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Absence of a four-body Efimov effect in the 2 + 2 fermionic problem

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In the free three-dimensional space, we consider a pair of identical $\uparrow$ fermions of some species or in some internal state and a pair of identical $\downarrow$ fermions of another species or in another state. There is a resonant s-wave interaction (that is, of zero range and infinite scattering length) between fermions in different pairs and no interaction within the same pair. We study whether this $2+2$ fermionic system can exhibit (as the $3+1$ fermionic system) a four-body Efimov effect in the absence of three-body Efimov effect, that is, the mass ratio α between $\uparrow$ and $\downarrow$ fermions and its inverse are both smaller than 13.6069 . . . . For this purpose, we investigate scale invariant zero-energy solutions of the four-body Schrödinger equation, that is, positively homogeneous functions of the coordinates of degree $s-7/2$, where s is a generalized Efimov exponent that becomes purely imaginary in the presence of a four-body Efimov effect. Using rotational invariance in momentum space, it is found that the allowed values of s are such that $M(s)$ has a zero eigenvalue; here the operator $M(s)$, that depends on the total angular momentum ℓ, acts on functions of two real variables (the cosine of the angle between two wave vectors and the logarithm of the ratio of their moduli), and we write it explicitly in terms of an integral matrix kernel. We have performed a spectral analysis of $M(s)$, analytical and for an arbitrary imaginary s for the continuous spectrum and numerical and limited to $s=0$ and $\ell \leqslant 12$ for the discrete spectrum. We conclude that no eigenvalue of $M(0)$ crosses zero over the mass ratio interval $\alpha \in [1; 13.6069\ldots]$, even if, in the parity sector $(-1)^\ell$, the continuous spectrum of $M(s)$ has everywhere a zero lower border. As a consequence, there is no possibility of a four-body Efimov effect for the $2+2$ fermions. We also enunciated a conjecture for the fourth virial coefficient of the unitary spin-1/2 Fermi gas, inspired from the known analytical form of the third cluster coefficient and involving the integral over the imaginary s axis of s times the logarithmic derivative of the determinant of $M(s)$ summed over all angular momenta. The conjectured value is in contradiction with the experimental results.

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I. INTRODUCTION

In three-dimensional cold atomic gases, thanks to magnetic Feshbach resonances, it is now possible to induce resonant s-wave interactions between the particles [1]. This means that the s-wave scattering length a is in absolute value much larger than the range (or the effective range) of the interaction. Essentially, one can assume that $1/a=0$, and since the de Broglie atomic wavelength is also much larger than the range of the interaction, one can replace the interactions by scaling invariant Wigner-Bethe-Peierls two-body contact conditions on the wave function [2]: One realizes the long-sought unitary limit.

Perhaps the most striking phenomenon that can take place in that regime is the Efimov effect, predicted for three particles with appropriate statistics and mass ratios [3]. It corresponds to the occurrence of an infinite number of bound states, with an asymptotically geometric spectrum close to the zero-energy accumulation point. The geometric part of the spectrum is characterized by a ratio, predicted by Efimov's zero-range theory, and a global energy scale that depends on the microscopic details of the interaction. The mere existence of such an energy scale forces us to supplement the two-body contact conditions by three-body ones that involve a length scale, the so-called three-body parameter, and that break the scale invariance at the three-body level. It is at this cost that the zero-range model becomes well defined and leads to a self-adjoint Hamiltonian. The Efimov effect is now observed experimentally with cold atoms [4], which gives access to the value of the three-body parameter [5].

A natural question is to know whether a four-body Efimov effect is possible [6,7], leading to an infinite, asymptotically geometric, spectrum of tetramers, with an energy ratio predicted by a zero-range theory and a global energy scale fixed by a four-body parameter appearing in four-body contact conditions. It is now understood that a prerequisite to the four-body Efimov effect is the absence of three-body Efimov effect: It is indeed expected that the introduction of three-body contact conditions (in terms of the three-body parameter) imposed by the three-body Efimov effect is sufficient to also render the four-body problem well defined, that is, without the need for a four-body parameter. As predicted in Ref. [6], no geometric sequence of tetramer states can then be found but, as shown numerically for four bosons [8], sequences of four-body complex energy resonances are expected in general, with the same geometric ratio as the trimer Efimov spectrum (see Refs. [9,10] for early studies not accessing the imaginary part of the energy). This prerequisite rules out systems with more than one boson [3] as possible candidates for a four-body Efimov effect and suggests use of fermions to counterbalance the Efimov effect by the Pauli exclusion principle, at least in three dimensions (what happens in lower dimensions or with resonant interactions in other channels than the s wave is discussed in Refs. [11,12]).

Consider then the so-called $p+q$ fermionic problem: p identical fermions of the same species or spin state resonantly interact in free space with q identical fermions of another species or spin state. It is assumed that there is no interaction between the identical fermions, since they cannot scatter in the

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s wave. It is convenient to adopt a pseudospin notation, with $\uparrow$ for the first species and $\downarrow$ for the second. The two species have in general different masses $m_\uparrow$ and $m_\downarrow$, and the crucial idea is to use their mass ratio as an adjustable parameter to search for the four-body Efimov effect without triggering the three-body one.

The $3+1$ or $\uparrow\uparrow\uparrow\downarrow$ resonant fermionic problem was investigated in Ref. [13]. A four-body Efimov effect was predicted for a mass ratio $13.384 < m_\uparrow/m_\downarrow < 13.6069\ldots$. Beyond $13.6069\ldots$ the three-body Efimov effect sets in as shown in Refs. [3,14,15], which blocks the four-body Efimov effect as discussed above: Apart from a finite number of tetramer states, one expects an infinite number of four-body resonances with the same geometric ratio as for the $2+1$ problem.

The main motivation of the present work is to determine the presence or the absence of a four-body Efimov effect in the $2+2$ or $\uparrow\uparrow\downarrow\downarrow$ fermionic problem. To our knowledge, no general and rigorous answer was given to this problem. One may think of attacking it with the Born-Oppenheimer approximation. We indeed expect (as for the three-body case) that the only possibility for a four-body Efimov effect is to have a large mass imbalance between the two species, for example, the $\uparrow$ fermions are much heavier than the $\downarrow$ ones. It is found that, in the presence of two $\uparrow$ fermions at fixed positions, there is a *single* bound state for the $\downarrow$ particle, which creates an effective $\propto -\hbar^2/(m_\downarrow r^2)$ attraction between the $\uparrow$ fermions. For a large-enough $m_\uparrow/m_\downarrow$ mass ratio, this indeed beats the centrifugal barrier $\propto\hbar^2/(m_\uparrow r^2)$ between the $\uparrow$ particles (they are fermions and approach each other with a nonzero angular momentum), which qualitatively explains the occurrence of a three-body Efimov effect in the $2+1$ problem, as pointed out in 1973 by Efimov [3]. However, as there is a *single* bonding orbital, one cannot put a *second* $\downarrow$ fermion in that orbital, but one can at best put one in the ground, zero-energy scattering state, which has two consequences: (i) the Born-Oppenheimer attractive potential between the $\uparrow$ particles is not lowered by the second $\downarrow$ fermion, so no four-body Efimov effect is predicted at a mass ratio strictly below the three-body Efimov effect threshold, and (ii) as emphasized in Ref. [16], the second $\downarrow$ fermion, being in a zero-energy eigenstate, does not have a fast motion as compared to the one of the heavy particles, which sheds doubts on the validity of the Born-Oppenheimer approximation. Alternatively, one may expect that this $2+2$ problem was already solved numerically in the literature; however, no convincingly dense coverage of the mass ratio interval between 1 and $13.6069\ldots$ seems to be available in the numerics [17] considering the narrowness of the above-mentioned mass interval. To obtain a firm answer to the question, we generalize the method of Refs. [13,18], deriving from the zero-range model momentum space integral equations for the $2+2$ fermionic problem at zero energy (see also the most general formulation of reference [19]) and using rotational symmetry and scale invariance to reduce them to a numerically tractable form.

Another motivation is to pave the way for the calculation of the fourth virial coefficient of a two-component unitary Fermi gas: This would make an interesting bridge between few-body and many-body physics. For a unit mass ratio $m_\uparrow/m_\downarrow = 1$, the value of this virial coefficient was already obtained experimentally from a measurement of the equation of state of a gas of ultracold atoms [20,21]. On the theory side, there exist two main techniques. First, there is the diagrammatic technique, used exactly (all diagrams are kept) for the third virial coefficient [22,23] and approximately (only some diagrams are kept: those relevant in the perturbative regime of a large effective range or a small scattering length) for the fourth virial coefficient [24] leading to a value different from but reasonably close to the experimental value. Second, there is the harmonic regulator technique [25], used with success for the third virial coefficient [26–29] and that requires us to determine the spectrum of up to four particles in an isotropic harmonic trap. A first, brute-force numerical solution of this trapped four-body problem [30] was not able to recover even the sign of the experimental value. In a more analytical way, this spectrum can be deduced from the solutions of the zero-energy free space problem [31,32], due to the SO(2,1) dynamical symmetry of the unitary Fermi gas [32–34], so the four-body integral equations written here may also be useful for the solution of the virial problem.

Our article is organized as follows. In Sec. II we derive the zero-energy momentum-space integral equations in general form. In Sec. III we successively use the rotational invariance, the scale invariance, and the parity invariance to put the integral equations in a maximally reduced form. This reduced form, written in Sec. IV A, exactly expresses the fact that some operator M, depending on the angular momentum ℓ and the scaling exponent s, has a zero eigenvalue, which motivates its spectral analysis; it allows us to show that two components of the continuous spectrum of M can be expressed exactly in terms of the Efimov transcendental functions appearing in the $\uparrow\uparrow\downarrow$ and $\uparrow\downarrow\downarrow$ three-body problems (see Sec. IV B) and that there is a third, unexpected continuum due to a term with no equivalent in the $3+1$ problem (see Sec. IV C). The question of the existence of the four-body Efimov effect in the $2+2$ fermionic problem is the subject of Sect. V, whereas the secondary motivation of this work, i.e., the fourth virial coefficient of the spin-1/2 unitary Fermi gas, is relegated to the Appendix B, where its expression in terms of the operator M is conjectured from a transposition of the known analytic expression of the third virial coefficient [28,29], and the conjectured value is compared to the experimental [20,21] and theoretical [24,30] values. We conclude in Sec. VI.

II. DERIVATION OF THE GENERAL FOUR-BODY INTEGRAL EQUATIONS

Particles 1 and 2, of positions $\mathbf{r}_1$ and $\mathbf{r}_2$, belong to species $\uparrow$. Particles 3 and 4, of positions $\mathbf{r}_3$ and $\mathbf{r}_4$, belong to species $\downarrow$. The four-body wave function ψ is subjected to the usual Wigner-Bethe-Peierls contact conditions, for a zero-range interaction of s-wave scattering length a between opposite-spin particles. For all $i \in \{1,2\}$ and all $j \in \{3,4\}$, when the distance r_{ij} between particles i and j tends to zero, at fixed position $\mathbf{R}_{ij} = (m_\uparrow \mathbf{r}_i + m_\downarrow \mathbf{r}_j)/(m_\uparrow + m_\downarrow)$ of their center of mass (different from the positions of the remaining two particles), one imposes

$$\psi_{\uparrow\uparrow\downarrow\downarrow}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \underset{r_{ij}\to 0}{=} \left(\frac{1}{r_{ij}} - \frac{1}{a}\right)\frac{\mu_{\uparrow\downarrow}}{2\pi\hbar^2} \times \mathcal{A}_{ij}((\mathbf{r}_k - \mathbf{R}_{ij})_{k\neq i,j}) + O(r_{ij}), \tag{1}$$

where the form of the regular part $\mathcal{A}_{ij}$ supposes that the center of mass of the four particles is at rest, and where $\mu_{\uparrow\downarrow} = m_\uparrow m_\downarrow/(m_\uparrow + m_\downarrow)$ is the reduced mass of two opposite-spin particles. Due to the fermionic antisymmetry, the regular parts are not independent functions:

$$\mathcal{A}_{13} = \mathcal{A}_{24} = -\mathcal{A}_{14} = -\mathcal{A}_{23} \equiv \mathcal{A}. \tag{2}$$

Schrödinger's equation at zero eigenenergy $E = 0$, written in the language of distributions, is then

$$\begin{aligned} H\psi_{\uparrow\uparrow\downarrow\downarrow} = &\,\mathcal{A}(\mathbf{r}_2 - \mathbf{R}_{13},\mathbf{r}_4 - \mathbf{R}_{13})\delta(\mathbf{r}_1 - \mathbf{r}_3)\\ &- \mathcal{A}(\mathbf{r}_2 - \mathbf{R}_{14},\mathbf{r}_3 - \mathbf{R}_{14})\delta(\mathbf{r}_1 - \mathbf{r}_4)\\ &- \mathcal{A}(\mathbf{r}_1 - \mathbf{R}_{23},\mathbf{r}_4 - \mathbf{R}_{23})\delta(\mathbf{r}_2 - \mathbf{r}_3)\\ &+ \mathcal{A}(\mathbf{r}_1 - \mathbf{R}_{24},\mathbf{r}_3 - \mathbf{R}_{24})\delta(\mathbf{r}_2 - \mathbf{r}_4) \end{aligned} \tag{3}$$

with the kinetic energy Hamiltonian

$$H = \sum_{n=1}^{4} -\frac{\hbar^2}{2m_n}\Delta_{\mathbf{r}_n} \tag{4}$$

and $\delta(\mathbf{r})$ is the Dirac distribution in three dimensions, stemming from the identity $\Delta_{\mathbf{r}}(1/r) = -4\pi\delta(\mathbf{r})$.

We now go to momentum space and we take the Fourier transform of Schrödinger's equation. In the left-hand side, each Laplace operator gives rise to a factor $-k_n^2$, where $\mathbf{k}_n$ is the wave vector of particle number n. In the right-hand side, one obtains, for example, for the first term:

$$\begin{aligned} &\int \prod_{n=1}^{4} d^3r_n e^{-i\sum_{n=1}^{4}\mathbf{k}_n\cdot\mathbf{r}_n}\mathcal{A}(\mathbf{r}_2 - \mathbf{R}_{13},\mathbf{r}_4 - \mathbf{R}_{13})\delta(\mathbf{r}_1 - \mathbf{r}_3)\\ &= (2\pi)^3\tilde{\mathcal{A}}(\mathbf{k}_2,\mathbf{k}_4)\delta\left(\sum_{n=1}^{4}\mathbf{k}_n\right), \end{aligned} \tag{5}$$

where the tilde indicates the Fourier transform. Introducing the function $D \equiv (2\pi)^3\tilde{\mathcal{A}}$, we obtain the four-body momentum space ansatz generalizing to the 2+2 fermionic problem the one of the 3+1 fermionic problem [13,18]:

$$\begin{aligned} \tilde{\psi}_{\uparrow\uparrow\downarrow\downarrow}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = &\frac{\delta\left(\sum_{n=1}^4 \mathbf{k}_n\right)}{\sum_{n=1}^4 \frac{\hbar^2k_n^2}{2m_n}}[D(\mathbf{k}_2,\mathbf{k}_4)\\ &- D(\mathbf{k}_2,\mathbf{k}_3) - D(\mathbf{k}_1,\mathbf{k}_4) + D(\mathbf{k}_1,\mathbf{k}_3)]. \end{aligned} \tag{6}$$

The ansatz obeys fermionic antisymmetry and Schrödinger's equation, not yet the contact condition (1), that it suffices to implement for $(i,j) = (1,3)$. One thus takes the inverse Fourier transform of $\tilde{\psi}$ at $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$, with the parametrization:

$$\mathbf{r}_1 = \mathbf{R}_{13} + \frac{m_3}{m_1+m_3}\mathbf{r}_{13}, \tag{7}$$

$$\mathbf{r}_3 = \mathbf{R}_{13} - \frac{m_1}{m_1+m_3}\mathbf{r}_{13}. \tag{8}$$

Only the contribution ψ_{24} of $D(\mathbf{k}_2,\mathbf{k}_4)$ to ψ diverges for $r_{13} \to 0$; in that inverse Fourier transform, we then take $\mathbf{K}_{13} = \mathbf{k}_1 + \mathbf{k}_3$, $\mathbf{k}_{13} = \mu_{13}(\mathbf{k}_1/m_1 - \mathbf{k}_3/m_3)$, and $\mathbf{k}_2,\mathbf{k}_4$ as integration variables (clearly $\mu_{13} = \mu_{\uparrow\downarrow}$), so $\mathbf{k}_1\cdot\mathbf{r}_1 + \mathbf{k}_3\cdot\mathbf{r}_3 = \mathbf{K}_{13}\cdot\mathbf{R}_{13} + \mathbf{k}_{13}\cdot\mathbf{r}_{13}$ and $\frac{\hbar^2k_1^2}{2m_1} + \frac{\hbar^2k_3^2}{2m_3} = \frac{\hbar^2k_{13}^2}{2\mu_{13}} + \frac{\hbar^2K_{13}^2}{2(m_1+m_3)}$; integration over $\mathbf{K}_{13}$ is straightforward, due to the momentum conservation, and integration over $\mathbf{k}_{13}$ also can be done using

$$u(\mathbf{r}) = \int \frac{d^3k_{13}}{(2\pi)^3}\frac{e^{i\mathbf{k}_{13}\cdot\mathbf{r}}}{k_{13}^2 + q_{13}^2} = \frac{e^{-q_{13}r}}{4\pi r}. \tag{9}$$

One obtains

$$\begin{aligned} \psi_{24}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = &\int \frac{d^3k_2d^3k_4}{(2\pi)^9}\frac{2\mu_{13}}{\hbar^2}u(r_{13})\\ &\times e^{i[\mathbf{k}_2\cdot(\mathbf{r}_2-\mathbf{R}_{13})+\mathbf{k}_4\cdot(\mathbf{r}_4-\mathbf{R}_{13})]}D(\mathbf{k}_2,\mathbf{k}_4) \end{aligned} \tag{10}$$

with $q_{13} \geqslant 0$ such that

$$\frac{\hbar^2q_{13}^2}{2\mu_{13}} = \frac{\hbar^2(\mathbf{k}_2+\mathbf{k}_4)^2}{2(m_1+m_3)} + \frac{\hbar^2k_2^2}{2m_2} + \frac{\hbar^2k_4^2}{2m_4}. \tag{11}$$

Taking $r_{13} \to 0$ in ψ_{24} is then elementary. In the contribution to ψ of $D(\mathbf{k}_2,\mathbf{k}_3)$, $D(\mathbf{k}_1,\mathbf{k}_4)$, and $D(\mathbf{k}_1,\mathbf{k}_3)$, noted as $\psi_{\neq 24}$, one can directly take $\mathbf{r}_{13} = \mathbf{0}$. Thanks to momentum conservation one can replace $\mathbf{k}_1 + \mathbf{k}_3$ by $-(\mathbf{k}_2 + \mathbf{k}_4)$ within the position-dependent phase factor, so the positions $\mathbf{r}_2 - \mathbf{R}_{13}$ and $\mathbf{r}_4 - \mathbf{R}_{13}$ appear as in Eq. (10):

$$\begin{aligned} &\psi_{\neq 24}(\mathbf{r}_1 = \mathbf{R}_{13},\mathbf{r}_2,\mathbf{r}_3 = \mathbf{R}_{13},\mathbf{r}_4)\\ &= \int \frac{d^3k_2d^3k_4}{(2\pi)^9}e^{i[\mathbf{k}_2\cdot(\mathbf{r}_2-\mathbf{R}_{13})+\mathbf{k}_4\cdot(\mathbf{r}_4-\mathbf{R}_{13})]}\int\frac{d^3k_1d^3k_3}{(2\pi)^3}\\ &\quad\times\frac{\delta\left(\sum_{n=1}^4\mathbf{k}_n\right)}{\sum_{n=1}^4\frac{\hbar^2k_n^2}{2m_n}}[-D(\mathbf{k}_2,\mathbf{k}_3) - D(\mathbf{k}_1,\mathbf{k}_4) + D(\mathbf{k}_1,\mathbf{k}_3)]. \end{aligned} \tag{12}$$

Finally, the contact condition at the unitary limit, that is, for $1/a = 0$, leads to the following integral equation for D:

$$\begin{aligned} 0 = &\frac{\mu_{\uparrow\downarrow}^{3/2}}{2\pi\hbar^2}\left[\frac{(\mathbf{k}_2+\mathbf{k}_4)^2}{m_\uparrow+m_\downarrow} + \frac{k_2^2}{m_\uparrow} + \frac{k_4^2}{m_\downarrow}\right]^{1/2}D(\mathbf{k}_2,\mathbf{k}_4)\\ &+\int\frac{d^3k_1d^3k_3}{(2\pi)^3}\frac{\delta\left(\sum_{n=1}^4\mathbf{k}_n\right)}{\sum_{n=1}^4\frac{\hbar^2k_n^2}{2m_n}}\\ &\times[D(\mathbf{k}_2,\mathbf{k}_3) + D(\mathbf{k}_1,\mathbf{k}_4) - D(\mathbf{k}_1,\mathbf{k}_3)], \end{aligned} \tag{13}$$

where the first term is simply $\frac{q_{13}\mu_{13}}{2\pi\hbar^2}D(\mathbf{k}_2,\mathbf{k}_4)$. Contrarily to the $3+1$ fermionic case [13,18], D is not subjected to any condition of exchange symmetry.

III. TAKING ADVANTAGE OF SYMMETRIES

A. Overview

The unknown function $D(\mathbf{k}_2,\mathbf{k}_4)$ in the integral equation (13) depends on six real variables. This is already a strong reduction, as compared to the 12 real variables of the original four-body wave function, but this still makes a numerical solution challenging.

Fortunately, one can use rotational invariance as in Sec. III B: the unknown function D can be considered, for example, as being the $m_z = 0$ component of a spinor of angular momentum ℓ. Then it is known how the various $2\ell+1$ components of the spinor transform under an arbitrary common rotation of $\mathbf{k}_2$ and $\mathbf{k}_4$, in terms of rotation matrices having spherical harmonics as matrix elements, so it suffices to know the value of the $2\ell+1$ component of the spinor in the particular configuration where vector $\mathbf{k}_2$ points along the x axis in the positive direction and $\mathbf{k}_4$ lies in the xy upper half-plane $y \geqslant 0$, at an angle $\theta_{24} \in [0,\pi]$ with respect to $\mathbf{k}_2$. As this particular configuration is characterized by the cosine of the angle θ_{24} and the two moduli k_2 and k_4, the unknown function $D(\mathbf{k}_2,\mathbf{k}_4)$ can be represented in terms of $2\ell+1$ unknown functions $f_{m_z}^{(\ell)}$ of these three real variables [18]:

$$D(\mathbf{k}_2,\mathbf{k}_4) = \sum_{m_z=-\ell}^{\ell} \left[Y_\ell^{m_z}(\mathbf{e}_2\cdot\mathbf{e}_z,\mathbf{e}_{4\perp 2}\cdot\mathbf{e}_z,\mathbf{e}_{24}\cdot\mathbf{e}_z)\right]^* \times f_{m_z}^{(\ell)}(k_2,k_4,u_{24}). \tag{14}$$

In this expression we have introduced the unit vectors

$$\mathbf{e}_2 = \frac{\mathbf{k}_2}{k_2}, \tag{15}$$

$$\mathbf{e}_{4\perp 2} = \frac{1}{v_{24}}\left(\frac{\mathbf{k}_4}{k_4} - u_{24}\mathbf{e}_2\right), \tag{16}$$

$$\mathbf{e}_{24} = \frac{\mathbf{k}_2\wedge\mathbf{k}_4}{|\mathbf{k}_2\wedge\mathbf{k}_4|}. \tag{17}$$

Here $\theta_{24}\in[0,\pi]$ is the angle between $\mathbf{k}_2$ and $\mathbf{k}_4$, and the notations

$$u_{24} \equiv \cos\theta_{24} \quad\text{and}\quad v_{24}\equiv \sin\theta_{24} \tag{18}$$

will be used throughout the paper. It is apparent that $\mathbf{e}_{4\perp 2}$ is obtained by projecting $\mathbf{e}_4 = \mathbf{k}_4/k_4$ orthogonally to $\mathbf{e}_2$ and by renormalizing the result to unity. Then $(\mathbf{e}_2,\mathbf{e}_{4\perp 2},\mathbf{e}_{24})$ forms a direct orthonormal basis. In that basis, an arbitrary (unit) vector $\mathbf{n}$ has uniquely defined spherical coordinates, that is, polar angle $\theta_{\mathbf{n}}\in[0,\pi]$ with respect to the axis $\mathbf{e}_{24}$ and the azimuthal angle $\phi_{\mathbf{n}}\in[0,2\pi[$ in the $\mathbf{e}_2-\mathbf{e}_{4\perp 2}$ plane with respect to axis $\mathbf{e}_2$. Then

$$Y_\ell^{m_z}(\mathbf{e}_2\cdot\mathbf{n},\mathbf{e}_{4\perp 2}\cdot\mathbf{n},\mathbf{e}_{24}\cdot\mathbf{n}) \equiv Y_\ell^{m_z}(\theta_{\mathbf{n}},\phi_{\mathbf{n}}), \tag{19}$$

where the right-hand side is the standard notation for the spherical harmonics [35]. Integral equations can then be obtained for the $f_{m_z}^{(\ell)}$, see Sec. III B.

For an infinite s-wave scattering length the Wigner-Bethe-Peierls contact conditions (1) are scale invariant. As the integral equation (13) was further specialized to the zero-energy case, its solution can be taken as scale invariant, which allows one to eliminate one more variable [13]:

$$f_{m_z}^{(\ell)}(k_2,k_4,u_{24}) = \left(k_2^2+k_4^2\right)^{-(s+7/2)/2}(\mathrm{ch}\,x)^{s+3/2} \times e^{im_z\theta_{24}/2}\Phi_{m_z}^{(\ell)}(x,u_{24}) \tag{20}$$

with

$$x \equiv \ln\frac{k_4}{k_2}. \tag{21}$$

The first factor contains the scaling exponent of the solution, which involves the unknown quantity s. By inserting the ansatz (20) into the linear integral equations of Sec. III B, one obtains linear integral equations for the unknown functions $\Phi_{m_z}^{(\ell)}(x,u)$, represented by a matrix $M^{(\ell)}(s)$ that depends parametrically on s, see Sec. III C; requiring that the functions $\Phi_{m_z}^{(\ell)}(x,u)$ are not identically zero, one gets an implicit equation for s, in the form [36]

$$\det M^{(\ell)}(s) = 0. \tag{22}$$

The way the first factor in Eq. (20) is parametrized by the quantity s ensures compatibility with the notation used by Efimov for the three-body problem [3]. In the three-body problem, the Efimov effect takes place if and only if one of the scaling exponents s is purely imaginary, and the geometric trimer energy spectrum has then a ratio $\exp(-2\pi/|s|)$. In the four-body problem, with our definition of s, the four-body Efimov effect occurs if and only if there is a purely imaginary s solving Eq. (22), in which case there exists a geometric sequence of tetramer eigenenergies with a ratio $\exp(-2\pi/|s|)$. A justification is given in Ref. [13]. The second factor in the ansatz (20) ensures that the matrix $M(s)$ is Hermitian for purely imaginary s, with bounded diagonal matrix elements, which is both mathematically and numerically advantageous. As compared to Ref. [13] it contains an additional term s in the exponent that for purely imaginary s suppresses phase oscillations in the matrix elements of $M(s)$ at large $|x|$ [37]. The third factor in Eq. (20) is a phase factor taking into account the fact that exchanging $\mathbf{k}_2$ and $\mathbf{k}_4$ in Eq. (14) transforms the spherical harmonics $Y_\ell^{m_z}$ into $(-1)^\ell e^{im_z\theta_{24}}Y_\ell^{-m_z}$ with the same values of the variables [18]; it ensures that the matrix $M(s)$ transforms in the simplest way under the exchange of $m_\uparrow$ and $m_\downarrow$, which must leave our $2+2$ problem invariant.

A last reduction of the problem can be obtained from parity invariance. It turns out that, under the transformation $(\mathbf{k}_2,\mathbf{k}_4)\to(-\mathbf{k}_2,-\mathbf{k}_4)$, the term of index m_z in the sum (14) acquires a factor $(-1)^{m_z}$ [18]. This shows that the odd-parity functions $\Phi_{m_z}^{(\ell)}$ (that is with m_z odd) are decoupled from the even-parity functions $\Phi_{m_z}^{(\ell)}$ (that is with m_z even) in the integral equations, and that $M^{(\ell)}(s)$ has zero matrix elements between the odd and the even channels.

B. Rotational invariance

To obtain the integral equations for the unknown functions $f_{m_z}^{(\ell)}$ in Eq. (14) we use a variational formulation: The integral equation (13) is equivalent to

$$\partial_{D^*(\mathbf{k}_2,\mathbf{k}_4)}\mathcal{E}[D,D^*] = 0, \tag{23}$$

where D and its complex conjugate D^* are taken as independent variables, ∂_{D^*} is the functional derivative with respect to D^*, and the functional $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{E}_{\mathrm{diag}} + \mathcal{E}_{24,23} + \mathcal{E}_{24,14} - \mathcal{E}_{24,13} \tag{24}$$

with the diagonal part

$$\mathcal{E}_{\mathrm{diag}} = \int d^3k_2 d^3k_4\, D^*(\mathbf{k}_2,\mathbf{k}_4)D(\mathbf{k}_2,\mathbf{k}_4) \times \frac{\mu_{\uparrow\downarrow}^{3/2}}{2\pi\hbar^2}\left[\frac{(\mathbf{k}_2+\mathbf{k}_4)^2}{m_\uparrow+m_\downarrow} + \frac{k_2^2}{m_\uparrow} + \frac{k_4^2}{m_\downarrow}\right]^{1/2} \tag{25}$$

and the generic off-diagonal part

$$\mathcal{E}_{24,ij} = \int \frac{d^3k_2 d^3k_4 d^3k_1 d^3k_3}{(2\pi)^3} D^*(\mathbf{k}_2,\mathbf{k}_4) D(\mathbf{k}_i,\mathbf{k}_j) \times \frac{\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\frac{\hbar^2}{2m_\uparrow}(k_1^2+k_2^2)+\frac{\hbar^2}{2m_\downarrow}(k_3^2+k_4^2)}. \tag{26}$$

Then one inserts the ansatz (14) into these functionals. Assuming that one is able to integrate over all variables other than k_2,k_4,θ_{24} and k_i,k_j,θ_{ij}, one obtains a functional of the $f_{m_z}^{(\ell)}$ and $f_{m_z}^{(\ell)*}$, which it remains to differentiate with respect to $f_{m_z}^{(\ell)*}$ to obtain the integral equations for the $f_{m_z}^{(\ell)}$.

Integration is simplified as follows: The final integral equations and their solutions $f_{m_z}^{(\ell)}$ cannot depend on the specific vector $\mathbf{e}_z$ introduced in Eq. (14). One can then replace $\mathbf{e}_z$ by an arbitrary unit vector $\mathbf{n}$ in the ansatz (14), and one can average the resulting functional $\mathcal{E}$ over $\mathbf{n}$ uniformly on the unit sphere for fixed $f_{m_z}^{(\ell)}$. The result of this average is particularly simple when the orthonormal basis of Eqs. (15)–(17) reduces to the usual Cartesian basis:

$$(\mathbf{e}_2,\mathbf{e}_{4\perp 2},\mathbf{e}_{24}) = (\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z). \tag{27}$$

Then [38]

$$\left\langle Y_\ell^{m_z}(\mathbf{e}_2\cdot\mathbf{n},\mathbf{e}_{4\perp 2}\cdot\mathbf{n},\mathbf{e}_{24}\cdot\mathbf{n})\left[Y_\ell^{m_z'}(\mathbf{e}_i\cdot\mathbf{n},\mathbf{e}_{j\perp i}\cdot\mathbf{n},\mathbf{e}_{ij}\cdot\mathbf{n})\right]^*\right\rangle_{\mathbf{n}} = \frac{1}{4\pi}(\langle \ell,m_z|R^{(ij)}|\ell,m_z'\rangle)^*, \tag{28}$$

where $\langle\ldots\rangle_{\mathbf{n}}$ indicates the average over the direction of $\mathbf{n}$ and the quantum operator $R^{(ij)}$ represents (in the usual spin-ℓ irreducible representation, with vectors $|\ell,m_z\rangle$ of angular momentum $m_z\hbar$ along z) the unique real space rotation $\mathcal{R}^{(ij)}$ that maps the Cartesian basis onto the basis $(\mathbf{e}_i,\mathbf{e}_{j\perp i},\mathbf{e}_{ij})$:

$$(\mathbf{e}_i,\mathbf{e}_{j\perp i},\mathbf{e}_{ij}) = \mathcal{R}^{(ij)}(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z). \tag{29}$$

After average over $\mathbf{n}$, and integration over $\mathbf{k}_1$ and $\mathbf{k}_3$ in $\mathcal{E}_{24,ij}$, it remains an integral over $\mathbf{k}_2$ and $\mathbf{k}_4$, with an integrand invariant by common rotation of $\mathbf{k}_2$ and $\mathbf{k}_4$. To evaluate that integrand, one can then indeed assume that $\mathbf{k}_2$ is along x (in the positive direction) and that $\mathbf{k}_4$ lies in the plane xy in the upper half $y\geqslant 0$:

$$\mathbf{k}_2 = k_2\mathbf{e}_x, \tag{30}$$

$$\mathbf{k}_4 = k_4(\cos\theta_{24}\mathbf{e}_x+\sin\theta_{24}\mathbf{e}_y) \text{ with } \theta_{24}\in[0,\pi], \tag{31}$$

so

$$(\mathbf{e}_2,\mathbf{e}_{4\perp 2},\mathbf{e}_{24}) = (\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z) \tag{32}$$

and one can use Eqs. (28) and (29). Then one pulls out a factor 4π (resulting from integration over the solid angle of $\mathbf{k}_2$) compensated by the 4π denominator in Eq. (28), and an uncompensated factor 2π (resulting from the integration over the azimuthal angle of $\mathbf{k}_4$ for the spherical coordinates of polar axis $\mathbf{k}_2/k_2=\mathbf{e}_x$ for $\mathbf{k}_4$), and one is left with an integration over the moduli k_2 and k_4 and over the angle θ_{24}.

For the functional $\mathcal{E}_{\rm diag}$, this gives a simple result: Since $i=2$ and $j=4$, the matrix $\mathcal{R}^{(ij)}$ is the identity matrix, $R^{(ij)}$ reduces to the identity operator; also, there is no $\mathbf{k}_1$ or $\mathbf{k}_3$ integration. One obtains

$$\mathcal{E}_{\rm diag} = \sum_{m_z=-\ell}^{\ell} 2\pi\int_0^\infty dk_2k_2^2dk_4k_4^2\int_{-1}^1 du_{24}|f_{m_z}(k_2,k_4,u_{24})|^2 \times\frac{\mu_{\uparrow\downarrow}^{3/2}}{2\pi\hbar^2}\left(\frac{k_2^2+k_4^2+2k_2k_4u_{24}}{m_\uparrow+m_\downarrow}+\frac{k_2^2}{m_\uparrow}+\frac{k_4^2}{m_\downarrow}\right)^{1/2} \tag{33}$$

with the same notation as in Eq. (18). For the generic off-diagonal part this leads to

$$\mathcal{E}_{24,ij} = \sum_{m_z,m_z'=-\ell}^{\ell} 2\pi\int_0^\infty dk_2k_2^2dk_4k_4^2\int_{-1}^1 du_{24}\int\frac{d^3k_1d^3k_3}{(2\pi)^3} \times(\langle\ell,m_z|R^{(ij)}|\ell,m_z'\rangle)^* f_{m_z}^{(\ell)*}(k_2,k_4,u_{24})f_{m_z'}^{(\ell)}(k_i,k_j,u_{ij}) \times\frac{\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\frac{\hbar^2(k_1^2+k_2^2)}{2m_\uparrow}+\frac{\hbar^2(k_3^2+k_4^2)}{2m_\downarrow}}. \tag{34}$$

The way to proceed with the integration over the directions of $\mathbf{k}_1$ and $\mathbf{k}_3$ depends on the indices i and j.

1. Case $(i,j)=(2,3)$

For $(i,j)=(2,3)$, one trivially integrates over $\mathbf{k}_1$ using the Dirac distribution that imposes $\mathbf{k}_1=-(\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$, and one integrates over $\mathbf{k}_3$ using spherical coordinates of polar axis $\mathbf{e}_x$ and of azimuthal axis $\mathbf{e}_y$; the azimuthal angle is called ϕ, and the polar angle is called θ_{23} since it is the angle between $\mathbf{k}_2$ and $\mathbf{k}_3$ [see Fig. 1(a)]. Then $\mathcal{R}^{(ij)}$ in Eq. (29) is the rotation of axis x and of angle ϕ:

$$\mathcal{R}^{(23)} = \mathcal{R}_x(\phi) \text{ and } R^{(23)} = e^{-i\phi L_x/\hbar}, \tag{35}$$

where L_x is the angular-momentum operator along x. Also

$$k_1^2 = k_2^2+k_3^2+k_4^2+2k_2k_3u_{23}+2k_2k_4u_{24} + 2k_3k_4(u_{23}u_{24}+v_{23}v_{24}\cos\phi) \tag{36}$$

with $u_{23}=\cos\theta_{23}$ and $v_{23}=\sin\theta_{23}$ as in Eq. (18). This gives

$$\mathcal{E}_{24,23} = \sum_{m_z,m_z'=-\ell}^{\ell} 2\pi\int_0^\infty dk_2dk_3dk_4k_2^2k_3^2k_4^2 \times\int_{-1}^1 du_{23}du_{24}\int_0^{2\pi}d\phi \times\frac{\langle\ell,m_z|e^{i\phi L_x/\hbar}|\ell,m_z'\rangle f_{m_z}^{(\ell)*}(k_2,k_4,u_{24})f_{m_z'}^{(\ell)}(k_2,k_3,u_{23})}{(2\pi)^3\left[\frac{\hbar^2(k_1^2+k_2^2)}{2m_\uparrow}+\frac{\hbar^2(k_3^2+k_4^2)}{2m_\downarrow}\right]}, \tag{37}$$

where k_1 is given by Eq. (36) and we used the fact that L_x has real matrix elements in the standard $|\ell,m_z\rangle$ basis.

2. Case $(i,j)=(1,4)$

For $(i,j)=(1,4)$, one integrates over $\mathbf{k}_3$ using the Dirac distribution that imposes $\mathbf{k}_3=-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_4)$ and one integrates over $\mathbf{k}_1$ using spherical coordinates in a rotated basis

$$(\mathbf{e}_X,\mathbf{e}_Y,\mathbf{e}_Z) = (\mathbf{e}_z,\mathbf{e}_4\wedge\mathbf{e}_z,\mathbf{e}_4) \text{ with } \mathbf{e}_4=\frac{\mathbf{k}_4}{k_4}. \tag{38}$$

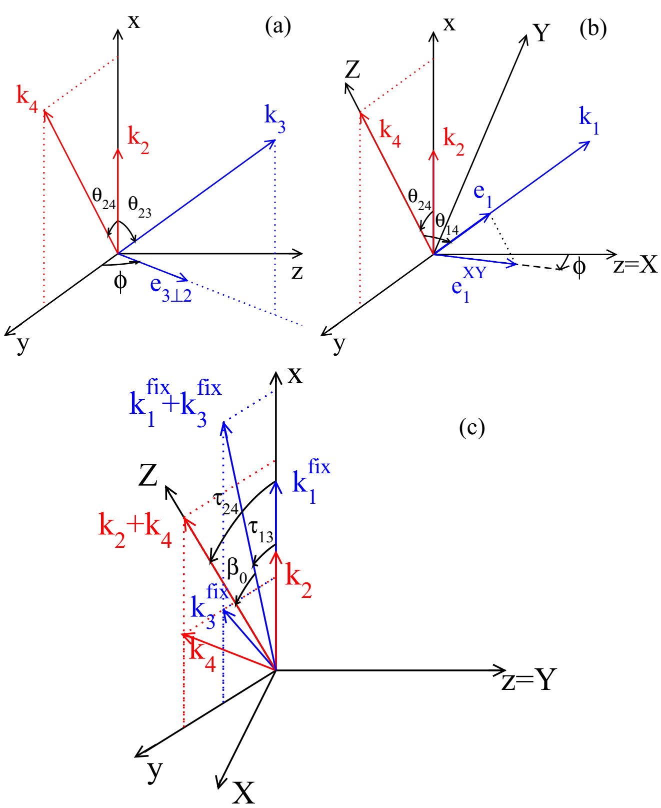

FIG. 1. (Color online) Positions and parametrizations of the wave vectors appearing in the angular integration in the functionals $\mathcal{E}_{24,ij}$. The vectors $\mathbf{k}_2$ and $\mathbf{k}_4$ are given by Eqs. (30) and (31). (a) For $(i,j)=(2,3)$, $\mathbf{k}_1=-(\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ and one integrates over $\mathbf{k}_3$ using spherical coordinates of polar axis x and azimuthal axis y. (b) For $(i,j)=(1,4)$, $\mathbf{k}_3=-(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_4)$ and one integrates over $\mathbf{k}_1$ using the polar axis Z (direction of $\mathbf{k}_4$) and the azimuthal axis X (direction of $\mathbf{e}_z$) as defined by Eq. (38), leading to the polar angle θ_{14} and the azimuthal angle ϕ (<0 in the figure). The dashed line gives the direction of the component $\mathbf{e}_1^{XY}$ of $\mathbf{e}_1=\mathbf{k}_1/k_1$ in the XY plane. (c) For $(i,j)=(1,3)$, one integrates over the rotation $\mathcal{R}$, such that $\mathbf{k}_1$ and $\mathbf{k}_3$ are given by the action of $\mathcal{R}$ on vectors $\mathbf{k}_1^{\rm fix}$ and $\mathbf{k}_3^{\rm fix}$ in the xy plane as in Eqs. (43) and (44), using the parametrization in terms of Euler angles associated to the convenient axes X, Y, and Z of Eq. (48).

The direction $\mathbf{e}_Z$ of $\mathbf{k}_4$ is taken as the polar axis, so the polar angle is θ_{14}; $\mathbf{e}_X$ is taken as the azimuthal axis, with the azimuthal angle called ϕ, see Fig. 1(b). Then the real space rotation $\mathcal{R}^{(ij)}$ in Eq. (29) is

$$\begin{aligned}\mathcal{R}^{(14)} &= \mathcal{R}_Z\Big(\phi-\frac{\pi}{2}\Big)\mathcal{R}_z(\theta_{24}-\theta_{14})\\ &= \mathcal{R}_z(\theta_{24})\mathcal{R}_x\Big(\phi-\frac{\pi}{2}\Big)\mathcal{R}_z(-\theta_{14})\end{aligned} \tag{39}$$

and the corresponding operator has matrix elements

$$\begin{aligned}\langle \ell,m_z|R^{(14)}|\ell,m_z'\rangle &= e^{-im_z\theta_{24}}\\ &\quad\times\langle \ell,m_z|e^{-i(\phi-\frac{\pi}{2})L_x/\hbar}|\ell,m_z'\rangle e^{im_z'\theta_{14}}.\end{aligned} \tag{40}$$

Using $\mathbf{e}_x=[\mathcal{R}^{(14)}]^{-1}\mathbf{e}_1=u_{24}\mathbf{e}_Z+v_{24}\mathbf{e}_Y$ and $\mathbf{e}_1=u_{14}\mathbf{e}_Z+v_{14}(\cos\phi\,\mathbf{e}_X+\sin\phi\,\mathbf{e}_Y)$ we get

$$\begin{aligned}k_3^2 &= k_1^2+k_2^2+k_4^2+2k_1k_2[u_{14}u_{24}+v_{14}v_{24}\cos(\phi-\pi/2)]\\ &\quad+2k_1k_4u_{14}+2k_2k_4u_{24}.\end{aligned} \tag{41}$$

This gives:

$$\begin{aligned}\mathcal{E}_{24,14} &= \sum_{m_z,m_z'=-\ell}^{\ell} 2\pi\int_0^\infty dk_1dk_2dk_4k_1^2k_2^2k_4^2\\ &\quad\times\int_{-1}^{1}du_{14}du_{24}\int_0^{2\pi}d\phi\\ &\quad\times\frac{e^{im_z\theta_{24}}\langle\ell,m_z|e^{i(\phi-\pi/2)L_x/\hbar}|\ell,m_z'\rangle e^{-im_z'\theta_{14}}}{(2\pi)^3\Big[\frac{\hbar^2(k_1^2+k_2^2)}{2m_\uparrow}+\frac{\hbar^2(k_3^2+k_4^2)}{2m_\downarrow}\Big]}\\ &\quad\times f_{m_z}^{(\ell)*}(k_2,k_4,u_{24})f_{m_z'}^{(\ell)}(k_1,k_4,u_{14}),\end{aligned} \tag{42}$$

where k_3 is given by Eq. (41).

3. Case $(i,j)=(1,3)$

For $(i,j)=(1,3)$, we find it convenient to replace the integration over $\mathbf{k}_1$ and $\mathbf{k}_3$ by an integration over the moduli k_1 and k_3, over the angle $\theta_{13}\in[0,\pi]$ and over a rotation matrix $\mathcal{R}$ uniformly distributed over the rotation group SO(3), the vectors $\mathbf{k}_1$ and $\mathbf{k}_3$ being given by the action of $\mathcal{R}$ on vectors fixed in the xy plane:

$$\mathbf{k}_1=\mathcal{R}\mathbf{k}_1^{\rm fix}\ \text{with}\ \mathbf{k}_1^{\rm fix}=k_1\mathbf{e}_x, \tag{43}$$

$$\mathbf{k}_3=\mathcal{R}\mathbf{k}_3^{\rm fix}\ \text{with}\ \mathbf{k}_3^{\rm fix}=k_3(u_{13}\mathbf{e}_x+v_{13}\mathbf{e}_y). \tag{44}$$

Then $\mathcal{R}$ is precisely the rotation matrix $\mathcal{R}^{(ij)}$ of Eq. (29) and

$$\begin{aligned}\mathcal{E}_{24,13} &= \sum_{m_z,m_z'=-\ell}^{\ell} 2\int_0^\infty\Big(\prod_{n=1}^4 dk_nk_n^2\Big)\int_{-1}^{1}du_{13}du_{24}\int_{\rm SO(3)}d\mathcal{R}\\ &\quad\times(\langle\ell,m_z|R|\ell,m_z'\rangle)^*f_{m_z}^{(\ell)*}(k_2,k_4,u_{24})f_{m_z'}^{(\ell)}(k_1,k_3,u_{13})\\ &\quad\times\frac{\delta\big[\mathbf{k}_2+\mathbf{k}_4+\mathcal{R}\big(\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix}\big)\big]}{\frac{\hbar^2(k_1^2+k_2^2)}{2m_\uparrow}+\frac{\hbar^2(k_3^2+k_4^2)}{2m_\downarrow}},\end{aligned} \tag{45}$$

where the factor 2 originates from $(4\pi\times2\pi)^2/[4\pi(2\pi)^3]$, R is the operator representing $\mathcal{R}$, and $d\mathcal{R}$ is the invariant measure over the group SO(3) normalized to unity (see Sec. 8.2 of Ref. [35]) [39]. To integrate over $\mathcal{R}$, we use the Euler parametrization as in Eq. (7.1-12) of Ref. [35]:

$$\mathcal{R}=\mathcal{R}_Z(\alpha)\mathcal{R}_Y(\beta)\mathcal{R}_Z(\gamma), \tag{46}$$

where the Euler angles α and γ run over an interval of length 2π and the Euler angle β runs over $[0,\pi]$, so the invariant measure is (see Sec. 8.2 of Ref. [35])

$$d\mathcal{R}=\frac{d\alpha\sin\beta d\beta d\gamma}{8\pi^2}. \tag{47}$$

Due to the occurrence of $\mathbf{k}_2+\mathbf{k}_4$ in the argument of the Dirac distribution in Eq. (45), the convenient direct orthonormal basis defining the rotation axes X, Y, and Z is now [see Fig. 1(c)]

$$(\mathbf{e}_X,\mathbf{e}_Y,\mathbf{e}_Z) = \left(\mathbf{e}_z \wedge \frac{\mathbf{k}_2+\mathbf{k}_4}{|\mathbf{k}_2+\mathbf{k}_4|},\mathbf{e}_z,\frac{\mathbf{k}_2+\mathbf{k}_4}{|\mathbf{k}_2+\mathbf{k}_4|}\right). \tag{48}$$

Then the Dirac distribution can be written as [40]

$$\begin{aligned}\delta\big[\mathbf{k}_2+\mathbf{k}_4+\mathcal{R}\big(\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big)\big] &= \delta(\sin\gamma)\delta[\sin(\beta_0-\beta\cos\gamma)]\\ &\quad\times\frac{\delta\big(|\mathbf{k}_2+\mathbf{k}_4|\cos(\beta_0-\beta\cos\gamma)+\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\big)}{|\sin\beta_0|\,\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\,|\mathbf{k}_2+\mathbf{k}_4|},\end{aligned} \tag{49}$$

where we have introduced the oriented angle β_0 between $\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}$ and $\mathbf{k}_2+\mathbf{k}_4$ such that [see Fig. 1(c)]

$$\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}} = \big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|(-\sin\beta_0\,\mathbf{e}_X+\cos\beta_0\,\mathbf{e}_Z). \tag{50}$$

There is no dependence on α in the right-hand side of Eq. (49): In the argument of δ, one can write $\mathbf{k}_2+\mathbf{k}_4$ as $\mathcal{R}_Z(\alpha)(\mathbf{k}_2+\mathbf{k}_4)$ and, due to the rotational invariance of the three-dimensional Dirac distribution, one can factor out and remove the rotation $\mathcal{R}_Z(\alpha)$. The integration over α in Eq. (45) then pulls out in the matrix element of R the orthogonal projector on the state of total angular momentum ℓ and of vanishing angular momentum along Z:

$$\int_0^{2\pi} d\alpha\, e^{-i\alpha L_Z/\hbar} = 2\pi|\ell,m_Z=0\rangle\langle\ell,m_Z=0|. \tag{51}$$

In the integral over γ, for example, over the interval $[-\pi/2,3\pi/2]$, only the points $\gamma=0$ and $\gamma=\pi$ contribute. The contribution of $\gamma=\pi$ can be deduced from the one of $\gamma=0$ by changing β into $-\beta$, due to $\mathcal{R}_Z(\pi)\mathcal{R}_Y(\beta)\mathcal{R}_Z(\pi)=\mathcal{R}_Y(-\beta)$ and to the invariance of $\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}$ and $|m_Z=0\rangle$ by rotation of axis Z. In the integral over $\beta\in[0,\pi]$, the $\gamma=\pi$ contribution can then be taken into account by extending the integration of the $\gamma=0$ contribution to $\beta\in[-\pi,0]$: One can take $\gamma=0$ in Eq. (49) and one faces

$$\begin{aligned}&\int_{-\pi}^{\pi} d\beta|\sin\beta|e^{i\beta m_z'}\delta[\sin(\beta_0-\beta)]\delta\big(|\mathbf{k}_2+\mathbf{k}_4|\cos(\beta_0-\beta)\\ &\quad+\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\big) = \delta\big(|\mathbf{k}_2+\mathbf{k}_4|-\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\big)\\ &\quad\times\int_{-\pi}^{\pi} d\beta|\sin\beta|e^{i\beta m_z'}\sum_{n\in\mathbb{Z}}\delta(\beta-\beta_0-\pi-2\pi n)\\ &= \delta\big(|\mathbf{k}_2+\mathbf{k}_4|-\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\big)|\sin\beta_0|(-1)^{m_z'}e^{i\beta_0 m_z'}.\end{aligned} \tag{52}$$

Due to the 2π periodicity of the integrand we have shifted the domain of integration to only keep, for example, the term $n=0$ of the Dirac comb. Finally, using $\beta_0=\tau_{24}-\tau_{13}$, where τ_{24} is the angle $\in[0,\pi]$ between $\mathbf{k}_2$ and $\mathbf{k}_2+\mathbf{k}_4$ and τ_{13} is the angle $\in[0,\pi]$ between $\mathbf{k}_1^{\text{fix}}$ and $\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}$ so (up to a phase factor)

$$|\ell,m_Z=0\rangle = e^{-i\tau_{24}L_z/\hbar}|\ell,m_x=0\rangle, \tag{53}$$

and using the property that [41]

$$\langle\ell,m_x=0|\ell,m_z'\rangle = 0 \text{ if } \ell+m_z' \text{ is odd}, \tag{54}$$

allowing one to replace $(-1)^{m_z'}$ with $(-1)^\ell$, we obtain

$$\begin{aligned}\mathcal{E}_{24,13} &= \sum_{m_z,m_z'=-\ell}^{\ell}\frac{(-1)^\ell}{2\pi}\int_0^\infty\left(\prod_{n=1}^4 dk_n k_n^2\right)\int_{-1}^1 du_{13}du_{24}\\ &\quad\times\frac{e^{im_z\tau_{24}}\langle\ell,m_z|\ell,m_x=0\rangle\langle\ell,m_x=0|\ell,m_z'\rangle e^{-im_z'\tau_{13}}}{|\mathbf{k}_2+\mathbf{k}_4|\big[\frac{\hbar^2(k_1^2+k_2^2)}{2m_\uparrow}+\frac{\hbar^2(k_3^2+k_4^2)}{2m_\downarrow}\big]\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|}\\ &\quad\times\delta\big(|\mathbf{k}_2+\mathbf{k}_4|-\big|\mathbf{k}_1^{\text{fix}}+\mathbf{k}_3^{\text{fix}}\big|\big)f_{m_z}^{(\ell)*}(k_2,k_4,u_{24})\\ &\quad\times f_{m_z'}^{(\ell)}(k_1,k_3,u_{13}),\end{aligned} \tag{55}$$

knowing that $|\ell,m_x=0\rangle$ has real components in the basis $|\ell,m_z\rangle$ up to a global phase and that $\mathbf{k}_2$ and $\mathbf{k}_4$ are given by Eqs. (30) and (31) and $\mathbf{k}_1^{\text{fix}}$ and $\mathbf{k}_3^{\text{fix}}$ by Eqs. (43) and (44).

C. Scale invariance

To take advantage of the scale invariance of the zero-energy solution, one uses the ansatz (20) with $s\in i\mathbb{R}$, as physically explained in Sec. III A, and one inserts it in the various terms (33), (37), (42), and (55) of the functional (24). In Eq. (33) one performs in the integral over k_4 the change of variable $k_4=e^x k_2$, where x ranges from $-\infty$ to $+\infty$, and one sets $u_{24}=u$ for conciseness, also introducing the mass ratio

$$\alpha \equiv \frac{m_\uparrow}{m_\downarrow}. \tag{56}$$

One pulls out a constant factor $\mathcal{F}$, which will be given and discussed later, to obtain

$$\begin{aligned}\mathcal{E}_{\text{diag}} &= \mathcal{F}\sum_{m_z=-\ell}^{\ell}\int_{\mathbb{R}}dx\int_{-1}^1 du\bigg[\frac{\alpha}{(1+\alpha)^2}\Big(1+\frac{u}{\operatorname{ch}x}\Big)\\ &\quad+\frac{e^{-x}+\alpha e^x}{2(\alpha+1)\operatorname{ch}x}\bigg]^{1/2}\big|\Phi_{m_z}^{(\ell)}(x,u)\big|^2.\end{aligned} \tag{57}$$

In Eq. (37) one performs the change of variable $k_4=e^x k_2$ and $k_3=e^{x'}k_2$ in the integrals over k_4 and k_3, also setting $\theta_{24}=\theta$, $u_{24}=u$, $v_{24}=v$ and $\theta_{23}=\theta'$, $u_{23}=u'$, $v_{23}=v'$ for concision. One then pulls out the same factor $\mathcal{F}$ to obtain

$$\begin{aligned}\mathcal{E}_{24,23} &= \mathcal{F}\sum_{m_z,m_z'=-\ell}^{\ell}\int_{\mathbb{R}}dxdx'\int_{-1}^1 dudu'\left(\frac{e^x\operatorname{ch}x'}{e^{x'}\operatorname{ch}x}\right)^{s/2}\\ &\quad\times\left(\frac{e^{x+x'}}{4\operatorname{ch}x\operatorname{ch}x'}\right)^{1/4}\Phi_{m_z}^{(\ell)*}(x,u)\Phi_{m_z'}^{(\ell)}(x',u')\\ &\quad\times\int_0^{2\pi}\frac{d\phi}{(2\pi)^2}\frac{e^{-im_z\theta/2}\langle l,m_z|e^{i\phi L_x/\hbar}|l,m_z'\rangle e^{im_z'\theta'/2}}{\mathcal{D}_{24,23}(\phi;x,u;x',u';\alpha)}.\end{aligned} \tag{58}$$

In the denominator, we have introduced the notation

$$\mathcal{D}_{24,23} = \frac{\frac{\hbar^2\left(k_1^2+k_2^2\right)}{2m_\uparrow}+\frac{\hbar^2\left(k_3^2+k_4^2\right)}{2m_\downarrow}}{\frac{\hbar^2 k_3 k_4}{\mu_{\uparrow\downarrow}}}, \tag{59}$$

where k_1 is given by Eq. (36) so

$$\begin{aligned}\mathcal{D}_{24,23}(\phi;x,u;x',u';\alpha) &= \mathrm{ch}(x-x') + \frac{1}{1+\alpha}(e^{-x-x'} + e^{-x'}u + e^{-x}u' + uu' \\ &\quad + vv'\cos\phi).\end{aligned} \tag{60}$$

In Eq. (42) one performs the change of variables $k_4 = e^x k_2$ and $k_1 = e^{x-x'}k_2$ (so $k_4/k_1 = e^{x'}$) in the integrals over k_4 and k_1, and the change of variable $\phi = \frac{\pi}{2} + \phi'$ in the integral over ϕ [42], also setting $\theta_{24} = \theta$, $u_{24} = u$, $v_{24} = v$ and $\theta_{14} = \theta'$, $u_{14} = u'$, $v_{14} = v'$. Again pulling out the factor $\mathcal{F}$ one gets

$$\begin{aligned}\mathcal{E}_{24,14} &= \mathcal{F}\sum_{m_z,m_z'=-\ell}^{\ell}\int_{\mathbb{R}} dx dx' \int_{-1}^{1} du du' \left(\frac{e^{-x}\,\mathrm{ch}\,x'}{e^{-x'}\,\mathrm{ch}\,x}\right)^{s/2} \\ &\quad\times\left(\frac{e^{-(x+x')}}{4\,\mathrm{ch}\,x\,\mathrm{ch}\,x'}\right)^{1/4}\Phi_{m_z}^{(\ell)*}(x,u)\Phi_{m_z'}^{(\ell)}(x',u') \\ &\quad\times\int_0^{2\pi}\frac{d\phi'}{(2\pi)^2}\frac{e^{im_z\theta/2}\langle l,m_z|e^{i\phi' L_x/\hbar}|l,m_z'\rangle e^{-im_z'\theta'/2}}{\mathcal{D}_{24,14}(\phi';x,u;x',u';\alpha)}.\end{aligned} \tag{61}$$

In the denominator we have introduced the notation

$$\mathcal{D}_{24,14} = \frac{\frac{\hbar^2\left(k_1^2+k_2^2\right)}{2m_\uparrow} + \frac{\hbar^2\left(k_3^2+k_4^2\right)}{2m_\downarrow}}{\frac{\hbar^2 k_1 k_2}{\mu_{\uparrow\downarrow}}} \tag{62}$$

with k_3 given by Eq. (41) so

$$\begin{aligned}\mathcal{D}_{24,14}(\phi';x,u;x',u';\alpha) &= \mathrm{ch}(x-x') + \frac{\alpha}{1+\alpha}(e^{x+x'} + e^{x'}u + e^{x}u' + uu' \\ &\quad + vv'\cos\phi').\end{aligned} \tag{63}$$

Finally, in Eq. (55), one performs the change of variables $k_4 = e^x k_2$ and $k_3 = e^{x'}k_1$ in the integrals over k_4 and k_3, also setting $\theta_{24} = \theta$, $u_{24} = u$, $\tau_{24} = \tau$ and $\theta_{13} = \theta'$, $u_{13} = u'$, $\tau_{13} = \tau'$. The integration over k_1 is straightforward due to the occurrence of a Dirac distribution in Eq. (55). Due to the phase factor in the ansatz (20), there naturally appear the angles $\gamma \equiv \tau - \theta/2$ and $\gamma' = \tau' - \theta'/2$. Since τ is the angle between $\mathbf{k}_2$ and $\mathbf{k}_2 + \mathbf{k}_4$ [see Fig. 1(c)], one has according to Eqs. (30) and (31) and using the usual representation of vectors in the xy plane by complex numbers:

$$e^{i\gamma} = \frac{1+e^x e^{i\theta}}{|1+e^x e^{i\theta}|}e^{-i\theta/2} = \frac{e^{(x+i\theta)/2}+e^{-(x+i\theta)/2}}{|e^{(x+i\theta)/2}+e^{-(x+i\theta)/2}|}. \tag{64}$$

As $\theta \in [0,\pi]$, the real part $\cos\gamma$ of this expression is nonnegative so one can choose $\gamma \in [-\frac{\pi}{2},\frac{\pi}{2}]$. Then forming the ratio of the imaginary part to the real part of the same expression gives the value of $\tan\gamma$ and

$$\gamma = \mathrm{atan}\left[\mathrm{th}\left(\frac{x}{2}\right)\tan\left(\frac{\theta}{2}\right)\right] \text{ with } \tan\left(\frac{\theta}{2}\right) = \left(\frac{1-u}{1+u}\right)^{1/2}. \tag{65}$$

One has the same expressions for γ', replacing the variables x, θ, and u with x', θ', and u'. This leads to

$$\begin{aligned}\mathcal{E}_{24,13} &= \mathcal{F}\sum_{m_z,m_z'=-\ell}^{\ell}\int_{\mathbb{R}} dx dx' \int_{-1}^{1} du du' \left[\frac{(u'+\mathrm{ch}\,x')\,\mathrm{ch}\,x'}{(u+\mathrm{ch}\,x)\,\mathrm{ch}\,x}\right]^{s/2} \\ &\quad\times\frac{(-1)^\ell\Phi_{m_z}^{(\ell)*}(x,u)\Phi_{m_z'}^{(\ell)}(x',u')}{4\pi[(u+\mathrm{ch}\,x)(u'+\mathrm{ch}\,x')\,\mathrm{ch}\,x\,\mathrm{ch}\,x']^{1/4}} \\ &\quad\times\frac{e^{im_z\gamma}\langle\ell,m_z|\ell,m_x=0\rangle\langle\ell,m_x=0|\ell,m_z'\rangle e^{-im_z'\gamma'}}{\left(\frac{e^{-x'}+\alpha e^{x'}}{1+\alpha}\right)(u+\mathrm{ch}\,x)+\left(\frac{e^{-x}+\alpha e^{x}}{1+\alpha}\right)(u'+\mathrm{ch}\,x')}.\end{aligned} \tag{66}$$

In all the results (57), (58), (61), and (66) there appears a factor

$$\mathcal{F} = \frac{\mu_{\uparrow\downarrow}}{8\hbar^2}\int_0^{+\infty}\frac{dk_2}{k_2}. \tag{67}$$

This factor contains a diverging integral, making these last calculations not entirely rigorous. We have checked, however, that always the same diverging integral is pulled out, even if one singles out a wave number other than k_2 [performing, for example, the change of variables $k_2 = e^{-x}k_4$ and $k_1 = e^{-x'}k_4$ in the integrals over k_2 and k_1 in Eq. (42)]. This is certainly due to the scale invariance of $dk_2/k_2 = d(\ln k_2)$. Alternatively, one can write the integral equation for the $f_{m_z}^{(\ell)}$ deduced from the functional derivatives of Eqs. (33), (37), (42), and (55) of the functional Eq. (24) with respect to $f_{m_z}^{(\ell)*}$; at this stage, one has only used rotational invariance. Then one inserts the scale invariant ansatz (20), and one obtains exactly the same integral equations for the $\Phi_{m_z}^{(\ell)}$ as those derived from the functional derivatives of Eqs. (57), (58), (61), and (66) with respect to $\Phi_{m_z}^{(\ell)*}$.

D. Parity invariance

The term of index m_z in the ansatz (14) is simply multiplied by $(-1)^{m_z}$ under the action of parity $(\mathbf{k}_2,\mathbf{k}_4) \to (-\mathbf{k}_2, -\mathbf{k}_4)$ [18]. This means that the odd-m_z components of $\Phi_{m_z}^{(\ell)}$ are decoupled from the even-m_z components of $\Phi_{m_z}^{(\ell)}$ in the integral equation. This property can also be obtained by an explicit calculation: first, for m_z and m_z' of different parities, the coupling amplitude between $|\ell,m_z\rangle$ and $|\ell,m_z'\rangle$ must vanish in Eq. (55); this can be seen from Eq. (54). Second, it also vanishes in Eqs. (58) and (61) after integration over ϕ or ϕ': L_x obeys the selection rule $\Delta m_z = \pm 1$, and, in an expansion of $e^{i\phi L_x}$ in powers of ϕ, only even powers of ϕ and L_x survive due to the parity of the denominator $\mathcal{D}$. In what follows, at a given angular momentum ℓ, we shall distinguish the manifold of parity $(-1)^{\ell+1}$, where $\mathcal{E}_{24,13}$ and the contribution of $D(\mathbf{k}_1,\mathbf{k}_3)$ in Eq. (13) are zero, and the manifold of parity $(-1)^\ell$ where they are *a priori* nonzero. Note that, in the particular case $\ell = 0$, there exists only the manifold of parity $(-1)^\ell$.

IV. FINAL FORM AND ASYMPTOTIC ANALYSIS

A. Explicit form of the integral equation

By taking the functional derivative of $\mathcal{E}$ of Eq. (24) with respect to $\Phi_{m_z}^{(\ell)}$, using the forms (57), (58), (61), and (66) of the various terms and not forgetting the minus sign in front of the last contribution in Eq. (24), we obtain the form of the integral equation (13) maximally reduced by use of the rotational symmetry and of the scale invariance:

$$0=\left[\frac{\alpha}{(1+\alpha)^2}\left(1+\frac{u}{\operatorname{ch}x}\right)+\frac{e^{-x}+\alpha e^{x}}{2(\alpha+1)\operatorname{ch}x}\right]^{1/2}\Phi^{(\ell)}_{m_z}(x,u)+\int_{\mathbb{R}}dx'\int_{-1}^{1}du'\sum_{m_z'=-\ell}^{\ell}K^{(\ell)}_{m_z,m_z'}(x,u;x',u';\alpha)\Phi^{(\ell)}_{m_z'}(x',u'), \tag{68}$$

with the following expression for the matrix kernel $K^{(\ell)}$:

$$\begin{aligned}
&K^{(\ell)}_{m_z,m_z'}(x,u;x',u';\alpha)\\
&=\left(\frac{e^{x}\operatorname{ch}x'}{e^{x'}\operatorname{ch}x}\right)^{s/2}\left(\frac{e^{x+x'}}{4\operatorname{ch}x\operatorname{ch}x'}\right)^{1/4}\int_0^{2\pi}\frac{d\phi}{(2\pi)^2}\frac{e^{-im_z\theta/2}\langle\ell,m_z|e^{i\phi L_x/\hbar}|\ell,m_z'\rangle e^{im_z'\theta'/2}}{\operatorname{ch}(x-x')+\frac{1}{1+\alpha}[(u+e^{-x})(u'+e^{-x'})+vv'\cos\phi]}\\
&+\left(\frac{e^{-x}\operatorname{ch}x'}{e^{-x'}\operatorname{ch}x}\right)^{s/2}\left(\frac{e^{-x-x'}}{4\operatorname{ch}x\operatorname{ch}x'}\right)^{1/4}\int_0^{2\pi}\frac{d\phi}{(2\pi)^2}\frac{e^{im_z\theta/2}\langle\ell,m_z|e^{i\phi L_x/\hbar}|\ell,m_z'\rangle e^{-im_z'\theta'/2}}{\operatorname{ch}(x-x')+\frac{\alpha}{1+\alpha}[(u+e^{x})(u'+e^{x'})+vv'\cos\phi]}\\
&-\frac{(-1)^{\ell}}{4\pi[(u+\operatorname{ch}x)(u'+\operatorname{ch}x')\operatorname{ch}x\operatorname{ch}x']^{1/4}}\left[\frac{(u'+\operatorname{ch}x')\operatorname{ch}x'}{(u+\operatorname{ch}x)\operatorname{ch}x}\right]^{s/2}\frac{e^{im_z\gamma}\langle\ell,m_z|\ell,m_x=0\rangle\langle\ell,m_x=0|\ell,m_z'\rangle e^{-im_z'\gamma'}}{\left(\frac{e^{-x'}+\alpha e^{x'}}{1+\alpha}\right)(u+\operatorname{ch}x)+\left(\frac{e^{-x}+\alpha e^{x}}{1+\alpha}\right)(u'+\operatorname{ch}x')}.
\end{aligned} \tag{69}$$

Here the scaling exponent s is purely imaginary, so a four-body Efimov takes place in our $2+2$ fermionic problem if Eq. (68) has a nonidentically zero solution $\Phi^{(\ell)}$ for some nonzero s. We recall that the angle $\theta\in[0,\pi]$ is such that $u=\cos\theta$ and $v=(1-u^2)^{1/2}=\sin\theta$, and that the angle γ is given by Eq. (65); the same relations hold among the primed variables.

The first, second, and third contributions in Eq. (69) originate, respectively, from the terms $D(\mathbf{k}_2,\mathbf{k}_3)$, $D(\mathbf{k}_1,\mathbf{k}_4)$, and $D(\mathbf{k}_1,\mathbf{k}_3)$ in the unreduced integral equation (13); the diagonal term in Eq. (68) emanates from the diagonal term of that equation. The integrals over ϕ can be evaluated after insertion of a closure relation in the eigenbasis of L_x [43]. Importantly, the third contribution in Eq. (69) vanishes when $\ell+m_z$ or $\ell+m_z'$ are odd, i.e., in the parity channel $(-1)^{\ell+1}$, as shown by the property (54) and as already pointed out in Sec. III D.

It is interesting to note the decoupled form of the prefactors in each contribution of Eq. (69) of the form $[f(x,u)]^{s/2+1/4}[f(x',u')]^{-s/2+1/4}$ with a function f given by $e^x/(2\operatorname{ch}x)$, $e^{-x}/(2\operatorname{ch}x)$, and $1/[(u+\operatorname{ch}x)\operatorname{ch}x]$, respectively. The fact that this function $f(x,u)$ is not common to all contributions prevents one from suppressing the s dependence of the matrix kernel $K^{(\ell)}$ by a simple gauge transform on $\Phi^{(\ell)}$: As expected, the s dependence of the problem (68) is nontrivial.

Our results (68) and (69) must obey the symmetry of the $2+2$ problem under the exchange of $\uparrow$ and $\downarrow$. First, this exchange has the effect of changing the mass ratio α into its inverse $1/\alpha$, see Eq. (56). Second, the momenta $\mathbf{k}_2$ and $\mathbf{k}_4$ in $D(\mathbf{k}_2,\mathbf{k}_4)$ are exchanged, so x of Eq. (21) is changed into its opposite; this also reverts the direction of the quantization axis $\mathbf{k}_2\wedge\mathbf{k}_4$ along which the angular momentum m_z is measured in Eq. (14): It changes m_z into $-m_z$ according to the identity [18]

$$e^{-i\pi L_x/\hbar}|\ell,m_z\rangle=(-1)^{\ell}|\ell,-m_z\rangle; \tag{70}$$

on the contrary, the nonoriented angle $\theta_{24}\in[0,\pi]$ between $\mathbf{k}_2$ and $\mathbf{k}_4$ is unchanged, so the variable u is unaffected. Hence, one must have

$$K^{(\ell)}_{m_z,m_z'}(x,u;x',u';\alpha)=K^{(\ell)}_{-m_z,-m_z'}(-x,u;-x',u';\alpha^{-1}) \tag{71}$$

for all values of the argument and of the indices of the kernel. It is clear that Eq. (69) indeed obeys the symmetry requirement (71): The first and second contributions are interchanged, whereas the third one is invariant since γ is changed into $-\gamma$, see Eq. (65). Note that our results also respect the parity invariance, see Sec. III D, and that the matrix kernel is Hermitian as our variational derivation guarantees:

$$K^{(\ell)}_{m_z,m_z'}(x,u;x',u';\alpha)=\left[K^{(\ell)}_{m_z',m_z}(x',u';x,u;\alpha)\right]^*. \tag{72}$$

B. Recovering the three-body problem from four-body asymptotics

The right-hand side of the integral equation (68) defines an operator $M^{(\ell)}(s)$ acting on the spinor functions $\Phi^{(\ell)}_{m_z}(x,u)$. The spectrum of this operator is physically relevant, since a four-body Efimov effect takes place with an Efimov scaling exponent $s\in i\mathbb{R}$ if and only if one of the eigenvalue Ω of $M^{(\ell)}(s)$ is zero. As $M^{(\ell)}(s)$ is a Hermitian operator, since s is here purely imaginary, its spectrum is real and in general includes a discrete part and a continuous part. The discrete spectrum corresponds to localized, square integrable eigenfunctions; we are able to determine it only numerically. The expected contribution to the continuous spectrum corresponds to extended functions that explore arbitrarily large values of $|x|$; as we now explain, it can be determined analytically from the asymptotic analysis of the kernel (69) when x and x' tend to $\pm\infty$ by a generalization of the discussion of reference [13]. There is also an unexpected contribution to the continuous spectrum, whose analysis is deferred to Sec. IV C.

1. Sector $x\to+\infty, x'\to+\infty$

Clearly, the diagonal part of $M^{(\ell)}(s)$ in Eq. (68) tends exponentially rapidly to a finite and nonzero value, and the second and third contributions to the kernel in Eq. (69)

tend exponentially fast to zero. In the first contribution in Eq. (69), the prefactor tends exponentially to unity since $e^x/(2\,\mathrm{ch}\,x) \to 1$, and in the denominator of the integrand, all the x- or x'-dependent terms are exponentially suppressed, except the first one, $\mathrm{ch}(x-x')$, since no hypothesis must be made on the magnitude of the difference $x-x'$. The eigenvalue problem then asymptotically reduces to

$$\Omega^{\to(\ell)}\tilde{\Phi}^{(\ell)}_{m_z}(x,u) = \left[\frac{\alpha(2+\alpha)}{(1+\alpha)^2}\right]^{1/2}\tilde{\Phi}^{(\ell)}_{m_z}(x,u) + \int_{\mathbb{R}} dx' \sum_{m_z'=-\ell}^{\ell}\int_{-1}^{1} du' \times \int_0^{2\pi}\frac{d\phi'}{(2\pi)^2}\frac{\langle \ell,m_z|e^{i\phi' L_x/\hbar}|\ell,m_z'\rangle}{\mathrm{ch}(x-x')+\frac{uu'+vv'\cos\phi'}{1+\alpha}}\tilde{\Phi}^{(\ell)}_{m_z'}(x',u'), \quad (73)$$

where the arrow in the exponent of Ω indicates that x and x' tend to positive infinity and the phase factors $e^{-im_z\theta/2}$ and $e^{im_z'\theta'/2}$ have been eliminated by a gauge transform on the spinor, $\Phi^{(\ell)}_{m_z}(x,u) = e^{-im_z\theta/2}\tilde{\Phi}^{(\ell)}_{m_z}(x,u)$. Then one performs a spin rotation by moving to the internal basis of the eigenstates $|\ell,m_x\rangle$ of L_x, in which $e^{i\phi L_x/\hbar}$ is diagonal: The components $\tilde{\Phi}^{(\ell)}_{m_x}(x,u)$ are all decoupled. For a given m_x, with $|m_x| \leqslant \ell$, the trick is to extend $\tilde{\Phi}^{(\ell)}_{m_x}(x,u)$ into a function of the real variable x and of the vector $\mathbf{n}$ on the two-dimensional unit sphere:

$$F^{(\ell)}_{m_x}(x,\mathbf{n}) \equiv \tilde{\Phi}^{(\ell)}_{m_x}(x,\cos\theta)e^{im_x\phi}, \quad (74)$$

where $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ are the polar and azimuthal angles of $\mathbf{n}$ in spherical coordinates, e.g., with respect to x and y axes, $\mathbf{n} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$. In the phase factor $e^{im_x\phi'}$ in the numerator and in $\cos\phi'$ in the denominator, one can then replace ϕ' by $\phi'-\phi$: The integrand is a periodic function of ϕ' of period 2π and its integral has the same value whatever the interval of length 2π over which ϕ' runs. Then one recognizes the scalar product $\mathbf{n}\cdot\mathbf{n}' = uu' + vv'\cos(\phi-\phi')$, where $\mathbf{n}' = (\cos\theta', \sin\theta'\cos\phi', \sin\theta'\sin\phi')$. The eigenvalue problem is now

$$\Omega^{\to(\ell)}_{m_x}F^{(\ell)}_{m_x}(x,\mathbf{n}) = \frac{[\alpha(2+\alpha)]^{1/2}}{1+\alpha}F^{(\ell)}_{m_x}(x,\mathbf{n}) + \int_{\mathbb{R}} dx' \int_{|\mathbf{n}|=1}\frac{d^2n}{(2\pi)^2}\frac{F^{(\ell)}_{m_x}(x',\mathbf{n}')}{\mathrm{ch}(x-x')+\frac{\mathbf{n}\cdot\mathbf{n}'}{1+\alpha}}. \quad (75)$$

The corresponding operator is invariant by translation along x and by rotation of $\mathbf{n}$ over the unit sphere. Its eigenfunctions $F^{(\ell)}_{m_x}(x,\mathbf{n})$ can therefore be taken as plane waves of the variable x and spherical harmonics of the variables (θ,ϕ), with the *same* quantum number m_x [this is imposed by the form (74)] but with *any* integer quantum number $L \geqslant |m_x|$ for the total angular momentum:

$$F^{(\ell)}_{m_x}(x,\mathbf{n}) = e^{ikx}Y_L^{m_x}(\theta,\phi). \quad (76)$$

As usual for a rotationally invariant operator, the eigenvalue does not depend on m_x. It only depends on L, so it suffices to specialize to $m_x = 0$, where $Y_L^0(\theta,\phi) \propto P_L(\cos\theta)$, where $P_L(X)$ is the Legendre polynomial of degree L. Then one gets the continuous spectrum "to the right" $(x,x' \to +\infty)$:

$$\Omega^{\to(\ell)}_{m_x}(\alpha) \in \{\Lambda_L(ik,\alpha^{-1}), \forall k \in \mathbb{R}, \forall L \geqslant |m_x|\}. \quad (77)$$

The function Λ_L of $s \in i\mathbb{R}$ and of the mass ratio was introduced and analytically calculated in Refs. [44,45], generalizing previous results [46,47]:

$$\Lambda_L(s,\beta) \equiv \frac{(1+2\beta)^{1/2}}{1+\beta} + \int_{-1}^{1}du\int_{\mathbb{R}}\frac{dx}{2\pi}\frac{e^{-sx}P_L(u)}{\mathrm{ch}\,x+\frac{\beta}{1+\beta}u}$$
$$= \cos\nu(\beta) + \frac{1}{\sin\nu(\beta)}\int_{\frac{\pi}{2}-\nu(\beta)}^{\frac{\pi}{2}+\nu(\beta)}d\theta \times P_L\left[\frac{\cos\theta}{\sin\nu(\beta)}\right]\frac{\sin(s\theta)}{\sin(s\pi)}, \quad (78)$$

where, in the second expression obtained after integration over x [45],

$$\nu(\beta) = \mathrm{asin}\,\frac{\beta}{1+\beta} \quad (79)$$

is a mass angle. For all $\beta > 0$, it is found numerically for even L that the maximal value of $\Lambda_L(s,\beta)$ over $s \in i\mathbb{R}^+$ is reached at $s=0$, and the minimal value is reached for $|s| \to +\infty$ [where $\Lambda_L(s,\beta)$ tends to $\cos\nu(\beta)$] . For odd L, the situation is found to be reversed: $\Lambda_L(s,\beta)$ is minimal at $s=0$ and maximal at infinity. To summarize, we expect that

$$\cos\nu(\beta) \leqslant \Lambda_L(s,\beta) \leqslant \Lambda_L(0,\beta) \quad \forall s \in i\mathbb{R}, L \text{ even},$$
$$\Lambda_L(0,\beta) \leqslant \Lambda_L(s,\beta) \leqslant \cos\nu(\beta) \quad \forall s \in i\mathbb{R}, L \text{ odd}. \quad (80)$$

This allows to determine the borders of the continuous component of quantum number L in Eq. (77), see Fig. 2. A physical explanation for the emergence of the function Λ_L is postponed to the end of the section.

2. Sector $x \to -\infty, x' \to -\infty$

The calculation closely resembles the previous one, except that it is now the second contribution in the right-hand side of Eq. (69) that survives. This was expected from the symmetry relation (71). We arrive at the continuous spectrum "to the left" $(x,x' \to -\infty)$:

$$\Omega^{\leftarrow(\ell)}_{m_x}(\alpha) \in \{\Lambda_L(ik,\alpha), \forall k \in \mathbb{R}, \forall L \geqslant |m_x|\} \quad (81)$$

that differs from (77) by the occurrence of α (rather than $1/\alpha$) in the argument of the Λ_L function [48]. The borders of the L components of that continuum are plotted in Fig. 2 for the first few values of L, using the numerically checked property (80).

3. Parity considerations

At fixed ℓ, the results (77) and (81) are expressed in terms of the quantum number m_x, whereas the original problem only distinguishes between an even-parity manifold (m_z is even) and an odd-parity manifold (m_z is odd). In practice, due to the property (54), the continua (77) and (81) with $L=0$ can be realized only in the manifold of parity $(-1)^\ell$ at any considered total angular momentum ℓ (obviously one must then take $m_x = 0$). The other continua (with $L \geqslant 1$) can all be realized, in both odd and even manifolds, for all values of $\ell \geqslant 0$ [49].

4. Physical discussion

The function $\Lambda_L(s,\beta)$ appears in the unitary three-body problem of two fermionic particles interacting with a single

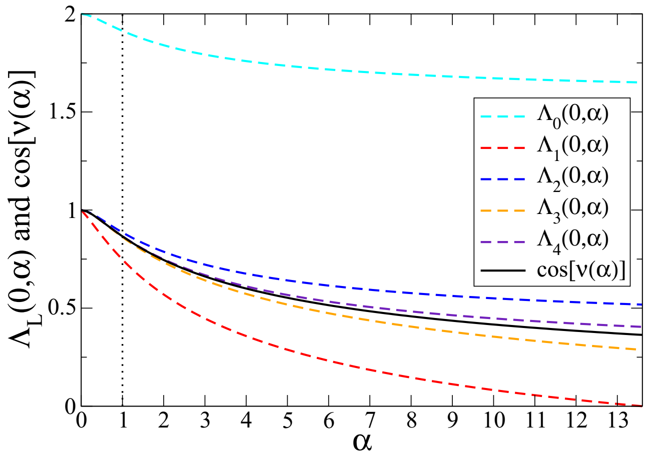

FIG. 2. (Color online) Analytically obtained borders of the continuous spectrum of $M(s)$ ($s \in i\mathbb{R}^+$) corresponding to the limits $x \to \pm\infty$. It is a collection of components characterized by an angular-momentum quantum number L of a three-body asymptotic problem (not to be confused with the total angular momentum ℓ of the four-body states), leading to a continuous set of eigenvalues ranging from $\Lambda_L(0,\alpha)$ to $\cos[\nu(\alpha)]$ for $x \to -\infty$ and ranging from $\Lambda_L(0,1/\alpha)$ to $\cos[\nu(1/\alpha)]$ for $x \to +\infty$, where α is the mass ratio given by Eq. (56), the Λ_L function is given by Eq. (78), and the mass angle ν is given by Eq. (79). We plot $\cos[\nu(\alpha)]$ (black solid line) and $\Lambda_L(0,\alpha)$ (colored dashed lines, with $L = 0,2,4$ from top to bottom above the solid line and $L = 1,3$ from bottom to top below the solid line) as functions of $\alpha \in [0,\alpha_c(2;1)]$, where $\alpha_c(2;1)$ is the critical mass ratio (83) for the Efimov effect in the $\uparrow\uparrow\downarrow$ three-body problem. Due to the $\alpha \leftrightarrow 1/\alpha$ symmetry of the $2+2$ fermionic problem, one can restrict to $\alpha \geqslant 1$ (that is, to the right of the vertical dotted line); the borders of the $x \to -\infty$ continuum can then be directly read on the figure, and the ones of the $x \to +\infty$ continuum can be obtained by mentally folding back the $\alpha \leqslant 1$ part of the figure into the $\alpha \geqslant 1$ part.

distinguishable particle, β being the mass ratio of the majority-to-minority species. For $s \in i\mathbb{R}$ this function is given by the first form in Eq. (78); it can be analytically extended to real values of s using, e.g., the second form in Eq. (78) [45]. The zero-energy solutions of this three-body problem have an Efimov scaling exponent s: The three-body wave function scales as R^{s-2}, with R being the three-particle hyperradius, and the allowed values of s at total angular momentum L must solve

$$\Lambda_L(s,\beta) = 0. \tag{82}$$

This three-body system exhibits an Efimov effect if and only if this equation has a purely imaginary solution $s \in i\mathbb{R}^*$. This occurs only at odd L, starting from a mass ratio [14]

$$\beta > \alpha_c(2;1) = 13.60696\ldots \tag{83}$$

for $L = 1$ and at increasingly larger critical mass ratios for $L = 3,5,\ldots$ [45,50].

It is thus apparent that the asymptotic analysis of the $2+2$ fermionic problem brings up the three-body problem. This is intuitive in position space: Imagine that at fixed position $\mathbf{r}_4 \neq \mathbf{0}$ of the fourth particle (of spin $\downarrow$), the positions $(\mathbf{r}_i)_{1\leqslant i\leqslant 3}$ of the other particles (of spin $\uparrow\uparrow\downarrow$) simultaneously tend to zero; then the four-body wave function $\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ must reproduce the behavior of the zero-energy scattering state of two $\uparrow$ particles and one $\downarrow$ particle, characterized by a mass ratio $\beta = m_\uparrow/m_\downarrow = \alpha$; in particular, it must exhibit the same scaling exponents s as the $2+1$ problem (see Sec. 5.3.6 in Ref. [34]). As these scaling exponents solve Eq. (82) with $\beta = \alpha$, this explains the occurrence of $\Lambda_L(s,\alpha)$ in the spectrum (81) [51]. Even if $\ell = 0$ for the four-body system, L can take any value, as the angular momentum can be distributed among particle 4 and the first three particles. The equivalent in momentum space of the considered limit is to have divergent $(\mathbf{k}_i)_{1\leqslant i\leqslant 3}$ at fixed $\mathbf{k}_4$, which, due to scale invariance, is equivalent to having $\mathbf{k}_4 \to \mathbf{0}$ at fixed $(\mathbf{k}_i)_{1\leqslant i\leqslant 3}$, that is, $x \to -\infty$ according to Eq. (21). This is why $\beta = \alpha$ corresponds to the spectrum (81). A similar reasoning with $\mathbf{r}_2$ fixed with $(\mathbf{r}_i)_{i\neq 2}$ tending to zero leads to $\beta = 1/\alpha$ and $x \to +\infty$, as for the spectrum (77).

C. A third, unexpected continuum

The first two contributions in Eq. (69) are innocuous: The denominator in their integrands cannot vanish, see Eqs. (59) and (60) and Eqs. (62) and (63), and, as we have seen, they have a short range in the (x,x') space. On the contrary, the third contribution in Eq. (69), which is nonzero only in the $(-1)^\ell$ parity sector, diverges when $(x,u) \to (0,-1)$ or $(x',u') \to (0,-1)$. This creates doubt about the bounded nature of the eigenvalues of $M(s)$, $s \in i\mathbb{R}$, for that parity. We investigate this problem mathematically in Appendix A and we conclude that $M(s)$ is bounded.

Physically, this divergence of the kernel leads to a quite interesting effect: the emergence of a third component of the continuous spectrum of $M(s)$, differing from the previously discussed $x \to \pm\infty$ continua. The intuitive idea is that one can turn the eigenvalue problem $\Omega\Phi = M(s)\Phi$ into an integral equation with a bounded kernel through an appropriate change of variables, with the consequence that one of the new variables, which we shall call t, can tend to $-\infty$, in which case the eigenvector Φ takes a plane wave structure $\propto \exp(ikt)$, $k \in \mathbb{R}$, with a spectrum:

$$\Omega^{\odot(\ell)} \in \left\{\frac{1}{\sqrt{2}}\left[1 - \frac{(-1)^\ell}{\mathrm{ch}(k\pi/2)}\right], \forall k \in \mathbb{R}\right\} \text{ [parity}(-1)^\ell]. \tag{84}$$

This is an unexpected feature of the $2+2$ fermionic problem, absent in the $3+1$ fermionic case [13].

To obtain this result, we construct a local approximation to the integral equation in the vicinity of $(x,u) = (0,-1),(x',u') = (0,-1)$, keeping only the leading diverging contributions. We use

$$y \equiv \pi - \theta \tag{85}$$

rather than $u = \cos\theta$ as integration variable, so $y,y' \to 0$ when $u,u' \to -1$. This pulls out a Jacobian $\sin y'$ that we absorb (in a way preserving the Hermiticity of the problem) with a change of function. We also take into account the fact that the third, diverging contribution in Eq. (69) involves a projector onto the $|\ell,m_x = 0\rangle$ state and that the phase factors $e^{im_z\gamma}$ and $e^{-im_z'\gamma'}$, $[(u + \mathrm{ch}\, x)\,\mathrm{ch}\, x]^{-s/2}$ and $[(u' + \mathrm{ch}\, x')\,\mathrm{ch}\, x']^{s/2}$, can be eliminated by a change of gauge. Hence the ansatz

$$\Phi^{(\ell)}_{m_z}(x,u) = \frac{(x^2+y^2)^{-s/2}}{y^{1/2}} e^{im_z\gamma}\langle \ell,m_z|\ell,m_x=0\rangle \Phi(x,y), \tag{86}$$

where $\sin y$ was linearized and $u+\mathrm{ch}\,x$ was quadratized. The resulting local eigenvalue problem is

$$\left(\Omega - \frac{1}{\sqrt{2}}\right)\Phi(x,y) = -\frac{(-1)^\ell}{2^{1/2}\pi}\int_D \frac{dx'dy'(yy')^{1/2}}{[(x^2+y^2)(x'^2+y'^2)]^{1/4}} \times \frac{\Phi(x',y')}{x^2+y^2+x'^2+y'^2}, \tag{87}$$

where we have conveniently restricted the integration to the upper half ($y>0$) of the disk D of radius $\rho_0 \ll 1$ centered in (0,0). In polar coordinates

$$(x,y) = (\rho\cos\psi, \rho\sin\psi) \tag{88}$$

only $(yy')^{1/2}$ depends on ψ in the kernel. It depends on ψ in a factorized way so Φ is also factorized:

$$\Phi(x,y) = \rho^{-1/2}\Phi(\rho)(\sin\psi)^{1/2} \tag{89}$$

and, since $\int_0^\pi d\psi' \sin\psi' = 2$,

$$\left(\Omega - \frac{1}{\sqrt{2}}\right)\Phi(\rho) = -\frac{(-1)^\ell 2^{1/2}}{\pi}\int_0^{\rho_0} d\rho' \frac{(\rho\rho')^{1/2}}{\rho^2+\rho'^2}\Phi(\rho'). \tag{90}$$

The scale invariance of this kernel motivates the logarithmic change of variable

$$t = \ln\frac{\rho}{\rho_0} \quad\text{and}\quad \phi(t) = \left(\frac{\rho}{\rho_0}\right)^{1/2}\Phi(\rho). \tag{91}$$

The resulting eigenvalue problem

$$\Omega\phi(t) = \frac{1}{2^{1/2}}\phi(t) - \frac{(-1)^\ell}{2^{1/2}\pi}\int_{-\infty}^0 \frac{dt'\phi(t')}{\mathrm{ch}(t-t')} \tag{92}$$

admits Eq. (84) as a continuous spectrum with eigenfunctions $\phi(t)$ that are for $t\to-\infty$ linear superpositions of e^{ikt} and e^{-ikt}, since $\int_{\mathbb{R}} dt e^{ikt}/\mathrm{ch}\,t = \pi/\mathrm{ch}(k\pi/2)$; we have checked numerically that it has no discrete eigenvalue [52].

1. Physical interpretation

We collect Eqs. (86), (89), and (91), taking as a particular solution of Eq. (92) at t large and negative the function $\phi(t) = 1$, which corresponds to an asymptotic plane wave in t space with a vanishing wave vector, that is, to $k=0$ in Eq. (84) [53]. Restricting for simplicity to a zero total angular momentum $\ell = 0$ [54], we then find that

$$\Phi^{(0)}_0(x,u) \underset{(x,u)\to(0,-1)}{\propto} \frac{1}{\rho^{s+3/2}}. \tag{93}$$

A more inspiring writing is obtained in terms of the center-of-mass and relative wave vectors $\mathbf{K}_{24} = \mathbf{k}_2+\mathbf{k}_4$ and $\mathbf{k}_{24} = (\mathbf{k}_2 - \alpha\mathbf{k}_4)/(1+\alpha)$ of particles 2 and 4:

$$\Phi^{(0)}(x,u) \underset{K_{24}/k_{24}\to 0}{\propto} \left(\frac{k_{24}}{K_{24}}\right)^{s+3/2}. \tag{94}$$

One has indeed $\rho^2 \simeq 2(u+\mathrm{ch}\,x)$ and $K_{24}^2 = 2k_2^2 e^x(u+\mathrm{ch}\,x)$, so K_{24} and ρ vanish in the same way when $(x,u)\to(0,-1)$; also the ratio K_{24}/k_{24} tends to zero if and only if $u+\mathrm{ch}\,x \to 0$ [55]. Restricting to a small neighborhood of the singularity, $K_{24} < \epsilon k_{24}$, where $\epsilon \ll 1$, we can in the ansatz (20) approximate the factor $(\mathrm{ch}\,x)^{s+3/2}$ by one and, in the denominator, approximate $k_2^2 + k_4^2 = 2k_{24}^2 + 2\frac{\alpha-1}{\alpha+1}\mathbf{k}_{24}\cdot\mathbf{K}_{24} + \frac{1+\alpha^2}{(1+\alpha)^2}K_{24}^2$ by its leading-order approximation $2k_{24}^2$ to isolate the singular behavior of $D(\mathbf{k}_2,\mathbf{k}_4)$:

$$D_{\mathrm{sing}}(\mathbf{k}_2,\mathbf{k}_4) \propto \frac{1}{k_{24}^{s+7/2}}\left(\frac{k_{24}}{K_{24}}\right)^{s+3/2}. \tag{95}$$

The key idea is then to see how this translates into a singularity of the regular part $\mathcal{A}_{13}$ of the four-body wave function that appears in the Wigner-Bethe-Peierls contact condition (1). As we have seen below Eq. (5), $\mathcal{A}_{13} = \mathcal{A}$ is related to $D(\mathbf{k}_2,\mathbf{k}_4)$ by a Fourier transform; using $(\mathbf{k}_{24},\mathbf{K}_{24})$ rather than $(\mathbf{k}_2,\mathbf{k}_4)$ as integration variables, and the fact that $\mathbf{k}_2\cdot\mathbf{r}_2 + \mathbf{k}_4\cdot\mathbf{r}_4 = \mathbf{k}_{24}\cdot\mathbf{r}_{24} + \mathbf{K}_{24}\cdot\mathbf{R}_{24}$, where $\mathbf{r}_{24} = \mathbf{r}_2 - \mathbf{r}_4$ and $\mathbf{R}_{24} = (m_2\mathbf{r}_2 + m_4\mathbf{r}_4)/(m_2+m_4)$ are the relative and center-of-mass coordinates of the particles 2 and 4, we obtain for the contribution to $\mathcal{A}$ of the singularity of D:

$$\mathcal{A}_{\mathrm{sing}}(\mathbf{r}_2 - \mathbf{R}_{13}, \mathbf{r}_4 - \mathbf{R}_{13}) \propto \int_{K_{24}<\epsilon k_{24}} d^3k_{24}d^3K_{24} e^{i\mathbf{K}_{24}\cdot(\mathbf{R}_{24}-\mathbf{R}_{13})}\frac{e^{i\mathbf{k}_{24}\cdot\mathbf{r}_{24}}}{k_{24}^{s+7/2}}\left(\frac{k_{24}}{K_{24}}\right)^{s+3/2}. \tag{96}$$

Integrating over the solid angles for $\mathbf{k}_{24}$ and $\mathbf{K}_{24}$, performing the change of variable $K_{24} = qk_{24}r_{24}/|\mathbf{R}_{24}-\mathbf{R}_{13}|$ at fixed k_{24}, changing the order of integration over k_{24} and q, and, finally, integrating over k_{24} [56] we obtain

$$\mathcal{A}_{\mathrm{sing}}(\mathbf{r}_2 - \mathbf{R}_{13}, \mathbf{r}_4 - \mathbf{R}_{13}) \propto \frac{|\mathbf{R}_{24}-\mathbf{R}_{13}|^{s-3/2}}{r_{24}} \times \int_0^{\epsilon|\mathbf{R}_{24}-\mathbf{R}_{13}|/r_{24}} \frac{dq}{q^{s+1/2}}[|q-1|^{s-1/2} - (q+1)^{s-1/2}]. \tag{97}$$

It then becomes obvious that the singularity in $D(\mathbf{k}_2,\mathbf{k}_4)$ at $\mathbf{k}_2+\mathbf{k}_4 = \mathbf{0}$ is linked to a $1/r_{24}$ divergence of the regular part $\mathcal{A}_{13}$ of the four-body wave function at $r_{24}=0$ [57,58]. This was physically expected: $\mathcal{A}_{13}(\mathbf{r}_2-\mathbf{R}_{13},\mathbf{r}_4-\mathbf{R}_{13})$ is essentially the wave function of particles 2 and 4 knowing that particles 1 and 3 have converged to the same location in the s wave; since 2 and 4 are in different spin states, they interact in the s wave and must be sensitive to the 2-4 Wigner-Bethe-Peierls contact conditions, which implies a $1/r_{24}$ divergence when $r_{24}\to 0$. Such a divergence of $\mathcal{A}_{13}$ was already pointed out in the scattering problem of two $\uparrow\downarrow$ dimers in Ref. [59] and in the general $N_\uparrow + N_\downarrow$ fermionic problem when $N_\uparrow \geqslant 2$ and $N_\downarrow \geqslant 2$ in Ref. [60] (see footnote 20 of that reference) [61].

This interpretation of the matrix kernel singularity at $(x,u) = (0,-1)$ has a simple, though illuminating, implication: The $1/r_{24}$ divergence in $\mathcal{A}_{13}$ can take place only when the particles 2 and 4 converge to the same location in the relative partial wave of zero angular momentum, since $\uparrow$ and $\downarrow$ particles resonantly interact only in the s wave. In such a configuration the angular momentum ℓ of the function $\mathcal{A}_{13}$ (that is of the whole system) is carried out by the center-of-mass motion of particles 2 and 4 with respect to $\mathbf{R}_{13}$; in this case there is a univocal link between the angular momentum ℓ and the parity, as for single particle systems, and the parity of $\mathcal{A}_{13}$ must be $(-1)^\ell$. This explains why the singularity at $(x,u)=(0,-1)$, and, ultimately, the third continuum (84), can appear only in that parity channel [62].

V. SEARCH FOR THE FOUR-BODY EFIMOV EFFECT

In the $3+1$ fermionic problem, the signature of a four-body Efimov effect was that an eigenvalue of the corresponding $M(s=0)$ operator crosses zero for some value of α below $\alpha_c(2;1)\simeq 13.6069$, specifically for $\alpha=\alpha_c(3;1)\simeq 13.384$ [13]. The question here is to know whether such a crossing can occur for the $2+2$ fermionic problem, that is, for $M(s=0)$ corresponding to Eqs. (68) and (69). We answer this question by a numerical calculation of the eigenvalues of $M(s=0)$.

A. Numerical implementation

In the $(-1)^{\ell+1}$ parity sector, we truncate the x variable in a symmetric way, that is, to $[x_{\min}=-x_{\max},x_{\max}]$, and we discretize it with a uniform step dx according to the usual midpoint integration method. We use θ rather than $u=\cos\theta$ as integration variable, so we use $\check{\Phi}(x,\theta)=(\sin\theta)^{1/2}\Phi(x,u)$ rather than $\Phi(x,u)$ as the unknown function. Multiplying the eigenvalue problem $\Omega\Phi=M(s=0)\Phi$ by $(\sin\theta)^{1/2}$, we get a Hermitian eigenvalue problem with the same eigenvalues, the same diagonal part, and a kernel $\check{K}^{(\ell)}_{m_z,m_z'}(x,\theta;x',\theta')=(\sin\theta\,\sin\theta')^{1/2}K^{(\ell)}_{m_z,m_z'}(x,u;x',u')$ with $u=\cos\theta$, $u'=\cos\theta'$. For better accuracy, we use the Gauss-Legendre integration scheme [63] for θ' as this angular variable is naturally bounded to $[0,\pi]$ [64]. In the $(-1)^\ell$ parity sector, there is the additional complication that the kernel diverges at the point $(x,\theta)=(0,\pi)$, making the previous (x,θ) discretization inefficient. At a small distance from this point, say, less than ρ_0, the optimal set of variables is (t,ψ), where $\psi\in[0,\pi]$ is defined by Eq. (88) and $t\in\mathbb{R}^-$ is given by Eq. (91), since the resulting kernel is bounded after a convenient change of the unknown function $\tilde{\Phi}^{(\ell)}_{m_z}(t,\psi)=\rho\,\check{\Phi}^{(\ell)}_{m_z}(x,\theta)$, see Eq. (92). So we resort to a mixed scheme: For $\rho=[x^2+(\pi-\theta)^2]^{1/2}>\rho_0$, we use the (x,θ) set of variables, with x uniformly discretized with a step dx submultiple of ρ_0 and θ discretized according to a Gauss-Legendre scheme over the interval $[0,\theta_{\max}]$ where $\theta_{\max}=\pi$ for $|x|>\rho_0$ and $\theta_{\max}=\pi-(\rho_0^2-x^2)^{1/2}$ for $|x|\leqslant\rho_0$, the number of angular points being scaled linearly with $\theta_{\max}$. For $\rho<\rho_0$, we use the (t,ψ) set of variables, with t truncated to $[t_{\min},0]$ and discretized with a uniform step dt according to the midpoint integration method, and $\psi\in[0,\pi]$ discretized according to the Gauss-Legendre scheme [65].

B. Results

The numerically obtained spectrum of $M(s=0)$ for the $2+2$ fermionic problem is plotted in Fig. 3 [left half for the $(-1)^\ell$ parity channels and right half for the $(-1)^{\ell+1}$ parity channels] for the first values $0\leqslant\ell\leqslant 3$ of the four-body internal angular-momentum quantum number ℓ, as a function of the mass ratio $\alpha=\frac{m_\uparrow}{m_\downarrow}$. It is a symmetric function under the exchange $\alpha\leftrightarrow 1/\alpha$ so the figure is restricted to $\alpha\geqslant 1$; as the starting integral equation (13) assumes scale invariance, which is broken by

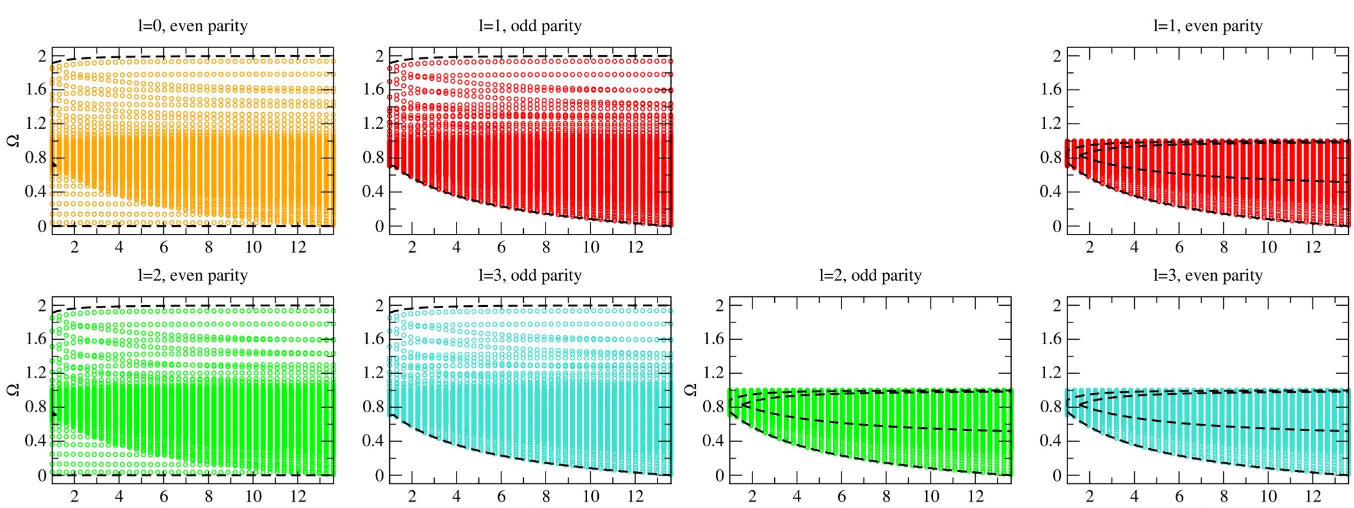

FIG. 3. (Color online) Eigenvalues of $M(s=0)$ as functions of the mass ratio $\alpha\in[1,13.6]$ for various values of the angular momentum ℓ and the parity [left half for the parity $(-1)^\ell$, right half for the parity $(-1)^{\ell+1}$], obtained numerically after discretization and truncation of the variables x and θ in the zone $\rho\equiv[x^2+(\pi-\theta)^2]^{1/2}>\rho_0$ and of their log-polar versions $t\equiv\ln(\rho/\rho_0)$ and ψ in the zone $\rho<\rho_0$ (see text): $x_{\max}=-x_{\min}=12$, $dx=1/5$, $d\theta\simeq\pi/15$, $\rho_0=2/5$, $t_{\min}=-12$, $dt=1/5$, $d\psi=\pi/15$, with the Gauss-Legendre integration scheme for the integrals over θ and ψ, and the midpoint integration formula for the integrals over x and t. For the $(-1)^{\ell+1}$ parity channels the $\rho<\rho_0$ zone is not useful and is not included in the numerics. The boundaries of the continuous spectrum of $M(s)$ are given by black thick dashed curves: For the $(-1)^{\ell+1}$ parity channels, this corresponds to the $x\to+\pm\infty$ continua of Eqs. (77) and (81) with all $L\geqslant 1$; for the $(-1)^\ell$ parity channels, this corresponds to the $x\to+\pm\infty$ continua of Eqs. (77) and (81) with all $L\geqslant 0$ and to the $(x,\theta)\to(0,\pi)$ (that is, $t\to-\infty$) continuum of Eq. (84). In contrast to the $3+1$ case, no eigenvalue of $M(s=0)$ is found to cross zero for $\alpha<\alpha_c(2;1)=13.6069\ldots$: No four-body Efimov effect is found for the $2+2$ fermionic problem.

the three-body Efimov effect beyond the threshold $\alpha_c(2;1) = 13.6069\ldots$, the figure is also restricted to $\alpha < \alpha_c(2;1)$.

For the $(-1)^\ell$ parity channels, the spectrum is entirely within the limits of the analytically predicted continuous spectrum, which are shown as dashed lines, except for even ℓ in a barely visible small triangle [66] close to $\Omega = 0.75$ with $1 \leqslant \alpha \lesssim 1.2$: This means that the necessarily discrete numerical spectrum must tend to a continuum when the truncations $x_{\rm max} = -x_{\rm min}$ and $t_{\rm min}$ tend to $+\infty$ and $-\infty$, respectively. The lower border of the continuum corresponds for even ℓ to the $k \to 0$ limit in Eq. (84), that is, zero, and for odd ℓ to the smallest of the two quantities, $1/\sqrt{2}$ [this is the $k \to +\infty$ limit of Eq. (84)] and $\Lambda_{L=1}(0,\alpha)$ [this is the minimal value of Eq. (81), see Fig. 2]. The upper border of the continuum corresponds, whatever the parity of ℓ, to $\Lambda_{L=0}(0,1/\alpha)$ [this is the maximal value of Eq. (77), see Fig. 2].

For the $(-1)^{\ell+1}$ parity channels, there are three differences. First, the continuum (84) cannot be realized, so the lower border of the continuous spectrum of $M(s=0)$, now given by $\Lambda_{L=1}(0,\alpha)$, reaches zero only at $\alpha = \alpha_c(2;1)$. Second, the $L = 0$ continua in Eqs. (77) and (81) cannot be realized so the upper border of the continuum of $M(s=0)$, given by $\Lambda_{L=2}(0,1/\alpha)$, is everywhere below $1 = \lim_{\alpha\to+\infty} \Lambda_{L=2}(0,1/\alpha)$. Third, the continuum presents, in the $\alpha - \Omega$ plane for $1.53 \lesssim \alpha$, a large void internal area, corresponding to $\Lambda_{L=2}(0,\alpha) < \Omega < \Lambda_{L=1}(0,1/\alpha)$. Still, many numerically found eigenvalues lay in this internal area: These eigenvalues must correspond to the discrete spectrum of $M(s=0)$, with localized (square-integrable) eigenfunctions [67]. We checked this numerically by calculating the density of states of $M(s=0)$, in practice the histogram of its eigenvalues, for increasing values of the numerical truncation $x_{\rm max} = -x_{\rm min}$: By doubling the values of $x_{\rm max}$ and $x_{\rm min}$, the spacing $\approx \pi/x_{\rm max}$ between successive k in Eqs. (77) and (81) is approximately divided by 2 so the density of states of the numerical quasicontinuum is approximately multiplied by 2, whereas the density of states of the discrete spectrum is essentially not (only exponentially weakly) affected as soon as $x_{\rm max}$ is much larger than the localized eigenfunctions width in x space. This is what is observed in Fig. 4, knowing that the locations of the internal and external borders of the continuum (corresponding to the dashed lines in Fig. 3) are indicated by vertical dashed lines in Fig. 4.

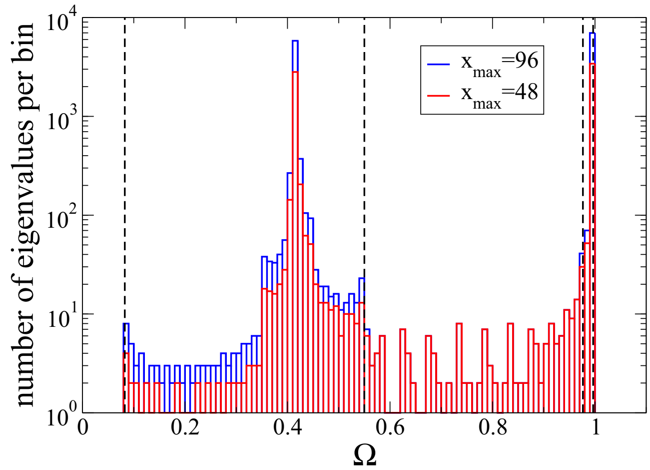

FIG. 4. (Color online) Histogram of the eigenvalues Ω of $M(s=0)$ for a mass ratio $\alpha = 10$ and an angular momentum $\ell = 1$ in the $(-1)^{\ell+1}$ parity channels, obtained numerically after discretization and truncation of the x and θ variables. The numerical grid is the same as in Fig. 3, except that much larger values of $x_{\rm max} = -x_{\rm min}$ are used to reveal the emergence of the continuous part of the spectrum in the $x_{\rm max} \to +\infty$ limit: $x_{\rm max} = 48$ (red bars in the foreground) and $x_{\rm max} = 96$ (blue bars in the background). The black vertical dashed lines indicate the analytically predicted borders of the continuous spectrum (as in Fig. 3); between the first two ones and between the last two ones, it is indeed observed that the number of eigenvalues per bin is approximately multiplied by 2 when $x_{\rm max}$ is doubled. On the contrary, the histogram is unaffected by the change of $x_{\rm max}$ in the bins strictly in between the second and third dashed lines, indicating that the corresponding eigenvalues belong to the discrete spectrum of $M(s=0)$, with localized eigenfunctions in x space.

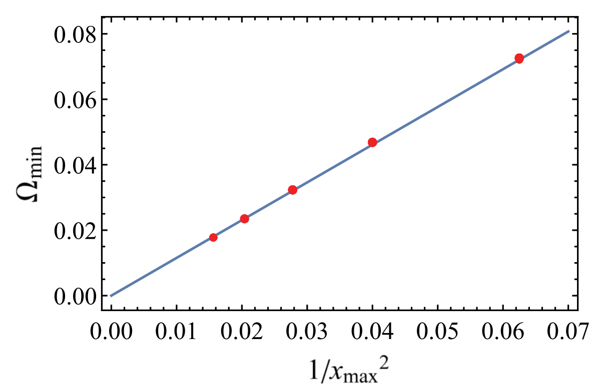

FIG. 5. (Color online) Numerically determined minimal eigenvalue $\Omega_{\rm min}$ of $M^{(\ell)}(s=0)$ almost at the three-body critical mass ratio, $\alpha = 13.6069 \simeq \alpha_c(2;1)$, as a function of the numerical cutoff $t_{\rm min} = -x_{\rm max}$. For each given cut-off value, each angular momentum ℓ from 0 to 12 and parity sector $(-1)^\ell$ and $(-1)^{\ell+1}$ contributes as a point in the figure: The fact that the points (in red) are superimposed and cannot be distinguished shows that $\Omega_{\rm min}$ does not depend on ℓ nor on the parity. Furthermore, $\Omega_{\rm min}$ is always positive; it is linear in $1/x_{\rm max}^2$ and extrapolates to zero for infinite cutoff (see the blue line): This is perfectly consistent with the fact that $\Omega_{\rm min}$ corresponds to the lower border of the $2+1$ continuum $\Lambda_{L=1}(ik,\alpha)$, where k has a minimal, discrete value scaling as $1/x_{\rm max}$ in the presence of the numerical cutoff, and $\Lambda_{L=1}(ik,\alpha_c(2;1))$ vanishes quadratically at $k=0$. In other words, there is no negative $\Omega_{\rm min}$ and no four-body Efimov effect for $2+2$ fermions.

C. Synthesis

In contrast to the $3+1$ case, no discrete eigenvalue of $M(s=0)$ (necessarily discrete because it would be below the lower border of the continuum) crosses zero for $\alpha < \alpha_c(2;1) = 13.6069\ldots$, that is, below the three-body Efimov effect threshold: No four-body Efimov effect is found for the $2+2$ fermionic problem [68]. This conclusion is apparent in Fig. 3, obtained for all the internal angular-momentum quantum numbers $0 \leqslant \ell \leqslant 3$. It extends to all the angular-momentum values that we were able to numerically explore, $4 \leqslant \ell \leqslant 12$, as we have shown with a dedicated careful spectral analysis almost perfectly at the three-body critical mass ratio, $\alpha = 13.6069$, see Fig. 5.

VI. CONCLUSION

We have studied in three dimensions a four-body $2+2$ fermionic system with resonant interactions and we have derived its momentum space integral equations at zero energy. By using rotational invariance and scale invariance, we have reduced them to a numerically tractable two-dimensional form (the unknown function depends on two variables only). With these equations we have numerically shown that no four-body Efimov effect occurs for the $2+2$ fermionic system in angular-momentum channels $0 \leqslant \ell \leqslant 12$. The $3+1$ fermionic system thus remains the only known one exhibiting a four-body Efimov effect [13].

A detailed treatment of the second motivation for deriving these integral equations, that is, a calculation of the already-measured [20,21] fourth cluster coefficient b_4 of the spin-1/2 unitary Fermi gas, is beyond the scope of this paper. Still we have numerically calculated an educated guess for b_4 inspired from the analytical form of the third cluster coefficient b_3 [28,29]: It is found that this guess does not reproduce the experimental value (see Appendix B) so a dedicated work is needed and left for the future.

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APPENDIX A: IS THE SPECTRUM OF $M(s)$ BOUNDED?

In the parity sector $(-1)^\ell$, the third contribution in Eq. (69) diverges when $(x,u) \to (0,-1)$ or $(x',u') \to (0,-1)$. The question is to know if this makes the operator $M(s)$ unbounded, for a purely imaginary $s = iS$.

To investigate this problem, we construct a simplified functional that focuses on the diverging part of the matrix kernel (69), replacing each nonzero limit expression by its limit and replacing the vanishing expressions by their leading-order (here quadratic) approximations:

$$u + \mathrm{ch}\, x \simeq \tfrac{1}{2}(x^2+y^2) \text{ with } y \equiv \pi - \theta. \tag{A1}$$

Dropping numerical factors and other bounded pieces (for example, the bit raised to the power s, of modulus one), we obtain the mean-Ω functional

$$\langle \Omega \rangle = \frac{\int dx du \int dx' du' \frac{\Phi^*(x,u)\Phi(x',u')}{[(x^2+y^2)(x'^2+y'^2)]^{1/4}(x^2+y^2+x'^2+y'^2)}}{\int dx du |\Phi(x,u)|^2}, \tag{A2}$$

where the integrals are taken over some convenient neighborhood of $(x,u) = (0,-1)$. It is convenient to use the angle θ rather than $u = \cos\theta$ as integration variable, which pulls out a Jacobian $\sin\theta \simeq y$; we absorb it in the integral in the denominator of Eq. (A2) with the change of function

$$\check{\Phi} = (\sin\theta)^{1/2}\Phi(x,u). \tag{A3}$$

A factor $(\sin\theta \sin\theta')^{1/2} \simeq (yy')^{1/2}$ remains in the integrand in the numerator. We restrict the integration over (x,y) to the upper half $y > 0$ of the disk $x^2+y^2 < 1$. Then it is natural to move to polar coordinates:

$$(x,y) = (\rho\cos\phi, \rho\sin\phi) \tag{A4}$$

so $x^2+y^2 = \rho^2$, $x'^2+y'^2 = \rho'^2$, and $(yy')^{1/2} = (\rho\rho')^{1/2}(\sin\phi\sin\phi')^{1/2}$. The occurrence of the Jacobians ρ and ρ' in the elements $\rho d\rho$ and $\rho' d\rho'$ motivates the change of variable in the radial integration:

$$X = \rho^2 \text{ and } X' = \rho'^2. \tag{A5}$$

Then, considering $\check{\Phi}$ as a function of X and ϕ, we obtain

$$\langle \Omega \rangle = \frac{\int_0^1 \frac{dX dX'}{X+X'} \int_0^\pi d\phi d\phi' (\sin\phi\sin\phi')^{1/2} \check{\Phi}^*(X,\phi)\check{\Phi}(X',\phi')}{2\int_0^1 dX \int_0^\pi d\phi\, |\check{\Phi}(X,\phi)|^2}. \tag{A6}$$

To get rid of the polar angle ϕ, we introduce

$$\Phi_a(X) \equiv \int_0^\pi d\phi\, (\sin\phi)^{1/2} \check{\Phi}(X,\phi) \tag{A7}$$

so the integral over ϕ and ϕ' in the numerator of Eq. (A6) reduces to the product $\Phi_a^*(X)\Phi_a(X')$. In that numerator, we use the fact that the modulus of the integral over X and X' is less than the integral of the modulus and that $\frac{1}{X+X'} \leqslant \frac{1}{(X^2+X'^2)^{1/2}}$. In the denominator of Eq. (A6), at fixed X, we apply over the interval $\phi \in [0,\pi]$ the Cauchy-Schwarz inequality $|\langle f|g\rangle|^2 \leqslant \langle f|f\rangle \langle g|g\rangle$ (in Dirac's notation) with $f(\phi) = (\sin\phi)^{1/2}$ and $g(\phi) = \check{\Phi}(X,\phi)$; after integration of the resulting inequality over X, we get:

$$\int_0^1 dX\, |\Phi_a(X)|^2 \leqslant 2\int_0^1 dX \int_0^\pi d\phi\, |\check{\Phi}(X,\phi)|^2, \tag{A8}$$

whose right-hand side is the denominator of Eq. (A6). We arrive at

$$|\langle \Omega \rangle| \leqslant \frac{\int_0^1 dX \int_0^1 dX' \frac{|\Phi_a(X)|\,|\Phi_a(X')|}{(X^2+X'^2)^{1/2}}}{\int_0^1 dX\, |\Phi_a(X)|^2}. \tag{A9}$$

We again move to polar coordinates

$$(X,X') = (r\cos\psi, r\sin\psi) \tag{A10}$$

to simplify the factor $\frac{1}{(X^2+X'^2)^{1/2}} = \frac{1}{r}$ with the Jacobian and to obtain

$$|\langle \Omega \rangle| \leqslant \frac{\int_0^{\pi/2} d\psi \int_0^{R(\psi)} dr |\Phi_a(r\cos\psi)||\Phi_a(r\sin\psi)|}{\int_0^1 dX\, |\Phi_a(X)|^2}. \tag{A11}$$

Since the domain of integration over (X,X') is the square $[0,1]^2$, ψ runs over $[0,\pi/2]$ and, at fixed ψ, r runs over $[0,R(\psi)]$ with

$$R(\psi) = \min\left(\frac{1}{\cos\psi}, \frac{1}{\sin\psi}\right). \tag{A12}$$

In the integral over r at fixed ψ, we again use the Cauchy-Schwarz inequality over the interval $r \in [0,R(\psi)]$ with $f(r) = |\Phi_a(r\cos\psi)|$ and $g(r) = |\Phi_a(r\sin\psi)|$:

$$\int_0^{R(\psi)} dr |\Phi_a(r\cos\psi)||\Phi_a(r\sin\psi)| \leqslant \left[\int_0^{R(\psi)} dr |\Phi_a(r\cos\psi)|^2\right]^{1/2} \left[\int_0^{R(\psi)} dr |\Phi_a(r\sin\psi)|^2\right]^{1/2}. \tag{A13}$$

In the first factor in the right-hand side of Eq. (A13), we perform the change of variable $X = r\cos\psi$, so

$$\int_0^{R(\psi)} dr|\Phi_a(r\cos\psi)|^2 = \frac{1}{\cos\psi}\int_0^{R(\psi)\cos\psi} dX|\Phi_a(X)|^2 \leqslant \frac{1}{\cos\psi}\int_0^1 dX|\Phi_a(X)|^2, \quad \text{(A14)}$$

where we used $R(\psi)\cos\psi \leqslant 1$ and the non-negativeness of $|\Phi_a|^2$. The last integral in Eq. (A14) is the denominator in the right-hand side of Eq. (A11). We proceed similarly in the second factor in the right-hand side of Eq. (A13), except that $\cos\psi$ is replaced with $\sin\psi$. Finally, the denominator in Eq. (A11) cancels out, so

$$|\langle\Omega\rangle| \leqslant \int_0^{\pi/2} \frac{d\psi}{(\cos\psi\,\sin\psi)^{1/2}} < +\infty \quad \text{(A15)}$$

and the spectrum of $M(s)$ is bounded, when $s \in i\mathbb{R}$.

APPENDIX B: ENUNCIATING AND TESTING A CONJECTURE FOR b_4

1. The cluster expansion

Consider a spatially uniform spin-1/2 Fermi gas at thermal equilibrium in the grand-canonical ensemble in the thermodynamic limit, with a temperature T, and a single chemical potential μ since the gas is unpolarized. The well-known cluster expansion is a series expansion of its pressure in powers of the fugacity $z = \exp(\beta\mu)$ in the nondegenerate limit $\mu \to -\infty$ for a fixed temperature T, with $\beta = 1/(k_BT)$ [70]. For our gas, it is generally written as

$$\frac{P\lambda^3}{k_BT} = 2\sum_{n\geqslant 1} b_n z^n, \quad \text{(B1)}$$

where the overall factor 2 accounts for the number of spin components and λ is the thermal de Broglie wavelength

$$\lambda = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2}. \quad \text{(B2)}$$

When reexpanded in terms of the small degeneracy parameter $\rho\lambda^3$, where ρ is the total density, the cluster expansion gives rise to the virial expansion with virial coefficients a_n [70]. In practice, one rather considers the deviation Δb_n of b_n from its ideal Fermi gas value, that is (for $n > 1$), from the mere effect of Fermi statistics:

$$b_n = \frac{(-1)^{n+1}}{n^{5/2}} + \Delta b_n. \quad \text{(B3)}$$

While the cluster expansion has been studied for a long time and the second cluster coefficient b_2 was obtained analytically in Ref. [71] (note that $b_1 = 1$ according to the ideal gas law), there is a renewed interest in the cluster coefficients for $n > 2$. First, the new challenge is to calculate the b_n for resonant s-wave interactions (with a scattering length a much larger in absolute value than the interaction range), whereas previous studies concentrated on the hard-sphere model [72]. Second, the b_n have been extracted up to $n = 4$ in the unitary limit from a measurement of the equation of state of ultracold atomic Fermi gases [20,21]. The two independent groups have reported consistent values of the fourth cluster coefficient:

$$\Delta b_4^{\rm ENS} = 0.096(15) \ \text{and}\ \Delta b_4^{\rm MIT} = 0.096(10). \quad \text{(B4)}$$

2. In the unitary limit

For zero-range interactions with infinite s-wave scattering length $a^{-1} = 0$, i.e., in the unitary limit, the harmonic regulator method used in Ref. [25], which introduces an isotropic harmonic trapping potential, is quite efficient, due to the SO(2,1) dynamical symmetry resulting from the scale invariance [33,73] and the subsequent separability of Schrödinger's equation in hyperspherical coordinates [32,46] in the trap. The value of b_n can be deduced from the canonical partition functions, that is, from the energy spectra, of all the possible k-body problems in the trap, with $k \leqslant n$. One has the following expansion of the grand potential Ω of the thermal equilibrium gas in the trap:

$$\frac{-\Omega}{k_BTZ_1} = \sum_{(n_\uparrow,n_\downarrow)\in\mathbb{N}^{2*}} B_{n_\uparrow,n_\downarrow}(\omega) z_\uparrow^{n_\uparrow} z_\downarrow^{n_\downarrow}, \quad \text{(B5)}$$

where Z_1 is the canonical partition function for one particle in the trap, and it is convenient at this stage to be general and introduce independent chemical potentials μ_σ for the various spin components σ, so $z_\sigma = \exp(\beta\mu_\sigma)$. Then, from the asymptotically exact local density approximation [26] (see also Ref. [25]), and introducing also the deviations $\Delta B_{n_\uparrow,n_\downarrow}(\omega)$ of $B_{n_\uparrow,n_\downarrow}(\omega)$ from the ideal Fermi gas value [74], one has

$$2\Delta b_n = n^{3/2}\sum_{n_\uparrow=1}^{n-1} \Delta B_{n_\uparrow,n_\downarrow=n-n_\uparrow}(0^+), \quad \text{(B6)}$$

where $\Delta B(0^+) = \lim_{\omega\to 0^+}\Delta B(\omega)$ and where we could restrict the sum to $n_\sigma \neq 0$, $\sigma = \uparrow, \downarrow$, since the fully polarized configurations are noninteracting and have zero deviations from the ideal gas.

For $n = 3$, extending to fermions the technique initially developed for bosons [28], the following analytical expression was obtained [29,75]:

$$\Delta B_{2,1}(0^+) = \sum_{\ell\in\mathbb{N}}\left(\ell+\frac{1}{2}\right)\int_0^{+\infty}\frac{dS}{\pi} S\frac{d}{dS}[\ln\Lambda_l(iS,\alpha)], \quad \text{(B7)}$$

where the function Λ_l is given by Eq. (78), and the mass ratio between the opposite spin component α is equal to 1 (so $\Delta B_{2,1} = \Delta B_{1,2}$). It gives

$$\Delta b_3 \simeq -0.355103, \quad \text{(B8)}$$

in agreement with previous numerical studies [26,27] and with the experimental values [20].

For $n = 4$, the problem is still open. A numerical attempt [30], with brute-force calculation of the four-body unitary spectrum in the trap, has produced the value

$$\Delta b_4^{\rm Blume} = -0.016(4). \quad \text{(B9)}$$

The disagreement with the experimental results (B4) is attributed to uncertainties in extrapolating to $\omega \to 0$ the numerical values of $\Delta B_{n_\uparrow,n_\downarrow}(\omega)$, in practice obtainable only for $\hbar\omega \gtrsim k_BT$. An approximate diagrammatic theory [24] (keeping *even* in the unitary limit only the diagrams that have leading contribution in the perturbative regime of a large effective range or a small scattering length) gives an estimate closer to the experimental values (B4),

$$\Delta b_4^{\text{Levinsen}} \approx 0.06. \tag{B10}$$

Extending the analytical method of Ref. [28] to the fermionic four-body problem is technically challenging and goes beyond the scope of the present work. On the contrary, it is reasonable here to propose and test a guess by direct transposition of Eq. (B7): The transcendental function $\Lambda_l(s)$ of the three-body problem is formally replaced by det $M^{(\ell)}(s)$ for the four-body problem, where det is the determinant and the operator $M^{(\ell)}(s)$, acting on the spinor functions $\Phi^{(\ell)}_{m_z}(x,u)$ as in the right-hand side of Eq. (68), was introduced and spectrally discussed in Sec. IV B for the $2+2$ fermionic problem and has a known equivalent for the $3+1$ fermionic problem, see Eq. (14) of Ref. [13]. Indeed, in both cases, the scaling exponents s (purely imaginary in the efimovian channels, real otherwise) allowed by Schrödinger's equation in the unitary Wigner-Bethe-Peierls model are such that $\Lambda_l(s) = 0$ for $n = 3$ or such that Eq. (68) has a nonzero solution $\Phi^{(\ell)}_{m_z}(x,u)$, that is, $M^{(\ell)}(s)$ admits a zero eigenvalue. Hence our conjecture:

$$\Delta B^{\text{conj}}_{n_\uparrow,n_\downarrow}(0^+) = \sum_{\ell\in\mathbb{N}} \left(\ell + \frac{1}{2}\right) \int_0^{+\infty} \frac{dS}{\pi} S \frac{d}{dS} \times \left[\ln \det M^{(\ell)}_{n_\uparrow,n_\downarrow}(iS)\right] \tag{B11}$$

with $(n_\uparrow,n_\downarrow) \in \{(1,3),(2,2),(3,1)\}$ and $M^{(\ell)}_{n_\uparrow,n_\downarrow}$ is the operator $M^{(\ell)}$ for the four-body problem with n_σ particles in each spin component σ.

3. Existence of the logarithmic derivative of the determinant

The conjecture (B11) is not as innocent as it may look at first sight. The difficulty is that $M^{(\ell)}$ is actually an operator and not a finite size matrix: It has a continuous spectrum, constituting an infinite, dense set of "eigenvalues"; even its discrete spectrum may present accumulation points, leading to an infinite but countable number of eigenvalues. In other words, the determinant of $M^{(\ell)}(iS)$ is not finite. Numerically, as we have already done in Sec. V, one of course truncates the unbounded variable x to the compact interval $[-x_{\max},x_{\max}]$, which amounts to imposing the boundary conditions to the spinor:

$$\Phi^{(\ell)}_{m_z}(x = \pm x_{\max},u) = 0 \ \forall u \in [-1,1], \forall m_z \in \{-\ell,\dots,\ell\}. \tag{B12}$$

After discretization of the x and u variables, $M^{(\ell)}(iS)$ is then replaced by a matrix, with a well-defined determinant; it is still unknown if there is convergence of the integrand in Eq. (B11) when $x_{\max} \to +\infty$. As we now see, the answer is positive.

The key point is that what appears in the integrand of Eq. (B11) is not the determinant itself but rather its logarithmic derivative, which can be written as

$$\frac{d}{dS} \ln \det M^{(\ell)}(iS) = \mathrm{Tr}\left\{[M^{(\ell)}(iS)]^{-1} \frac{d}{dS} M^{(\ell)}(iS)\right\}, \tag{B13}$$

where Tr is the trace and M^{-1} the inverse of M.

a. Parity $(-1)^{\ell+1}$. In the parity sector $(-1)^{\ell+1}$, the spectrum of $M^{(\ell)}(iS)$ is at nonzero distance from 0 for a mass ratio $\alpha = 1$, as there is no four-body Efimov effect, see Fig. 3. So the inverse of $M^{(\ell)}(iS)$ is well defined. Also the operator $M^{(\ell)}(iS)$ is local in the x basis, meaning that the off-diagonal matrix elements of the operator $\mathcal{D}^{-1/2} K^{(\ell)} \mathcal{D}^{-1/2}$ are rapidly decreasing functions of $|x - x'|$, for example, there exists a constant $A^{(\ell)}$ such that

$$\frac{|\langle x,u,\ell,m_z|K^{(\ell)}(iS)|x',u',\ell,m_z'\rangle|}{[d(x,u)d(x',u')]^{1/2}} \leqslant \frac{A^{(\ell)}}{\mathrm{ch}(x-x')} \tag{B14}$$

for all x,x',u,u' and all m_z,m_z' of parity opposite to ℓ and for all $S \in \mathbb{R}$. Here we have used Dirac's notation and singled out as in Eq. (68) a diagonal part and a kernel part,

$$M^{(\ell)}(iS) = \mathcal{D} + K^{(\ell)}(iS), \tag{B15}$$

where the operator $\mathcal{D}$ is positive and defined by the diagonal-element function $d(x,u)$,

$$\mathcal{D}|x,u,\ell,m_z\rangle = d(x,u)|x,u,\ell,m_z\rangle \ \text{ with}$$
$$d(x,u) = \left[\frac{\alpha}{(1+\alpha)^2}\left(1 + \frac{u}{\mathrm{ch}\,x}\right) + \frac{e^{-x} + \alpha e^{x}}{2(\alpha+1)\,\mathrm{ch}\,x}\right]^{1/2}. \tag{B16}$$

This locality is apparent for the first two contributions in the right-hand side of Eq. (69): Each contribution is bounded and is consistent with Eq. (B14) at the four infinities $(x,x') = (\pm\infty, \pm\infty)$ (see reasoning in Sec. IV B). We then expect that the inverse of $M^{(\ell)}(iS)$, which can be written as

$$[M^{(\ell)}(iS)]^{-1} = \mathcal{D}^{-1} + K^{(\ell)}_{\text{inv}}(iS), \tag{B17}$$

is also local from the geometric series expansion:

$$(\mathcal{D}+K)^{-1} = \mathcal{D}^{-1/2}(\mathbb{1} + \mathcal{D}^{-1/2}K\mathcal{D}^{-1/2})^{-1}\mathcal{D}^{-1/2}$$
$$= \mathcal{D}^{-1} + \mathcal{D}^{-1/2}\sum_{n\geqslant 1}(-1)^n(\mathcal{D}^{-1/2}K\mathcal{D}^{-1/2})^n\mathcal{D}^{-1/2} \tag{B18}$$

each term of the series being local [for simplicity, we omit the exponent (ℓ) and the argument iS]. This holds, of course, if the operator $\mathcal{D}^{-1/2}K\mathcal{D}^{-1/2}$ is small enough. For $\ell = 1$ in the $2+2$ fermionic problem, this can be made rigorous: The best constant in Eq. (B14) is

$$A = \frac{2(2-\sqrt{3})}{3\pi} \simeq 0.05686. \tag{B19}$$

Then [76]

$$\frac{\left|\langle x,u,\ell=1,m_z=0\middle|K^{(\ell=1)}_{\text{inv}}(iS)\middle|x',u',\ell=1,m_z'=0\rangle\right|}{[d(x,u)d(x',u')]^{-1/2}} \leqslant \frac{2A}{\sqrt{1-(2\pi A)^2}} \frac{\mathrm{sh}\left(\frac{2\delta}{\pi}|x-x'|\right)}{\mathrm{sh}(2|x-x'|)} \tag{B20}$$

with $\delta = \arccos(-2\pi A) \in]\pi/2,\pi[$.

This locality *per se* is not enough to ensure the convergence of the trace in Eq. (B13). Making the trace explicit in that equation and injecting a closure relation leads to writing

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS) = \int_{\mathbb{R}} dx dx' \int_{-1}^{1} du du' \times \sum_{m_z,m_z'}^{(-1)^{\ell+1}} \langle x,u,\ell,m_z|[M^{(\ell)}(iS)]^{-1}|x',u',\ell,m_z'\rangle \times \langle x',u',\ell,m_z'|\frac{d}{dS}M^{(\ell)}(iS)|x,u,\ell,m_z\rangle, \quad (B21)$$

where the sum is restricted to m_z and m_z' of parity opposite to that of ℓ as the exponent $(-1)^{\ell+1}$ indicates. The locality of $M(iS)^{-1}$, and even of $\frac{d}{dS}M(iS)$, exponentially bounds the excursion of $|x-x'|$ in the integral over x', but still there remains the integral over the unbounded variable x. One must take advantage of the structure of $K(iS)$ and of its derivative: splitting

$$K(iS) = K_1(iS) + K_2(iS), \quad (B22)$$

where K_1 and K_2 respectively correspond to the first term and the second term in the right-hand side of Eq. (69), one finds

$$\frac{d}{dS}K(iS) = i[D_1,K_1(iS)] + i[D_2,K_2(iS)], \quad (B23)$$

where $[A,B]=AB-BA$ is the commutator of two operators and the diagonal operators D_j are defined by the following diagonal functions:

$$d_1(x) = \frac{1}{2}\ln\frac{e^{+x}}{2\,\mathrm{ch}\,x}, \quad (B24)$$

$$d_2(x) = \frac{1}{2}\ln\frac{e^{-x}}{2\,\mathrm{ch}\,x}, \quad (B25)$$

such that

$$D_j|x,u,\ell,m_z\rangle = d_j(x)|x,u,\ell,m_z\rangle. \quad (B26)$$

Clearly the diagonal term $\mathcal{D}^{-1}$ in M^{-1}, see Eq. (B17), has a zero contribution to the trace, as $[\mathcal{D},D_j]=0$. Equation (B21) is correspondingly rewritten as

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS) = \int_{\mathbb{R}} dx dx' \int_{-1}^{1} du du' \times \sum_{m_z,m_z'}^{(-1)^{\ell+1}} \sum_{j=1}^{2} i[d_j(x')-d_j(x)] \times \langle x,u,\ell,m_z|K_{\mathrm{inv}}^{(\ell)}(iS)|x',u',\ell,m_z'\rangle \times \langle x',u',\ell,m_z'|K_j^{(\ell)}(iS)|x,u,\ell,m_z\rangle. \quad (B27)$$

Then $d_1(x)$ tends exponentially rapidly to 0 when $x\to+\infty$, whereas it diverges linearly with x when $x\to-\infty$; the contrary holds for $d_2(x)$. A second property is that there exists a constant B such that

$$|\langle x,u,\ell,m_z|K_1(iS)|x',u',\ell,m_z'\rangle| \leqslant \left(\frac{e^{+x+x'}}{4\,\mathrm{ch}\,x\,\mathrm{ch}\,x'}\right)^{1/4}\frac{B}{\mathrm{ch}(x-x')}, \quad (B28)$$

$$|\langle x,u,\ell,m_z|K_2(iS)|x',u',\ell,m_z'\rangle| \leqslant \left(\frac{e^{-x-x'}}{4\,\mathrm{ch}\,x\,\mathrm{ch}\,x'}\right)^{1/4}\frac{B}{\mathrm{ch}(x-x')}. \quad (B29)$$

This is due to the fact, evident from Eqs. (59) and (60), that the denominator in the integral over ϕ in Eq. (69) is always larger than $(\mu_{\uparrow\downarrow}/m_\uparrow)\,\mathrm{ch}(x-x')$. Then, for $|x-x'|=O(1)$, the upper bound for the matrix elements of K_1 (respectively, K_2) tends exponentially fast to 0 when $x\to-\infty$ (respectively, $x\to+\infty$), due to the first factor in Eqs. (B28) and (B29), which suppresses the linear divergence in $d_1(x)$ [respectively, in $d_2(x)$]. Then the integral over x and x' in the trace converges exponentially rapidly at infinity, and the logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ in the $(-1)^{\ell+1}$ parity channel is well defined [77] and its value can be calculated with a rapidly vanishing error in the truncation $x_{\max}$ when $x_{\max}\to+\infty$. The numerics agree with this conclusion and indicate that the surprisingly low value $x_{\max}=5$ is sufficient.

b. Parity $(-1)^\ell$. The situation differs physically for the $(-1)^\ell$ parity sector, at least for even ℓ: The third contribution in Eq. (69) is nonzero, and it leads to a continuous part in the spectrum of $M^{(\ell)}(iS)$ that reaches zero for even ℓ, see Eq. (84). Then the spectrum of the inverse $[M^{(\ell)}(iS)]^{-1}$ is no longer bounded, and its matrix elements are not bounded even if one uses the optimal (t,ψ) representation in which the matrix elements of $M^{(\ell)}(iS)$ are bounded, see Eq. (92), when the lower cut-off $t_{\min}$ on the t variable tends to $-\infty$. Then, as we shall see, there is no exponential locality in the t basis but still the logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ has a finite limit when $t_{\min}\to-\infty$, which is approached with an error vanishing linearly with $1/t_{\min}$.

To derive this property, a spectral or "Fourier-space" analysis is more appropriate than the "real"-space analysis of the previous parity case. After the gauge transform and the change of functions performed in Eqs. (86), (89), and (91), the asymptotic $t\to-\infty$ part of the eigenstates of $M^{(\ell)}(iS)$ of the continuum (84) can be written as

$$\phi_k(t) = e^{ikt} - e^{i\theta(k,S)}e^{-ikt}, \quad k>0, \quad (B30)$$

see Fig. 6. The plane waves e^{ikt} and e^{-ikt} are indeed two linearly independent solutions of the eigenvalue problem (92)

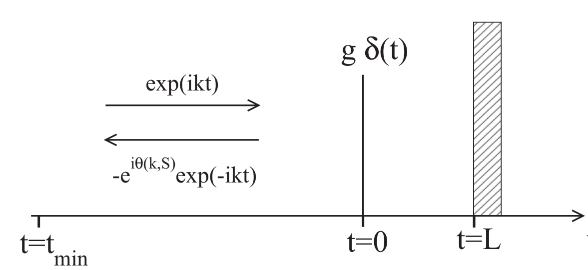

FIG. 6. Diagram giving the structure of the eigenstates of the third continuum (84), in terms of the variable t of Eq. (91): $k>0$ is the wave vector of the incoming wave and $-k$ that of the reflected wave with a phase shift $\theta(k,S)$. The reflection due to the physics at $t=O(1)$ is, in a toy model, represented by a Dirac scattering potential at $t=0$ and a hard wall acting as a mirror at $t=L$; the toy model exemplifies the expected low-k behavior (B32) of $\theta(k,S)$.

with

$$\Omega = \Omega_k = \frac{1}{\sqrt{2}}\left[1 - \frac{1}{\mathrm{ch}(k\pi/2)}\right] \tag{B31}$$

[see Eq. (84), here ℓ is even]. The right solution is some specific superposition of these two degenerate solutions, with a relative amplitude determined by the physics at $t = O(1)$, that is, for (x,u) not extremely close to $(0,-1)$. Analytically, the value of this relative amplitude is an unknown function of k and S, but we know that it must be of modulus one, so we can express it as in Eq. (B30) in terms of a mere phase shift $\theta(k,S) \in \mathbb{R}$: (i) the "Hamiltonian" $M^{(\ell)}(iS)$ for the spinor is Hermitian, so the corresponding evolution operator is unitary and conserves probability, and (ii) the third continuum (84) is not degenerate with the other continua for the considered mass ratio $\alpha = 1$, so the wave e^{ikt} incoming from $t = -\infty$ has no channel to escape and must fully get out by the incoming channel.

The key property that we shall use is that, as is common in one-dimensional scattering problems, the phase shift $\theta(k,S)$ vanishes linearly at low k:

$$\theta(k,S) \underset{k\to 0}{=} kb(S) + o(k), \tag{B32}$$

where $b(S)$ is a S-dependent effective scattering length. We present two plausible arguments to establish this. The first argument results from the assumption that the writing (B30) can be smoothly extended from $k > 0$ to $k < 0$: This implies that if one directly replaces k by $-k$ in Eq. (B30), the resulting wave $t \mapsto e^{-ikt} - e^{i\theta(-k,S)}e^{ikt}$ must reproduce the physical solution $e^{ikt} - e^{i\theta(k,S)}e^{-ikt}$ up to a global phase factor, so $e^{i\theta(k,S)} = e^{-i\theta(-k,S)}$ and there exists an integer q such that

$$-\theta(-k,S) = \theta(k,S) + 2q\pi. \tag{B33}$$

The fact that, in one dimension, the arbitrarily low energy waves are generically fully retroreflected (no matter how small, but nonzero, the scattering potential is) leads to $\theta(k,S) \to 0$ for $k \to 0$ and to $q = 0$ in Eq. (B33); then, if $\theta(k,S)$ is a smooth function of k, Eq. (B32) holds. The second argument utilizes some model for the scattering potential in the region $t = O(1)$, introducing on purpose most singular potentials, see Fig. 6: a pointlike fixed scattering center of coupling constant $g = \hbar^2/(2m_{\mathrm{eff}}a)$ placed at $t = 0$ at a distance L from a hard wall (the second element acts as a mirror and ensures that the wave is fully reflected at all energies). At low k, the dispersion relation Ω_k can be quadratized, $\Omega_k \approx \hbar^2k^2/(2m_{\mathrm{eff}})$ with $m_{\mathrm{eff}} > 0$, leading to an effective Schrödinger equation and scattering problem, so at fixed S

$$e^{i\theta(k)} = \frac{(ka)^{-1} + (\tan kL)^{-1} + i}{(ka)^{-1} + (\tan kL)^{-1} - i}. \tag{B34}$$

The phase shift $\theta(k,S)$ is indeed an odd function of k, and at low k one indeed obtains the linear law (B32) with $2/b(S) = 1/a(S) + 1/L(S)$.

Then the property (B32) leads to the conclusion that the logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ has a finite limit when the lower cut-off value $t_{\min}$ tends to $-\infty$, as we now see. Similarly to Eq. (B12), this lower cutoff corresponds to the boundary condition

$$\phi(t_{\min}) = 0, \tag{B35}$$

which, considering (B30), leads to the quantization condition for k [78]:

$$2k|t_{\min}| + \theta(k,S) = 2n\pi, \quad \forall n \in \mathbb{N}^*. \tag{B36}$$

Then the contribution of the corresponding eigenvalues to the logarithmic derivative of the determinant is

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS)|_{\odot} = \sum_{n>0}\frac{d}{dS}\ln\Omega_k = \sum_{n>0}\Omega_k^{-1}\frac{d\Omega_k}{dk}\frac{dk}{dS}. \tag{B37}$$

Taking the derivative of Eq. (B36) with respect to S at fixed n we obtain

$$\frac{dk}{dS} = -\frac{\partial_S\theta(k,S)}{2|t_{\min}| + \partial_k\theta(k,S)}. \tag{B38}$$

In the large-$|t_{\min}|$ limit, one can neglect $\partial_k\theta(k,S)$ in the denominator and one can replace in Eq. (B37) the sum over n by an integral $\int dn$. According to Eq. (B36),

$$2\frac{dk}{dn}|t_{\min}| \underset{t_{\min}\to-\infty}{\to} 2\pi \tag{B39}$$

so

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS)|_{\odot} \underset{t_{\min}\to-\infty}{\to} -\int_0^{+\infty}\frac{dk}{2\pi}\frac{1}{\Omega_k}\frac{d\Omega_k}{dk}\partial_S\theta(k,S). \tag{B40}$$

This is finite even if Ω_k vanishes quadratically in $k = 0$, i.e., it is saved from a naïvely expected logarithmic divergence, because the phase shift $\theta(k,S)$ vanishes linearly with k and so does its derivative with respect to S.

This main point being established, there remains a problem of practical interest, the speed of the convergence with $|t_{\min}|$. The answer is provided by Poisson's summing formula:

$$\sum_{n\in\mathbb{Z}} f(\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}}\hat{f}(2\pi n/\lambda) \tag{B41}$$

for any $\lambda > 0$ and for an arbitrary function $f(k)$, $\hat{f}(x) = \int_{\mathbb{R}} dk\,\exp(-ikx)f(k)$ being its Fourier transform. For simplicity, we give details in the case where $\theta(k,S)$ is linear in k at all k, that is $\theta(k,S) = kb(S)$. From the quantization condition (B36) one has

$$k = \lambda n \quad \text{with} \quad \lambda = \frac{2\pi}{2|t_{\min}| + b(S)} \tag{B42}$$

so

$$\frac{dk}{dS} = -\frac{db(S)}{dS}\frac{\lambda}{2\pi}k. \tag{B43}$$

This, together with Eqs. (B37), leads to a function f given by

$$f(k) = \frac{k\frac{db(S)}{dS}}{\Omega_k}\frac{d\Omega_k}{dk} \tag{B44}$$

such that

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS)|_{\odot} = -\frac{\lambda}{2\pi}\sum_{n>0} f(\lambda n). \tag{B45}$$

Then using the fact that the function f is even, one can express the sum over $\mathbb{N}^*$ in terms of the sum over $\mathbb{Z}$ and then in terms of $f(0)$ and $\hat{f}$:

$$\sum_{n>0} f(\lambda n) = -\frac{1}{2}f(0) + \frac{1}{2}\sum_{n\in\mathbb{Z}} f(\lambda n) = -\frac{1}{2}f(0) + \frac{1}{2\lambda}\sum_{n\in\mathbb{Z}} \hat{f}(2\pi n/\lambda). \quad \text{(B46)}$$

The function f is a smooth function of k, in particular in $k=0$, that rapidly decreases at infinity, so its Fourier transform $\hat{f}(q)$ is also rapidly decreasing when $|q|\to+\infty$. In the large-$|t_{\min}|$ limit, $1/\lambda$ diverges linearly in $|t_{\min}|$ and one commits an exponentially small error $O[\exp(-C|t_{\min}|)]$ (C is some constant) in neglecting the $n\neq 0$ terms in the last sum over n in Eq. (B46). As a consequence,

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS)|_{\odot} \underset{t_{\min}\to-\infty}{=} -\frac{1}{2\pi}\int_0^{+\infty} dk \frac{k\frac{db(S)}{dS}}{\Omega_k}\frac{d\Omega_k}{dk} + \frac{\frac{db(S)}{dS}}{2|t_{\min}|+b(S)} + O[\exp(-C|t_{\min}|)], \quad \text{(B47)}$$

where we have replaced λ, $f(0)$, and $\hat{f}(0)$ by their values. When $\theta(k,S)$ is not a linear function of k, we obtain the general result

$$\frac{d}{dS}\ln\det M^{(\ell)}(iS)|_{\odot} \underset{t_{\min}\to-\infty}{=} -\frac{1}{2\pi}\int_0^{+\infty} dk \frac{\partial_S\theta(k,S)}{\Omega_k}\frac{d\Omega_k}{dk} + \lim_{k\to 0}\frac{\frac{1}{2}\frac{d\Omega_k}{dk}\partial_S\theta(k,S)}{\Omega_k[2|t_{\min}|+\partial_k\theta(k,S)]} + O[\exp(-C|t_{\min}|)]. \quad \text{(B48)}$$

In any case, when $t_{\min}\to-\infty$, the limiting value of the logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ is approached with an error that vanishes only polynomially with $1/t_{\min}$ [79].

On the contrary, if the dispersion relation Ω_k nowhere approaches zero, as for an odd ℓ in the parity sector $(-1)^\ell$, the $\lim_{k\to 0}$ term in the right-hand side of Eq. (B48) is zero and the convergence of the logarithmic derivative of the determinant is exponentially fast with $|t_{\min}|$, as also observed numerically; this last situation is then similar to the exponentially fast convergence of $\frac{d}{dS}\ln\det M^{(\ell)}(iS)$ when $x_{\max}\to+\infty$, which is always achieved for the continua (77) and (81), even when the third term in Eq. (69) is active.

4. Other convergence issues

To show that the conjectured values (B11) are finite, one must also check that the integral over S is convergent at infinity and that the sum over the angular momenta ℓ is convergent. This we have initially explored numerically. First, at a given ℓ, we found that the logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ rapidly decreases when $S\to+\infty$, presumably exponentially fast, see Fig. 7(a). Second, after the integration over S is taken, one observes also a rapid convergence of the series over ℓ, see Fig. 7(b), if one takes the precaution to be accurate enough in the discretization of the integral over u [80].

These numerical results suggest that the contribution of angular momentum ℓ to $\Delta B_{2,2}^{\rm conj}(0^+)$ and to $\Delta B_{3,1}^{\rm conj}(0^+)$ can be obtained, when ℓ is large enough, from a perturbative calculation in Eq. (B13), limited to leading order in the operators $K_j^{(\ell)}$ defined by Eqs. (B15) and (B22), at least for the $2+2$ problem in the parity channel $(-1)^{\ell+1}$ where $K_{\rm inv}$ in Eq. (B17) has a chance of being bounded. This idea was implemented with success at the three-body level in Ref. [28], treating the integral term in Eq. (78) as a perturbation of the constant term [81].

Let us implement the idea for the $2+2$ problem in the parity sector $(-1)^{\ell+1}$ of the subspace of angular momentum ℓ. We truncate Eq. (B18) to order one included in the operator K, to obtain

$$\frac{d}{dS}\ln\det M \simeq \mathrm{Tr}\left[(\mathcal{D}^{-1}-\mathcal{D}^{-1}K\mathcal{D}^{-1})\frac{d}{dS}K\right]. \quad \text{(B49)}$$

Then we split K as in Eq. (B22) and we use the commutator structure (B23). Using the invariance of the trace in a cyclic permutation and the fact that the diagonal operators $\mathcal{D}$ of Eq. (B16) and D_j of Eq. (B26) commute, we find that only the crossed quadratic contributions in K_1 and K_2 survive, so

$$\frac{d}{dS}\ln\det M \simeq \mathrm{Tr}\left[-\mathcal{D}^{-1}K_1\mathcal{D}^{-1}\frac{d}{dS}K_2-(1\leftrightarrow 2)\right] = \frac{d}{dS}\mathrm{Tr}(-\mathcal{D}^{-1}K_1\mathcal{D}^{-1}K_2). \quad \text{(B50)}$$

Integrating by parts in Eq. (B11) and using the fact that the integrand is an even function of S we obtain the approximation

$$\int_0^{+\infty}\frac{dS}{\pi}S\frac{d}{dS}\ln\det M \simeq \int_{\mathbb{R}}\frac{dS}{2\pi}\mathrm{Tr}(\mathcal{D}^{-1}K_1\mathcal{D}^{-1}K_2). \quad \text{(B51)}$$

Calculating the trace in the $|x,u,\ell,m_z\rangle$ basis (with $\ell+m_z$ odd) and injecting a closure relation in that basis as, e.g., in Eq. (B21), we realize that the integrand has a very simple dependence with S, due to simplifications as follows:

$$\left(\frac{e^x\,\mathrm{ch}\,x'}{e^{x'}\,\mathrm{ch}\,x}\right)^{s/2}\left(\frac{e^{-x'}\,\mathrm{ch}\,x}{e^{-x}\,\mathrm{ch}\,x'}\right)^{s/2} = e^{i(x-x')S}, \quad \text{(B52)}$$

where we wrote the phase factor of the first term as it is in Eq. (69) and the phase factor of the second term of Eq. (69) with $x\leftrightarrow x'$, and used $s=iS$ with S real. So the integral over x or x' takes the form of a Fourier transform with respect to x or x', with S as the conjugate variable; this is the Fourier transform of a smooth rapidly decreasing function of x or x', so, as a function of S, it is a rapidly decreasing function. This gives a reason for the numerically observed fast decay of $\frac{d}{dS}\ln\det M$ at large S. Also, integration over S is straightforward due to

$$\int_{\mathbb{R}}\frac{dS}{2\pi}e^{i(x-x')S} = \delta(x-x'). \quad \text{(B53)}$$

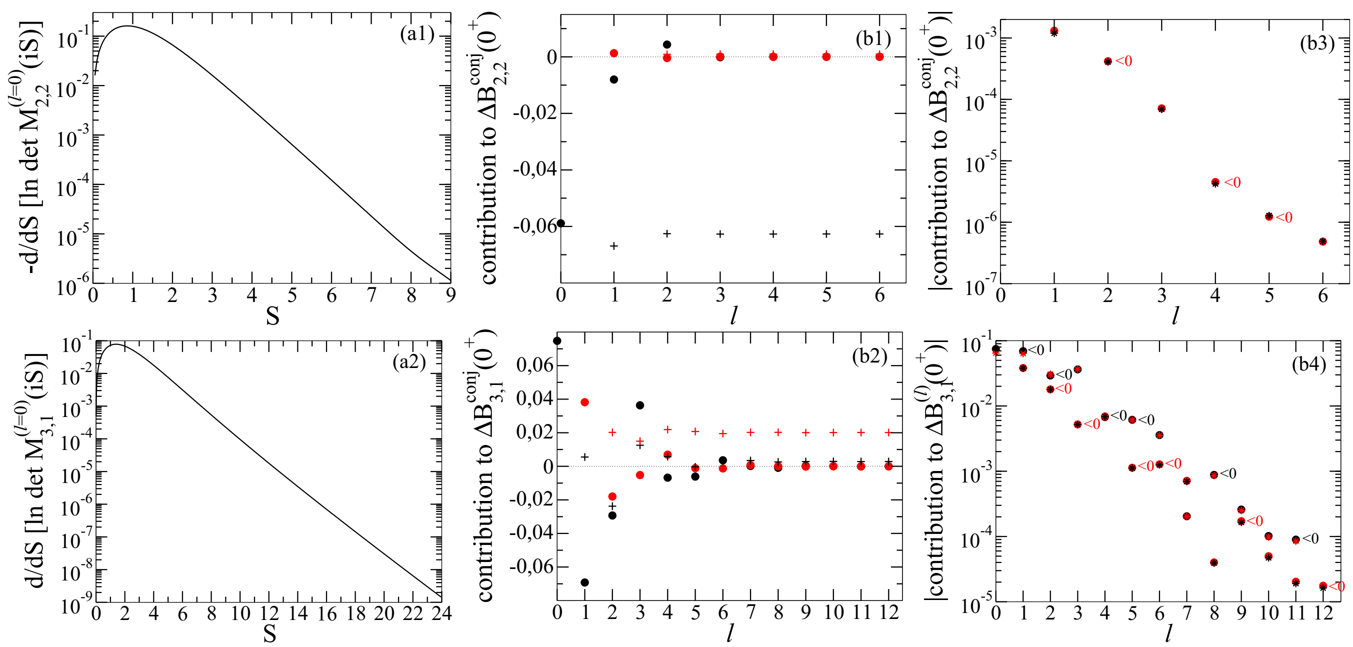

FIG. 7. (Color online) Convergence in the integration over S and summation over ℓ in Eq. (B11) (we recall that the mass ratio is $\alpha = 1$). (a) The logarithmic derivative of the determinant of $M^{(\ell)}(iS)$ is a rapidly decreasing function of S; the figure takes as an example (a1) the $\ell = 0$ channel of the $2+2$ fermionic problem (for a numerical cutoff $t_{\rm min} = -9$, thus without extrapolation; note the minus sign in the vertical axis) and (a2) the $\ell = 0$ channel of the $3+1$ fermionic problem. (b) The sum over ℓ also seems to converge well, see the contribution of each angular-momentum channel to the result (B11) for (b1) the $2+2$ problem and (b2) the $3+1$ problem: black disks for the parity sector $(-1)^\ell$ and red disks for the parity sector $(-1)^{\ell+1}$. The plus signs indicate the corresponding cumulative sums. Convincing evidence is even shown in (b3) for the $2+2$ problem in the $(-1)^{\ell+1}$ parity sector and in (b4) for the $3+1$ problem in both parity sectors, where the numerical results [black disks for parity $(-1)^\ell$ and red disks for parity $(-1)^{\ell+1}$] are compared to the perturbative results (B54) and (B60) [red asterisks for parity $(-1)^\ell$ and black asterisks for parity $(-1)^{\ell+1}$] that extend to four bodies a technique developed for three bodies in Ref. [28] and are expected to be exact asymptotic equivalents for $\ell \to +\infty$ [what is actually plotted is the absolute value of the results to allow for a log scale, but their sign is indicated with the label "< 0" of the same color as the corresponding disks when they are negative: The negative black (red) disks are indicated with a black (red) "< 0" label. Note that the black (red) asterisks always have the same sign as the corresponding red (black) disks].

We finally obtain the leading-order approximation

$$\Delta B_{2,2}^{\rm conj}(0^+)|^{(\ell)}_{{\rm parity}(-1)^{\ell+1}} \simeq \frac{2\ell+1}{(4\pi)^2}\int_{\mathbb{R}} dx \int_0^\pi d\theta \int_0^\pi d\theta' \int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}\frac{vv'}{d(x,u)d(x,u')\,{\rm ch}\,x}$$
$$\times \frac{\frac{1}{4}\sum_{n=0}^{1}\sum_{n'=0}^{1}(-1)^{(\ell+1)(n+n')}\mathcal{T}_\ell(\theta+n\pi,\theta'+n'\pi,\phi,\phi')}{\left\{1+\frac{1}{1+\alpha}[(u+e^{-x})(u'+e^{-x})+vv'\cos\phi]\right\}\left\{1+\frac{\alpha}{1+\alpha}[(u+e^{x})(u'+e^{x})+vv'\cos\phi']\right\}}. \quad \text{(B54)}$$

In the integrand of Eq. (B54), $d(x,u)$ is given by Eq. (B16), we again use the notations $u = \cos\theta$ and $v = \sin\theta$ and the same for θ', and we introduced the function

$$\mathcal{T}_\ell(\theta,\theta',\phi,\phi') \equiv \sum_{m_z,m_z'=-\ell}^{\ell} e^{-im_z\theta}\langle \ell,m_z|e^{i\phi L_x/\hbar}|\ell,m_z'\rangle e^{im_z'\theta'} \times \langle \ell,m_z'|e^{i\phi' L_x/\hbar}|\ell,m_z\rangle \quad \text{(B55)}$$
$$= {\rm Tr}_\ell[e^{-i\theta L_z/\hbar}e^{i\phi L_x/\hbar}e^{i\theta' L_z/\hbar}e^{i\phi' L_x/\hbar}]$$
$$= \frac{\sin[(2\ell+1)\delta/2]}{\sin(\delta/2)}, \quad \text{(B56)}$$

where the trace is taken over the whole subspace $\{|\ell,m_z\rangle, -\ell \leqslant m_z \leqslant \ell\}$ of angular momentum ℓ without any parity restriction and the angle $\delta \in [0,\pi]$ is such that [82]

$$1+2\cos\delta = uu'(1+\cos\phi\cos\phi') - (u+u')\sin\phi\sin\phi' + vv'(\cos\phi+\cos\phi') + \cos\phi\cos\phi'. \quad \text{(B57)}$$

The sum over n and n' in the numerator of the integrand of Eq. (B54) suppresses the contribution to $\mathcal{T}_\ell$ of the states $|\ell,m_z\rangle$ and $|\ell,m_z'\rangle$ of the wrong parity, $(-1)^{m_z} = (-1)^{m_z'} = (-1)^\ell$. We expect the approximation (B54) to be an exact asymptotic equivalent for $\ell \to +\infty$, and this is also what the comparison to the numerical results in Fig. 7(b3) indicates. Amazingly it is already good for $\ell = 1$, as it deviates from the numerical value by about 9% only.

This perturbative treatment can also be applied to the $3+1$ problem, using the integral equations of Ref. [13]. The main difference is that the spinor $\Phi_{m_z}^{(\ell)}(x,u)$ is now subjected to a condition reflecting the fermionic exchange symmetry of the two $\downarrow$ particles that are spectators of the interacting $\uparrow\downarrow$ pair [18],

$$\Phi_{-m_z}^{(\ell)}(-x,u) = (-1)^{\ell+1}\Phi_{m_z}^{(\ell)}(x,u). \tag{B58}$$

This means that the kernel $K^{(\ell)}$ must be restricted to the corresponding subspace, hence the occurrence of a projector $P = (1+U)/2$ on that subspace, where the unitary operator U such that in Dirac's notation

$$\begin{aligned} U|x,u,\ell,m_z\rangle &= (-1)^{\ell+1}|-x,u,\ell,-m_z\rangle \\ &= -e^{i\pi L_x/\hbar}|-x,u,\ell,m_z\rangle \end{aligned} \tag{B59}$$

is an involution ($U^2 = \mathbb{1}$) [83]. The interesting point is now that, even if $\frac{d}{dS}K$ is a sum of commutators as in Eq. (B23), the corresponding D_j do not commute with the projector P. As a consequence, when one expands M^{-1} up to first order in K, $\frac{d}{dS}\ln\det M$ contains both a contribution of order one in K and two contributions of order two in K. Here is the result in the subspace of angular momentum ℓ and parity ε [84]:

$$\begin{aligned}
\Delta B_{3,1}^{\rm conj}(0^+)|_{\text{parity }\varepsilon}^{(\ell)} \simeq{}& \frac{2\ell+1}{2\pi\sqrt{2}}\int_0^\pi d\theta\int_0^{2\pi}\frac{d\phi}{2\pi}\,\frac{v}{d_{31}(0,u)}\,\frac{\frac{1}{2}\sum_{n=0}^{1}\varepsilon^n\mathcal{T}_\ell(\theta+n\pi,0,\phi+\pi,0)}{3+\frac{2\alpha}{1+\alpha}(2u+u^2+v^2\cos\phi)} \\
&+\frac{2\ell+1}{8\pi^2}\int_{\mathbb{R}}dx\int_0^\pi d\theta\int_0^\pi d\theta'\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}\,\frac{vv'}{d_{3,1}(x,u)d_{3,1}(x,u')\,\mathrm{ch}\,x} \\
&\times\frac{\frac{1}{4}\sum_{n=0}^{1}\sum_{n'=0}^{1}\varepsilon^{n+n'}\mathcal{T}_\ell(\theta+n\pi,\theta'+n'\pi,\phi,\phi')}{\left\{2+e^{-2x}+\frac{2\alpha}{1+\alpha}[e^{-x}(u+u')+uu'+vv'\cos\phi]\right\}\left\{2+e^{2x}+\frac{2\alpha}{1+\alpha}[e^{x}(u+u')+uu'+vv'\cos\phi']\right\}} \\
&-\frac{2\ell+1}{4\pi^2}\int_{\mathbb{R}}dx'\int_0^\pi d\theta\int_0^\pi d\theta'\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}\left(\frac{e^{x'}}{\mathrm{ch}\,x'}\right)^{1/2}\frac{vv'}{d_{31}(0,u)d_{31}(x',u')} \\
&\times\frac{\frac{1}{4}\sum_{n=0}^{1}\sum_{n'=0}^{1}\varepsilon^{n+n'}\mathcal{T}_\ell(\theta+n\pi,n'\pi,\phi,\phi'+\pi)}{\left[2e^{-x'}+e^{x'}+\frac{2\alpha}{1+\alpha}(ue^{-x'}+u'+uu'+vv'\cos\phi)\right]\left[2e^{-x'}+e^{x'}+\frac{2\alpha}{1+\alpha}(ue^{-x'}+u'+uu'+vv'\cos\phi')\right]},
\end{aligned} \tag{B60}$$

where $d_{31}(x,u)$ defines the diagonal part $\mathcal{D}$ of the operator M for the $3+1$ problem [as $d(x,u)$ did for the $2+2$ problem], see Ref. [13]:

$$d_{31}(x,u) = \left[\frac{1+2\alpha}{(1+\alpha)^2}+\frac{\alpha u}{(1+\alpha)^2\,\mathrm{ch}\,x}\right]^{1/2}. \tag{B61}$$

As can be checked in Fig. 7(b4), this approximation is in good agreement with the numerical results even for $\ell = 0$, where it deviates from the exact result only by $\simeq$13%. In the large-ℓ limit the first contribution in the right-hand side of Eq. (B60) rapidly dominates over the other two; summing over the two parity sectors $\varepsilon = \pm 1$ and restricting for simplicity to a mass ratio $\alpha = 1$, one can integrate it over θ and ϕ at fixed $\delta \in [0,\pi]$, where $1 + 2\cos\delta = u + \cos\phi + u\cos\phi$ as shown by Eq. (B57) taken with $\theta' = \phi' = 0$, to obtain the rapidly decreasing large-ℓ equivalent [85]

$$\begin{aligned}\Delta B_{3,1}^{\rm conj}(0^+)|^{(\ell)} \underset{\ell\to+\infty}{\sim}{}& \frac{2\ell+1}{2\pi^2}\int_0^\pi d\delta\,\sin[(\ell+1/2)\delta] \\ &\times\frac{\arccos\frac{8\cos^2\delta+5\cos\delta-1}{3(3+\cos\delta)}}{[(5+4\cos\delta)(1+\cos\delta+\cos^2\delta)]^{1/2}}.\end{aligned} \tag{B62}$$

5. The verdict

The numerical results for our conjecture (B11) are

$$\Delta B_{2,2}^{\rm conj}(0^+) = -0.0617(2), \tag{B63}$$

$$\Delta B_{3,1}^{\rm conj}(0^+) = +0.02297(4), \tag{B64}$$

leading, after use of Eq. (B6), to

$$\Delta b_4^{\rm conj} = -0.063(1). \tag{B65}$$

This clearly disagrees with the experimental values (B4). Remarkably, for $\Delta B_{3,1}(0^+)$ our conjectured value is very close to the approximate diagrammatic result 0.025 of Ref. [24], whereas for $\Delta B_{2,2}(0^+)$ it widely differs from the (still approximate) result -0.036 of Ref. [24] (these values were communicated to us privately by Jesper Levinsen).

A useful complementary test is to compare to the theoretical results of Ref. [30]. As mentioned above and in that reference, these results, obtained with the harmonic regulator technique, are trustable at *nonzero* values of $\beta\hbar\omega$ without extrapolation to $\beta\hbar\omega = 0$ [ω is the angular oscillation frequency in the trap and $\beta = 1/(k_B T)$]. It is actually straightforward to extend with the same notations the conjecture (B11) to a nonzero value of ω,

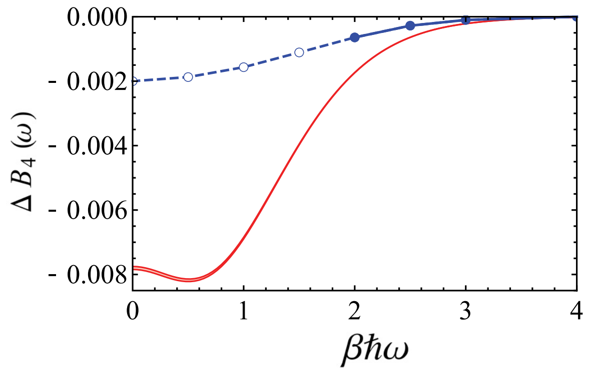

FIG. 8. (Color online) Fourth-order cluster coefficient (B67) for a harmonically trapped unpolarized spin-1/2 unitary Fermi gas at temperature T, as a function of $\beta\hbar\omega$, with $\beta = 1/(k_B T)$, ω the angular oscillation frequency of a fermion in the trap, and $\alpha = m_\uparrow/m_\downarrow = 1$. Blue line with symbols: results of Ref. [30] obtained by brute-force numerical calculation of the up-to-four-body spectra in the trap (disks: actually calculated values; circles: values resulting from an extrapolation). Red lines: our conjecture (B66) (the values slightly differ depending on the linear or cubic extrapolation to the numerical cut-off limit $1/t_{\min} \to 0^-$).

see Eq. (38) of Ref. [28]:

$$\Delta B^{\rm conj}_{n_\uparrow,n_\downarrow}(\omega) = \sum_{\ell\in\mathbb{N}} \left(\ell + \frac{1}{2}\right) \int_0^{+\infty} \frac{dS}{\pi} \frac{\sin(S\beta\hbar\omega)}{\mathrm{sh}(\beta\hbar\omega)} \times \frac{d}{dS}\left[\ln\det M^{(\ell)}_{n_\uparrow,n_\downarrow}(iS)\right]. \tag{B66}$$

Since $|\sin(S\beta\hbar\omega)/\,\mathrm{sh}(\beta\hbar\omega)| \leqslant S$, this does not raise new convergence issues and the numerical evaluation of $\Delta B^{\rm conj}_{2,2}(\omega)$ and $\Delta B^{\rm conj}_{3,1}(\omega)$ is straightforward once the logarithmic derivatives of the determinant of M are known. The resulting value of the fourth in-trap cluster coefficient

$$\Delta B_4(\omega) \equiv \tfrac{1}{2}[\Delta B_{3,1}(\omega) + \Delta B_{2,2}(\omega) + \Delta B_{1,3}(\omega)] \tag{B67}$$

(with $\Delta B_{3,1} = \Delta B_{1,3}$ for the mass ratio $\alpha = 1$) is plotted as a function of $\beta\hbar\omega$ in Fig. 8. It clearly disagrees with the results of Ref. [30], not only with the ones resulting from the extrapolation to $\beta\hbar\omega = 0$ but also with the actually calculated ones.

The conjecture is thus invalidated, and more theoretical work is needed to derive the correct analytical expression for Δb_4 of the unitary Fermi gas.

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[36] In principle, $M^{(\ell)}(s)$ is an operator, with an infinite number of eigenvalues, and its determinant is infinite. The mathematically correct form is to require that one of the eigenvalues of $M^{(\ell)}(s)$ vanishes. In practice, in the numerical calculation, one discretizes and truncates the variables so the determinantal form can be used.

[37] This additional term leads to the factor $(\mathrm{ch}\,x)^s$ in the ansatz (20) and to an additional factor $(\mathrm{ch}\,x'/\mathrm{ch}\,x)^s$ in the kernel K of Eq. (69). This does not change the determinant of the matrix $M(s)$. This also preserves the Hermitian nature of $M(s)$ for $s\in i\mathbb{R}$.

[38] With the notation $Y_\ell^{m_z}(\mathbf{n})=Y_\ell^{m_z}(\theta_\mathbf{n},\phi_\mathbf{n})$, where $\theta_\mathbf{n}$ and $\phi_\mathbf{n}$ are the polar and azimuthal angles of $\mathbf{n}$ in the usual spherical coordinates attached to the xyz Cartesian coordinates, one faces the integral over the unit sphere $\int\frac{d^2n}{4\pi}Y_\ell^{m_z}(\mathbf{n})[Y_\ell^{m_z'}(\mathcal{R}^{-1}\mathbf{n})]^*$. Since $Y_\ell^{m_z'}(\mathcal{R}^{-1}\mathbf{n})=\langle\mathbf{n}|R|\ell,m_z'\rangle=\sum_{m_z''}Y_\ell^{m_z''}(\mathbf{n})\langle\ell,m_z''|R|\ell,m_z'\rangle$ and the spherical harmonics form an orthonormal basis, $\int d^2n\,Y_\ell^{m_z}(\mathbf{n})[Y_\ell^{m_z''}(\mathbf{n})]^*=\delta_{m_z,m_z''}$, we get (28). Here $\mathcal{R}$ is a rotation in the three-dimensional space and the operator R is its representation in the Hilbert space.

[39] For an arbitrary function ϕ, one defines $I\equiv\int d^3k_1d^3k_3\phi(\mathbf{k}_1,\mathbf{k}_3)$. Then one has $I=(4\pi\times2\pi)\int_{\mathrm{SO}(3)}d\mathcal{R}\int_0^{+\infty}dk_1dk_3k_1^2k_3^2\int_{-1}^1du_{13}\phi(\mathcal{R}\mathbf{k}_1^{\rm fix},\mathcal{R}\mathbf{k}_3^{\rm fix})$. To show this one makes in the integral defining I the unit Jacobian change of variable $\mathbf{k}_1\to\mathcal{R}\mathbf{k}_1$ and $\mathbf{k}_3\to\mathcal{R}\mathbf{k}_3$, where $\mathcal{R}$ is any rotation: $I=\int d^3k_1d^3k_3\phi(\mathcal{R}\mathbf{k}_1,\mathcal{R}\mathbf{k}_3)$. As the result does not depend on $\mathcal{R}$, we can average it over SO(3), with the normalized invariant measure. Exchanging the order of integration over $\mathcal{R}$ and over $\mathbf{k}_1$ and $\mathbf{k}_3$, we get $I=\int d^3k_1d^3k_3J_{\mathbf{k}_1,\mathbf{k}_3}$ with $J_{\mathbf{k}_1,\mathbf{k}_3}\equiv\int_{\mathrm{SO}(3)}d\mathcal{R}\phi(\mathcal{R}\mathbf{k}_1,\mathcal{R}\mathbf{k}_3)$. In $J_{\mathbf{k}_1,\mathbf{k}_3}$ one then performs the change of variable $\mathcal{R}\to\mathcal{R}\rho$, where ρ is any rotation. As the measure is invariant, $J_{\mathbf{k}_1,\mathbf{k}_3}=\int_{\mathrm{SO}(3)}d\mathcal{R}\phi(\mathcal{R}\rho\mathbf{k}_1,\mathcal{R}\rho\mathbf{k}_3)$. Then for any given $\mathbf{k}_1$ and $\mathbf{k}_3$, one chooses ρ such that $\rho\mathbf{k}_1=\mathbf{k}_1^{\rm fix}$ and $\rho\mathbf{k}_3=\mathbf{k}_3^{\rm fix}$, so $J_{\mathbf{k}_1,\mathbf{k}_3}=\int_{\mathrm{SO}(3)}d\mathcal{R}\phi(\mathcal{R}\mathbf{k}_1^{\rm fix},\mathcal{R}\mathbf{k}_3^{\rm fix})$. Inserting this expression of $J_{\mathbf{k}_1,\mathbf{k}_3}$ into I and exchanging again the order of integration gives $I=\int_{\mathrm{SO}(3)}d\mathcal{R}\int d^3k_1d^3k_3\phi(\mathcal{R}\mathbf{k}_1^{\rm fix},\mathcal{R}\mathbf{k}_3^{\rm fix})$. At fixed $\mathbf{k}_1$ one integrates over $\mathbf{k}_3$ in spherical coordinates of polar axis $\mathbf{k}_1$; as the integrand does not depend on the azimuthal angle, we pull out a factor 2π. The resulting integral over k_3 and θ_{13} does not depend on the direction of $\mathbf{k}_1$ so, after integration over $\mathbf{k}_1$ in spherical coordinates of arbitrary polar axis, one pulls out an additional factor 4π and gets the desired relation.

[40] A three-dimensional $\delta(\mathbf{k})$ is the product of three one-dimensional $\delta(\mathbf{u}_n\cdot\mathbf{k})$, where $(\mathbf{u}_n)$ is an orthonormal basis. As explained in the text one can take $\mathbf{k}=\mathbf{k}_2+\mathbf{k}_4+\mathcal{R}_Y(\beta)\mathcal{R}_Z(\gamma)(\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix})$. We first take $\mathbf{u}_1=\mathbf{e}_Y$ so $\mathbf{u}_1\cdot\mathbf{k}=(\mathcal{R}_Z(-\gamma)\mathbf{e}_Y)\cdot(\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix})=\sin\gamma\,\mathbf{e}_X\cdot(\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix})=-\sin\gamma\sin\beta_0|\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix}|$, where we used Eq. (50). This gives the factor $\delta(\sin\gamma)$ in Eq. (49). As explained in the text we can restrict to the case $\gamma=0$ (up to a change $\beta\leftrightarrow-\beta$) and we are left with a two-dimensional Dirac $\delta(\mathbf{k}_\perp)$ in the plane orthogonal to $\mathbf{e}_Y$. In principle $\mathbf{k}_\perp=\mathbf{k}_2+\mathbf{k}_4+\mathcal{R}_Y(\beta)(\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix})$ but we can equivalently take $\mathbf{k}_\perp=\mathcal{R}_Y(-\beta)(\mathbf{k}_2+\mathbf{k}_4)+\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix}$ due to the rotational invariance of the Dirac distribution. Using $\mathcal{R}_Y(-\beta)\mathbf{e}_Z=\cos\beta\,\mathbf{e}_Z-\sin\beta\,\mathbf{e}_X$ and taking $\mathbf{u}_2=\frac{\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix}}{|\mathbf{k}_1^{\rm fix}+\mathbf{k}_3^{\rm fix}|}=\cos\beta_0\,\mathbf{e}_Z-\sin\beta_0\,\mathbf{e}_X$ and its orthogonal counterpart $\mathbf{u}_3=\sin\beta_0\,\mathbf{e}_Z+\cos\beta_0\,\mathbf{e}_X$ in the ZX plane, we justify Eq. (49).

[41] Similarly to Eq. (53), $|\ell,m_x=0\rangle=s_+e^{-i(\pi/2)L_y/\hbar}|\ell,m_z=0\rangle=s_-e^{-i(-\pi/2)L_y/\hbar}|\ell,m_z=0\rangle$, where $s_\pm$ are just signs since $|\ell,m_x=0\rangle$ can be taken with real components in the $|\ell,m_z\rangle$ basis (L_y has purely imaginary matrix elements). Then $s_-s_+\langle\ell,m_z=0|e^{-i\pi L_y/\hbar}|\ell,m_z=0\rangle=1$. The action of $e^{-i\pi L_y/\hbar}$ in Cartesian coordinates is $(x,y,z)\to(-x,y,-z)$; in spherical coordinates of polar axis z it is $(\theta,\phi)\to(\pi-\theta,\pi-\phi)$. For $Y_\ell^{m_z=0}(\theta,\phi)$, which does not depend on ϕ, this is equivalent to the action of parity $(\theta,\phi)\to(\pi-\theta,\pi+\phi)$ and it pulls out a factor $(-1)^l$ so $s_-=(-1)^\ell s_+$ and $|\ell,m_x=0\rangle=(s_+/2)(e^{-i(\pi/2)L_y/\hbar}+(-1)^\ell e^{-i(-\pi/2)L_y/\hbar})|\ell,m_z=0\rangle$. Series expanding the exponentials in this last expression, and using the fact that L_y only couples states of different m_z parity, one gets Eq. (54).

[42] One can keep $[0,2\pi]$ as the range of integration over ϕ' since the integrand is a periodic function of ϕ' of period 2π.

[43] For $b_0>b_1>0$, $\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{e^{im_x\phi}}{b_0+b_1\cos\phi}=z_0^{|m_x|}/[(b_0-b_1)(b_0+b_1)]^{1/2}$ with $z_0=-b_1/\{b_0+[(b_0-b_1)(b_0+b_1)]^{1/2}\}$.

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[48] If $x\to-\infty,x'\to+\infty$ or $x\to+\infty,x'\to-\infty$, then the matrix kernel (69) entirely tends exponentially to zero, which does not bring any significant new information.

[49] This is trivial for $\ell=0$. For $\ell\geqslant1$, this results from the fact that, for $L\geqslant1$, one can take $m_x=1$. Then $\langle\ell,m_z|\ell,m_x=1\rangle\neq0$, except if ℓ is even and $m_z=0$ [in agreement with Eq. (54), considering the $x\leftrightarrow y$ symmetry], in which case one may return

to the choice $m_x = 0$ and use the fact that $\langle \ell, m_z = 0|\ell, m_x = 0\rangle \neq 0$ for even ℓ.

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[51] Let us explain more physically why the Λ_L function appears in the expression of the continuous spectrum. The idea is to consider a physical state of the $2+2$ fermionic system, corresponding to a nonzero square integrable solution $\Phi^{(\ell)}_{m_z}(x,u)$ of Eq. (68), and to see how the four-body wave function scales when three particles, say, 1, 2, and 3, converge to the same location, the fourth particle being at some other fixed location. As we have seen, an extended eigenstate of the continuum that varies for $x \to -\infty$ as $e^{ikx} e^{im_z\theta/2}\langle \ell, m_z|\ell, m_x = 0\rangle P_L(u)$, $k \in \mathbb{R}$, has an eigenvalue $\Omega = \Lambda_L(ik,\alpha)$. According to the analytic continuation argument of Ref. [13], this implies that the $\Omega = 0$ localized eigenstate $\Phi^{(\ell)}_{m_z}(x,u)$ vanishes for $x \to -\infty$ as $e^{\kappa x} e^{im_z\theta/2}\langle \ell, m_z|\ell, m_x = 0\rangle P_L(u)$, where the real quantity κ is the minimal positive root of

$$\Lambda_L(\kappa,\alpha) = 0$$

with L chosen to minimize κ (minimizing κ amounts to selecting the most slowly decreasing exponential function $e^{\kappa x}$, that is, the leading contribution for $x \to -\infty$). This implies that κ is one of the possible scaling exponents s_3 of the $2+1$ fermionic problem, see Eq. (82). In order to determine the limit of Eq. (10) when r_{13} and $|\mathbf{r}_2 - \mathbf{R}_{13}| \to 0$ tend to zero with the same scaling law, that is, both are proportional to the vanishing hyperradius R_{123} of particles 1, 2, and 3, we determine the large-$\mathbf{k}_2$ limit of the integrand of ψ_{24} in Eq. (10) at fixed $\mathbf{k}_4$: Omitting to write the angular part for simplicity, we find that $D(\mathbf{k}_2,\mathbf{k}_4)$ scales as $k_2^{-2-\kappa} k_4^{\kappa-s-3/2}$, so its Fourier transform, according to the usual power-law counting argument, scales as $|\mathbf{r}_2 - \mathbf{R}_{13}|^{\kappa-1}|\mathbf{r}_4 - \mathbf{R}_{13}|^{s-\kappa-3/2}$. The same reasoning applies to ψ_{14}. At fixed $|\mathbf{r}_4 - \mathbf{R}_{13}| > 0$, the four-body wave function therefore scales as $R_{123}^{\kappa-2} = R_{123}^{s_3-2}$, exactly as predicted by Eq. (5.179) of Ref. [34]. This whole discussion is formal for the $2+2$ fermionic problem since, as we shall see, there is no four-body Efimov effect, but it explicitly applies to the $3+1$ fermionic problem and nicely completes Ref. [13].

[52] The continuous spectrum $\Omega^{\odot(\ell)}$ can be recovered by keeping only but exactly the last contribution in Eq. (69) to the matrix kernel $K^{(\ell)}_{m_z,m_z'}$ of Eq. (68), that is, without resorting to a local approximation of this contribution around $(x,u) = (0,-1)$. The explicit calculation remains simple for a unit mass ratio $\alpha = 1$. The eigenvectors of the resulting operator are then of the form $\Phi^{(\ell)}_{m_z}(x,u) = e^{im_z\gamma}\langle \ell, m_z|\ell, m_x = 0\rangle\Phi(x,u)$ with the ansatz $\Phi(x,u) = (\mathrm{ch}\, x)^{-(s-1/2)/2}(u + \mathrm{ch}\, x)^{-(s+7/2)/2}\chi(\sqrt{2}k_{24}/K_{24})$, k_{24} and K_{24} being the relative and center-of-mass wave numbers of particles 2 and 4, so $2k_{24}/K_{24} = (\frac{\mathrm{ch}\, x - u}{\mathrm{ch}\, x + u})^{1/2}$. One then obtains the integral equation for $\chi(k)$: $\Omega(2k^2+1)^{1/2}\chi(k) = (k^2+1)^{1/2}\chi(k) - \frac{2(-1)^\ell}{\pi}\int_0^{+\infty} dk' k'^2 \chi(k')/(1+k^2+k'^2)$. Further setting $\chi(k) = k^{-3/2}(1+2k^2)^{-1/4}\psi(t)$ with $k = \exp(-t)$, one obtains $\Omega\psi(t) = (\frac{1+e^{-2t}}{1+2e^{-2t}})^{1/2}\psi(t) - \frac{2(-1)^\ell}{\pi}\int_{\mathbb{R}} dt' \frac{\psi(t')\exp[-3(t+t')/2]}{(1+e^{-2t}+e^{-2t'})[(1+2e^{-2t})(1+2e^{-2t'})]^{1/4}}$. The $t \to -\infty$ continuum of that eigenvalue problem solves $(\sqrt{2}\Omega - 1)\psi_\infty(t) = -\frac{(-1)^\ell}{\pi}\int_{\mathbb{R}} dt' \frac{\psi_\infty(t')}{\mathrm{ch}(t-t')}$, with plane-wave solutions $\psi_\infty(t') = e^{ikt}$ reproducing (84). The ansatz for $\Phi(x,u)$ results from the fact that $D(\mathbf{k}_2,\mathbf{k}_4)$ is of the form $K_{24}^{-(s+7/2)}P_\ell(\mathbf{e}_z\cdot\mathbf{K}_{24}/K_{24})\chi(\sqrt{2}k_{24}/K_{24})$ (P_ℓ is a Legendre polynomial), which is apparent if one turns back to Eq. (13) and realizes that its last contribution conserves the total wave vector $\mathbf{K}_{24}$ (up to a sign).

[53] In reality, the physical solution for the tensor $\Phi^{(\ell)}_{m_z}(x,u)$ must correspond to a zero-eigenvalue of the matrix $M^{(\ell)}(s)$. In this respect, $k = 0$ is acceptable only for ℓ even. Furthermore, as we shall see in Appendix B [see the note [78] called above Eq. (B36)], when $\Omega = 0$, $\phi(t)$ actually scales linearly in t for $t \to -\infty$, so $D(\mathbf{k}_2,\mathbf{k}_4)$ actually diverges as $\ln(|\mathbf{k}_2 + \mathbf{k}_4|/k_{24})/(|\mathbf{k}_2+\mathbf{k}_4|/k_{24})^{s+3/2}$ when $\mathbf{k}_2+\mathbf{k}_4 \to \mathbf{0}$. This would result in a $\ln(r_{24}/|\mathbf{R}_{24}-\mathbf{R}_{13}|)/r_{24}$ divergence of the function $\mathcal{A}_{13}(\mathbf{r}_2-\mathbf{R}_{13},\mathbf{r}_4-\mathbf{R}_{13})$ at $r_{24}=0$, which is physically opaque. Let us keep in mind, however, that, according to Sec. V there is no four-body Efimov effect in the $2+2$ problem, so Eq. (68) does not actually support any nonidentically zero solution $\Phi^{(\ell)}$ if $s \in i\mathbb{R}$.

[54] For $\ell > 0$, one can use the identity: $\sum_{m_z=-\ell}^{\ell}[Y_\ell^{m_z}(\mathbf{e}\cdot\mathbf{e}_z,\mathbf{e}_{4\perp 2}\cdot\mathbf{e}_z,\mathbf{e}_{24}\cdot\mathbf{e}_z)]^* e^{im_z(\gamma_{24}+\theta_{24}/2)}\langle \ell, m_z|\ell, m_x = 0\rangle = s_+ Y_\ell^0(\hat{\mathbf{K}}_{24})$, where $\hat{\mathbf{K}}_{24} = (\mathbf{k}_2+\mathbf{k}_4)/|\mathbf{k}_2+\mathbf{k}_4|$, we recall that $\gamma_{24} = \tau_{24} - \theta_{24}/2$ and τ_{24} is the angle between $\mathbf{k}_2$ and $\mathbf{k}_2+\mathbf{k}_4$, the spherical harmonics notation is as in note [38] and the sign s_+ is defined in note [41]. This identity is also useful for note [52].

[55] If $k_{24} \to +\infty$, then $|x|$ necessarily tends to $+\infty$ since $|u| \leqslant 1$, in which case K_{24} and k_{24} both diverge as $e^{|x|}$ and have a nonvanishing ratio.

[56] We used $\int_0^{+\infty} dy \sin(ay)\sin(by)/y^{s+1/2} = \frac{1}{2}\Gamma(\frac{1}{2}-s)\sin[\frac{\pi}{4}(1-2s)][|a-b|^{s-\frac{1}{2}} - |a+b|^{s-\frac{1}{2}}]$, for $s \in i\mathbb{R}$, a and b real numbers that differ in absolute value.

[57] The integrand in Eq. (97) is $O(1/q^2)$ when $q \to +\infty$, so the integral converges when $r_{24} \to 0$ at fixed nonzero $\mathbf{R}_{24} - \mathbf{R}_{13}$. Furthermore, it converges to a nonzero value (e.g., to π for $s = 0$).

[58] If one takes a function $D(\mathbf{k}_2,\mathbf{k}_4)$ with no singularity in $\mathbf{k}_2+\mathbf{k}_4 = \mathbf{0}$, for example, $\Phi_0^{(0)}(x,u) = \exp(-\kappa|x|)$ in the ansatz (20) as in note [51] for $\ell = L = 0$, one finds by an explicit calculation that $\mathcal{A}(\mathbf{r}_2-\mathbf{R}_{13},\mathbf{r}_4-\mathbf{R}_{13})$ is finite for $\mathbf{r}_2 = \mathbf{r}_4 = \mathbf{R}_{24}$.

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[61] There is a paradox here. As shown by Eq. (3), the action of the Hamiltonian H on $\psi_{24}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ leads to a $\delta(\mathbf{r}_1-\mathbf{r}_3)$ distribution, not to a $\delta(\mathbf{r}_2-\mathbf{r}_4)$ distribution. This shows that ψ_{24} is the so-called 1-3 Faddeev component, and it cannot have any $1/r_{24}$ singularity. How can $\mathcal{A}_{13}$ then have such a singularity? The answer as usual lies in the order of the limits. At fixed nonzero r_{13}, it is apparent that the function $u(r_{13})$ in Eq. (10), through its dependence on q_{13} given by Eq. (11), provides an ultraviolet cutoff of order $1/r_{13}$ in the $(\mathbf{k}_2,\mathbf{k}_4)$ wave-vector space, so ψ_{24} cannot diverge when $r_{24} \to 0$. But if one first takes the $r_{13} \to 0$ limit, the function $u(r_{13})$ is replaced by its equivalent $1/(4\pi r_{13})$ which has no momentum dependence: The wave-vector cutoff is set to infinity and a $1/r_{24}$ divergence in $\lim_{r_{13}\to 0}(r_{13}\psi_{24})$ can now take place at $r_{24} = 0$.

[62] This reasoning can be transposed to the case of four identical bosons, when three of them converge to the same location in the relative three-body channel where the Efimov effect takes place. As this channel has a zero angular momentum and an even

parity, this implies that, in such a configuration, the total internal angular momentum ℓ of the four-body system is carried by the relative motion of the fourth boson with respect to the center of mass of the first three bosons, leading to a global parity $(-1)^\ell$. This indicates that, for $\ell \neq 0$, the four-boson unitary system in an isotropic harmonic trap should have interacting states in the $(-1)^{\ell+1}$ parity sector that are immune to the three-body Efimov effect. Such "universal" states have indeed been observed numerically in Ref. [69] but for a total internal angular momentum $\ell = 0$: This observation cannot be explained by our reasoning.

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[64] If for simplicity one omits to write the other variables, the integral over θ', of the form $\int_0^\pi d\theta' \check{K}(\theta,\theta')\check{\Phi}(\theta')$, is approximated by $\sum_{j=1}^{n_\theta} w(\theta_j)\check{K}(\theta_i,\theta_j)\check{\Phi}(\theta_j)$, where $(\theta_i)_{1\leqslant i\leqslant n_\theta}$ is the set of (nonequispaced) discrete values of θ proposed by the n_θ points Gauss-Legendre method and $w(\theta_i)$ the corresponding weights. To render the resulting discretized form of $\check{M}(s=0)$ Hermitian, it suffices to take as unknowns $w(\theta_j)^{1/2}\check{\Phi}(\theta_j)$ and to multiply the eigenvalue equation by $w(\theta_i)^{1/2}$, which leads to the kernel $[w(\theta_i)w(\theta_j)]^{1/2}\check{K}(\theta_i,\theta_j)$ without modifying the spectrum.

[65] With the notations of the previous note [64], one takes $[dx\, w_x(\theta_i)]^{1/2}\check{\Phi}^{(\ell)}_{m_z}(x,\theta_i)$ as unknowns in the zone $\rho > \rho_0$ and $[dt\, w(\psi_i)]^{1/2}\rho_0 e^t\check{\Phi}^{(\ell)}_{m_z}(x(t,\psi_i),\theta(t,\psi_i))$ as unknowns in the zone $\rho < \rho_0$. In this way, after multiplication of the eigenvalue equation by $[dx w_x(\theta_i)]^{1/2}$ or $[dt w(\psi_i)]^{1/2}\rho_0 e^t$, one obtains a Hermitian matrix. The x dependence of the weight $w_x(\theta_i)$ results from the x dependence of $\theta_{\max}$.

[66] The triangle corresponds to the zone $1/\sqrt{2} \leqslant \Omega \leqslant \Lambda_{L=1}(0,\alpha)$.

[67] The same conclusion must hold for the eigenvalues Ω in Fig. 3 [parity $(-1)^{\ell+1}$] that are above the upper external border of the continuum, as well as for those below the lower external border of the continuum (there are some for α close to unity).

[68] It is explicitly supposed here that a potential purely imaginary root s_4 of Eq. (22) would exist above some threshold value $\alpha_c(2;2)$ of the mass ratio α, with $s_4 = 0$ at threshold. One can, however, imagine another scenario, with s_4 still a continuous function of α: s_4 would exist for all mass ratio $\alpha \in [1,\alpha_c(2;1)]$, with $\alpha_c(2,1) = 13.6069\ldots$ the three-body Efimov effect threshold, in which case s_4 would not need to cross zero for some α. This scenario is, however, excluded (i) by the experimental results for the spin-1/2 unitary Fermi gas, which has a mass ratio $\alpha = 1$ (no significant four-body losses are observed) and (ii) by the numerical calculations in the conjecture on the fourth cluster coefficient b_4 of that unitary Fermi gas in Appendix B [it is found for $\alpha = 1$ that the operator $M(s)$ is positive for all purely imaginary s, which excludes the existence of a root s_4].

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[74] Note that $\Delta B_{n_\uparrow,n_\downarrow}$ is actually equal to $B_{n_\uparrow,n_\downarrow}$ as soon as the two indices differ from zero, since the ideal gas grand potential is the sum of the grand potential of each spin component, and no $\uparrow$ - $\downarrow$ crossed term can appear in the resulting cluster expansion.

[75] It can be deduced from Eq. (7) of Ref. [29] by integration by parts.

[76] For $\ell = 1$ within the even parity sector, the minimal value of A given by Eq. (B19) corresponds to $x = x' \to +\infty$, $u = u' = 0$ in Eq. (B14). Let us use the notation $O = \langle \ell = 1, m_z = 0|\mathcal{D}^{-1/2}K^{(\ell=1)}\mathcal{D}^{-1/2}|\ell = 1, m'_z = 0\rangle$ and introduce the operator T in the space of functions of the single variable x such that, in Dirac's notation, $\langle x|T|x'\rangle = A/\mathrm{ch}(x-x')$. Then Eq. (B14) can be rewritten as $|\langle x,u|O|x',u'\rangle| \leqslant \langle x|T|x'\rangle$. For any integer n greater than 1, we inject $n-1$ closure relations and use the triangular inequality to obtain $|\langle x,u|O^n|x',u'\rangle| \leqslant \int_{\mathbb{R}} dx_1 \ldots dx_{n-1}\int_{-1}^1 du_1 \ldots du_{n-1}\langle x|T|x_1\rangle \ldots \langle x_{n-1}|T|x'\rangle = 2^{n-1}\langle x|T^n|x'\rangle$. Then $|\langle x,u|[(\mathbb{1}+O)^{-1}-\mathbb{1}]|x',u'\rangle| \leqslant \langle x|\frac{T}{\mathbb{1}-2T}|x'\rangle = \int_{\mathbb{R}}\frac{dk}{2\pi}e^{ik(x-x')}\frac{t_k}{1-2t_k}$, where we have used a series expansion in powers of O and where $t_k = \int_{\mathbb{R}} dy \frac{A}{\mathrm{ch}\, y}e^{iky} = \pi A/\mathrm{ch}(k\pi/2)$ is the eigenspectrum of T. Calculating the integral over k and combining the result with the identity $\langle \ell = 1, m_z = 0|K^{(\ell=1)}_{\mathrm{inv}}|\ell = 1, m'_z = 0\rangle = \mathcal{D}^{-1/2}[(\mathbb{1}+O)^{-1}-\mathbb{1}]\mathcal{D}^{-1/2}$ leads to Eq. (B20).

[77] To finish the calculation, one can use the mild hypothesis that the matrix elements of K_{inv} are uniformly bounded, that $|d_1(x) - d_1(x')|(\frac{e^{x+x'}}{4\,\mathrm{ch}\,x\,\mathrm{ch}\,x'})^{1/4} \leqslant |d_1(x)|(\frac{e^x}{2\,\mathrm{ch}\,x})^{1/4} + |d_1(x')|(\frac{e^{x'}}{2\,\mathrm{ch}\,x'})^{1/4}$, due to $\exp(x') \leqslant 2\,\mathrm{ch}\,x'$ or $\exp(x) \leqslant 2\,\mathrm{ch}\,x$ and to the triangular inequality $|d_1(x) - d_1(x')| \leqslant |d_1(x)| + |d_1(x')|$. Then one can integrate over x' or over x (depending on the term), using $\int_{\mathbb{R}}\frac{dx'}{\mathrm{ch}(x-x')} = \pi$, and one finally faces the integral $\int_{\mathbb{R}} dx|d_1(x)|(\frac{e^x}{2\,\mathrm{ch}\,x})^{1/4} < +\infty$.

[78] The value $k = 0$, that is, $n = 0$, should not be included. If one directly takes the limit $k \to 0$ in Eq. (B30) one gets the absurd result $\phi_{k=0}(t) = 0$. The correct way of taking the limit is to first divide Eq. (B30) by ik. One then finds that $\phi_{k=0}(t)$ diverges as $2t - b(S)$ when $t \to -\infty$, so it does not satisfy the boundary condition (B35).

[79] In the numerics, we extrapolate to $1/t_{\min} = 0$ using a cubic fit in $1/t_{\min}$, with data down to minimal values $1/|t_{\min}| = 1/200$ for $\ell = 0$ and $1/|t_{\min}| = 1/30$ for $\ell > 0$. For $\ell = 0$, as a test of the finite-$t_{\min}$ formalism, we have used Eq. (B48) to predict the leading numerical error on $\Delta B^{\mathrm{conj}(\ell=0)}_{2,2}(0^+)$ due to the $t_{\min}$ truncation, that is, $(8\pi|t_{\min}|)^{-1}\int_{\mathbb{R}} dS[b(\infty) - b(S)] = 2.3(1)/(8\pi|t_{\min}|)$, which agrees with the direct numerical calculation. To obtain $b(S)$ at any given S, and hence the integral of $b(\infty) - b(S)$, we calculated numerically the eigenvectors corresponding to the first few eigenvalues Ω_n ($n \geqslant 1$) of $M^{(\ell=0)}(s = iS)$, and we fitted the corresponding functions $\phi_n(t)$ [defined as in Eq. (91)] with a three-parameter sine function $t \mapsto A_n \sin(k_n t - \theta_n/2)$ as suggested by Eq. (B30), where A_n is a complex amplitude, k_n an effective wave number, and θ_n a phase shift. The fits are very good, and the obtained values of k_n agree very well with the dispersion relation (84). Setting $\theta(k_n,S) = \theta_n$, we also find that the quantization condition (B36)

is well obeyed. Finally, extrapolating θ_n/k_n to $n=0$ linearly in k_n^2 gives $b(S)$. To be complete, we note that $b(S)$ looks like a negative-amplitude Gaussian on a nonzero background $b(\infty)$, that is, $b(\infty)-b(S)\simeq 1.05\times\exp(-0.668S^2)$. As the variable t in Eq. (91) depends on ρ_0, so does $b(\infty)$. In our numerics, $\rho_0=2/5$ and we find $b(\infty)=3.84(1)$. More analytically, one expects at large S that the phase shift $\theta(k,S)$ is imposed by the third contribution in Eq. (69), the first two ones becoming rapidly oscillating and negligible. Then one can use the analytical results of note [52]: introducing the phase shift $\theta_\psi(k)$ such that $\psi(t)=\sin[kt-\theta_\psi(k)/2]+o(1)$ for $t\to-\infty$, one expects that $\theta(k,S)\to\theta_\psi(k)-2k\ln(\rho_0/\sqrt{2})$ for $S\to\infty$, so that $b(\infty)=b_\psi-2\ln(\rho_0/\sqrt{2})$ with $b_\psi\simeq 1.33$. Our numerics fulfill these expectations, which constitutes a good test.

[80] In practice, we used a Gauss-Legendre scheme with up to 59 points, using θ rather than $u=\cos\theta$ as the integration variable, with the change of function (A3) and the inclusion of the extra Jacobian $(\sin\theta\sin\theta')^{1/2}$ in the matrix kernel.

[81] The integration over $S\in\mathbb{R}$ of $\Lambda_\ell(iS)/\cos\nu-1$ using the first line of Eq. (78) leads to $\int_{-1}^{1}du\,P_\ell(u)/(1+u\sin\nu)$. The large-$\ell$ limit of that integral reproduces exactly Eq. (42) of Ref. [28], as we have checked using $P_\ell(u)=(2^\ell\ell!)^{-1}\frac{d^\ell}{du^\ell}[(u^2-1)^\ell]$ and then integrating ℓ times by parts then using Laplace's method.

[82] The product of the four unitary operators under the trace represents in the single-particle Hilbert space a rotation of angle δ around some axis. It is easy to explicitly evaluate this trace as a function of the angles $\theta,\theta',\phi,\phi'$ in the case $\ell=1$, where each operator can be replaced by a well-known 3×3 rotation matrix in the usual, three-dimensional space. This leads to the expression (B57).

[83] Furthermore, the Hilbert space was limited in Ref. [13] to the kets $|x,u,\ell,m_z\rangle$ with $x>0$, as the symmetry condition (B58) allows, which amounts to adding an extra projector $\mathcal{P}_{x>0}$. This complicates things because $\mathcal{P}_{x>0}$ and U do not commute. Fortunately, in calculating operator traces, one can use the properties $UP=PU=P$, $\mathcal{P}_{x<0}=U\mathcal{P}_{x>0}U$, and $\mathcal{P}_{x<0}+\mathcal{P}_{x>0}=\mathbb{1}$, as well as the invariance of the trace under a cyclic permutation of the operators, so $\mathrm{Tr}(\mathcal{P}_{x<0}PAP)=\mathrm{Tr}(U\mathcal{P}_{x>0}UPAP)=\mathrm{Tr}(\mathcal{P}_{x>0}PAP)=\frac{1}{2}\mathrm{Tr}(PAP)=\frac{1}{2}\mathrm{Tr}(AP)$ and $\mathrm{Tr}(AP\mathcal{P}_{x<0}PB)=\mathrm{Tr}(AP\mathcal{P}_{x>0}PB)=\frac{1}{2}\mathrm{Tr}(APB)$, where A and B are arbitrary operators.

[84] If one writes the operator M of Ref. [13] before its restriction to the subspace of symmetry (B58) as $\mathcal{D}+K_0+UK_0U$, then after implementation of the symmetry and restriction to the Hilbert space of kets $|x,u,\ell,m_z\rangle$ with $x>0$, it becomes $\mathcal{P}_{x>0}[\mathcal{D}+(1+U)K_0(1+U)]\mathcal{P}_{x>0}$. The first, second, and third contributions in the right-hand side of Eq. (B60) are, respectively, given by $-(\ell+1/2)/(2\pi)$ times the integral over $S\in\mathbb{R}$ of $\mathrm{Tr}_{\ell,\varepsilon}(\tilde{K}_0U)$, of $-\frac{1}{2}\mathrm{Tr}_{\ell,\varepsilon}[\tilde{K}_0(U\tilde{K}_0U)]$, and of $-\mathrm{Tr}_{\ell,\varepsilon}(\tilde{K}_0^2U)$, where we have set $\tilde{K}_0\equiv\mathcal{D}^{-1}K_0$ and the index ℓ,ε means that the trace is restricted to the states $|\ell,m\rangle$ with $(-1)^m=\varepsilon$. Note that $U\mathcal{D}U=\mathcal{D}$ and $[\mathcal{P}_{x>0},\mathcal{D}]=0$.

[85] The ℓ independent function in factor of the sine function in the integrand of Eq. (B62) is a smooth function of δ over $[0,\pi]$, with all its even order derivatives (including the zeroth order) vanishing at $\delta=0$ and all its odd order derivatives vanishing at $\delta=\pi$. Under repeated integration by parts (always integrating the sine function), the fully integrated term vanishes at the boundaries and one pulls out at each step a factor $(\ell+1/2)^{-1}$. So Eq. (B62) is $O[(\ell+1/2)^{-n}]$ when $\ell\to+\infty$, for all integers n.