the remaining interaction between the laser-dressed particle and the QED vacuum is weak, thus allowing calculations in a perturbative scheme similar to the conventional QED. Various strong-field QED processes in a plane-wave field (or an approximated plane-wave field with the spot radius of the laser beam being much larger than the laser central wavelength [10]) have been investigated by this method, including laser-assisted bremsstrahlung [11], multiphoton Compton scattering [7,12,13], electron-positron pair production [3,14,15], and so on.

For an arbitrary non-plane-wave field, exact analytical solutions for the above equations are generally out of reach. However, several laser fields of particular interest in study are of this kind, e.g., the field composed of two counterpropagating laser waves. The colliding wave configuration can provide higher field intensity and the electron classical trajectory is distinctly different compared to that in the plane-wave field. Extensive studies suggest that this kind of laser field is efficient in producing electron-positron pairs and supporting avalanche pair production, and thus is ideal for vacuum cascade observation [16–20]. Although it seems to be the simplest case in the category of non-plane-wave fields, analytical solutions of the Klein-Gordon equation or the Dirac equation have not been obtained in this field yet. It is interesting to note that already in Ref. [5] the analytical solutions were constructed for a complex plane-wave field with component waves of arbitrary polarizations and frequencies. However, if one tries to generalize the method in Ref. [5] to the counterpropagating wave case, one would find extra terms hard to deal with which are proportional to the nonvanishing dot product of the two wave vectors and are thus nothing else but the very manifestation of the fundamental feature of non-plane-wave fields.

Without the analytical Volkov solution of Dirac equation in this field, the pair production rate has been calculated under the

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important quasistationarity approximation [19]: the reaction rate for each pair production is calculated with the value of the external field fixed at a given time, and finally these rates are time-averaged over the total interaction time. This approximation is justified if $\xi \gg 1$ since the formation length of a QED process in a plane-wave field is ξ times smaller than the laser wavelength [1,21]. In Ref. [22] a complete QED calculation based on the Volkov solution in a plane-wave field is carried out for the electron-positron pair production in the impact of a 50 MeV electron with two colliding 10 keV x-ray laser beams each of the intensity 10^{20} W/cm^2. This treatment takes the advantage that $\xi \ll 1$ though the absolute intensity is high. Although it is by this measure not a strong-field problem, novel features in the pair production process compared to the plane-wave case are identified. Therefore, it is meaningful to investigate analytical solutions for the particle dynamical equations in non-plane-wave fields to calculate more accurate strong-field QED reaction rates in these fields, especially in the regime with intermediate values of ξ. The recent breakthrough in solving the Dirac equation in propagating laser waves is worked out by Di Piazza in Refs. [23,24]. Applying the WKB method and looking for a solution of the form $\psi(x) = \exp[iS(x)/\hbar]\varphi(x)$, the action function $S(x)$ is derived first by solving the classical electron dynamical equation in the background electromagnetic field and then the bi-spinor $\varphi(x)$ is constructed via the method of characteristics. In this way the electron wave functions in the presence of a background electromagnetic field of a general space-time structure are constructed in particular inertial frames where the initial energy of the electron is the largest dynamical energy scale. In Ref. [24] it is argued that even in view of the future strong laser facilities, such as the Extreme Light Infrastructure (ELI) [25] and the Exawatt Center for Extreme Light Studies (XCELS) [26], it is still necessary to employ ultrarelativistic electrons in experimental studies on strong-field QED problems in the quantum nonlinearity regime where not only $\xi \gtrsim 1$ but also the quantum nonlinearity parameter $\chi = \xi \frac{\omega_b}{m_e} \gtrsim 1$ with ω_b being the photon energy in the rest frame of the electron.

In this paper, we develop a method and analytically solve the Klein-Gordon equation in a background electromagnetic field composed of two counterpropagating laser waves in a Lorentz-covariant manner. The method is inspired by the observation that if the coefficients of the Fourier expansion of the solution in a plane-wave field to different photon modes are written in the form of a Bessel function, all parameters can be determined by simple rules, as demonstrated in Sec. II. For the solution in the non-plane-wave field under study, the coefficients of the Fourier expansion of the solution to different modes of the two kinds of photons are written as a multiplication of Bessel functions, and the parameters are determined by rules in analogy or as a development to the simple rules found in the plane-wave case. The applicable range is obtained by examining the validity of approximations used in derivation steps, and a largest dynamical parameter condition is imposed, that there exists an inertial frame in which the particle asymptotic momentum is far larger than the other field dynamical parameters. We can have in mind the case of an energetic particle obliquely impacting on two counterpropagating optical or x-ray intense laser waves in the laboratory frame, while the calculation can be conducted in arbitrary boosted frames. This is the content of Sec. III. By solving the Klein-Gordon equation, the action function $S(x)$ of the solution of the Dirac equation is obtained, because if the solution $\psi = \exp[iS'(x)/\hbar]$ satisfies the Klein-Gordon equation

$$[(\hat{p}_\mu - eA_\mu)^2 - m_e^2]\psi = 0, \tag{1}$$

with $\hat{p}_\mu = i\hbar\partial_\mu$, then except for a term proportional to $\hbar$ it results in

$$(\partial_\mu S' + eA_\mu)(\partial^\mu S' + eA^\mu) - m_e^2 = 0, \tag{2}$$

which is just the equation determining the action function $S(x)$ for Dirac equation [23,24].

In Sec. IV the newly obtained solution is justified by showing that solutions in a plane-wave field can be recovered naturally from it. In Sec. V the error induced by this solution to calculations of scattering matrix terms is discussed. Besides, the solution is simplified and takes a form in close similarity to the solution in a plane-wave field. Finally it is validated by substituting the action function into the basic equation (2) and proving the consistency.

In the following the natural units $c = \hbar = 1$ are used unless claimed otherwise. In this context the electron charge e should not be confused with the exponential constant in exponential expressions.

II. RECONSTRUCTION OF THE SOLUTION FOR THE KLEIN-GORDON EQUATION IN A CIRCULARLY POLARIZED PLANE WAVE

We consider the solution for the Klein-Gordon equation (1) in a circularly polarized plane wave $A = a_1\cos(k\cdot x) + a_2\sin(k\cdot x)$, where a_1 and a_2 are the field vector potential components in two orthogonal directions, k is the wave-vector of the field, and $\cdot$ denotes the four-vector multiplication. Up to a normalization constant the Fourier expansion of the solution to different photon modes takes the form

$$\psi = e^{\pm iq_\pm\cdot x}\sum_{n=-\infty}^{\infty} c_n J_n(\alpha)e^{in\beta}e^{ink\cdot x}, \tag{3}$$

where the dressed momentum is $q_\pm = p_\pm + \frac{m_e^2\xi^2}{2k\cdot p_\pm}k$ with $p_\pm$ being the electron's (with "−" subscript) or positron's (with "+" subscript) asymptotic momentum outside the electromagnetic field, and J_n is the Bessel function. Our question is, if we only know the form of the solution as given in Eq. (3), how do we determine the values of c_n, α, and β so that ψ can satisfy Eq. (1)? The answer seems to be trivial, but useful information can be drawn from the approach to actually obtain these parameters.

Substituting Eq. (3) into Eq. (1), we obtain that

$$\sum_{n=-\infty}^{\infty} c_n J_n(\alpha)e^{in\beta}[2nk\cdot p_\pm + ep_\pm\cdot(a_1 - ia_2)e^{ik\cdot x} + ep_\pm\cdot(a_1 + ia_2)e^{-ik\cdot x}]e^{ink\cdot x} = 0. \tag{4}$$

This results in the sequence of equations related to different photon modes

$$c_n J_n(\alpha)2nk\cdot p_\pm + c_{n-1}J_{n-1}(\alpha)e^{-i\beta}ep_\pm\cdot(a_1 - ia_2) + c_{n+1}J_{n+1}(\alpha)e^{i\beta}ep_\pm\cdot(a_1 + ia_2) = 0. \tag{5}$$

Compared to the identity of the Bessel function

$$\frac{\alpha}{2n}[J_{n-1}(\alpha)+J_{n+1}(\alpha)]=J_n(\alpha), \tag{6}$$

it can be found that in Eq. (5) if we let

$$c_n=1 \tag{7}$$

and

$$\mathrm{Im}[e^{-i\beta}ep_\pm\cdot(a_1-ia_2)]=0, \tag{8}$$

the equation becomes

$$\frac{-\mathrm{Re}[e^{-i\beta}ep_\pm\cdot(a_1-ia_2)]}{2nk\cdot p_\pm}[J_{n-1}(\alpha)+J_{n+1}(\alpha)]=J_n(\alpha), \tag{9}$$

and thus

$$\alpha=-\frac{\mathrm{Re}[e^{-i\beta}ep_\pm\cdot(a_1-ia_2)]}{k\cdot p_\pm}. \tag{10}$$

From Eqs. (8) and (10) α and β can be written in the familiar form

$$\cos\beta=\frac{p_\pm\cdot a_1}{\sqrt{(p_\pm\cdot a_1)^2+(p_\pm\cdot a_2)^2}}, \tag{11}$$

$$\sin\beta=-\frac{p_\pm\cdot a_2}{\sqrt{(p_\pm\cdot a_1)^2+(p_\pm\cdot a_2)^2}}, \tag{12}$$

$$\alpha=-\frac{e\sqrt{(p_\pm\cdot a_1)^2+(p_\pm\cdot a_2)^2}}{k\cdot p_\pm}. \tag{13}$$

Another solution is got by changing the sign of the above quantities, corresponding to $\beta\to\beta+\pi$ and $\alpha\to-\alpha$, but this leads to the same wave function (3) since $e^{in(\beta+\pi)}J_n(-\alpha)=(-1)^{2n}e^{in\beta}J_n(\alpha)$.

In this section the coefficients in the Fourier expansion terms of the solution for the Klein-Gordon equation are acquired in an empirical way, in which the identity (6) plays a crucial role. In the following it is shown that the experiences and observations gained here can be used to tackle nontrivial problems.

III. CONSTRUCTION OF THE SOLUTION FOR THE KLEIN-GORDON EQUATION IN COUNTERPROPAGATING WAVES

Our aim is to find the solution for the Klein-Gordon equation and equivalently the action function for the Dirac equation in a non-plane-wave field. Consider

$$\left[(\hat{p}-eA-eA')^2-m_e^2\right]\psi=0, \tag{14}$$

where $A=a[\epsilon_1\cos(k\cdot x)+\epsilon_2\sin(k\cdot x)]$ and $A'=a'[\epsilon_1'\cos(k'\cdot x)+\epsilon_2'\sin(k'\cdot x)]$ respectively represent a circularly polarized plane wave with the non-plane-wave condition $k\cdot k'\neq 0$. As a special case, consider that the two waves counterpropagate with each other. Without loss of generality, suppose $k=(\omega,0,0,k_z)$ and $k'=(\omega',0,0,k_z')$, where k_z is positive and k_z' is negative, and let $\epsilon_1=\epsilon_1'=(0,1,0,0)$ and $\epsilon_2=\epsilon_2'=(0,0,1,0)$.

Based on the observation of the structure of the solution (3) in a plane-wave field, we assume the Fourier expansion of the solution here takes the form

$$\psi=e^{\pm ip_\pm'\cdot x}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}C_{nm}e^{ink\cdot x}e^{imk'\cdot x}, \tag{15}$$

where

$$C_{nm}=c_{nm}J_n(\alpha_m)e^{in\beta}J_m(\alpha_n')e^{im\beta'}, \tag{16}$$

and the formal dressed momentum is written as

$$p_\pm'=p_\pm+\frac{e^2a^2}{2k\cdot p_\pm}k+\frac{e^2a'^2}{2k'\cdot p_\pm}k'. \tag{17}$$

The arguments of the Bessel functions are assumed to be index dependent, the reason of which will be shown later. The quantity $p_\pm'$ is not the physical dressed momentum, which can only be determined after the acquisition of C_{nm}. The solution (15) would be obtained by determining c_{nm}, α_m, α_n', β, and β'.

Substituting Eq. (15) into Eq. (14), a sequence of equations related to different photon modes can be derived. Considering the possibility of c_{nm} to be a function of the time-space coordinates, we get

$$\begin{aligned}&c_{nm}J_n(\alpha_m)J_m(\alpha_n')(r_0+r_1n+r_2m+r_3nm)\\&\quad+c_{n-1,m}J_{n-1}(\alpha_m)J_m(\alpha_{n-1}')e^{-i\beta}b\\&\quad+c_{n+1,m}J_{n+1}(\alpha_m)J_m(\alpha_{n+1}')e^{i\beta}b^*\\&\quad+c_{n,m-1}J_n(\alpha_{m-1})J_{m-1}(\alpha_n')e^{-i\beta'}d\\&\quad+c_{n,m+1}J_n(\alpha_{m+1})J_{m+1}(\alpha_n')e^{i\beta'}d^*=0,\end{aligned} \tag{18}$$

with

$$\begin{aligned}r_0=&\frac{e^4a^2a'^2k\cdot k'}{2(k\cdot p_\pm)(k'\cdot p_\pm)}+2e^2A\cdot A'\\&+\frac{(\hat{p}^2-2eA\cdot\hat{p}-2eA'\cdot\hat{p})c_{nm}}{c_{nm}}-\frac{2}{c_{nm}}(\hat{p}_\mu c_{nm})(\pm p_\pm'^\mu),\end{aligned} \tag{19}$$

$$r_1=\pm 2k\cdot p_\pm\pm\frac{e^2a'^2}{k'\cdot p_\pm}k\cdot k'-\frac{2}{c_{nm}}(\hat{p}_\mu c_{nm})k^\mu, \tag{20}$$

$$r_2=\pm 2k'\cdot p_\pm\pm\frac{e^2a^2}{k\cdot p_\pm}k\cdot k'-\frac{2}{c_{nm}}(\hat{p}_\mu c_{nm})k'^\mu, \tag{21}$$

$$r_3=2k\cdot k', \tag{22}$$

$$b=\pm eap_\pm\cdot(\epsilon_1-i\epsilon_2), \tag{23}$$

$$d=\pm ea'p_\pm\cdot(\epsilon_1'-i\epsilon_2'). \tag{24}$$

It is found that the empirical method (7)–(10) based on the identity (6) can be applied. Let

$$r_0=0, \tag{25}$$

$$\mathrm{Im}(e^{-i\beta}b)=\mathrm{Im}(e^{-i\beta'}d)=0. \tag{26}$$

Moreover, assuming c_{nm} is index independent and using the relation

$$\begin{aligned}J_n(\alpha)J_m(\alpha')nm=&\frac{\alpha}{4}[J_{n+1}(\alpha)+J_{n-1}(\alpha)]J_m(\alpha')m\\&+\frac{\alpha'}{4}[J_{m+1}(\alpha')+J_{m-1}(\alpha')]J_n(\alpha)n,\end{aligned} \tag{27}$$

Eq. (18) can be written as

$$\begin{aligned}0 = {} & J_n(\alpha_m)J_m(\alpha'_n)r_1 n\\ & + \frac{r_3 m\alpha_m}{4}[J_{n-1}(\alpha_m)+J_{n+1}(\alpha_m)]J_m(\alpha'_n)\\ & + \mathrm{Re}(e^{-i\beta}b)[J_{n-1}(\alpha_m)J_m(\alpha'_{n-1})+J_{n+1}(\alpha_m)J_m(\alpha'_{n+1})]\\ & + J_n(\alpha_m)J_m(\alpha'_n)r_2 m\\ & + \frac{r_3 n\alpha'_n}{4}[J_{m-1}(\alpha'_n)+J_{m+1}(\alpha'_n)]J_n(\alpha_m)\\ & + \mathrm{Re}(e^{-i\beta'}d)[J_{m-1}(\alpha'_n)J_n(\alpha_{m-1})+J_{m+1}(\alpha'_n)J_n(\alpha_{m+1})].\end{aligned} \tag{28}$$

If we take the assumption that

$$\alpha_{m\pm1} \approx \alpha_m \quad \text{and} \quad \alpha'_{n\pm1} \approx \alpha'_n, \tag{29}$$

Eq. (28) can be considerably simplified and reduced into two equations

$$\begin{aligned}0 = {} & J_n(\alpha_m)r_1 n + \left[\frac{r_3 m\alpha_m}{4} + \mathrm{Re}(e^{-i\beta}b)\right]\\ & \times [J_{n-1}(\alpha_m)+J_{n+1}(\alpha_m)],\end{aligned} \tag{30}$$

$$\begin{aligned}0 = {} & J_m(\alpha'_n)r_2 m + \left[\frac{r_3 n\alpha'_n}{4} + \mathrm{Re}(e^{-i\beta'}d)\right]\\ & \times [J_{m-1}(\alpha'_n)+J_{m+1}(\alpha'_n)],\end{aligned} \tag{31}$$

and thus it is straightforward to use the identity (6) to obtain

$$\alpha_m = -\frac{4\mathrm{Re}(e^{-i\beta}b)}{2r_1 + r_3 m}, \tag{32}$$

$$\alpha'_n = -\frac{4\mathrm{Re}(e^{-i\beta}d)}{2r_2 + r_3 n}. \tag{33}$$

This result shows explicitly how the arguments of the Bessel functions depend on the indices and automatically explains the assumption in Eq. (16).

As a brief summary, the coefficients c_{nm}, β, β', α_m, and α'_n needed to determine the solution (15) can be obtained respectively from Eqs. (25), (26), (32), and (33). The remaining task is to solve Eq. (25) for the explicit form of c_{nm} and finally check the validity of the assumption (29).

In the derivation for solving Eq. (25), we keep $\hbar$ explicitly for the benefit of indicating the perturbation orders. Presume the solution takes the form

$$c_{nm} = \eta e^{-\frac{i}{\hbar}[f(k\cdot x)+g(k'\cdot x)+q(k_d\cdot x)]}, \tag{34}$$

where $k_d = k - k'$ and η is a constant; f, g, and q are defined as real functions with the argument $k\cdot x$, $k'\cdot x$, and $k_d\cdot x$, respectively. Thus, there is $\hat{p}_\mu c_{nm} = i\hbar\partial_\mu c_{nm} = c_{nm}[k_\mu f' + k'_\mu g' + (k_\mu - k'_\mu)q']$ with f', k', and q' being the total differential of the corresponding functions with respect to the variables $k\cdot x$, $k'\cdot x$, and $k_d\cdot x$, respectively. Substituting this expression into Eq. (25), it can be derived that

$$\begin{aligned}& k\cdot k'[2f'g' + 2(g'-f')q' - 2q'^2 - 2i\hbar q'' + \varrho_1 \mp 2\varrho_2 g'\\ & \mp 2\varrho_3 f' \mp 2q'(\varrho_3 - \varrho_2)] - 2e^2 aa'\cos(k_d\cdot x)\\ & \mp 2(f'k + g'k' + q'k_d)\cdot p_\pm = 0,\end{aligned} \tag{35}$$

with $\varrho_1 = 2\varrho_2\varrho_3$, $\varrho_2 = e^2a^2/(2k\cdot p_\pm)$, and $\varrho_3 = e^2a'^2/(2k'\cdot p_\pm)$. Equation (35) can be simplified by taking $f' = \pm\varrho_2$ and $g' = \pm\varrho_3$ for corresponding $p_\pm$, such that

$$\begin{aligned}& k\cdot k'(-2q'^2 - 2i\hbar q'') - 2e^2aa'\cos(k_d\cdot x)\\ & - 2(\varrho_2 k + \varrho_3 k' \pm q'k_d)\cdot p_\pm = 0.\end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned}& \pm q'k_d\cdot p_\pm + e^2aa'\cos(k_d\cdot x) + \tfrac{1}{2}e^2(a^2 + a'^2)\\ & = -(q'^2 + i\hbar q'')k\cdot k'.\end{aligned} \tag{37}$$

According to Eq. (37) if $k_d\cdot p_\pm = 0$, for example in a standing-wave case $k'_z = -k_z$ and $a = a'$ with the particle beam shooting perpendicular to the $\hat{z}$ direction, the function

$$y(\phi) = e^{-\frac{i}{\hbar}q(2\phi)}, \tag{38}$$

with $\phi = k_d\cdot x/2$, satisfies the Mathieu differential equation

$$\frac{d^2y}{d\phi^2} + [c_1 - 2c_2\cos 2\phi]y = 0, \tag{39}$$

with

$$c_1 = -\frac{2e^2a^2}{\hbar^2\omega^2},\ c_2 = \frac{e^2a^2}{\hbar^2\omega^2}. \tag{40}$$

A Mathieu equation is also found in solving the Klein-Gordon equation in a rotating electric field [27]. The formal resemblance is reasonable since a particle located in the vicinity of an antinode of the standing wave experiences a rotating electric field.

In the following we consider an oblique incidence of the particle into the laser field and suppose the condition

$$|\lambda| = \left|\frac{k\cdot k'}{\pm k_d\cdot p_\pm}\right| \ll 1. \tag{41}$$

Then Eq. (37) can be solved by the perturbative method and the solution is obtained as a sum of terms proportional to different orders of λ, such that

$$q = q_0 + \sum_{n=1}^{\infty}\lambda^n q_n, \tag{42}$$

where the zeroth-order solution takes a simple form

$$q_0 = \mp\frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm} \mp \frac{(\varrho_2 k + \varrho_3 k')\cdot p_\pm}{k_d\cdot p_\pm}k_d\cdot x. \tag{43}$$

Therefore, the solution c_{nm} reads

$$c_{nm} = \eta e^{\pm\frac{i}{\hbar}[-\varrho_2 k\cdot x - \varrho_3 k'\cdot x + \frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm} + \frac{e^2(a^2+a'^2)}{2k_d\cdot p_\pm}k_d\cdot x + O(\lambda)]}. \tag{44}$$

Note that

$$e^{\pm ip'_\pm\cdot x}c_{nm} = \eta e^{\pm i(p_\pm + \frac{e^2a^2+e^2a'^2}{2k_d\cdot p_\pm}k_d)\cdot x}e^{\pm i[\frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm} + O(\lambda)]}, \tag{45}$$

where the natural units $c = \hbar = 1$ are used again.

The condition (41) is satisfied in many scenarios. This means that there exists at least one inertial frame; here for the sake of convenient illustration we assume it to be the laboratory frame, in which the particle asymptotic energy satisfies $\varepsilon \gg \omega(\omega')$ and we further require that $\varepsilon \gg m_e\xi(m_e\xi')$ with $\xi = \frac{ea}{m_e}$ and $\xi' = \frac{ea'}{m_e}$. This is referred to as the largest dynamical parameter condition in the paper similarly to that in Ref. [23]. Besides this energy requirement, the geometry of the setup of the particle-laser system can become significant in special cases, which will be discussed later. Then it can be calculated that

$$r_1 = \pm 2k \cdot p_\pm \left(1 \pm q' \frac{k \cdot k'}{k \cdot p_\pm}\right) \approx \pm 2k \cdot p_\pm, \tag{46}$$

$$r_2 = \pm 2k' \cdot p_\pm \left(1 \mp q' \frac{k \cdot k'}{k' \cdot p_\pm}\right) \approx \pm 2k' \cdot p_\pm, \tag{47}$$

since the terms $q'\frac{k\cdot k'}{k\cdot p_\pm}, q'\frac{k\cdot k'}{k'\cdot p_\pm} \sim \frac{m_e^2(\xi^2+\xi'^2+\xi\xi')}{\varepsilon^2}$. As stated earlier, the approximation presented in Eq. (29) needs to be checked. It is validated by

$$|\alpha_{m\pm1} - \alpha_m| \approx \frac{m_e \xi \omega' \sin\theta_p}{\varepsilon\omega(1-\cos\theta_p)^2} \sim \frac{m_e\xi}{\varepsilon} \ll 1, \tag{48}$$

$$|\alpha'_{n\pm1} - \alpha'_n| \approx \frac{m_e \xi' \omega \sin\theta_p}{\varepsilon\omega'(1+\cos\theta_p)^2} \sim \frac{m_e\xi'}{\varepsilon} \ll 1, \tag{49}$$

where for illustration simplicity $\omega \approx \omega'$ is assumed and θ_p is the azimuth angle of the particle's asymptotic momentum to the $\hat{z}$ axis.

Therefore, the solution for the Klein-Gordon equation (14) is obtained, which reads

$$\psi = \eta e^{\pm i(p_\pm + \frac{e^2a^2+e^2a'^2}{2k_d\cdot p_\pm}k_d)\cdot x} e^{\pm i[\frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm} + O(\lambda)]} \times \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\alpha_m)e^{in\beta}J_m(\alpha'_n)e^{im\beta'}e^{ink\cdot x}e^{imk'\cdot x}, \tag{50}$$

where

$$\cos\beta = \cos\beta' = \frac{p_\pm\cdot\epsilon_1}{\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}, \tag{51}$$

$$\sin\beta = \sin\beta' = -\frac{p_\pm\cdot\epsilon_2}{\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}, \tag{52}$$

$$\alpha_m = -\frac{ea\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k\cdot p_\pm \pm \frac{m}{2}k\cdot k'}, \tag{53}$$

$$\alpha'_n = -\frac{ea'\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k'\cdot p_\pm \pm \frac{n}{2}k\cdot k'}, \tag{54}$$

and the constant η can be acquired by the normalization of the wave function.

Finally the applicable range of the solution (50)–(54) is addressed. First, let us scrutinize the steps (41), (46), (47), (48), and (49) which have taken approximations regarding four-vector multiplications. As mentioned previously, the validity of these approximations relies not only on the largest dynamical parameter condition but also on the geometric relations among the constituents' momenta. By checking the denominators of the expressions in these five equations, specifically speaking $|k_d\cdot p_\pm|$, $|k\cdot p_\pm|$, and $|k'\cdot p_\pm|$, the conclusion is that to justify these steps the angle θ_p should satisfy

$$\cos\theta_p \neq \frac{\omega-\omega'}{\omega+\omega'} \tag{55}$$

and

$$\cos\theta_p \neq \pm 1. \tag{56}$$

Second, since the magnitudes of α_m and α'_n scale as $m_e\xi/\omega$ and $m_e\xi'/\omega'$, respectively, and thus can be large, the usage of Eqs. (46) and (47) in Eqs. (53) and (54) can only ensure that the neglected part is relatively much smaller than the remaining one. To guarantee the absolute magnitude of the neglected part to be $\ll 1$, extra constraints on the parameters are taken, such that

$$\left|q'\frac{k\cdot k'}{k\cdot p_\pm}\right|\alpha_m \ll 1, \quad \left|q'\frac{k\cdot k'}{k'\cdot p_\pm}\right|\alpha'_n \ll 1. \tag{57}$$

These leads to the constraint on the field intensity

$$\xi \ll \frac{(\varepsilon^2\omega)^{\frac{1}{3}}}{m_e}, \quad \xi' \ll \frac{(\varepsilon^2\omega')^{\frac{1}{3}}}{m_e}. \tag{58}$$

As an example, consider the two waves coming from the tunable Ti : Sa lasers with the photon energy $\omega = \omega' \sim 2$ eV and the particle being an electron accelerated by the laser-plasma wake field to the energy 10 GeV [28] with $\theta_p = \pi/4$. Take $\xi \sim \xi' \sim 1$ which corresponds to the laser intensity about 10^{19} W/cm^2. Then it can be calculated that $|\lambda| \sim 10^{-10}$, $|q'\frac{k\cdot k'}{k\cdot p_\pm}| \sim |q'\frac{k\cdot k'}{k'\cdot p_\pm}| \sim 10^{-8}$, $|\alpha_{m\pm1}-\alpha_m| \sim |\alpha'_{n\pm1}-\alpha'_n| \sim 10^{-4}$, and $|q'\frac{k\cdot k'}{k\cdot p_\pm}|\alpha_m \sim |q'\frac{k\cdot k'}{k'\cdot p_\pm}|\alpha'_n \sim 10^{-2}$, and thus it is in the applicable range of the solution (50)–(54). Consider, as another example, the same particle condition but the waves are 1 keV x-ray lasers with $\xi \sim \xi' \sim 1$ corresponding to the laser intensity about 10^{25} W/cm^2, which is even higher than that available by the present technology, it can be found that $|\lambda| \sim 10^{-7}$, $|q'\frac{k\cdot k'}{k\cdot p_\pm}| \sim |q'\frac{k\cdot k'}{k'\cdot p_\pm}| \sim 10^{-8}$, $|\alpha_{m\pm1}-\alpha_m| \sim |\alpha'_{n\pm1}-\alpha'_n| \sim 10^{-4}$, and $|q'\frac{k\cdot k'}{k\cdot p_\pm}|\alpha_m \sim |q'\frac{k\cdot k'}{k'\cdot p_\pm}|\alpha'_n \sim 10^{-5}$. Therefore it is also in the applicable range of the solution (50)–(54).

IV. RECOVERING SOLUTIONS IN A PLANE-WAVE FIELD

By taking $A' = 0$, it is found that the solution (50)–(54) is reduced to the familiar solution of the Klein-Gordon equation in a plane-wave field as given in Eqs. (3)–(13).

It is more interesting to note that by taking k' in the same direction as k and thus denoting $k' = rk$ with $r = \omega'/\omega$, the solution (50)–(54) becomes

$$\psi = \eta e^{\pm i(p_\pm + \frac{e^2a^2}{2k\cdot p_\pm}k + \frac{e^2a'^2}{2k'\cdot p_\pm}k')\cdot x} e^{\pm i\frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm}} \times \sum_{n=-\infty}^{\infty} J_n(\alpha)e^{in\beta}e^{ink\cdot x}\sum_{m=-\infty}^{\infty} J_m(\alpha')e^{im\beta'}e^{imk'\cdot x}, \tag{59}$$

where

$$\cos\beta = \cos\beta' = \frac{p_\pm\cdot\epsilon_1}{\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}, \tag{60}$$

$$\sin\beta = \sin\beta' = -\frac{p_\pm\cdot\epsilon_2}{\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}, \tag{61}$$

$$\alpha = -\frac{ea\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k\cdot p_\pm}, \tag{62}$$

$$\alpha' = -\frac{ea'\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k'\cdot p_\pm}. \tag{63}$$

Therefore

$$\psi = \eta e^{\pm i p_\pm \cdot x - i\int_{-\infty}^{y} dy' \frac{1}{2k\cdot p_\pm}[2ep_\pm \cdot A_t(y') \pm e^2 A_t^2(y')]} \tag{64}$$

with $y = k \cdot x$ and the total field potential $A_t(y) = A(y) + A'(ry)$, and thus the solution of the Klein-Gordon equation in a two-color plane-wave field is recovered.

V. VALIDATION AND SIMPLIFICATION OF THE SOLUTION

In the derivation for the solution, we have solved Eq. (25) and obtained the arguments of the Bessel functions in an approximate manner. The errors they induce to the result can be well estimated. In the solution (50), there is an error proportional to $\lambda \sim \omega/\varepsilon$ in the phase which can be diminished by calculating higher order terms in Eq. (42). The error caused by the approximated arguments of the Bessel functions to the coefficients of each photon mode is proportional to $m_e\xi/\varepsilon \ll 1$ or $m_e\xi'/\varepsilon \ll 1$. Therefore the error of a scattering matrix term computed by using this wave function (50)–(54) can be estimated accordingly.

Let us consider the spectral width of the solution. The asymptotic formula of the Bessel function for $n \gg \alpha > 0$ gives

$$J_n(\alpha) \approx \frac{1}{\sqrt{2\pi n}}\left(\frac{e\alpha}{2n}\right)^n. \tag{65}$$

Thus the cutoff takes place at $n = \alpha$ and $J_n(\alpha)$ drops sharply as n increases beyond α [29]. For $J_n(\alpha)J_m(\alpha')$ with

$$\alpha = \frac{ea\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k\cdot p_\pm}, \tag{66}$$

$$\alpha' = \frac{ea'\sqrt{(p_\pm\cdot\epsilon_1)^2 + (p_\pm\cdot\epsilon_2)^2}}{k'\cdot p_\pm}, \tag{67}$$

the cutoff indices are $n_c = \pm\alpha$ and $m_c = \pm\alpha'$. These turn out to be approximately also the cutoff indices of the term $J_n(\alpha_m)J_m(\alpha'_n)$ in Eq. (50) since there are $|\frac{n_c k\cdot k'}{k\cdot p_\pm}| \sim \frac{m_e\xi}{\varepsilon} \ll 1$ and $\frac{m_c k\cdot k'}{k'\cdot p_\pm} \sim \frac{m_e\xi'}{\varepsilon} \ll 1$.

To estimate how well the approximated solution (50)–(54) satisfies the Klein-Gordon equation (14), let us calculate the expectation value of the left-hand side of Eq. (14) with Dirac bra-ket notation

$$\delta\varepsilon^2 = \langle\psi|(\hat{p} - eA - eA')^2 - m_e^2|\psi\rangle. \tag{68}$$

This is to compute the temporal-spatial integration of the multiplication of the conjugate of the solution (50) and the right-hand side of Eq. (28). It is reasonable to compare this residual energy with the characteristic energy of the particle, for example ε^2 in the laboratory frame or Lorentz invariants such as $k\cdot p_\pm$. Without going into the details and focusing only on the energy scales, we get

$$\frac{\delta\varepsilon^2}{\varepsilon^2} \sim \frac{m_e^2\xi\xi' n_c m_c}{\varepsilon^2} \sim \frac{(m_e^2\xi\xi')^2}{\omega\omega'\varepsilon^2}. \tag{69}$$

It is tempting to write the solution (50) in a more concise form via using the identity

$$e^{ix\sin\theta} = \sum_{-\infty}^{\infty} J_n(x)e^{in\theta}, \tag{70}$$

like what has been done to the solution (59) and (64). But due to the index-dependent arguments α_m and α'_n of the Bessel functions in the solution (50), not only the direct application of this identity is incorrect, but also the sum there cannot be decomposed as a product of two sums like in Eq. (59). This illustrates the complex manner of the particle coupling with the non-plane-wave field composed of two counterpropagating laser waves.

However, if at the cutoff there is

$$|\alpha_{m_c} - \alpha| \sim \left|\alpha\alpha'\frac{\omega'}{\varepsilon}\right| \sim \frac{m_e^2\xi\xi'}{\omega\varepsilon} \ll 1, \tag{71}$$

$$|\alpha'_{n_c} - \alpha'| \sim \left|\alpha\alpha'\frac{\omega}{\varepsilon}\right| \sim \frac{m_e^2\xi\xi'}{\omega'\varepsilon} \ll 1, \tag{72}$$

which compared to Eq. (69) just means the energy scale of $\delta\varepsilon/\varepsilon$ is small, then the following approximation is reasonable:

$$\alpha_m \approx \alpha, \quad \alpha'_n \approx \alpha'. \tag{73}$$

Therefore the solution is simplified as

$$\psi = e^{iS}, \tag{74}$$

where the action function reads

$$\begin{aligned} S = &\pm\left(p_\pm + \frac{e^2a^2 + e^2a'^2}{2k_d\cdot p_\pm}k_d\right)\cdot x \pm \frac{e^2aa'\sin(k_d\cdot x)}{k_d\cdot p_\pm} \\ &- \left[\frac{eap_\pm\cdot\epsilon_1}{k\cdot p_\pm}\sin(k\cdot x) - \frac{eap_\pm\cdot\epsilon_2}{k\cdot p_\pm}\cos(k\cdot x)\right] \\ &- \left[\frac{ea'p_\pm\cdot\epsilon_1}{k'\cdot p_\pm}\sin(k'\cdot x) - \frac{ea'p_\pm\cdot\epsilon_2}{k'\cdot p_\pm}\cos(k'\cdot x)\right], \end{aligned} \tag{75}$$

or in an integral form

$$\begin{aligned} S = &\pm\left(p_\pm + \frac{e^2a^2 + e^2a'^2}{2k_d\cdot p_\pm}k_d\right)\cdot x \pm \int_{-\infty}^{y} dy'\frac{1}{k_d\cdot p_\pm}e^2F(y') \\ &- \int_{-\infty}^{y} dy'\frac{1}{k\cdot p_\pm}ep_\pm\cdot A(y') \\ &- \int_{-\infty}^{y} dy'\frac{1}{k'\cdot p_\pm}ep_\pm\cdot A'(y'), \end{aligned} \tag{76}$$

with $F = -A\cdot A'$ and $y = k_d\cdot x$. In analogy to the plane-wave solution, the dressed momentum of the particle in this non-plane-wave field can be identified as

$$q = p_\pm + \frac{e^2a^2 + e^2a'^2}{2k_d\cdot p_\pm}k_d, \tag{77}$$

and accordingly the dressed mass is

$$m_* = \sqrt{q^2} \approx m_e\sqrt{1 + \xi^2 + \xi'^2}. \tag{78}$$

The validity of action (75) can be checked by substituting it into Eq. (2). Then the left-hand side of the equation reads

$$\begin{aligned} &(\partial_\mu S + eA_\mu + eA'_\mu)(\partial^\mu S + eA^\mu + eA'^\mu) - m_e^2 \\ &\sim \frac{(m_e\xi)^3}{\varepsilon} + m_e^2\xi\xi', \end{aligned} \tag{79}$$

and with $\frac{m_e\xi^3}{\varepsilon} \ll 1$ it is

$$(\partial_\mu S + eA_\mu + eA'_\mu)(\partial^\mu S + eA^\mu + eA'^\mu) - m_e^2[1 - O(\xi\xi')] = 0. \tag{80}$$

Like above, the ratio of the extra term over the characteristic energy of the particle $\sim m_e^2\xi\xi'/\varepsilon^2 \ll 1$. Note also that there are meaningful strong-field problems with $\xi \sim 1$ and $\xi' \ll 1$, which results in $\xi\xi' \ll 1$.

In fact, it is possible to construct other forms of expressions which can also make Eq. (2) established up to a certain perturbation order, but this paper shows how this particular action function (75) or (76) is derived step by step and proves its reasonableness. It takes the form analogous to that of the plane-wave solution and laser-dressed physical quantities such as the dressed momentum and the dressed mass can be directly identified.

It is worth noticing that neglecting in Eq. (28) the terms multiplied by $r_3 m\alpha_m$ and $r_3 n\alpha'_n$, the arguments of the Bessel functions would be index independent and the action function (75) can be obtained. However, if the error brought to the Klein-Gordon equation is estimated based on Eq. (28), an energy scale of $\frac{m_e^3\xi^2\xi'}{\omega}$ or $\frac{m_e^3\xi\xi'^2}{\omega'}$ is found which is more severe than $m_e^2\xi\xi'$ found in Eq. (80) by direct calculation with the explicit expression of the action function. This also indicates that the amplitude of $\delta\varepsilon$ is overestimated in Eq. (69).

VI. CONCLUSIONS

The analytical solution of the Klein-Gordon equation and equivalently the action function of a Dirac particle in a non-plane-wave electromagnetic field are investigated. The method is developed based on the idea that the coefficients of the Fourier expansion terms of the solution to different photon modes mainly adopt the form of multiplications of Bessel functions.

For the field composed of two counterpropagating circularly polarized plane waves, a detailed derivation is illustrated. The coefficients are determined by the rules (25), (26), (32), and (33) as a development or direct analogy to the rules observed in the plane wave case; see Eqs. (7), (8), and (10). In order to solve Eq. (25) explicitly and justify the assumption (29), the largest dynamical parameter condition is imposed, that there exists an inertial frame in which the particle asymptotic momentum is far larger than the other field dynamical parameters. As already mentioned in Sec. I, this condition is of realistic meaning in view of strong-field experimental campaigns into the quantum nonlinearity regime.

The solution for the Klein-Gordon equation is obtained analytically; see Eqs. (50)–(54). Discussions on its applicable range and examples can be found at the end of Sec. III.

It is clearly shown in Sec. IV that the solutions for the Klein-Gordon equation in the one-color and two-color plane-wave fields can be recovered. In Sec. V the error of the solution and the error it can induce to the scattering matrix term calculation are discussed. Considering the cutoff property of the Bessel function, it is found that the solution can be simplified and the action function takes an integral form (76) similar to that in the plane-wave case. The laser-dressed momentum and mass of the particle are identified. The validity of the simplified action is justified by directly calculating the basic equation (2) that defines the action function. Comparing the simplified action (75) and Eq. (44), it can be found that the non-plane-wave feature of the coupling of the particle to the two waves is mainly determined by Eq. (25).

The solution (50)–(54) as well as the action (76) is Lorentz invariant. It has the advantage of clear structure and close similarity to the solution in a plane-wave case, and thus favors convenient application. This solution for scalar particles can be used to calculate QED processes of fermions (electrons and positrons) if the spin effect can be neglected, which indicates the laser field of optical or lower frequency [30] and detailed analysis can be found in Ref. [13]. Besides, with the solution of the Klein-Gordon equation, the solution of the corresponding Dirac equation in the same electromagnetic field can be derived, for example, by the characteristic method shown in the paper [23]. It then can provide more exact reaction rates of multiphoton pair production process, Compton scattering process, and so on in this non-plane-wave field. By comparing the results calculated from the plane-wave solution and the non-plane-wave solution, features particularly related to the non-plane-wave field can be identified and used in experimental design and explanation.

This paper indicates the special convenience of using Bessel functions in describing the dressed states of particles in electromagnetic fields. Extending this method to solve problems in other field configurations shall be investigated in the future work.

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