Anticoherence of spin states with point-group symmetries

D. Baguette,[1] F. Damanet,[1] O. Giraud,[2] and J. Martin[1]

[1]*Institut de Physique Nucléaire, Atomique et de Spectroscopie, Université de Liège, Bâtiment B15, B-4000 Liège, Belgium*

[2]*LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France*

We investigate multiqubit permutation-symmetric states with maximal entropy of entanglement. Such states can be viewed as particular spin states, namely anticoherent spin states. Using the Majorana representation of spin states in terms of points on the unit sphere, we analyze the consequences of a point-group symmetry in their arrangement on the quantum properties of the corresponding state. We focus on the identification of anticoherent states (for which all reduced density matrices in the symmetric subspace are maximally mixed) associated with point-group-symmetric sets of points. We provide three different characterizations of anticoherence and establish a link between point symmetries, anticoherence, and classes of states equivalent through stochastic local operations with classical communication. We then investigate in detail the case of small numbers of qubits and construct infinite families of anticoherent states with point-group symmetry of their Majorana points, showing that anticoherent states do exist to arbitrary order.

I. INTRODUCTION

Geometrical representations in science have a long history. They give further insight in many different contexts, ranging from classical mechanics to graph theory and quantum information [1]. One such representation is the Bloch sphere picture for spin-$\frac{1}{2}$ states, which has been widely used in the latter context. Several geometrical generalizations of the Bloch representation to higher spin systems have been proposed [2–6]. In his seminal paper [2], Majorana introduced a particularly convenient way of visualizing pure spin-j states as a set of $2j$ points on the Bloch sphere. This representation has been used in various contexts, such as in the study of spinor Bose gases [7–9] or entanglement quantification and classification in multiqubit systems [10–13]. It can be used to highlight symmetries of spinor wave functions and to determine the inert states of spin systems, which are stationary for generic energy functionals [9,14], and also to study the Berry phase acquired during a cyclic evolution of a spin system [15].

Spin-j coherent states $|\mathbf{n}\rangle$ take a very simple form in the Majorana representation. These states are defined as the pure states verifying $\langle\mathbf{n}|\mathbf{J}|\mathbf{n}\rangle = j\mathbf{n}$ for some unit vector $\mathbf{n}$ ($\hbar$ is set to 1), where $\mathbf{J} = (J_x, J_y, J_z)$ is the spin angular momentum operator associated with a spin-j system. In the Majorana representation, they are depicted by a single point ($2j$-fold degenerated) on the Bloch sphere and are, in this sense, highly directional. For this reason, spin-coherent states are also called polarized states. By contrast, pure spin-j states $|\psi_j\rangle$ that depart the most from coherent states could be defined as the states which do not display any directional properties. However, for any finite j, the moments $\langle(\mathbf{J}\cdot\mathbf{n})^k\rangle \equiv \langle\psi_j|(\mathbf{J}\cdot\mathbf{n})^k|\psi_j\rangle$ of the spin components, from a certain order, depend on $\mathbf{n}$. At best, one can ask for the absence of directionality in the moments up to a given order. This motivates the following definition [16]: A spin-j state $|\psi_j\rangle$ is said to be anticoherent to order t (or t-anticoherent) if $\langle(\mathbf{J}\cdot\mathbf{n})^k\rangle$ is independent of the unit vector $\mathbf{n}$ for $k = 1, \ldots, t$. For example, 1-anticoherence means that the expectation value of the spin operator $\mathbf{J}$ vanishes.

The problem of identifying anticoherent states can be tackled from a quantum information perspective. Quantum information mainly deals with the manipulation of information stored on a set of two-level quantum systems, or qubits. It is possible to rephrase the above definition of anticoherence in the qubit language. The key element is the one-to-one mapping between the Hilbert space of a single spin-j (of dimension $2j + 1$) and the symmetric subspace of the Hilbert space associated with a system of N qubits (of dimension $N + 1$) when $N = 2j$. In this picture, spin-coherent states are fully separable, whereas symmetric N-qubit t-anticoherent states are characterized by the fact that their t-qubit reduced density matrices correspond to the maximally mixed state [6,17]. As a consequence, t-anticoherent states are the most entangled symmetric states, in the sense that for any $(k, N - k)$ bipartition with $k = 1, \ldots, t$ the entanglement entropy is maximal. Note that the maximization of the entropy of entanglement over the *whole* Hilbert space for all possible bipartitions is not reached for symmetric states [18] and thus the states that realize this maximum are not t-anticoherent; however, states maximizing the entanglement entropy for $(1, N - 1)$ bipartitions can be symmetric and thus 1-anticoherent. These states, with maximally mixed one-qubit reductions, have been the subject of much research [19–26], notably due to their importance in two-party communication tasks. Similarly, the notion of nonclassicality of spin states based on the Glauber-Sudarshan P representation [27,28] has been translated to multiqubit symmetric states in [29], where it has been shown to imply entanglement.

The aim of this paper is to investigate anticoherence from a geometrical point of view, using the Majorana representation. As previously mentioned, for coherent states, Majorana points all lie at the same place. By contrast, states whose Majorana points are spread out as far away as possible on the sphere have been shown to be highly entangled [10,11,30]. For instance, it was found that states obtained by considering Majorana points with a configuration identical to that of electric charges at equilibrium on a sphere maximize the geometric measure of entanglement for certain spin values. Such geometric arrangements are known to display point-group symmetries.

We can therefore expect that arrangements of Majorana points with a certain point-group symmetry lead to highly entangled states. In this paper, we investigate along those lines the consequences of point-group symmetries on anticoherence properties of spin states. As we will see, this allows us to construct states with high order of anticoherence.

The paper is organized as follows. In Sec. II, we present various characterizations of anticoherence, both from a spin and a multiqubit perspective. In particular, we derive a set of equalities that the components of a permutation-symmetric state must satisfy in order to be t-anticoherent. In Sec. III, we establish the consequences of the symmetries of Majorana points on the components of the corresponding state and determine which symmetries lead to anticoherence. In Sec. IV, we show that any state displaying cyclic symmetry can be transformed under stochastic local operations and classical communications (SLOCC) to an anticoherent state, and establish a link between symmetries, anticoherence and SLOCC classes. We also identify all anticoherent states with cyclic symmetry for five qubits. In Sec. V, we present a systematic method allowing us to find t-anticoherent states and identify infinite families of anticoherent states with cyclic group symmetry. In Sec. VI, we prove that there exists anticoherent states to all orders.

II. DEFINITION AND CHARACTERIZATIONS OF ANTICOHERENCE

A. Correspondence between spin-j states and multiqubit symmetric states

In this section we formalize the correspondence between single spin-j states and multiqubit symmetric states and transpose the definition of anticoherence given in the introduction to multiqubit symmetric states. We first describe spin-j states, then N-qubit symmetric states, and finally their one-to-one mapping.

1. Single spin-j states

Any pure spin-j state $|\psi_j\rangle$ can be expanded in the standard angular momentum basis $\{|j,m\rangle : -j \leqslant m \leqslant j\}$ of joint eigenstates of $\mathbf{J}^2$ and J_z as

$$|\psi_j\rangle = \sum_{m=-j}^{j} c_m |j,m\rangle, \qquad (1)$$

with c_m complex coefficients such that $\sum_m |c_m|^2 = 1$. Coherent states are eigenstates of $\mathbf{J} \cdot \mathbf{n}$ with eigenvalue j, where $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is the unit vector pointing along the direction specified by the angles (θ,φ) on the unit sphere. For spin-$\frac{1}{2}$, their general expression reads

$$|\mathbf{n}\rangle = \cos\left(\frac{\theta}{2}\right)\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}\left|\frac{1}{2},-\frac{1}{2}\right\rangle. \qquad (2)$$

More generally, a spin-j coherent state $|\mathbf{n}\rangle$ has expansion

$$|\mathbf{n}\rangle = \sum_{m=-j}^{j} \sqrt{C_{2j}^{j-m}}\left[\cos\left(\frac{\theta}{2}\right)\right]^{j+m}\left[\sin\left(\frac{\theta}{2}\right)e^{i\varphi}\right]^{j-m}|j,m\rangle, \qquad (3)$$

where C_{2j}^{j-m} is a binomial coefficient.

The Husimi function of $|\psi_j\rangle$, defined by

$$Q(\theta,\varphi) = |\langle\mathbf{n}|\psi_j\rangle|^2, \qquad (4)$$

is the probability of finding the spin in the coherent state pointing along the direction (θ,φ) [31].

2. N-qubit symmetric states

Any pure symmetric state $|\psi_S\rangle$ of N qubits can be expressed as

$$|\psi_S\rangle = \sum_{k=0}^{N} d_k |D_N^{(k)}\rangle, \qquad (5)$$

with d_k complex coefficients such that $\sum_k |d_k|^2 = 1$ and $\{|D_N^{(k)}\rangle : 0 \leqslant k \leqslant N\}$ the (orthonormal) symmetric Dicke states defined in the computational basis as

$$|D_N^{(k)}\rangle = \frac{1}{\sqrt{C_N^k}} \sum_\sigma |\underbrace{0\dots0}_{N-k}\underbrace{1\dots1}_{k}\rangle,$$

where $k = 0,\dots,N$ is the number of 1's and the sum runs over all permutations of the qubits. In the particular case of a symmetric *separable* state $|\Phi_S\rangle = |\phi\rangle^{\otimes N}$, where

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad (6)$$

is a single-qubit state with $\alpha,\beta \in \mathbb{C}$, the expansion (5) takes the form

$$|\Phi_S\rangle = \sum_{k=0}^{N} \sqrt{C_N^k}\, \alpha^{N-k}\beta^k |D_N^{(k)}\rangle. \qquad (7)$$

The probability of finding the pure symmetric state $|\psi_S\rangle$ in the separable state $|\Phi_S\rangle$ is given by

$$Q(\alpha,\beta) = |\langle\Phi_S|\psi_S\rangle|^2. \qquad (8)$$

3. One-to-one mapping

A one-to-one correspondence between Eqs. (1)–(4) and (5)–(8) can be made by setting $k = j - m$, $N = 2j$, $\alpha = \cos(\frac{\theta}{2})$, and $\beta = \sin(\frac{\theta}{2})e^{i\varphi}$, and allows a formal equivalence between the standard basis and the Dicke basis,

$$\begin{aligned} |D_N^{(k)}\rangle &\leftrightarrow \left|\frac{N}{2}, \frac{N}{2} - k\right\rangle, \\ |j,m\rangle &\leftrightarrow |D_{2j}^{(j-m)}\rangle. \end{aligned} \qquad (9)$$

The spin-j angular momentum operator $\mathbf{J}$ is formally equivalent to the collective spin operator $\mathbf{S} = (S_x, S_y, S_z)$ associated with the N-qubit system. Here we define $S_l = \frac{1}{2}\sum_{i=1}^{N} \sigma_l^{(i)}$ ($l = x,y,z$) and $\sigma_l^{(i)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_l \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$, where σ_l are the Pauli operators $\sigma_x = |0\rangle\langle1| + |1\rangle\langle0|$, $\sigma_y = i(|1\rangle\langle0| - |0\rangle\langle1|)$, and $\sigma_z = |0\rangle\langle0| - |1\rangle\langle1|$ and appear as the ith factor in the tensor product. More precisely, the matrix elements $\langle j,m|\mathbf{J}|j,m'\rangle$ and $\langle D_N^{(k)}|\mathbf{S}|D_N^{(k')}\rangle$ are equal for $k = j - m$, $k' = j - m'$, and $N = 2j$. Spin-coherent states coincide with symmetric separable states, $|\mathbf{n}\rangle \leftrightarrow |\Phi_S\rangle$, in the sense that the spin-j coherent state given by Eq. (3) can be seen as the $2j$-fold tensor product of the spin-$\frac{1}{2}$ coherent state (2).

B. General conditions of anticoherence

A t-anticoherent pure spin-j state $|\psi_j\rangle$ is defined by the fact that $\langle(\mathbf{J} \cdot \mathbf{n})^k\rangle$ is independent of the unit vector $\mathbf{n}$ for $k = 1, \ldots, t$. A criterion for anticoherence has been given by Bannai and Tagami [32] for any j in terms of correlators of J_x, J_y, and J_z and their powers. In particular, it was found as a corollary that a state is 1-anticoherent if and only if $\langle J_x \rangle = \langle J_y \rangle = \langle J_z \rangle = 0$. It is 2-anticoherent if and only if it is 1-anticoherent and fulfills the additional conditions $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \langle J_z^2 \rangle$ and $\langle J_k J_l \rangle = 0$ for $k, l \in \{x, y, z\}$ and $k \neq l$, meaning that both the expectation value and the variance of the spin components are equal in all directions. For larger order of anticoherence t, these conditions quickly become cumbersome.

This definition of anticoherence can be naturally translated to the multiqubit setting. The mapping of the previous subsection leads to the following definition: A multiqubit symmetric state $|\psi_S\rangle$ is t-anticoherent if $\langle(\mathbf{S} \cdot \mathbf{n})^k\rangle$ is independent of $\mathbf{n}$ for $k = 1, \ldots, t$.

A simple characterization has been given in [6,17]. Let ρ_t be the $(t+1) \times (t+1)$ reduced density matrix of $\rho = |\psi_S\rangle\langle\psi_S|$ obtained by tracing over $N - t$ qubits and expressed in the Dicke basis $\{|D_t^{(k)}\rangle : k = 0, \ldots, t\}$ spanning the t-qubit symmetric subspace. Note that since symmetric states are by definition invariant under permutation of the qubits, the t-qubit reduced states do not depend on the choice of the t qubits. A state is t-anticoherent if and only if ρ_t is proportional to the identity matrix, namely

$$\rho_t = \frac{\mathbb{1}_{t+1}}{t+1}, \tag{10}$$

where $\mathbb{1}_{t+1}$ is the $(t+1) \times (t+1)$ identity matrix. This implies that t-anticoherent states maximize the entanglement entropy $S = -\mathrm{tr}(\rho_t \log \rho_t)$ among symmetric states for any bipartition $(t, N - t)$ of the N qubits.

In this section we derive three different characterizations of anticoherence to any order t. The first one is in terms of expectation values of convenient combinations of spin operators, the second is in terms of Dicke coefficients of a state, and the third is based on the multipolar expansion of the Husimi function associated with a state.

1. In terms of expectation value of spin operators

Let $A(q)$ be the real numbers defined by

$$A(q) = \frac{1}{N+1} \sum_{k=0}^{N} \left(\frac{N}{2} - k\right)^q \tag{11}$$

for integers q. For $q = 2$, we have $A(2) = \frac{1}{12}N(N+2)$ and for $q = 4$, $A(4) = \frac{1}{20}[3N(N+2) - 4]A(2)$. Note that for all odd q, $A(q) = 0$. We now show that a symmetric state $|\psi_S\rangle$ is t-anticoherent if and only if expectation values of the operators $S_+^r S_z^q$ with $S_+ = S_x + i S_y$ verify

$$\langle\psi_S|S_+^r S_z^q|\psi_S\rangle \equiv \langle S_+^r S_z^q\rangle = A(q)\,\delta_{r0} \tag{12}$$

for $r = 0, \ldots, t$ and $q = 0, \ldots, t - r$, with δ_{r0} the Kronecker symbol. This set of equalities is much more compact than similar conditions found in [32]. They are equivalent to the fact that the t-qubit reduced density matrix ρ_t is maximally mixed

for t-anticoherent states: Eq. (12) expresses equality of the diagonal entries of ρ_t ($r = 0$) and vanishing of all off-diagonal entries ($r \neq 0$).

To show this, we write each entry of the reduced density matrix ρ_t as [17]

$$(\rho_t)_{\ell,\ell+r} = \sum_{q=0}^{t-r} C_{\ell q}^{(r)} \langle S_+^r S_z^q\rangle, \tag{13}$$

where $C^{(r)}$ for $r = 0, \ldots, t$ are invertible square matrices of dimension $t + 1 - r$, which do not depend on the state $|\psi_S\rangle$. Equation (13) follows from three facts: (i) Any operator which is a linear combination of $|D_N^{(k+r)}\rangle\langle D_N^{(k)}|$ for $k = 0, \ldots, N - r$ can be expressed as a linear combination of the operators $S_+^r S_z^q$ with $q = 0, \ldots, N - r$ (see Appendix B of [17]); (ii) any entry $(\rho_t)_{\ell,\ell+r}$ can be expressed as the expectation value $(\rho_t)_{\ell,\ell+r} = \langle\psi_S|\hat{O}_{\ell,\ell+r}|\psi_S\rangle$ of an operator $\hat{O}_{\ell,\ell+r}$ which is a linear combination of the operators $|D_N^{(k+r)}\rangle\langle D_N^{(k)}|$ for $k = 0, \ldots, N - r$ (see Appendix C of [17]); and (iii) the decomposition of $\hat{O}_{\ell,\ell+r}$ into operators $S_+^r S_z^q$, which, according to point (i), involves powers of spin operators such that $q \leqslant N - r$, in fact, only involves terms with $q \leqslant t - r$. Since $C^{(r)}$ is invertible, it follows from Eq. (13) that the vanishing of $\langle S_+^r S_z^q\rangle$ for all $q = 0, \ldots t - r$ is equivalent to the vanishing of $(\rho_t)_{\ell,\ell+r}$. Thus, Eq. (12) for $r \neq 0$ is equivalent to the fact that ρ_t is diagonal.

According to Eq. (13) for $r = 0$, diagonal entries of ρ_t are all equal to $1/(t+1)$ if and only if the vector of components $\langle S_z^q\rangle$ (with $q = 0, \ldots, t$) is given by the constant vector $[C^{(0)}]^{-1}(1, \ldots, 1)^T/(t+1)$. It remains to show that this vector is equal to the vector $[A(0), \ldots, A(t)]$. We do this by evaluating $\langle\psi_S^A|(\mathbf{S} \cdot \mathbf{n})^q|\psi_S^A\rangle$ for an arbitrary anticoherent state $|\psi_S^A\rangle$. For any state $|\psi_S\rangle$, one has $\langle\psi_S|(\mathbf{S} \cdot \mathbf{n})^q|\psi_S\rangle = \mathrm{tr}[\rho U_\mathbf{n} S_z^q U_\mathbf{n}^\dagger] = \mathrm{tr}[U_\mathbf{n}^\dagger \rho U_\mathbf{n} S_z^q]$, where $U_\mathbf{n} = u_\mathbf{n}^{\otimes N}$, with $u_\mathbf{n} = \exp(-i\theta\,\mathbf{n} \cdot \boldsymbol{\sigma}/2)$ the rotation operator of angle θ around the $\mathbf{n}$ axis. For a t-anticoherent state $|\psi_S^A\rangle$ with density matrix ρ^A, the latter expression does not depend on the unit vector $\mathbf{n}$ for $q = 0, \ldots, t$, and thus we can replace it with its average over all $\mathbf{n}$, so that

$$\langle\psi_S^A|(\mathbf{S} \cdot \mathbf{n})^q|\psi_S^A\rangle = \frac{1}{4\pi} \int \mathrm{tr}\left[U_{\mathbf{n}'}^\dagger \rho^A U_{\mathbf{n}'} S_z^q\right] d\mathbf{n}'. \tag{14}$$

Using the completeness relation for coherent states, it is easy to show that for any density matrix ρ, we have $\langle D_N^{(k)}| \int U_\mathbf{n}^\dagger \rho U_\mathbf{n} d\mathbf{n}|D_N^{(k')}\rangle = 4\pi\,\delta_{kk'}/(N+1)$ [1]. Thus, for any unit vector $\mathbf{n}$ and any t-anticoherent state $|\psi_S^A\rangle$ $(t \geqslant q)$, we have

$$\langle\psi_S^A|(\mathbf{S} \cdot \mathbf{n})^q|\psi_S^A\rangle = \frac{\mathrm{tr}\left[S_z^q\right]}{N+1} = A(q), \tag{15}$$

which completes the proof.

2. In terms of Dicke coefficients

The conditions for anticoherence of a state in terms of its Dicke coefficients d_k [see Eq. (5)] are obtained by expressing Eqs. (12) in the Dicke basis. This gives the "diagonal"

conditions ($r = 0$)

$$\sum_{k=0}^{N} \left(\frac{N}{2} - k \right)^q |d_k|^2 = A(q) \tag{16}$$

for $q = 0, \ldots, t$ and the "off-diagonal" conditions

$$\sum_{k=0}^{N-r} B(k,r) \left(\frac{N}{2} - k \right)^q d_k^* d_{k+r} = 0 \tag{17}$$

for $r = 1, \ldots, t$ and $q = 0, \ldots, t - r$, where

$$\begin{aligned} B(k,r) &= \left\langle D_N^{(k+r)} \middle| S_+^r \middle| D_N^{(k)} \right\rangle \\ &= i^r \sqrt{(k+1)(k-N)(r-1)_{k+2}(r-1)_{k+N+1}}, \end{aligned}$$

with $(x)_n = x(x+1) \cdots (x+n-1)$ the Pochhammer symbol.

For instance, Eqs. (16) and (17) yield, for 1-anticoherence, the conditions

$$\sum_{k=0}^{N} (N - 2k) |d_k|^2 = 0, \tag{18}$$

$$\sum_{k=0}^{N-1} \sqrt{(N-k)(k+1)} \, d_k^* d_{k+1} = 0, \tag{19}$$

which were first derived in [17]. For 2-anticoherence, in addition to Eqs. (18) and (19), the following conditions must be fulfilled:

$$\sum_{k=0}^{N} \left(\frac{N}{2} - k \right)^2 |d_k|^2 = \frac{N}{12}(N + 2), \tag{20}$$

$$\sum_{k=0}^{N-1} \left(\frac{N}{2} - k \right) \sqrt{(N-k)(k+1)} \, d_k^* d_{k+1} = 0, \tag{21}$$

$$\sum_{k=0}^{N-2} \sqrt{(N-k)(N-k-1)(k+1)(k+2)} \, d_k^* d_{k+2} = 0. \tag{22}$$

More generally, conditions of anticoherence to order t are obtained by adding to those of order $t - 1$ a set of $t + 1$ equations, one given by (16) with $q = t$ and the t other equations given by (17) with $q + r = t$. So the total number of (real) equations for t-anticoherence is $(t + 1)^2 - 1$, as expected from condition (10).

3. Based on Q and P functions

This characterization of anticoherence provides a more direct physical insight. The density matrix ρ of a spin-j state can always be expanded into state multipole operators T_ℓ^m [31] as

$$\rho = \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} c_{\ell m} T_\ell^m. \tag{23}$$

It can be further described in terms of its Husimi function $Q(\theta, \varphi) = \langle \mathbf{n} | \rho | \mathbf{n} \rangle$, or in terms of its Glauber-Sudarshan P function defined by

$$\rho = \int P(\theta, \varphi) |\mathbf{n}\rangle \langle \mathbf{n}| d\mathbf{n}. \tag{24}$$

Both Q and P functions can be expanded over spherical harmonics $Y_\ell^m(\theta, \varphi)$ with coefficients $Q_{\ell m}$ and $P_{\ell m}$ proportional to $c_{\ell m}$, see, e.g., [31]. As shown in [6], a state is t-anticoherent if and only $c_{\ell m}$ vanishes for all ℓ, m with $0 < \ell \leqslant t$ and $-\ell \leqslant m \leqslant \ell$. From the proportionality of expansion coefficients of Q and P with $c_{\ell m}$, we deduce that for a t-anticoherent state all spherical multipole moments $Q_{\ell m}$ and $P_{\ell m}$ of order ℓ with $0 < \ell \leqslant t$ of the Husimi and P functions vanish. As the Husimi function is the probability of finding the state in a coherent state with a specific direction, and as the spherical harmonics $Y_\ell^m(\theta, \varphi)$ are more and more oscillating as ℓ increases, directionality of a t-anticoherent state is less and less pronounced as t increases because the Husimi function is more and more uniform on the sphere.

III. POINT GROUPS FOR MAJORANA POINTS

A. Majorana representation

A particularly convenient way of visualizing pure spin-j states or N-qubit symmetric states is the Majorana or stellar representation [2]. Any pure symmetric state $|\psi_S\rangle$ can be represented by a set of $N = 2j$ points with angles $\{(\theta_i, \varphi_i), i = 1, \ldots, N\}$ on the Bloch sphere. These N points are associated with N pure single-qubit states $|\phi_1\rangle, \ldots, |\phi_N\rangle$, where $|\phi_i\rangle = \cos\left(\frac{\theta_i}{2}\right)|0\rangle + e^{i\varphi_i} \sin\left(\frac{\theta_i}{2}\right)|1\rangle$, such that

$$|\psi_S\rangle = \mathcal{N} \sum_\sigma |\phi_{\sigma(1)} \ldots \phi_{\sigma(N)}\rangle, \tag{25}$$

where the sum runs over all permutations $\sigma \in S_N$ (the permutation group of N elements) and $\mathcal{N}$ is a normalization constant. Let n_S and $N - n_N$ be, respectively, the first and the last indices of nonvanishing Dicke coefficients d_k for a state of the form (5). The Majorana representation [2] of such a state is obtained by finding the roots of the polynomial

$$P(z) = \sum_{k=n_S}^{N-n_N} (-1)^k \sqrt{C_N^k} \, d_k z^k. \tag{26}$$

This polynomial can be put in the form

$$P(z) = (-1)^{N-n_N} \sqrt{C_N^{n_N}} d_{N-n_N} z^{n_S} \prod_{k=1}^{N-n_N-n_S} (z - z_k), \tag{27}$$

where z_k are the nonzero roots of $P(z)$. Applying the (inverse) stereographic projection from the complex plane onto the Bloch sphere to the zeros of $P(z)$ through the relation $z_m = \cot(\theta_m/2)e^{-i\varphi_m}$ yields $N - n_S - n_N$ points (θ_m, φ_m), and n_S points at the South pole of the Bloch sphere (corresponding to the roots located at $z = 0$). Adding n_N points at the North pole of the Bloch sphere yields the Majorana representation of $|\psi_S\rangle$ as N points on the sphere, called Majorana points. For instance, for a Dicke state $|D_N^{(k)}\rangle$, the Majorana representation is given by $N - k$ points at the North pole and k points at the South pole of the Bloch sphere. For a spin-coherent (or symmetric separable N-qubit) state $|\mathbf{n}\rangle$ [see Eq. (3)], the Majorana points are N points located at (θ, φ).

One of the advantages of the Majorana representation is its behavior under local unitary transformations (LUs). Symmetric LUs are transformations of the form $U^{\otimes N}$, where U is a unitary operator acting on a single qubit. It corresponds,
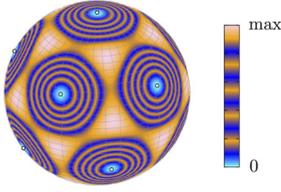
FIG. 1. (Color online) Husimi function of the 12-qubit 5-anticoherent state $|\psi_S\rangle = \frac{\sqrt{7}}{5}|D_{12}^{(1)}\rangle + \frac{\sqrt{11}}{5}|D_{12}^{(6)}\rangle - \frac{\sqrt{7}}{5}|D_{12}^{(11)}\rangle$. The Majorana points, depicted by white points on the sphere, display an icosahedral symmetry. The color code is chosen so as to highlight the contour lines of the Husimi function.

up to a global phase, to a rigid rotation of the Bloch sphere, and thus of the Majorana points, thereby preserving the symmetries of their arrangement.

B. Majorana points for anticoherent states

From its definition, anticoherence of a state $|\psi_S\rangle$ is preserved by symmetric LU. In particular, this means that it is only a feature of the relative arrangement of Majorana points. We expect Majorana points of an anticoherent state to be spread out over the sphere as evenly as possible. Indeed, anticoherent states should be as different as possible from any directional (polarized) state and thus have a Majorana representation as distinct as possible from N points located at the same place.

As shown in Sec. II B 3, anticoherence is related to the vanishing of multipole moments of lower order and thus to the uniformity of the Husimi function over the sphere. A natural way of constructing states with vanishing multipole moments is to consider Majorana points arranged symmetrically on the sphere, as the zeros of the Husimi function are diametrically opposite to the Majorana points on the Bloch sphere. For example, 12 points taken at the vertices of an icosahedron yield a 12-qubit state which is 5-anticoherent. Its Husimi function is depicted in Fig. 1. In the next section, we consider arrangements of points with symmetries belonging to the seven infinite point-group families. Note that anticoherent states with symmetries corresponding to the exceptional point groups can also be constructed, as the example just mentioned shows.

As a consequence of these geometrical features, one may expect the barycenter of the Majorana points to coincide with the center of the Bloch sphere. In fact, this is not the case [17]. The distance from the barycenter to the center (which can serve to define an entanglement measure [33]) can be surprisingly large, as we show in the following example. States of the form $|\psi_S\rangle = \mathcal{N}(\sqrt{N-2}|D_N^{((N-1)/2)}\rangle + |D_N^{(N-1)}\rangle)$ for odd $N > 3$ have a Majorana representation corresponding to $(N-1)/2$ points at the south pole, 1 point at the north pole and $(N-1)/2$ points arranged in a regular polygon parallel to the equatorial plane (a similar family can be found for even N). They are 1-anticoherent, as can be checked from Eqs. (18) and (19); however, for large N, the barycenter of their Majorana points goes in norm to $1/5$.

C. Point-group-symmetric states

Point groups are discrete subgroups of $O(3)$. There are seven infinite families of axial groups indexed by an integer

n and seven exceptional point groups. The seven infinite point-group families are generated by rotations and reflections. We can always bring by LU any point configuration with axial rotation symmetry to a configuration where the symmetry axis is the z axis. We denote by r_n the rotation with axis Oz and angle $2\pi/n$. Similarly, we can always bring a point configuration with reflection symmetry with respect to a plane containing the z axis to a configuration where this plane is the plane Oxz containing both the x and the z axes. We denote by σ_v the reflection with respect to this "vertical" plane and by σ_h the reflection with respect to the "horizontal" plane Oxy. Additional symmetry planes, traditionally denoted by σ_d, are obtained by rotating σ_v by an angle π/n around Oz.

The most fundamental family is that of cyclic groups C_n. The group C_n is generated by the rotation r_n; i.e., its elements are the r_n^k with $0 \leqslant k \leqslant n-1$. The other groups can be described as follows (see, e.g., [34]): C_{nh}, generated by r_n and σ_h; C_{nv}, generated by r_n and σ_v for odd n and by r_n, σ_v, and σ_d for even n; S_{2n}, generated by the product $r_{2n}\sigma_h$; D_n, dihedral group, generated by r_n and by a rotation of angle π around the x axis which can be expressed as the product $\sigma_h\sigma_v$; D_{nh}, generated by D_n and σ_h; D_{nd}, generated by D_n and S_{2n}, or equivalently, by $r_{2n}\sigma_h$ and σ_v.

A given symmetry of the Majorana points of the state (5) reflects on its Dicke coefficients $d_k \in \mathbb{C}$. We first determine the consequences of a symmetry on the roots z_k of the polynomial (27). We recall that the mapping between the complex plane and the Bloch sphere is chosen as $z = \cot(\theta/2)e^{-i\varphi}$. If the Majorana representation of a state is invariant under a symmetry, then the product $z^{n_S} \prod_{k=1}^{N-n_N-n_S}(z - z_k)$ in (27) must be left unchanged, and thus the polynomial (26) remains unchanged up to a multiplicative constant. This induces relations between the Dicke coefficients.

A rotation r_n transforms z into $z \exp(-2i\pi/n)$. Points at the poles are not affected by a rotation r_n. If a configuration of points is invariant under r_n, then

$$\prod_{k=1}^{N-n_N-n_S} (z - z_k) = \prod_{k=1}^{N-n_N-n_S} (z - z_k e^{-2i\pi/n})$$
$$= e^{-\frac{2i\pi}{n}(N-n_N-n_S)} \prod_{k=1}^{N-n_N-n_S} \left(z e^{\frac{2i\pi}{n}} - z_k \right).$$
(28)

From Eq. (27) we then get

$$P(z) = e^{-\frac{2i\pi}{n}(N-n_N)} P\left(z e^{\frac{2i\pi}{n}} \right).$$
(29)

Using the expansion Eq. (26) and identifying the coefficients in the polynomials on both sides of Eq. (29) we get that for all k with $n_S \leqslant k \leqslant N - n_N$

$$d_k = e^{-\frac{2i\pi}{n}(N-n_N-k)} d_k,$$
(30)

unless $k = N - n_N + qn$. In particular, for all k, we have that $d_k = 0$ unless $k = n_N + qn$ with $q \in \mathbb{N}$. Because of the symmetry, $N - n_N - n_S$ has to be a multiple of n, so that the latter condition is equivalent to $k = n_S \pmod{n}$.

A similar approach can be followed for all other symmetry operations (see the Appendix). The results are summarized in Table I.

TABLE I. Constraints on the Dicke coefficients of a state $|\psi_S\rangle$ invariant under a certain symmetry operation on its Majorana points.

Symmetry operation	Constraints on Dicke coefficients
r_n	$d_k = 0 \quad \forall\, k \neq n_S (\mathrm{mod}\, n)$
σ_h	$n_N = n_S$, and $\exists\, \xi \in \mathbb{R}$: $d_{N-k} = e^{i\xi} d_k^* \quad \forall\, k$
σ_v	$d_k \in \mathbb{R} \quad \forall\, k$
σ_d	$d_k = \pm|d_k| e^{\frac{ik\pi}{2n}} \quad \forall\, k$
$\sigma_h \sigma_v$	$d_{N-k} = d_k \quad \forall\, k$, or $d_{N-k} = -d_k \quad \forall\, k$
s_n	$n_N = n_S$, and $\exists\, \xi \in \mathbb{R}$: $d_k = 0 \quad \forall\, k \neq n_S (\mathrm{mod}\, n)$ $d_{N-k} = (-1)^q e^{i\xi} d_k^* \quad$ otherwise

D. Canonical form of point-group-symmetric states

From the results of Table I, we obtain a canonical form for the Dicke coefficients of a state whose Majorana points display a certain symmetry. That is, all states with point-group-symmetric Majorana point arrangements can be brought to one of the forms (31)–(40), depending on the symmetry.

1. Cyclic groups

A state with C_n symmetry is characterized by an arrangement of Majorana points invariant under r_n. From Table I, we obtain for the vector of Dicke coefficients $(d_0, d_1, \ldots, d_N)$ the canonical form

$$\mathbf{d}_{C_n} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots, \mathbf{0}_{n_N}\right), \quad (31)$$

where $\mathbf{0}_m$ stands for a string of m zeros. The consequences of this symmetry are particularly interesting. Indeed, due to the particular form of the Dicke coefficients in Eq. (31), the t-qubit reduced density matrix is a diagonal matrix for $t < n$ since all off-diagonal conditions (17) are satisfied. As all the symmetry groups described in Sec. III C have C_n as a subgroup, t-anticoherence with $t < n$ for point-group-symmetric states only requires the additional diagonal conditions (16) for $q = 0, \ldots, t$.

For a state with C_{nh} symmetry, the arrangement of points is additionally invariant under σ_h. The form (31) together with the condition of invariance under σ_h in Table I leads to the canonical form

$$\mathbf{d}_{C_{nh}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots \right.$$
$$\left. \ldots, \mathbf{0}_{n-1}, d_{n_S+n}^* e^{i\xi}, \mathbf{0}_{n-1}, d_{n_S}^* e^{i\xi}, \mathbf{0}_{n_S}\right), \quad (32)$$

with $\xi \in \mathbb{R}$. For $n \geq 2$, states of the form (32) satisfy Eqs. (18) and (19) and are thus all 1-anticoherent.

For a state with C_{nv} symmetry, the conditions of invariance under σ_v given in Table I and the form (31) lead to the canonical form

$$\mathbf{d}_{C_{nv}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \right.$$
$$\left. \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots, \mathbf{0}_{n_N}\right), \quad d_k \in \mathbb{R}. \quad (33)$$

2. Rotation-reflection group

As C_n is a subgroup of S_{2n}, any state with Majorana points displaying S_{2n} symmetry is of the form (31). The more general condition is that of invariance under s_n, which can be expressed as $n_S = n_N$, $N = 2n_S + mn$ with m even (for odd m, there is no way to achieve S_{2n} symmetry); see Table I. The canonical form is thus

$$\mathbf{d}_{S_{2n}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots \right.$$
$$\left. \ldots, \mathbf{0}_{n-1}, -d_{n_S+n}^* e^{i\xi}, \mathbf{0}_{n-1}, d_{n_S}^* e^{i\xi}, \mathbf{0}_{n_S}\right), \quad (34)$$

with $\xi \in \mathbb{R}$. For $n \geq 2$, states of the form (34) satisfy Eqs. (18) and (19) and are thus all 1-anticoherent.

3. Dihedral groups

Because C_n is a subgroup of D_n, the Dicke coefficients of a configuration with symmetry group D_n are of the form (31), with the additional invariance under $\sigma_h \sigma_v$ given in Table I. These conditions give the canonical forms

$$\mathbf{d}_{D_n} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots \right.$$
$$\left. \ldots, d_{n_S+2n}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S}, \mathbf{0}_{n_S}\right) \quad (35)$$

or

$$\mathbf{d}_{D_n} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots \right.$$
$$\left. \ldots, -d_{n_S+2n}, \mathbf{0}_{n-1}, -d_{n_S+n}, \mathbf{0}_{n-1}, -d_{n_S}, \mathbf{0}_{n_S}\right). \quad (36)$$

For $n \geq 2$, states of the form (36) satisfy Eqs. (18) and (19) and are thus all 1-anticoherent.

The D_{nh} symmetry additionally imposes invariance under σ_h given in Table I, which leads to the canonical forms

$$\mathbf{d}_{D_{nh}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots \right.$$
$$\left. \ldots, d_{n_S+2n}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S}, \mathbf{0}_{n_S}\right), \quad d_k \in \mathbb{R}, \quad (37)$$

or

$$\mathbf{d}_{D_{nh}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots, \right.$$
$$\left. -d_{n_S+2n}, \mathbf{0}_{n-1}, -d_{n_S+n}, \mathbf{0}_{n-1}, -d_{n_S}, \mathbf{0}_{n_S}\right), \quad d_k \in \mathbb{R}. \quad (38)$$

The 5-anticoherent states that will be given in Eq. (51) are examples of states displaying D_{nh} symmetry.

The D_{nd} symmetry imposes the form (34) together with the invariance under σ_d given in Table I, which implies the general condition that the d_k be real and $d_{N-k} = (-1)^q d_k$ for indices $k = n_S + qn$, that is, the forms

$$\mathbf{d}_{D_{nd}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots, d_{n_S+2n}, \right.$$
$$\left. \mathbf{0}_{n-1}, -d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S}, \mathbf{0}_{n_S}\right), \quad d_k \in \mathbb{R}, \quad (39)$$

or

$$\mathbf{d}_{D_{nd}} = \left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, d_{n_S+2n}, \ldots, \right.$$
$$\left. -d_{n_S+2n}, \mathbf{0}_{n-1}, d_{n_S+n}, \mathbf{0}_{n-1}, -d_{n_S}, \mathbf{0}_{n_S}\right), \quad d_k \in \mathbb{R}. \quad (40)$$

The 7-anticoherent state presented in Fig. 5 is an example of a state displaying D_{7d} symmetry.

To summarize, C_n-symmetric states, all have a diagonal t-qubit reduced density matrix for $t < n$, but are not necessarily 1-anticoherent, whereas C_{nh}-, S_{2n}-, and D_n-symmetric states are always a least 1-anticoherent. This includes states with Majorana points invariant under inversion (which transforms a point into its symmetric with respect to the center of the sphere), which corresponds to the symmetry group S_2.

IV. ANTICOHERENT STATES IN SLOCC CLASSES

Local unitaries are a special case of more general transformations used in the context of quantum information theory, namely stochastic local operation with classical communication (SLOCC). By definition, two states $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent under SLOCC if there exists a local protocol with classical communication which transforms one state into the other with a finite probability of success. Mathematically, this corresponds to the requirement that these two states can be related by an invertible local operation (ILO) $A_1 \otimes \cdots \otimes A_N$ with A_k invertible operators [35]. This defines an equivalence relation which partitions the Hilbert space into different SLOCC classes. For symmetric states, the A_k can be chosen equal [36], so that two symmetric states $|\Phi_S\rangle$ and $|\Psi_S\rangle$ belong to the same SLOCC class if and only if there exists an invertible operator A acting on a single qubit such that $|\Psi_S\rangle = A^{\otimes N}|\Phi_S\rangle$. Each SLOCC class contains, at most, one state (up to LU) with a maximally mixed one-qubit reduced density matrix [37]. Thus, each SLOCC class contains, at most, one anticoherent state. The unicity (up to LU) of 1-anticoherent states within their SLOCC class makes them natural representatives of SLOCC classes [21]. As a corollary, it follows that two anticoherent states which are not LU-equivalent necessarily belong to different SLOCC classes. Moreover, the union of SLOCC classes containing a 1-anticoherent state is dense in Hilbert space [21].

The various SLOCC classes of symmetric states can be gathered into families denoted by $\mathcal{D}_{m_1,m_2,\dots,m_d}$ [38], which are defined by their diversity degree d (the number of distinct Majorana points on the Bloch sphere), and by their degeneracy configuration $m_1, m_2, \dots, m_d$ (the degeneracy of each Majorana points). For instance, the family $\mathcal{D}_{1,1,\dots,1}$ contains SLOCC classes with states such that all Majorana points are nondegenerate and the family $\mathcal{D}_N$ contains a single SLOCC class (that of separable symmetric states). All families with a diversity degree $d \leqslant 3$ contain a single SLOCC class, since any set of three distinct points can be transformed into any other set of three distinct points by SLOCC transformation [11,38].

A. C_n symmetry, 1-anticoherence, and SLOCC classes

We now provide a way of finding, given a C_n-symmetric state, its SLOCC-equivalent 1-anticoherent state (when it exists). Our result provides a clear link between symmetry and entanglement classes. We show that any SLOCC class belonging to a family $\mathcal{D}_{m_1,m_2,\dots,m_d}$ with all $m_k < N/2$ and containing states with C_n symmetry possesses a 1-anticoherent state of C_n symmetry. Moreover, any SLOCC class belonging to a family with at least one $m_k \geqslant N/2$ $(k = 1, \dots, d)$ does not contain any anticoherent state, except for the family $\mathcal{D}_{N/2,N/2}$, for which $|D_N^{(N/2)}\rangle$ is 1-anticoherent (but not 2-anticoherent)

[17]. This statement has the following corollary: If a SLOCC class contains a 1-anticoherent state which does not display C_n symmetry, then the class does not contain any state displaying C_n symmetry (as anticoherent states are unique up to LU within their SLOCC class). Note that the possibility of a connection between symmetry and types of entanglement has been pointed out in [12].

A state displaying C_n symmetry can always be brought by LU to the canonical form (31). This state can be converted via a diagonal ILO $A^{\otimes N}$ with $A = \mathrm{diag}(y,1)$ to a SLOCC-equivalent state

$$\tilde{\mathbf{d}}_{C_n} = \mathcal{N}\left(\mathbf{0}_{n_S}, d_{n_S}, \mathbf{0}_{n-1}, d_{n_S+n}\, y^{\frac{n}{2}},\, \mathbf{0}_{n-1}, d_{n_S+2n}\, y^n, \dots, \mathbf{0}_{n_N}\right) \tag{41}$$

with the same C_n symmetry (throughout this work, $\mathcal{N}$ will denote a normalization constant). The state (41) can at least be made 1-anticoherent. It suffices to take y equal to the only strictly positive root of the polynomial

$$\sum_{k=0}^{N}(N-2k)|d_k|^2\, y^k. \tag{42}$$

The existence and unicity of the root follows from Descartes' rule of signs: If the degeneracy of Majorana points is smaller than $N/2$, we are ensured that at least one d_k and one $d_{k+N/2}$ for $k = 0, \dots, N/2$ are nonzero, so that there is exactly one change of sign in the coefficients of the polynomial (42).

Thus, from any C_n-symmetric state belonging to a family $\mathcal{D}_{m_1,m_2,\dots,m_d}$ with all $m_k < N/2$, our procedure allows us to construct the SLOCC-equivalent C_n-symmetric 1-anticoherent state. This is particularly useful since 1-anticoherence is a requirement for t-anticoherence. Moreover, it provides a practical way of determining whether two states displaying cyclic symmetries in their Majorana points (with degeneracies $m_k < N/2$) belong to the same SLOCC class. It suffices to determine the point arrangements of their SLOCC-equivalent anticoherent states (41) and check whether they are identical up to rigid rotation of the sphere.

B. Illustrations

1. Case $N \leqslant 4$

All anticoherent states of up to $N = 4$ qubits have been identified in [17]. Remarkably, they all display C_n symmetry, and coincide with anticoherent states found using the method presented in the preceding section. For 2 and 3 qubits, the only 1-anticoherent states (up to LU) are the C_2-symmetric Bell state $\mathbf{d}_{C_2} = \frac{1}{\sqrt{2}}(1,0,1)$ and the C_3-symmetric GHZ state $\mathbf{d}_{C_3} = \frac{1}{\sqrt{2}}(1,0,0,1)$. The case of four qubits is more interesting as there is an infinite number of SLOCC-inequivalent 1-anticoherent states of the form $\mathbf{d}_{C_2} = \mathcal{N}(1,0,\tau,0,1)$, with $\tau \in \mathbb{C}$ [17]. Their Majorana representation displays a dihedral D_2 symmetry. Our method allows us to recover these states starting from the general form (31) of a four-qubit C_2-symmetric state, $\mathbf{d}_{C_2} = \mathcal{N}(1,0,\mu,0,\nu)$ with $\nu > 0$ [39], and then using the SLOCC-equivalent state (41), with $y = 1/\sqrt{\nu}$ the only positive root of the polynomial (42).
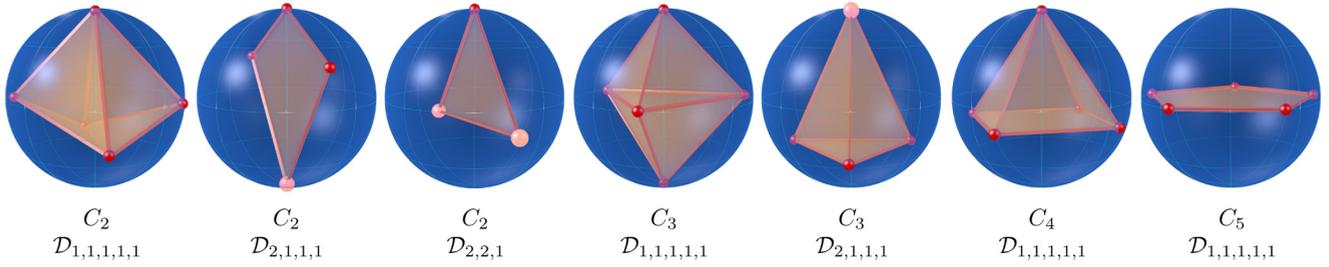
FIG. 2. (Color online) Majorana representation of five-qubit 1-anticoherent states displaying C_n symmetry. The small and large points on the spheres correspond to nondegenerate and twice-degenerate Majorana points, respectively. The corresponding states are those given in the main text (see Sec. IV B 2).

2. Case N = 5

We consider here exhaustively the case of five-qubit states with any possible C_n symmetry. As discussed in Sec. IV A, classes belonging to families characterized by points with multiplicity $m \geqslant N/2$ do not contain any anticoherent states. On the other hand, all other classes belonging to the three families $\mathcal{D}_{1,1,1,1,1}$, $\mathcal{D}_{2,1,1,1}$, and $\mathcal{D}_{2,2,1}$ and containing C_n-symmetric states have exactly one (up to LU) anticoherent state. We now consider all SLOCC classes containing a C_n-symmetric state for some order n and explicitly construct the related 1-anticoherent state. It can be checked that none of these states is 2-anticoherent.

C_2 *symmetry.* If the C_2-symmetric state belongs to the family $\mathcal{D}_{1,1,1,1,1}$, all points are nondegenerate, which implies, in particular, that exactly one point must lie at a pole in order to fulfill the symmetry requirement. If we choose this point to lie at the north pole, Eq. (31) reduces to

$$\mathbf{d}_{C_2} = \mathcal{N}(1,0,\mu,0,\nu,0), \tag{43}$$

where $\mu > 0$ and $\nu \in \mathbb{C}$ [39]. From Eq. (41) the 1-anticoherent state which is SLOCC-equivalent to (43) is of the form

$$\mathbf{d}_{C_2} = \mathcal{N}(1,0,\mu\, y,0,\nu\, y^2,0), \tag{44}$$

with

$$y = \frac{1}{\sqrt{6}|\nu|}\sqrt{\mu^2 + \sqrt{\mu^4 + 60|\nu|^2}} \tag{45}$$

the only positive root of the polynomial (42). The Majorana representation of this state is shown in Fig. 2 for $\mu = 2$ and $\nu = 10\,i$.

The most general state with symmetry C_2 in the family $\mathcal{D}_{2,1,1,1}$ takes (up to LU) the form $\mathbf{d} = \mathcal{N}(0,0,1,0,\mu,0)$ with $\mu \geqslant 0$, as the degenerate point is necessarily at a pole, which can be chosen as the south pole. As follows from Eq. (41) with $y = 1/(\sqrt{3}\mu)$ the positive root of Eq. (42), it can be brought to the 1-anticoherent state $\mathbf{d}_{C_2} = (0,0,\sqrt{3},0,1,0)/2$, whose Majorana representation is shown in Fig. 2. Thus, states with C_2 symmetry in the family $\mathcal{D}_{2,1,1,1}$ all belong to the same SLOCC class.

The family $\mathcal{D}_{2,2,1}$ contains states which have five Majorana points, two of which are doubly degenerate, leaving only three distinct points. Therefore, there is only one SLOCC class in the family $\mathcal{D}_{2,2,1}$ (see Sec. IV), which contains (up to LU) a 1-anticoherent state displaying C_2 symmetry, namely $\mathbf{d}_{C_2} = \mathcal{N}(1,\mu,0,\nu,0)$, with $\mu = \sqrt{\frac{2}{15}(1 + 2\sqrt{19})}$ and $\nu = \frac{1}{15}(\sqrt{5} +$

$2\sqrt{95})$. Indeed, it is easy to check that this state is of the form (31) and verifies Eqs. (18) and (19) (see Fig. 2).

C_3 *symmetry.* For the family $\mathcal{D}_{2,2,1}$, there is no way to satisfy the C_3 symmetry. Any C_3-symmetric state belonging to the family $\mathcal{D}_{1,1,1,1,1}$ has to be of the form $\mathbf{d}_{C_3} = \mathcal{N}(0,1,0,0,\mu,0)$, with one point at the north pole and one point at the south pole. Using Eq. (41), the SLOCC-equivalent 1-anticoherent state is given by $\mathbf{d}_{C_3} = \frac{1}{\sqrt{2}}(0,1,0,0,1,0)$, actually displaying D_{3h} symmetry by virtue of Eq. (37) (see Fig. 2).

Similarly, one can show that C_3-symmetric states belonging to the family $\mathcal{D}_{2,1,1,1}$ are SLOCC equivalent to the 1-anticoherent state $\mathbf{d}_{C_3} = \frac{1}{\sqrt{6}}(1,0,0,\sqrt{5},0,0)$ (see Fig. 2).

C_4 *symmetry.* The only family with states having C_4 symmetry is $\mathcal{D}_{1,1,1,1,1}$ and their most general canonical form reads $\mathbf{d}_{C_4} = \mathcal{N}(1,0,0,0,\mu,0)$, which can be brought by SLOCC to the 1-anticoherent state $\mathbf{d}_{C_4} = \frac{1}{\sqrt{8}}(\sqrt{3},0,0,0,\sqrt{5},0)$ (see Fig. 2).

C_5 *symmetry.* The only family with states having C_5 symmetry is $\mathcal{D}_{1,1,1,1,1}$ and their most general canonical form reads $\mathbf{d}_{C_5} = \mathcal{N}(1,0,0,0,0,\mu)$, which can be brought by SLOCC to the 1-anticoherent state $\mathbf{d}_{C_5} = \frac{1}{\sqrt{2}}(1,0,0,0,0,1)$, actually displaying D_{5h} symmetry as follows from Eq. (37) (see Fig. 2).

3. Example for N = 6

The case of six qubits can be treated in the same way as the five-qubit case. As for $N = 5$, families with fourfold or higher degeneracy of Majorana points do not contain any anticoherent state. Family $\mathcal{D}_{3,3}$ contains a single SLOCC class, in which $|D_6^{(3)}\rangle$ is the only (up to LU) 1-anticoherent state. All other classes containing a C_n-symmetric state, belonging to the four families $\mathcal{D}_{1,1,1,1,1,1}$, $\mathcal{D}_{2,1,1,1,1}$, $\mathcal{D}_{2,2,1,1}$, and $\mathcal{D}_{2,2,2}$, each contain a unique (up to LU) 1-anticoherent state. Moreover, for $N = 6$, it is possible to find higher-order anticoherent states, as we now show as an example.

C_3 *symmetry.* We consider a state of the form $\mathbf{d}_{C_3} = \mathcal{N}(1,0,0,\mu,0,0,\nu)$ which is SLOCC-equivalent to the 1-anticoherent state $\mathbf{d}_{C_3} = \mathcal{N}(1,0,0,\mu\, y^{3/2},0,0,\nu\, y^3)$, with $y = 1/|\nu|^{1/3}$. For the state to be 2-anticoherent, it must additionally verify Eq. (20), i.e., $\mu = \sqrt{\frac{5}{2}|\nu|}$, which leads to

$$\mathbf{d}_{C_3} = \frac{1}{3}(\sqrt{2},0,0,\sqrt{5},0,0,\sqrt{2}\,e^{i\varphi_\nu}). \tag{46}$$

The only parameter left is φ_ν, resulting in a one-parameter family of SLOCC-inequivalent 2-anticoherent states

(the SLOCC-inequivalence is a consequence of the LU-inequivalence which can be checked from the relative positions of the Majorana points). Among these states, the one with $\varphi_v = \pi$ is the only 3-anticoherent state [it verifies the additional off-diagonal conditions (17) for $t = 3$]. It corresponds to an octahedral arrangement of Majorana points.

C_4 symmetry. A C_4-symmetric state of six qubits can be constructed by putting one Majorana point at the north pole and another at the south pole. The Dicke coefficients are then of the form $\mathbf{d}_{C_4} = \mathcal{N}(0,1,0,0,0,\mu,0)$, with $\mu > 0$. This state is SLOCC equivalent to the 1-anticoherent state $\mathbf{d}_{C_4} = \frac{1}{\sqrt{2}}(0,1,0,0,0,1,0)$ [obtained by using $y = 1/\sqrt{\mu}$ the positive root of Eq. (42)]. This state is 3-anticoherent, as can be easily checked through Eqs. (16) and (17). It corresponds to an octahedral arrangement of Majorana points and is thus LU equivalent to state (46), with $\varphi_v = \pi$. Note that this state has both C_3 and C_4 symmetry and coincides with both the most quantum spin-3 state [40] and the most entangled six-qubit symmetric state with respect to the geometric measure of entanglement [10].

V. INFINITE FAMILIES OF HIGHER-ORDER ANTICOHERENT STATES WITH C_n SYMMETRY

A natural question is to ask whether there exist t-anticoherent states for arbitrary number of qubits N and order t. For larger numbers of qubits, it becomes difficult to find analytical solutions to Eqs. (16) and (17). It is, however, possible to construct infinite families of anticoherent states. An example was already given in Sec. III A. Another family of 1-anticoherent states was proposed in [17] and reads

$$\frac{1}{\sqrt{2N-2}}\left(\sqrt{N-2}\,\big|D_N^{(0)}\big\rangle + \sqrt{N}\,\big|D_N^{(N-1)}\big\rangle\right). \qquad (47)$$

A family of 2-anticoherent states was proposed in [16] for even N with $N \geqslant 6$,

$$\sqrt{\frac{N+2}{6N}}\left(\big|D_N^{(0)}\big\rangle + 2\sqrt{\frac{N-1}{N+2}}\big|D_N^{(N/2)}\big\rangle + \big|D_N^{(N)}\big\rangle\right). \qquad (48)$$

The states (48) are, in fact, 3-anticoherent for $N \geqslant 8$, as can be checked from Eq. (16), given that Eq. (12) for $r \neq 0$ is automatically satisfied because of the $C_{N/2}$ symmetry.

We now present a systematic method which allows us, for a given order t of anticoherence, to find N-qubit anticoherent states displaying C_n symmetry with $n > t$ when they exist. As shown in Sec. III D, C_n symmetry ensures that the off-diagonal equations (17) are satisfied for $t < n$. The remaining

set of diagonal equations (16) can be reformulated as a linear system of equations in the variables $|d_k|^2$ with an additional positivity constraint on the solutions. Such a system can be solved analytically using linear programming.

More specifically, for an N-qubit state with C_n symmetry, the (normalized) vector of Dicke coefficients takes the form (31). Equation (16) becomes

$$\sum_{k=0}^{r}\left(\frac{N}{2} - kn - n_S\right)^q |d_{n_S+kn}|^2 = A(q), \qquad (49)$$

where $r = (N - n_S - n_N)/n$ is the number of free Dicke coefficients, n_S and n_N are the number of points at the south and north poles, respectively, and $A(q)$ is given by Eq. (11). By setting $x_k \equiv |d_{n_S+kn}|^2$ and $u_k = \frac{N}{2} - kn - n_S$, the set of equations becomes

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ u_0 & u_1 & \cdots & u_{r-1} & u_r \\ u_0^2 & u_1^2 & & u_{r-1}^2 & u_r^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_0^{t-1} & u_1^{t-1} & \cdots & u_{r-1}^{t-1} & u_r^{t-1} \\ u_0^t & u_1^t & \cdots & u_{r-1}^t & u_r^t \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r \end{pmatrix} = \begin{pmatrix} A(0) \\ A(1) \\ A(2) \\ \vdots \\ A(t-1) \\ A(t) \end{pmatrix},$$
$$(50)$$

with $x_k \geqslant 0$. This system of $t+1$ equations and $r+1$ unknowns can be solved with linear programming, which guarantees to either find an analytic expression of a feasible solution or prove that there is no solution.

For fixed order of anticoherence $t = 1, \ldots, 20$, we applied this method systematically for each $N = 1, \ldots, 500$, with all possible numbers of points n_S and n_N at the poles, and for all values of n such that $t + 1 \leqslant n \leqslant N$ and $r = (N - n_S - n_N)/n$ be an integer. For a fixed order of anticoherence t, we found that a positive solution to Eq. (50) exists only for N larger than a certain value N_t. Examples of states with $N = N_t$ are represented in Fig. 3 for $t = 2, \ldots, 7$ through their Husimi function, showing that it becomes more and more uniform as t increases.

Figure 4 displays all pairs (t, N) for which t-anticoherent states of N qubits with C_n symmetry, $n > t$, are found to exist. As mentioned, no t-anticoherent states are found whenever N is smaller than the threshold value N_t. As the figure indicates, the value of N_t goes as t^2; this is to be expected since an N-qubit state has $N + 1$ complex Dicke coefficients, which have to satisfy $(t + 1)^2 - 1$ conditions for t-anticoherence. Above this minimal value N_t, t-anticoherent states can be found for almost all values of N. We stress that the bound
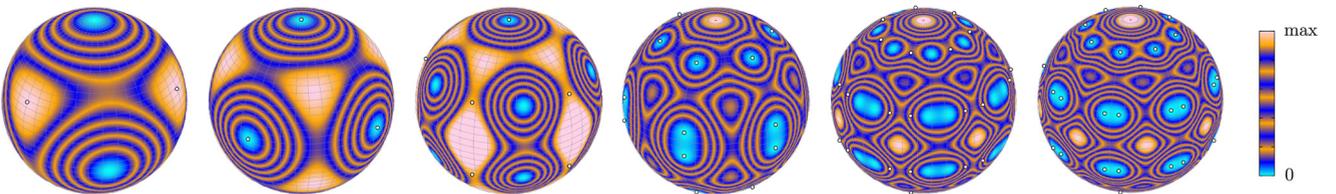

FIG. 3. (Color online) Husimi function of C_n-symmetric states corresponding to the smallest possible number N_t of qubits for order of anticoherence $t = 2, \ldots, 7$, with $n > t$. From left to right: 2-anticoherent state of 4 qubits (tetrahedron, C_3 symmetry), 3-anticoherent state of 6 qubits (octahedron, C_4 symmetry), 4-anticoherent state of 12 qubits (C_5 symmetry), 5-anticoherent state of 24 qubits (C_6 symmetry), 6-anticoherent state of 42 qubits (C_7 symmetry), 7-anticoherent state of 75 qubits (C_8 symmetry).
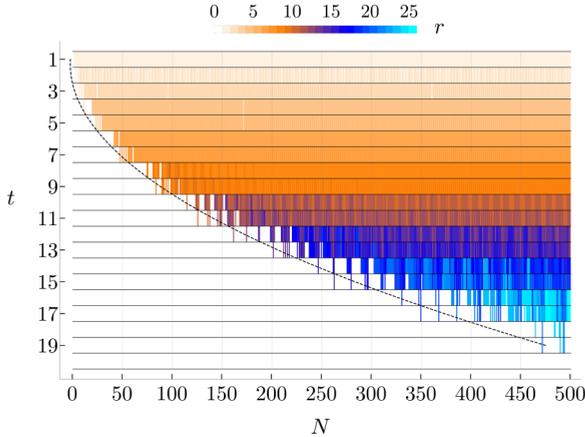
FIG. 4. (Color online) Each bar corresponds to a couple (t,N) for which a C_n-symmetric N-qubit t-anticoherent state with $t < n$ exists. In particular, the minimum number of qubits N_t for which such a state exists is given by the position of the first bar on each line. This value N_t is well fitted by $N_t = t(at + b)$, with $a = 1.52$ and $b = -3.93$ (dashed line). Color (grayness) represents the number $r = (N - n_S - n_N)/n$ of free Dicke coefficients (all others being zero) on which the linear optimisation is realized.

N_t is for states with C_n symmetry such that $n > t$, ensuring that Eqs. (12) for $r \neq 0$ are trivially satisfied. These equations may, however, be satisfied in a nontrivial way when $n \leqslant t$. For instance, for $t = 7$ we found a 42-qubit state with D_{7d} symmetry, displayed in Fig. 5.

This approach enables us to identify infinite families of t-anticoherent states for higher values of t. For instance, a solution to (50) for $t = 5$ and $N = 4(m + 1)$ with integer $m \geqslant 5$ is given by the $D_{m+1,h}$-symmetric states

$$\mathbf{d}_{D_{m+1,h}} = \mathcal{N}(d_0, \mathbf{0}_m, d_{N/4}, \mathbf{0}_m, d_{N/2}, \mathbf{0}_m, d_{3N/4}, \mathbf{0}_m, d_N), \quad (51)$$

with

$$\begin{aligned} d_0 = d_N &= \sqrt{(2 + N)(4 + N)(7N - 4)}, \\ d_{N/4} = d_{3N/4} &= 4\sqrt{2}\sqrt{(N - 2)(N - 1)(N + 2)}, \\ d_{N/2} &= 2\sqrt{3}\sqrt{(N - 1)(N^2 + 16)}. \end{aligned} \quad (52)$$

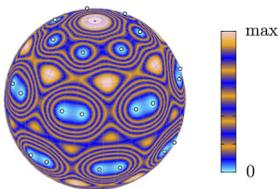We identified similar families for higher values of t.

FIG. 5. (Color online) Husimi function of the 42-qubit 7-anticoherent state $|\psi_S\rangle = (\sqrt{7062}|D_{42}^{(0)}\rangle + \sqrt{29315}|D_{42}^{(7)}\rangle + 3\sqrt{451}|D_{42}^{(14)}\rangle + \sqrt{36777}|D_{42}^{(21)}\rangle - 3\sqrt{451}|D_{42}^{(28)}\rangle + \sqrt{29315}|D_{42}^{(35)}\rangle - \sqrt{7062}|D_{42}^{(42)}\rangle)/343$. This state displays D_{7d} symmetry; hence, $n = t = 7$.

VI. ANTICOHERENCE AT ALL ORDERS

It is, in fact, possible, using the approach presented in this paper, to prove the existence of t-anticoherent states for arbitrary t. As we show here, one can construct t-anticoherent states with at most $t + 1$ nonzero components. Let us consider a quantum state whose Dicke coefficients are zero for $k \neq k_i, 0 \leqslant i \leqslant t$. We set $x_i = |d_{k_i}|^2$, so that the vector of Dicke coefficients takes the form $(0, \ldots, 0, x_0, 0, \ldots, 0, x_1, 0, \ldots, 0, x_t, 0, \ldots, 0)$. The off-diagonal equations (17) are fulfilled as soon as there are t or more zeros between each x_i. Defining the functions $g_q(u) = (\frac{1}{2} - u)^q$, we can rewrite Eq. (16) as

$$\sum_{i=0}^{t} x_i\, g_q\left(\frac{k_i}{N}\right) = \frac{1}{N + 1} \sum_{k=0}^{N} g_q\left(\frac{k}{N}\right). \quad (53)$$

We want to find values of k_i and $x_i > 0$ such that (53) holds, that is, sample points k_i and positive weights x_i such that the sum on the right-hand side, which runs from $k = 0$ to N, can be evaluated by calculating the function g_q at only $t + 1$ points. As the right-hand side of Eq. (53) is a Riemann sum, which converges to the integral of g_q over $[0,1]$ at large N, the problem almost appears like a quadrature problem.

The well-known Gauss-Legendre integration method [41] states that the integral of an arbitrary function g which is continuous over $[-1,1]$ can be approximated by evaluating g at a finite number of points with a certain weight. The quadrature at order t reads

$$\sum_{i=0}^{t} w_i\, g(u_i) \approx \int_{-1}^{1} g(u)du, \quad (54)$$

where the u_i, $0 \leqslant i \leqslant t$, are the $t + 1$ roots of the Legendre polynomial $P_{t+1}(u)$, and weights are given by

$$w_i = \frac{2}{(t + 1)P'_{t+1}(u_i)P_t(u_i)}. \quad (55)$$

The Gauss-Legendre quadrature has the property that the approximation (54) becomes an exact equality when g is a polynomial of degree $\leqslant 2t + 1$. For functions defined over $[0,1]$ and for polynomials g_q defined above, which are of degree less than t, the quadrature can be expressed, after a simple change of variables, as

$$\sum_{i=0}^{t} \frac{w_i}{2}\, g_q\left(\frac{1 + u_i}{2}\right) = \int_0^1 g_q(u)du. \quad (56)$$

As the weights (55) are such that $\sum_{i=0}^{t} w_i = 2$, the weights $w_i/2$ in (56) sum up to 1. The right-hand side of Eq. (56) is exactly the $N \to \infty$ limit of the right-hand side of Eq. (53). Thus, choosing k_i/N as close as possible to the roots $(1 + u_i)/2$ should make it possible to solve (53).

To prove this, let us rewrite the system of equations (53) in matrix form. In order to make the N-dependance clear we denote $X^{(N)} = (x_0, \ldots, x_t)$, $B^{(N)} = [A(0), \ldots, A(t)]/N^q$, and define the matrix

$$M_{qi}^{(N)} = g_q\left(\frac{k_i}{N}\right), \quad 0 \leqslant q, i \leqslant t. \quad (57)$$

Positions k_i are chosen as

$$k_i = \lfloor N(1 + u_i)/2 \rfloor, \quad 0 \leqslant i \leqslant t, \qquad (58)$$

with $\lfloor \cdot \rfloor$ the floor function. Equation (53) reduces to $M^{(N)} X^{(N)} = B^{(N)}$. Similarly, we can set $X^{(\infty)} = (w_0/2, \ldots, w_t/2)$,

$$B^{(\infty)} = \left(\int_0^1 g_0(u)du, \ldots, \int_0^1 g_t(u)du \right), \qquad (59)$$

and

$$M_{qi}^{(\infty)} = g_q \left(\frac{1 + u_i}{2} \right) = \left(-\frac{u_i}{2} \right)^q, \quad 0 \leqslant q, i \leqslant t, \quad (60)$$

so that Eq. (56) for $q = 0, \ldots, t$ can be reexpressed in matrix form as $M^{(\infty)} X^{(\infty)} = B^{(\infty)}$. The matrix $M^{(\infty)}$ is invertible since it is the Vandermonde matrix of the $-u_i/2$, which are all distinct, and thus $X^{(\infty)} = (M^{(\infty)})^{-1} B^{(\infty)}$. Since $k_i/N \to (1 + u_i)/2$ for $N \to \infty$, we have $M^{(N)} \to M^{(\infty)}$, and thus for sufficiently large N, the matrix $M^{(N)}$ is also invertible (note that the size $t + 1$ of the matrix is fixed independently of N). Thus, for k_i given by (58) the system (53) has a unique solution given by $X^{(N)} = (M^{(N)})^{-1} B^{(N)}$. In order to solve the diagonal equations (16), it suffices to show that this solution is such that $x_i > 0$. However, for $N \to \infty$ we have $X^{(N)} = (M^{(N)})^{-1} B^{(N)} \to (M^{(\infty)})^{-1} B^{(\infty)} = X^{(\infty)}$, or equivalently, component by component, $x_i \to w_i/2$. Since the weights (55) of the Gauss-Legendre quadrature are all strictly positive, then for sufficiently large N all x_i will be positive.

The off-diagonal equations (17) are fulfilled as soon as there are t or more zeros between each x_i. Since t, and thus the positions of the zeros of $P_{t+1}(u)$, are fixed, then for sufficiently large N, indices defined by (58) will be such that $\min_i |k_{i+1} - k_i| \geqslant t + 1$. Thus, a t-anticoherent state exists for any t.

Note that Gauss-Legendre quadrature with $t + 1$ points is exact for any polynomial of degree less than or equal to $2t + 1$. Thus, the solution we construct is, in fact, $(2t + 1)$-anticoherent. This means that we do not need to impose $t + 1$ positions: Only half of them would suffice. We can thus look for a solution with D_{nh} symmetry, i.e., such that the k_i verify $k_i = k_{t-i}$ and $x_i = x_{t-i}$. Since the equations for odd q are automatically fulfilled, there remains a set of (for t even) $t/2 + 1$ equations and $t/2 + 1$ variables $x_0, \ldots, x_{t/2}$. The systems then corresponds to a quadrature to order $t/2$, which is exact for polynomials up to degree $t + 1$. It is thus possible to obtain D_{nh}-symmetric t-anticoherent states for any t.

The proof above only states the existence of some N sufficiently large such that a t-anticoherent state exists. We were able to find solutions using $N = t(t + 1)(t + 2)/6$ up to $t = 120$. More specifically, using linear programming, we looked for solutions of the form $(0, \ldots, 0, x_0, 0, \ldots, 0, x_1, 0, \ldots, 0, x_t, 0, \ldots, 0)$ with $x_i = x_{t-i}$ and $k_i = k_{t-i}$, taking k_i as in (58) for $0 \leqslant i \leqslant t/2$. We checked that for $t \geqslant 30$ the conditions $\min_i |k_{i+1} - k_i| \geqslant t + 1$ [and thus the off-diagonal equations (17)] are always fulfilled; for even t up to $t = 120$, linear programming found an analytical solution satisfying all diagonal equations for $q = 0, \ldots, t$.

VII. CONCLUSION

In this paper, the problem of identifying t-anticoherent states of a spin-j system (or equivalently $2j$-qubit permutation-symmetric states with maximally mixed t-qubit reductions in the symmetric subspace) has been tackled from a geometric point of view. We have provided three different characterizations of t-anticoherence. A first one is given in terms of a finite number of expectation values of spin operators and is independent of any basis, a second one in terms of Dicke coefficients taking the form of a set of $t(t + 2)$ real equations, and a third one in terms of spherical multipole moments of the Husimi and Glauber-Sudarshan functions. Using the Majorana representation of a spin-j state in terms of points on the sphere, we analyzed the consequence of a point-group symmetry of the Majorana points on the coefficients of the quantum state, for the seven infinite families of point-group symmetries. Our results are summarized in Table I. A consequence is that states with C_{nh}, S_{2n}, or D_n symmetry are always at least 1-anticoherent, in contrast to C_n-symmetric states which are not necessarily 1-anticoherent. However, we were able to show that any SLOCC class containing C_n-symmetric states does contain a C_n-symmetric anticoherent state (provided the maximal degeneracy of its Majorana points is not too high). Because a SLOCC class contains, at most, one anticoherent state (up to LU), our results provide a clear link between symmetry and SLOCC classes. This approach allowed us to identify all anticoherent states with C_n symmetry for $N = 5$ qubits and to construct families of C_n-symmetric states with higher order of anticoherence. We discussed a method based on linear programming allowing us to obtain N-qubit C_n-symmetric t-anticoherent states with $t < n$, when they exist. Finally, we constructed explicitly t-anticoherent states with arbitrary order t, proving that such states exist at all order. Note that our results about anticoherence of spin states are of interest in the context of quantum information as the states we identified are maximally entanglement symmetric states with respect to the entanglement entropy.

APPENDIX: CONSEQUENCES OF SYMMETRIES OF THE SET OF MAJORANA POINTS ON THE DICKE COEFFICIENTS

Reflection σ_h

The reflection σ_h transforms z into $1/z^*$ and exchanges the north and south poles, so that a configuration of points invariant under σ_h must be such that $n_N = n_S$ and

$$\prod_{k=1}^{N-n_N-n_S} (z - z_k) = \prod_{k=1}^{N-2n_N} \left(z - \frac{1}{z_k^*} \right). \qquad (A1)$$

We have

$$\prod_{k=1}^{N-2n_N} \left(z - \frac{1}{z_k^*} \right) = \frac{(-z)^{N-2n_N}}{\prod_{k=1}^{N-2n_N} z_k^*} \prod_{k=1}^{N-2n_N} \left(\frac{1}{z} - z_k^* \right). \qquad (A2)$$

The coefficient of the lowest-order term in z in Eqs. (26) and (27) yields (for $n_S = n_N$) the identity

$$\prod_{k=1}^{N-2n_N} z_k = \frac{d_{n_N}}{d_{N-n_N}}, \tag{A3}$$

so that Eq. (A1) can be rewritten as

$$\frac{(-1)^N}{d_{n_N}^*} z^N P^*\left(\frac{1}{z}\right) = \frac{1}{d_{N-n_N}} P(z) \tag{A4}$$

(here P^* is the polynomial whose coefficients are the complex conjugates of those of P). Identifying the coefficients of z^k in the expansion yields that for all k with $n_N \leqslant k \leqslant N - n_N$

$$d_{N-k} = \frac{d_{n_N}}{d_{N-n_N}^*} d_k^*. \tag{A5}$$

In particular, taking $k = n_N$ yields $|d_{N-n_N}|^2 = |d_{n_N}|^2$. If we let $d_{N-n_N}/d_{n_N}^* = \exp(i\xi)$, then $d_{N-k} = e^{i\xi} d_k^*$. The condition of invariance under σ_h is thus that $n_N = n_S$ and that there exists $\xi \in \mathbb{R}$ such that $d_{N-k} = e^{i\xi} d_k^* \ \forall \, k$.

a. Reflection σ_v

The reflection σ_v transforms z into z^*. A configuration of points invariant under σ_v must be such that

$$\prod_{k=1}^{N-n_N-n_S} (z - z_k) = \prod_{k=1}^{N-n_N-n_S} (z - z_k^*). \tag{A6}$$

Following the same lines as above, this yields the condition

$$\frac{1}{d_{N-n_N}^*} P^*(z) = \frac{1}{d_{N-n_N}} P(z), \tag{A7}$$

which by identification of the coefficients gives that there exists $\xi \in \mathbb{R}$ such that for all k with $n_S \leqslant k \leqslant N - n_N$ one has $d_k^* = \exp(i\xi) d_k$. In particular, this implies (iterating twice the condition) that $\xi = 0$ or π, independently of k. One can further remove the overall sign of the d_k, leading to the conditions given in Table I.

b. Reflection σ_d

The reflection σ_d transforms z into $z^* \exp(-i\pi/n)$. Equation (29) becomes

$$P(z) = e^{-\frac{i\pi}{n}(N-n_N)} \frac{d_{N-n_N}}{d_{N-n_N}^*} P^*\left(z e^{\frac{i\pi}{n}}\right) \tag{A8}$$

so that

$$d_k = e^{-\frac{i\pi}{n}(N-n_N-k)} \frac{d_{N-n_N}}{d_{N-n_N}^*} d_k^* \tag{A9}$$

for all k with $n_S \leqslant k \leqslant N - n_N$. If we let $e^{-\frac{i\pi}{n}(N-n_N)} d_{N-n_N}/d_{N-n_N}^* = e^{i\xi}$, then Eq. (A9) yields

$$d_k = e^{i\xi} e^{\frac{ik\pi}{n}} d_k^*. \tag{A10}$$

This imposes for the phase θ_k of d_k that $\theta_k = \xi/2 + k\pi/n + q_k\pi$ for some integer q_k. Removing the overall constant phase, the d_k can be taken as

$$d_k = \pm |d_k| e^{\frac{ik\pi}{2n}}. \tag{A11}$$

c. Rotation $\sigma_h\sigma_v$

This is a rotation of π around the x axis. It transforms z into $1/z$. Invariance under $\sigma_h\sigma_v$ implies that $n_N = n_S$ and

$$\prod_{k=1}^{N-n_N-n_S} (z - z_k) = \prod_{k=1}^{N-2n_N} \left(z - \frac{1}{z_k}\right). \tag{A12}$$

Following the same steps as for σ_h, we get the condition

$$d_{N-k} = \frac{d_{n_N}}{d_{N-n_N}} d_k \tag{A13}$$

for all k with $n_N \leqslant k \leqslant N - n_N$. Taking this expression at $k = n_N$ we have $d_{n_N}^2 = d_{N-n_N}^2$, so that $d_{n_N}/d_{N-n_N} = \pm 1$, leading to the conditions given in Table I.

d. Transformation $s_n = r_{2n}\sigma_h$

Such an operation transforms z into $\frac{1}{z^*} \exp(-i\pi/n)$. A configuration of points invariant under s_n must be such that $n_N = n_S$ and

$$\prod_{k=1}^{N-n_N-n_S} (z - z_k) = \prod_{k=1}^{N-n_N-n_S} \left(z - \frac{1}{z_k^*} e^{-i\pi/n}\right). \tag{A14}$$

The identity that corresponds to this case can be derived as above and gives, for all k, $n_N \leqslant k \leqslant N - n_N$,

$$d_{N-k} = \frac{d_{n_N}}{d_{N-n_N}^*} d_k^* e^{i\pi(N-k-n_N)/n}. \tag{A15}$$

Applying this condition to $k = n_N$, we get that $N - 2n_N = 0$ mod n. Iterating twice condition (A15) we get that $d_k = 0$ unless $k = n_N$ mod n, which is the condition of invariance under r_n. This is to be expected as one has precisely $s_n^2 = r_n$. If $k = n_N + qn$ for some integer n, and if we let $d_{N-n_N}/d_{n_N}^* = \exp(i\xi)$, then (A15) becomes $d_{N-k} = (-1)^q \exp(i\xi) d_k^*$.

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