Intrinsic formation of electromagnetic divergence and rotation by parabolic focusing

Kazunori Shibata,[1,*] Mayuko Takai,[2] Mitsuharu Uemoto,[3] and Shinichi Watanabe[2,†]

[1]*RIKEN SPring-8 Center, Theory Team, 1-1-1 Kouto, Sayo-cho, Sayo-gun, Hyogo 679-5148, Japan*

[2]*Department of Physics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, Kanagawa 223-8522, Japan*

[3]*Department of Materials Engineering Science, Graduate School of Engineering Science, Osaka University, Machikaneyama 1-3, Toyonaka, Osaka 560-8531, Japan*

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A linearly polarized electromagnetic wave is found to form divergent and rotation fields after focused by an off-axis parabolic mirror. These distributions are generated within a subwavelength scale around the focus and at times when the electric field at the focus vanishes. We theoretically and experimentally show that not only the direction but also the structure of the distributions varies with the incident polarization. In addition, the distributions move in one direction with a phase velocity faster than the speed of light.

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Tight focusing of broadband electromagnetic (EM) waves ultimately realizes an intense and single-cycle pulse [1], which has opened a new avenue in higher-order nonlinear optical processes in condensed matter physics [2,3], molecular physics [4,5], attosecond science [6,7], and the vacuum decay and related phenomena [8–10]. The polarization distribution at the vicinity of the focus plays a crucial role in such nonlinear optical phenomena. For example, the light-vacuum interaction strongly depends on the focusing angle through the direction of the focused EM field [9]. On the other hand, recent progress of the high-field terahertz-pump optical-probe experiments [11–14] provides a deep insight on understanding the carrier and spin dynamics within the temporal scale of the single cycle. The precise knowledge of the polarization distribution of such a focused EM field is of a fundamental importance and it is indeed necessary, e.g., for the precise control of electrons and spins. In particular, focusing with a parabolic mirror (PM) with large numerical aperture (NA) should be established, since it is routinely used for a tight focusing of broadband EM waves [15]. However, there is no sufficient theoretical expression and experimental demonstration of the time evolution of the polarization distribution.

In this paper we discuss the EM field focused by a PM both theoretically and experimentally. For a linearly polarized incident EM wave, the focused EM field is found to form non-negligible and symmetric divergent and rotation spatial distributions at the vicinity of the focus, depending on the direction of the incident polarization. These distributions appear at the zero-crossing time, i.e., the time when the focused electric field at the focus vanishes. These divergent and rotation distributions emerge strongly for a PM with large NA and move with a phase velocity faster than the speed of light. These features are theoretically derived and experimentally observed using the time-domain spectroscopy in terahertz frequency range. Our finding provides a leading theory for the future polarimetry imaging system [16,17] and precise control of a spatially and polarization-resolved light-matter interaction.

We begin with the theoretical calculation. The experimental results are subsequently shown. Figure 1 shows the coordinate system used in this study. The PM is expressed by $z = (x^2 + y^2)/(4f)$, where f is the focal length and the focus is given by $(0,0,f)$. The incident EM wave is assumed to be a linearly polarized plane wave with wavelength λ and it propagates toward the $-Z$ direction. We refer to the incident EM wave as "$X(Y)$ polarization" if the electric field is parallel to the $X(Y)$ axis. For the reflection on the mirror surface, we employ the physical optics (PO) method [18]. In this method, the surface current is approximated by an equivalent current, which is proportional to the tangential component of the incident magnetic field.

We first consider the X polarization. We calculate the steady solution. A physical quantity $F(\boldsymbol{x},t)$ can be given by $F(\boldsymbol{x},t) = \mathrm{Re}[\bar{F}(\boldsymbol{x})e^{-i\omega t}]$, where $\omega = ck = 2\pi c/\lambda$. The spatial part of the focused electric field at $\boldsymbol{x} = (x,y,f+z)$ is calculated by an integral [19] over the mirror surface

$$\bar{\boldsymbol{E}}(\boldsymbol{x}) = 2iE_0k\int g^{(3)}(k;\boldsymbol{x},\tilde{\boldsymbol{x}})\begin{pmatrix}1\\0\\\tilde{x}/(2f)\end{pmatrix}e^{-ik\tilde{z}}d\tilde{x}d\tilde{y}, \quad (1)$$

where $E_0 > 0$ is the amplitude of the incident electric field, $g^{(3)}(k;\boldsymbol{x},\tilde{\boldsymbol{x}})$ is the dyadic Green function, and the longitudinal vector is derived by the PO method. Since we have employed the PO method, the curvature radius of the PM must be sufficiently longer than the wavelength, i.e., we have to suppose $\lambda/f \ll 1$. For the integration range, we treat the PM as a segment. We suppose the projection of the mirror over the XY plane is a circle whose center is $(x_1,0)$, with radius l, and $l/f \ll 1$. This is the reason we use a term "segment" for the mirror. The effective focal length is given by $\alpha = f/\cos^2(\theta/2)$, where θ is the offset angle. We discuss the EM field at position $\boldsymbol{x}' = (x',y',z')$ in a new coordinate system $X'Y'Z'$ defined in Fig. 1. The origin $\boldsymbol{x}' = \boldsymbol{0}$ is the focus.

We now approximate the integral (1) as follows. We have already assumed $\lambda/f, l/f \ll 1$. In order to see a subwavelength scale around the focus, let us assume $x'/f, y'/f, z'/f \ll 1$. The integrand can be expressed as a power series of these ratios. A power of the ratios is sometimes represented by a power of $1/f$, e.g., λ^2/f^2 and $x'y'/f^2$ are represented

*ka-shibata@spring8.or.jp

†watanabe@phys.keio.ac.jp

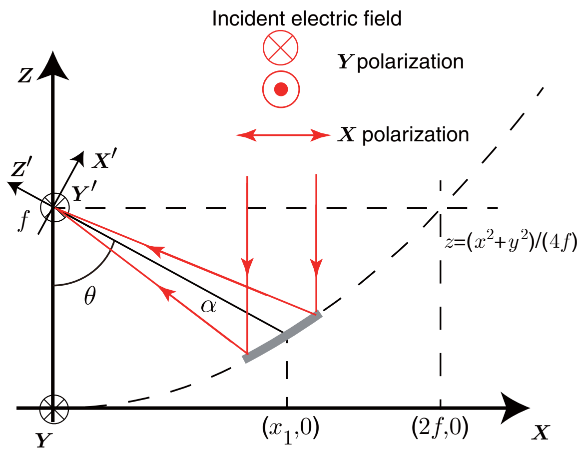

FIG. 1. (Color online) The coordinate system for the parabolic mirror. The incident EM wave propagates toward the $-Z$ direction. The effective focal length is given by $\alpha = f/\cos^2(\theta/2)$. Z' is the direction from the center of the mirror to the focus. The Y' axis is parallel to the Y axis. The unit vector of the X' axis is given by the outer product of the unit vectors of the Y' and Z' axes.

by $O(1/f^2)$. The integrand contains the exponential part of $\exp[ik(|\boldsymbol{x}-\tilde{\boldsymbol{x}}|-\tilde{z})]$. This part is calculated as $\exp[ik(|\boldsymbol{x}-\tilde{\boldsymbol{x}}|-\tilde{z})] = \exp[ik(f+z') + O(1/f)]$, where $\exp[ik(f+z')]$ can be extracted from the integral. Then, $\exp[O(1/f)]$ and the rest terms of the integrand can be expressed as a Taylor series. Since we have assumed the integration ranges of $\tilde{x}$ and $\tilde{y}$ are sufficiently shorter than the length of f, the expansion is valid. In this way we obtain $\bar{\boldsymbol{E}}(\boldsymbol{x}')$ as a power series of $1/f$. The electric field is obtained by multiplying $e^{-i\omega t}$ to it and using the real part. The magnetic field is calculated using Faraday's law. The detail for all the theoretical processes and results are given in the Appendix.

The X' component of the focused electric field at the focus is given by

$$E_{X'}^{(X)}(\boldsymbol{0},t) = \frac{kl^2 E_0}{2\alpha}\left[\cos\omega t + \frac{1}{k\alpha}\sin\omega t - \frac{1}{\alpha^2}\left(\frac{1}{k^2}+\frac{l^2}{4}\right)\cos\omega t\right] + O\left(\frac{1}{f^4}\right). \quad (2)$$

Since α is proportional to f, the first term in the bracket is the main component or the leading term of the focused electric field. We define the zero-crossing time t_0 as the time when $E_{X'}^{(X)}(\boldsymbol{0},t_0)$ drops to $O(1/f^4)$ because the electric field is expressed as a power series of $1/f$. The condition is satisfied if the following holds:

$$\cos\omega t_0 + \frac{1}{k\alpha}\sin\omega t_0 = 0. \quad (3)$$

Therefore we define the zero-crossing time using this equation. It can be seen that $|\sin\omega t_0|$ is almost unity under the assumption that $\lambda/f \ll 1$.

At the zero-crossing time and the vicinity of the focus, by discarding the second or higher order of $\boldsymbol{x}'$, the X' and Y' components of the EM field in the focal plane ($z'=0$) are given by

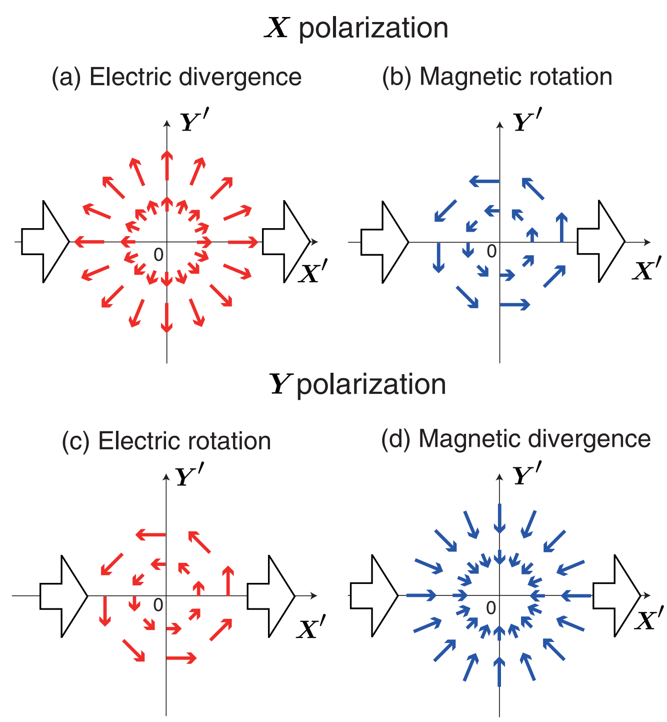

FIG. 2. (Color online) The EM divergent and rotation fields at the zero-crossing time t_0 in the focal plane for $\sin\omega t_0 > 0$. For the X polarization, (a) and (b) represent Eqs. (4) and (5), respectively. Similarly, for the Y polarization, (c) and (d) represent Eqs. (6) and (7), respectively. The large arrows indicate the direction that the divergent and rotation fields move. The phase velocity $V > c$ is given by Eq. (9).

$$\begin{pmatrix} E_{X'}^{(X)}(x',y',0,t_0) \\ E_{Y'}^{(X)}(x',y',0,t_0) \end{pmatrix} \approx \frac{kl^2 E_0 \sin\omega t_0}{2f^3}\left(\frac{1}{k}+\frac{kl^2}{8}\right) \times \sin\theta\cos^4(\theta/2)\begin{pmatrix} x' \\ y' \end{pmatrix} \quad (4)$$

and

$$\begin{pmatrix} B_{X'}^{(X)}(x',y',0,t_0) \\ B_{Y'}^{(X)}(x',y',0,t_0) \end{pmatrix} \approx \frac{k^2 l^4 E_0 \sin\omega t_0}{16cf^3}\sin\theta\cos^4(\theta/2)\begin{pmatrix} -y' \\ x' \end{pmatrix}. \quad (5)$$

The equations indicate an electric divergent and a magnetic rotation field for every wavelength $\lambda \ll f$. Figures 2(a) and 2(b) show the schematic of these EM fields. These distributions are generated even though the incident EM wave is linearly polarized along the X axis. This is the consequence of the breaking of the axisymmetry since the electric divergent and magnetic rotation fields vanish at $\theta = 0^\circ$ by $\sin\theta$. The other part of θ dependence is $\cos^4(\theta/2)$. It is the result of expanding the effective focal length. The function $\sin\theta\cos^4(\theta/2)$ has a maximum value at $\theta \approx 48.2^\circ$. For a zero-crossing time after the half period, $\sin\omega t_0$ changes its sign and the directions of the EM field vectors are reversed. Note that the definition of the zero-crossing time (3) is important. If the zero-crossing time is defined by a condition in which only the leading term vanishes, the electric field around the focus is just a small portion of the X' component and the electric divergence never occurs.

For the Y polarization, the vector $(1,0,\tilde{x}/(2f))$ in Eq. (1) is replaced by $(0,1,\tilde{y}/(2f))$. The Y' components of the focused electric field at the focus $E_{Y'}^{(Y)}(\mathbf{0},t)$ is also expressed by the right side of Eq. (2). Therefore, the leading term is a part of the Y' component and we can define the zero-crossing time as the time when $E_{Y'}^{(Y)}(\mathbf{0},t_0)$ drops to $O(1/f^4)$, i.e., using Eq. (3). Finally, in the focal plane, by discarding the second or higher order of $\boldsymbol{x}'$, we obtain

$$\begin{pmatrix} E_{X'}^{(Y)}(x',y',0,t_0) \\ E_{Y'}^{(Y)}(x',y',0,t_0) \end{pmatrix} \approx \frac{k^2 l^4 E_0 \sin\omega t_0}{16 f^3} \sin\theta \cos^4(\theta/2) \begin{pmatrix} -y' \\ x' \end{pmatrix} \tag{6}$$

and

$$\begin{pmatrix} B_{X'}^{(Y)}(x',y',0,t_0) \\ B_{Y'}^{(Y)}(x',y',0,t_0) \end{pmatrix} \approx -\frac{k^2 l^4 E_0 \sin\omega t_0}{16 c f^3} \sin\theta \cos^4(\theta/2) \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{7}$$

The equations indicate an electric rotation and a magnetic divergent field [see Figs. 2(c) and 2(d)]. From Eqs. (4), (5), (6), and (7), it is clear that not only the direction but also the structure of the EM divergent and rotation fields varies for the X and Y polarizations. This indicates that the focused EM field of higher orders strongly depends on the incident polarization.

We would like to comment on the origin of these divergent and rotation fields. In general, an oscillating current generates the electric field and the directions of the current and the electric field can be different. This effect appears as the nondiagonal terms in the dyadic Green function. It is a reason for the vector field distributions at the zero-crossing time. The other reason is intrinsic to the PM. For the $X(Y)$ polarization, the surface current contains not only $X(Y)$ component but also Z component, since the PM is curved and the motion of the electrons on the mirror surface is restricted. The Z component of the surface current contributes to the symmetric distributions.

For further quantitative analysis on these distributions, we consider the Y polarization. Similar results are obtained for the X polarization. The magnitudes of Eqs. (6) and (7) are proportional to the square of the area of the mirror segment and f^{-3}. This dependency is obviously different from that of the leading term; this term is proportional to the area of the mirror segment and decreases with f^{-1}. Let us consider the ratio of the magnitude of the electric rotation to the leading term. Suppose the electric rotation occurs in an area of diameter of the wavelength λ, we tentatively substitute (x',y') with $(0,\lambda/2)$ for the X' component of Eq. (6). Using $|\sin\omega t_0| \approx 1$ we obtain

$$\left|E_{X'}^{(Y)}(0,\lambda/2,0,t_0)\right| \Big/ \left(\frac{kl^2 E_0}{2\alpha}\right) \approx \frac{\pi l^2 \sin\theta \cos^2(\theta/2)}{8 f^2}. \tag{8}$$

The ratio is proportional to the area of the mirror segment. It implies that the electric rotation is considerable for a PM with large NA, although it is derived under the assumption $l \ll f$.

Next, let us consider a time t sufficiently close to t_0. In this case, the electric rotation exists and its center (x_c,y_c) in the $X'Y'$ plane is defined by a point at which both $E_{X'}^{(Y)}(x_c,y_c,0,t)$ and $E_{Y'}^{(Y)}(x_c,y_c,0,t)$ drop to $O(1/f^4)$. The center is estimated to be $(x_c,y_c) = (V(t-t_0),0)$, where

$$V = c\frac{8 f^2}{l^2 \sin\theta \cos^2(\theta/2)}. \tag{9}$$

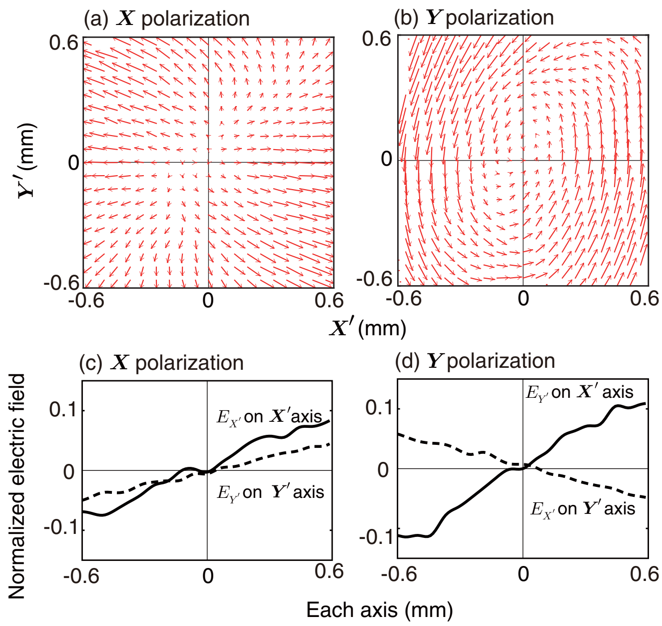

FIG. 3. (Color online) (a) and (b) The spatial distributions of the electric field in the focal plane at the zero-crossing time t_0. The experimental results for $\lambda = 602\,\mu$m and $\sin\omega t_0 > 0$ are shown. The polarization of the incident terahertz pulse is along the X axis for (a) and the Y axis for (b). (c) $E_{X'}^{(X)}(x',0,0,t_0)$ along the X' axis and $E_{Y'}^{(X)}(0,y',0,t_0)$ along the Y' axis for the electric divergence in (a). (d) $E_{X'}^{(Y)}(0,y',0,t_0)$ along the Y' axis and $E_{Y'}^{(Y)}(x',0,0,t_0)$ along the X' axis for the electric rotation in (b). In (c) and (d), each electric field is normalized by the maximum electric field at the focus.

The center and the electric rotation move along the X' axis in the positive direction regardless of the rotation direction. This is the same for the magnetic divergence. We found that the phase velocity V exceeds c because of the assumption $l \ll f$. In addition, V is inversely proportional to the ratio in Eq. (8).

For the focusing by a large mirror whose characteristic length is of the order of f, the above equations are not applicable directly. However, using numerical calculations, we confirmed that the essential features of the EM field at the zero-crossing time remain the same for this case.

We experimentally observed the electric field by a polarization-resolved terahertz-TDS imaging setup [20]. The linearly polarized incident terahertz pulse was focused by an off-axis PM with an offset angle of 90°, NA of about 0.45, and focal length of 25.4 mm. The projection of the mirror over the XY plane is a circle with a diameter of 50.8 mm.

Figures 3(a) and 3(b) show the spatial distribution of the focused electric field vectors on the electro-optic crystal at the zero-crossing time for the X- and Y-polarized incident terahertz EM wave of $\lambda = 602\,\mu$m. The divergence for the X polarization is shown in Fig. 3(a) and the rotation for the Y polarization is shown in Fig. 3(b). Figures 3(c) and 3(d) show the position dependence of the electric fields along each axis. It can be seen that the fields vary linearly at the vicinity of the focus, in accordance with Eqs. (4) and (6). The ratio at the distance of $\lambda/2$ is about 0.05. The distributions move along the X' axis and their phase velocities are 1.9×10^{10} and 9.3×10^9 m/s for the divergence and rotation, respectively [21].

We now compare the theoretical and experimental values. We derived the ratio of the magnitude of the electric rotation to the leading term at the focus in Eq. (8) and the velocity of the moving center in Eq. (9). Although the equations are derived under the assumption $l \ll f$ so that they cannot be applicable to the experiments, we tentatively substitute $f = l = 25.4$ mm and $\theta = 90^\circ$. We obtain the ratio of Eq. (8) as 0.20 and the velocity of Eq. (9) as 4.8×10^9 m/s. The difference between the theoretical and experimental results will be caused by the effect of the large mirror or aberrations since the experimental results are asymmetric though Eqs. (4) and (6) are symmetric.

In summary, we theoretically and experimentally investigated the linearly polarized EM wave focused by a PM. We found EM divergent and rotation fields at the zero-crossing time. These distributions appear within the subwavelength scale around the focus. The structure of the distributions varies with the incident polarization because the focused EM field of higher orders strongly depends on the incident polarization. In addition, we found that the distributions move along the X' axis with a phase velocity $V > c$. Our findings will be useful for constructing a broadband polarimetry imaging system [16,17] as well as a vector field imaging system [22].

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APPENDIX

System

We show the detail of our analytic calculation. The PM is expressed by $z = (x^2 + y^2)/(4f)$, where f is the focal length of the PM. The focus is given by $(0,0,f)$.

The incident EM wave propagating toward the $-Z$ direction is reflected by the PM and the reflected wave travels toward the focus. We assume the incident EM wave is a monochromatic plane wave with wavelength λ. The wave number is given by $\boldsymbol{k} = (0,0,-k)$, where $k = |\boldsymbol{k}| = 2\pi/\lambda$. The angular frequency is given by $\omega = ck$. We consider the steady solution. The spatial part of the focused electric field at $\boldsymbol{x} = (x,y,f+z)$ is expressed by an integral over the mirror surface [19]

$$\bar{\boldsymbol{E}}(\boldsymbol{x}) = i\omega\mu_0 \int_{\mathrm{PM}} g^{(3)}(k;\boldsymbol{x},\tilde{\boldsymbol{x}})\bar{\boldsymbol{j}}(\tilde{\boldsymbol{x}})d\tilde{\boldsymbol{x}}, \tag{A1}$$

where $\bar{\boldsymbol{j}}$ is the surface current on the PM and $g^{(3)}$ is the dyadic Green function. It is given by

$$g^{(3)}_{mn}(k;\boldsymbol{x},\tilde{\boldsymbol{x}}) = \left(\delta_{mn} - \frac{1}{k^2}\frac{\partial^2}{\partial x_m \partial \tilde{x}_n}\right)\frac{e^{ik|\boldsymbol{x}-\tilde{\boldsymbol{x}}|}}{4\pi|\boldsymbol{x}-\tilde{\boldsymbol{x}}|}, \tag{A2}$$

where $m,n = 1,2,3$ represent the X,Y, and Z components, respectively. The surface current $\bar{\boldsymbol{j}}(\tilde{\boldsymbol{x}})$ can be expressed as

$$\bar{\boldsymbol{j}}(\tilde{\boldsymbol{x}}) = \frac{2}{\mu_0\omega}\boldsymbol{n} \times [\boldsymbol{k} \times \bar{\boldsymbol{E}}^{(\mathrm{inc})}(\tilde{\boldsymbol{x}})]\delta[\tilde{z} - (\tilde{x}^2 + \tilde{y}^2)/(4f)] \tag{A3}$$

by the physical optics method [18]. $\bar{\boldsymbol{E}}^{(\mathrm{inc})}(\tilde{\boldsymbol{x}})$ is the incident electric field. The vector $\boldsymbol{n}$ is an outward normal vector at $\tilde{\boldsymbol{x}}$ and it is given by

$$\boldsymbol{n} = \frac{1}{\sqrt{1 + (\tilde{x}^2 + \tilde{y}^2)/(4f^2)}}\begin{pmatrix} -\tilde{x}/(2f) \\ -\tilde{y}/(2f) \\ 1 \end{pmatrix}. \tag{A4}$$

The area element is given by

$$dS = \sqrt{1 + \frac{\tilde{x}^2 + \tilde{y}^2}{4f^2}}d\tilde{x}d\tilde{y}. \tag{A5}$$

We assume the incident EM wave is linearly polarized. If the electric field is parallel to the $X(Y)$ axis, we refer to the incident EM wave as "$X(Y)$ polarization." For the X polarization, the focused electric field is given by

$$\bar{\boldsymbol{E}}(\boldsymbol{x}) = 2iE_0k \int g^{(3)}(k;\boldsymbol{x},\tilde{\boldsymbol{x}})\begin{pmatrix} 1 \\ 0 \\ \tilde{x}/(2f) \end{pmatrix}e^{-ik\tilde{z}}d\tilde{x}d\tilde{y}, \tag{A6}$$

where E_0 is the amplitude of the incident electric field [Eq. (1)]. For the Y polarization, the vector $(1,0,\tilde{x}/(2f))$ in the integrand is replaced by $(0,1,\tilde{y}/(2f))$. Hereafter, the expressed EM field is divided by E_0.

We consider the focusing by a small segment of the PM. The segment can be characterized by the projection over the XY plane. We suppose the projection is a circle whose center is $(x_1,0)$, with radius l. A position on the segment can be expressed by $\tilde{\boldsymbol{x}} = (x_1 + \xi,\eta,\tilde{z})$, where ξ and η are the new variables of the integration. They are restricted by $\xi^2 + \eta^2 \leqslant l^2$. We define the offset angle θ by the angle defined by the vector from the center of the mirror to the focus and the unit vector of the Z axis. In the following, for the sake of saving space, we often use S_θ and C_θ, instead of $\sin\theta$ and $\cos\theta$, respectively. The distance between the center of the mirror and the focus is the effective focal length $\alpha = f/C^2_{\theta/2}$.

A new coordinate system

We define a new coordinate system $X'Y'Z'$ by

$$\begin{aligned} \boldsymbol{X}' &= C_\theta\boldsymbol{X} + S_\theta\boldsymbol{Z}, \\ \boldsymbol{Y}' &= \boldsymbol{Y}, \\ \boldsymbol{Z}' &= -S_\theta\boldsymbol{X} + C_\theta\boldsymbol{Z}, \end{aligned} \tag{A7}$$

where the bold types are unit vectors of the axes. The Z' axis is the direction from the center of the mirror to the focus. The Y' axis is parallel to the Y axis and the unit vector of the X' axis is given by the outer product of the unit vectors of the Y' and Z' axes. The focus is taken to be the origin. The position at the vicinity of the focus in the new coordinate system $\boldsymbol{x}' = (x',y',z')$ is given by $x' = C_\theta x + S_\theta z$, $y' = y$, and $z' = -S_\theta x + C_\theta z$.

Expansion of the integrand

We make an assumption that the wavelength and the mirror radius are both sufficiently shorter than the focal length, i.e., $\lambda/f, l/f \ll 1$. In addition, we consider only at the vicinity of the focus and suppose $x/f, y/f, z/f \ll 1$. Then we can divide all the variables with unit of length into two groups. One is the "long" group. Its elements are the focal length f, α, and x_1. The other is the "short" group. Its elements are the wavelength λ, the radius l, $\boldsymbol{x}$, $\boldsymbol{x}'$, ξ, and η. We can consider ratios of one of the short group over f, such as λ/f. By the assumptions, such a ratio is basically small. Therefore we can express the integrand of Eq. (A6) as a power series of the ratios. We sometimes represent the power of the ratios by the power of $1/f$. For example, λ/f and l/f are represented by $O(1/f)$. Only the exponential part of $\exp[ik(|\boldsymbol{x}-\tilde{\boldsymbol{x}}|-\tilde{z})]$ requires a careful treatment.

In the case of the X polarization

The spatial part of the focused electric field $\bar{\boldsymbol{E}}(\boldsymbol{x}')$ is given by

$$\begin{aligned}
\bar{E}_{X'}(\boldsymbol{x}') &= i e^{ik(f+z')}\frac{kl^2}{2\alpha}\biggl\{1-\frac{1}{k^2\alpha^2}-\frac{l^2}{4\alpha^2}+\frac{S_\theta x'}{2f}-\frac{z'}{\alpha}-\frac{S_\theta x'z'}{f\alpha}+\frac{Q(\boldsymbol{x}')}{f^2}-\frac{x'^2}{\alpha^2}\\
&\quad + i\left[\frac{1}{k\alpha}+\frac{S_\theta x'}{2f\alpha}\left(\frac{3}{k}+\frac{kl^2}{4}\right)-\left(\frac{2}{k}+\frac{kl^2}{4}\right)\frac{z'}{\alpha^2}+k\left(1+\frac{S_\theta x'}{2f}-\frac{2z'}{\alpha}\right)\frac{x'^2+y'^2}{2\alpha}\right]\biggr\}+O\left(\frac{1}{f^4}\right),\\
\bar{E}_{Y'}(\boldsymbol{x}') &= i e^{ik(f+z')}\frac{kl^2}{2\alpha}y'\left[\frac{S_\theta}{2f}-\frac{x'}{\alpha^2}-\frac{S_\theta z'}{f\alpha}+i\frac{S_\theta}{2f\alpha}\left(\frac{kl^2}{4}+\frac{3}{k}+k\frac{x'^2+y'^2}{2}\right)\right]+O\left(\frac{1}{f^4}\right),\\
\bar{E}_{Z'}(\boldsymbol{x}') &= i e^{ik(f+z')}\frac{kl^2}{2\alpha}\biggl\{-\frac{S_\theta}{k^2f\alpha}-\frac{l^2S_\theta}{4f\alpha}-\frac{x'}{\alpha}+\frac{2x'z'}{\alpha^2}-S_\theta\frac{x'^2+y'^2}{f\alpha}\\
&\quad + i\left[\frac{S_\theta}{kf}-\left(\frac{3}{k}+\frac{kl^2}{4}\right)\frac{x'}{\alpha^2}-\frac{2S_\theta}{kf\alpha}z'-k\frac{x'^2+y'^2}{2\alpha^2}x'\right]\biggr\}+O\left(\frac{1}{f^4}\right),
\end{aligned} \tag{A8}$$

where

$$Q(\boldsymbol{x}') = -\frac{f^2}{\alpha^2}\left[x'^2+y'^2-z'^2+k^2\frac{x'^2+y'^2}{8}(x'^2+y'^2+l^2)\right]. \tag{A9}$$

The zero-crossing time

The electric field is obtained by multiplying $e^{-i\omega t}$ to $\bar{\boldsymbol{E}}(\boldsymbol{x}')$ and using the real part, e.g., $E_{X'}(\boldsymbol{x}',t) = \mathrm{Re}[\bar{E}_{X'}(\boldsymbol{x}')e^{-i\omega t}]$. To simplify the argument of the trigonometric functions, we replace the time t by $t+(\pi/2+kf)/\omega$. At the focus ($\boldsymbol{x}'=\boldsymbol{0}$), $E_{Y'}(\boldsymbol{0},t)=O(1/f^4), E_{Z'}(\boldsymbol{0},t)=O(1/f^2)$, and the X' component [Eq. (2)] is given by

$$E_{X'}(\boldsymbol{0},t) = \frac{kl^2}{2\alpha}\left[\cos\omega t+\frac{1}{k\alpha}\sin\omega t-\frac{1}{\alpha^2}\left(\frac{1}{k^2}+\frac{l^2}{4}\right)\cos\omega t\right]+O\left(\frac{1}{f^4}\right). \tag{A10}$$

The zero-crossing time is defined as in the main text. Considering the very vicinity of the focus and discarding the second or higher order of $\boldsymbol{x}'$, in the focal plane and at the zero-crossing time, we obtain

$$\begin{aligned}
E_{X'}(x',y',0,t_0) &\approx \frac{kl^2S_{\omega t_0}}{2f^3}\left(\frac{1}{k}+\frac{kl^2}{8}\right)S_\theta C_{\theta/2}^4 x', \quad E_{Y'}(x',y',0,t_0) \approx \frac{kl^2S_{\omega t_0}}{2f^3}\left(\frac{1}{k}+\frac{kl^2}{8}\right)S_\theta C_{\theta/2}^4 y',\\
E_{Z'}(x',y',0,t_0) &\approx \frac{kl^2S_{\omega t_0}}{2f}\left[\frac{S_\theta}{kf}-\left(\frac{2}{k}+\frac{kl^2}{4}\right)\frac{x'}{\alpha^2}\right]C_{\theta/2}^2.
\end{aligned} \tag{A11}$$

If $S_\theta \neq 0$, the X' and Y' components are a divergent field in the focal plane [Eq. (4)]. The factor of $\sin\omega t_0$ can be regarded as unity (or -1) and the divergent field depends on the offset angle θ almost by $\sin\theta\cos^4(\theta/2)$. For $0 \leqslant \theta \leqslant \pi/2$, this function has the maximum value at $\theta_0 = \sin^{-1}\sqrt{5}/3 = \cos^{-1}2/3 \approx 0.841$ rad $\approx 48.2^\circ$.

The ratio of the magnitude of the electric divergent field to the leading term is given by

$$R_{x'}^{(X)} = \frac{kl^2}{2f^3}\left(\frac{1}{k}+\frac{kl^2}{8}\right)S_\theta C_{\theta/2}^4|S_{\omega t_0}x'| \bigg/ \left(\frac{kl^2}{2\alpha}\right) = \frac{S_\theta C_{\theta/2}^2}{f^2}\left(\frac{1}{k}+\frac{kl^2}{8}\right)|S_{\omega t_0}x'|. \tag{A12}$$

Suppose the divergent field occurs in an area of diameter of the wavelength λ, we tentatively substitute $x'=\lambda/2$. Furthermore, recall the fact that $|S_{\omega t_0}|$ is almost unity, we obtain

$$R_{\lambda/2}^{(X)} \approx \frac{S_\theta C_{\theta/2}^2}{f^2}\left(\frac{1}{k}+\frac{kl^2}{8}\right)\frac{\lambda}{2}. \tag{A13}$$

The ratio is a linear function of the area of the mirror segment.

The move of the electric divergence

Let us consider the electric field at the vicinity of the focus for a time t sufficiently close to t_0. We introduce t' by $t = t_0 + t'$ and regard it as a small amount. We take the terms of only zeroth or first order of $\boldsymbol{x}'$ and t' for the electric field. The X' and Y' components of the electric field is approximated by

$$\begin{aligned} E_{X'}(x',y',0,t_0+t') &\approx \frac{kl^2 S_{\omega t_0}}{2\alpha}\left[\frac{S_\theta x'}{2f\alpha}\left(\frac{2}{k}+\frac{kl^2}{4}\right)-\left(1-\frac{l^2}{4\alpha^2}\right)\omega t'\right],\\ E_{Y'}(x',y',0,t_0+t') &\approx \frac{kl^2 S_{\omega t_0}}{4f\alpha^2}S_\theta\left(\frac{2}{k}+\frac{kl^2}{4}\right)y'. \end{aligned} \tag{A14}$$

The center (x_c, y_c) of the divergence can be defined as a point that the right sides of these equations disappear. If the order of $\omega t'$ is higher than or equal to $1/f$, the bracket of the X' component remains for every x'. It means the center can be defined if and only if $\omega t' = O(1/f^2)$. Under this limitation we obtain $(x_c, y_c) = (V_{Xe}t', 0)$, where

$$V_{Xe} = c\frac{8f^2}{l^2 S_\theta C_{\theta/2}^2}\frac{k^2 l^2}{8+k^2 l^2}. \tag{A15}$$

As a matter of fact, the X' and Y' components are given by

$$\begin{pmatrix} E_{X'}(x',y',0,t_0+t') \\ E_{Y'}(x',y',0,t_0+t') \end{pmatrix} \approx \frac{kl^2 S_{\omega t_0}}{2f^3}\left(\frac{1}{k}+\frac{kl^2}{8}\right)S_\theta C_{\theta/2}^4\begin{pmatrix} x'-x_c \\ y'-y_c \end{pmatrix}, \tag{A16}$$

and (x_c, y_c) is surely the center of the divergence.

The magnetic field

The magnetic field is calculated using Faraday's law. As a result we obtain

$$\begin{aligned} B_{X'}(\boldsymbol{x}',t) &= \frac{kl^2 S_\theta}{4f\alpha^2}\left(1+\frac{k^2l^2}{4}+k^2\frac{x'^2+y'^2}{2}\right)y'\frac{S_\delta}{\omega}+\frac{k^2l^2S_\theta}{2f\alpha}\left(\frac{1}{2}-\frac{z'}{\alpha}\right)y'\frac{C_\delta}{-\omega}+O\left(\frac{1}{f^4}\right),\\ B_{Y'}(\boldsymbol{x}',t) &= \frac{kl^2}{2\alpha^2}\left[-1-\left(1+\frac{k^2l^2}{4}\right)\frac{S_\theta x'}{2f}+\left(2+\frac{k^2l^2}{4}\right)\frac{z'}{\alpha}-k^2\left(1+\frac{S_\theta x'}{2f}-\frac{2z'}{\alpha}\right)\frac{x'^2+y'^2}{2}\right]\\ &\quad\times\frac{S_\delta}{\omega}+\frac{k^2l^2}{2\alpha}\left(-1+\frac{l^2}{4\alpha^2}-\frac{S_\theta x'}{2f}+\frac{z'}{\alpha}-\frac{Q(\boldsymbol{x}')}{f^2}+\frac{x'^2+y'^2}{2\alpha^2}+\frac{S_\theta x'z'}{f\alpha}\right)\frac{C_\delta}{-\omega}+O\left(\frac{1}{f^4}\right),\\ B_{Z'}(\boldsymbol{x}',t) &= \frac{kl^2y'}{2\alpha^3}\left[1+\frac{k^2}{2}\left(x'^2+y'^2+\frac{l^2}{2}\right)\right]\frac{S_\delta}{\omega}+\frac{k^2l^2y'}{2\alpha^2}\left(1-\frac{2z'}{\alpha}\right)\frac{C_\delta}{-\omega}+O\left(\frac{1}{f^4}\right), \end{aligned} \tag{A17}$$

where $\delta = kz' - \omega t$.

Therefore, in the focal plane and at the zero-crossing time, the magnetic field at the vicinity of the focus is given by

$$B_{X'}(x',y',0,t_0) \approx -\frac{k^2l^4S_{\omega t_0}}{16cf^3}S_\theta C_{\theta/2}^4 y',\quad B_{Y'}(x',y',0,t_0) \approx \frac{k^2l^4S_{\omega t_0}}{16cf^3}S_\theta C_{\theta/2}^4 x',\quad B_{Z'}(x',y',0,t_0) \approx -\frac{k^2l^4S_{\omega t_0}}{8c\alpha^3}y'. \tag{A18}$$

If $S_\theta \neq 0$, the X' and Y' components are a magnetic rotation field in the focal plane [Eq. (5)].

The move of the magnetic rotation

Let $t = t_0 + t'$ and t' be a sufficiently small amount. The center of the magnetic rotation field can be approximated by $(x_c, y_c) = (V_{Xm}t', 0)$, where

$$V_{Xm} = c\frac{8f^2}{l^2 S_\theta C_{\theta/2}^2}. \tag{A19}$$

As a matter of fact, the X' and Y' components are given by

$$\begin{pmatrix} B_{X'}(x',y',0,t_0+t') \\ B_{Y'}(x',y',0,t_0+t') \end{pmatrix} \approx \frac{k^2l^4S_{\omega t_0}}{16cf^3}S_\theta C_{\theta/2}^4\begin{pmatrix} -(y'-y_c) \\ x'-x_c \end{pmatrix}. \tag{A20}$$

In the case of the *Y* polarization

The spatial part of the focused electric field $\bar{\boldsymbol{E}}(\boldsymbol{x}')$ is given by

$$\bar{E}_{X'}(\boldsymbol{x}') = -ie^{ik(f+z')}\frac{kl^2}{2\alpha^2}\left(\frac{x'}{\alpha} + \frac{ikl^2}{8f}S_\theta\right)y' + O\left(\frac{1}{f^4}\right),$$

$$\bar{E}_{Y'}(\boldsymbol{x}') = ie^{ik(f+z')}\frac{kl^2}{2\alpha}\left[1 + \frac{i}{k\alpha} - \frac{1}{k^2\alpha^2} - \frac{l^2}{4\alpha^2} - \frac{1}{\alpha}\left(1 + \frac{2i}{k\alpha} + \frac{ikl^2}{4\alpha}\right)z' + ik\left(1 - \frac{2z'}{\alpha}\right)\frac{x'^2 + y'^2}{2\alpha} + i\frac{kl^2S_\theta}{8f\alpha}x' + \frac{Q(\boldsymbol{x}')}{f^2} - \frac{y'^2}{\alpha^2}\right] + O\left(\frac{1}{f^4}\right),$$

$$\bar{E}_{Z'}(\boldsymbol{x}') = -ie^{ik(f+z')}\frac{kl^2}{2\alpha^2}y'\left(1 + \frac{3i}{k\alpha} + \frac{ikl^2}{4\alpha} - \frac{2z'}{\alpha} + ik\frac{x'^2 + y'^2}{2\alpha}\right) + O\left(\frac{1}{f^4}\right). \quad \text{(A21)}$$

The zero-crossing time

At the focus ($\boldsymbol{x}' = \boldsymbol{0}$) we obtain $E_{X'}(\boldsymbol{0},t), E_{Z'}(\boldsymbol{0},t) = O(1/f^4)$ and

$$E_{Y'}(\boldsymbol{0},t) = \frac{kl^2}{2\alpha}\left[\cos\omega t + \frac{1}{k\alpha}\sin\omega t - \frac{1}{\alpha^2}\left(\frac{1}{k^2} + \frac{l^2}{4}\right)\cos\omega t\right] + O\left(\frac{1}{f^4}\right). \quad \text{(A22)}$$

It is apparent the leading term is a part of the Y' component.

Discarding the second or higher order of $\boldsymbol{x}'$, the electric field in the focal plane and at the zero-crossing time is given by

$$E_{X'}(x',y',0,t_0) \approx -\frac{k^2l^4}{16f^3}S_{\omega t_0}S_\theta C_{\theta/2}^4 y', \quad E_{Y'}(x',y',0,t_0) \approx \frac{k^2l^4}{16f^3}S_{\omega t_0}S_\theta C_{\theta/2}^4 x', \quad E_{Z'}(x',y',0,t_0) \approx -\frac{kl^2}{2\alpha^3}S_{\omega t_0}\left(\frac{2}{k} + \frac{kl^2}{4}\right)y'. \quad \text{(A23)}$$

If $S_\theta \neq 0$, the X' and Y' components form an electric rotation field in the focal plane [Eq. (6)].

The ratio [Eq. (8)] of the magnitude of the rotation at $y' = \lambda/2$ to the leading term is given by

$$R_{\lambda/2}^{(Y)} \approx \frac{\pi S_\theta C_{\theta/2}^2}{8}\frac{l^2}{f^2}. \quad \text{(A24)}$$

Note that if the wavelength is sufficiently shorter than the mirror radius, the ratio $R_{\lambda/2}^{(X)}$ of Eq. (A13) accords with Eq. (A24).

The move of the electric rotation

Let $t = t_0 + t'$ and t' be a small amount. The X' and Y' components of the electric field in the focal plane are approximated by

$$E_{X'}(x',y',0,t_0 + t') \approx -\frac{k^2l^4S_{\omega t_0}}{16f\alpha^2}S_\theta y', \quad E_{Y'}(x',y',0,t_0 + t') \approx \frac{kl^2}{2\alpha}S_{\omega t_0}\left[\frac{kl^2S_\theta}{8f\alpha}x' - \left(1 - \frac{l^2}{4\alpha^2}\right)\omega t'\right]. \quad \text{(A25)}$$

The center (x_c, y_c) of the rotation at the time t can be defined as a point that the right sides of these equations disappear. Finally, we obtain $(x_c, y_c) = (V_{Ye}t', 0)$, where

$$V_{Ye} = c\frac{8f^2}{l^2S_\theta C_{\theta/2}^2}. \quad \text{(A26)}$$

It is Eq. (9). As a matter of fact, the X' and Y' components are given by

$$\begin{pmatrix} E_{X'}(x',y',0,t_0 + t') \\ E_{Y'}(x',y',0,t_0 + t') \end{pmatrix} \approx \frac{k^2l^4S_{\omega t_0}}{16f^3}S_\theta C_{\theta/2}^4\begin{pmatrix} -(y' - y_c) \\ x' - x_c \end{pmatrix}, \quad \text{(A27)}$$

and (x_c, y_c) is surely the center of the rotation.

The magnetic field

The magnetic field is calculated as

$$B_{X'}(\boldsymbol{x}',t) = \frac{kl^2}{2c\alpha}\left\{\left[1 + \frac{k^2l^2S_\theta}{8f}x' - \left(2 + \frac{k^2l^2}{4}\right)\frac{z'}{\alpha} + k^2\left(1 - \frac{2z'}{\alpha}\right)\frac{x'^2 + y'^2}{2}\right]\frac{S_\delta}{k\alpha} - \left(1 - \frac{l^2}{4\alpha^2} - \frac{z'}{\alpha} + \frac{Q(\boldsymbol{x}')}{f^2} - \frac{x'^2 + y'^2}{2\alpha^2}\right)C_\delta\right\} + O\left(\frac{1}{f^4}\right),$$

$$B_{Y'}(\boldsymbol{x}',t) = \frac{k^2 l^4 S_\theta y'}{16 c f \alpha^2} S_\delta + O\left(\frac{1}{f^4}\right),$$

$$B_{Z'}(\boldsymbol{x}',t) = \frac{k l^2}{2 c \alpha^2}\left\{ -\left[1 + \frac{k^2}{2}\left(x'^2 + y'^2 + \frac{l^2}{2}\right)\right]\frac{x' S_\delta}{k\alpha} + \left[\frac{l^2 S_\theta}{4f} + \left(1 - \frac{2z'}{\alpha}\right)x'\right]C_\delta\right\} + O\left(\frac{1}{f^4}\right), \tag{A28}$$

where $\delta = kz' - \omega t$. At the zero-crossing time, discarding the second or higher order of $\boldsymbol{x}'$, we obtain

$$B_{X'}(x',y',0,t_0) \approx -\frac{k^2 l^4 S_{\omega t_0} S_\theta C_{\theta/2}^4}{16 c f^3} x', \quad B_{Y'}(x',y',0,t_0) \approx -\frac{k^2 l^4 S_{\omega t_0} S_\theta C_{\theta/2}^4}{16 c f^3} y', \quad B_{Z'}(x',y',0,t_0) \approx \frac{k^2 l^4 S_{\omega t_0}}{8 c \alpha^3} x'. \tag{A29}$$

If $S_\theta \neq 0$, the X' and Y' components in the focal plane are a magnetic divergent field [Eq. (7)].

The move of the magnetic divergence

Let $t = t_0 + t'$ and t' be a sufficiently small amount. The center of the magnetic divergence can be approximated by $(x_c, y_c) = (V_{Ym} t', 0)$, where

$$V_{Ym} = c \frac{8 f^2}{l^2 S_\theta C_{\theta/2}^2}. \tag{A30}$$

As a matter of fact, the X' and Y' components are given by

$$\begin{pmatrix} B_{X'}(x',y',0,t_0+t') \\ B_{Y'}(x',y',0,t_0+t') \end{pmatrix} \approx -\frac{k^2 l^4 S_{\omega t_0} S_\theta C_{\theta/2}^4}{16 c f^3} \begin{pmatrix} x' - x_c \\ y' - y_c \end{pmatrix}. \tag{A31}$$

Finally, from Eqs. (A15), (A19), (A26), and (A30), it can be seen that only the velocity of the electric divergence for the X polarization V_{Xe} is different from the other three velocities.

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