Conservation relation of nonclassicality and entanglement for Gaussian states in a beam splitter

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We study the relation between single-mode nonclassicality and two-mode entanglement in a beam splitter. We show that single-mode nonclassicality (the entanglement potential) of incident light cannot be transformed into two-mode entanglement completely after a single beam splitter. Some of the entanglement potential remains as single-mode nonclassicality in the two entangled output modes. Two-mode entanglement generated in the process can be equivalently quantified as an increase in the minimum uncertainty widths (or decrease in the squeezing) of the output states compared to the input states. We use the nonclassical depth and logarithmic negativity as single-mode nonclassicality and entanglement measures, respectively. We realize that a conservation relation between the two quantities can be adopted for Gaussian states, if one works in terms of uncertainty width. This conservation relation is extended to many sets of beam splitters.

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I. INTRODUCTION

Quantum entanglement is an essential source for quantum information processing [1]. Entanglement between two-mode Gaussian states [2,3] is of considerable interest because of its availability and controllability in experiments, and its applications such as quantum teleportation [4] and dense coding [5].

The entanglement of two-mode Gaussian states can be generated in an experiment via a nonlinear optical device, such as a parametric down converter [6]. On the other hand, a beam splitter (BS) as a linear passive device has been studied extensively to generate quantum entanglement [7–16]. In particular, Kim *et al.* [10] studied the properties of different input states, such as squeezed states, in order to have the output fields be entangled. They conjectured that the nonclassicality of input fields is a necessary condition for entangling the output via a BS, which was proved by Wang [17]. Wolf *et al.* [11] proved a necessary and sufficient condition for entangling bipartite Gaussian states with passive optical devices and found a close relation between input squeezing and output entanglement of the Gaussian state. Furthermore, Tahira *et al.* [13] investigated the generation of Gaussian entanglement from a single-mode squeezed state mixed with a thermal state at a BS, where detailed experimental conditions are analyzed.

Recently, a new measure for nonclassicality, the entanglement potential, was introduced by Asbóth *et al.* [18] based on its inherent relation to two-mode entanglement. The entanglement potential is the maximum amount of two-mode entanglement extractable from a single-mode nonclassical state using linear optical devices. More recently, Vogel and Sperling [19] arrived at a more intimate connection between nonclassicality and two-mode entanglement. The rank of two-mode entanglement that a nonclassical state can generate is equal to the number of terms needed in the coherent-state expansion of this nonclassical state. It was also pointed out in Ref. [20] that such a connection can exist in many-particle entanglement. With these results, an interesting question arises: Is there a way to quantify single-mode nonclassicality and two-mode entanglement so that the summation of the two quantities is conserved via linear passive devices, such as a BS?

In this paper, we address this question for arbitrary two-mode Gaussian states. First, we show that not all of the nonclassicality, which is present in an input single-mode state, is converted to two-mode entanglement, even for an optimum BS mixing angle (see Fig. 2). We calculate the remaining single-mode nonclassicalities by partially tracing the two output modes. This wipes out two-mode entanglement and enables the calculation of single-mode nonclassical depths.

Second, we realize an interesting relation between the generated two-mode entanglement and the change in the logarithm of the initial and final nonclassical depths (or uncertainty widths). The change in the logarithm of the initial and final uncertainity widths [see Eq. (11)] displays the same behavior with the logarithmic negativity measure [21–23] for the generated two-mode entanglement (Fig. 4). We use this observation for treating nonclassicality and entanglement on an equal footing, relying on the soul of the entanglement potential [18,19]. Hence, a conservation relation for the sum of single-mode nonclassicality and generated two-mode entanglement can be deduced.

This paper is organized as follows. In Sec. II, we start with the basic theory of input and output states at a BS. In Sec. II B, we introduce the definition for the entanglement depth. In Sec. III, we show that nonclassicality in an input single-mode state cannot be extracted completely using only a single beam splitter. When two-mode entanglement is wiped out via a partial trace operation, the nonclassical depths of the output modes do not vanish. In Sec. IV, we outline the relation between the nonclassical depth and the uncertainty width of a single-mode state. We define the generated two-mode entanglement as the difference between the input and output nonclassicalities in a BS. We show that this definition is

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qualitatively equivalent to the logarithmic negativity measure of entanglement. It enables us to convert the logarithmic negativity for the two-mode entanglement to single-mode nonclassicality and vice versa. A conservation relation of nonclassicality and entanglement is derived at a BS. We illustrate this relation with examples and extend the relation to many sets of BSs. A summary of this paper is given in Sec. V. In the Appendixes, detailed derivations are provided.

II. INPUT AND OUTPUT GAUSSIAN STATES OF A BS

A. Input-output relation

We consider a lossless BS with two single-mode Gaussian fields as the input. The complex amplitudes β_1,β_2 of the output fields are related to those α_1,α_2 of the input fields as [24,25]

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = M\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \tag{1}$$

where

$$M = \begin{pmatrix}\cos\theta & \sin\theta e^{i\varphi}\\ -\sin\theta e^{-i\varphi} & \cos\theta\end{pmatrix} \tag{2}$$

is the beam-splitter transformation matrix with transmittance $\cos^2\theta$ and phase difference φ between the reflected and the transmitted fields.

For a single-mode Gaussian state, the characteristic function of the state is given by

$$\chi(\alpha_i,\alpha_i^*) = \exp\left(-\tfrac{1}{2}\boldsymbol{x}_i^\dagger V_i \boldsymbol{x}_i\right), \tag{3}$$

where $\boldsymbol{x}_i^\dagger = (\alpha_i^*,\alpha_i)$, and V_i is the covariance matrix of the single-mode state ($i = 1,2$). For two separable single-mode Gaussian states, the characteristic function of the states is given by

$$\begin{aligned}\chi_{\text{in}}(\alpha_1,\alpha_1^*,\alpha_2,\alpha_2^*) &= \chi(\alpha_1,\alpha_1^*)\chi(\alpha_2,\alpha_2^*)\\ &= \exp\left(-\tfrac{1}{2}\boldsymbol{y}^\dagger V_{\text{in}}\boldsymbol{y}\right),\end{aligned} \tag{4}$$

where $\boldsymbol{y}^\dagger = (\alpha_1^*,\alpha_1,\alpha_2^*,\alpha_2)$ and $V_{\text{in}} = \left(\begin{smallmatrix}V_1 & 0\\ 0 & V_2\end{smallmatrix}\right)$ is the input covariance matrix. For a general input two-mode Gaussian, $V_1 = \left(\begin{smallmatrix}a & b\\ b^* & a\end{smallmatrix}\right)$ and $V_2 = \left(\begin{smallmatrix}c & d\\ d^* & c\end{smallmatrix}\right)$. A physical quantum system implies $a^2 \geqslant |b|^2 + \frac{1}{4}$, $c^2 \geqslant |d|^2 + \frac{1}{4}$ from the uncertainty principle [26].

By expressing the characteristic function in terms of the output complex amplitudes β_1,β_2 using the transformation Eqs. (1), (2), and (4), the output covariance matrix is given by a unitary transformation of V_{in} as

$$V_{\text{out}} = U^\dagger(\theta,\varphi)V_{\text{in}}U(\theta,\varphi) = \begin{pmatrix}A & C\\ C^\dagger & B\end{pmatrix}, \tag{5}$$

where $U(\theta,\varphi)$ is related to M and it is given by

$$U(\theta,\varphi) = \begin{pmatrix}\cos\theta & 0 & -\sin\theta e^{i\varphi} & 0\\ 0 & \cos\theta & 0 & -\sin\theta e^{-i\varphi}\\ \sin\theta e^{-i\varphi} & 0 & \cos\theta & 0\\ 0 & \sin\theta e^{i\varphi} & 0 & \cos\theta\end{pmatrix}. \tag{6}$$

The expressions of matrices A, B, and C are given in Appendix A.

B. Single-mode nonclassicality

For any quantum state ρ, it can be represented in the Glauber-Sudarshan representation as

$$\rho = \int P(\alpha,\alpha^*)\,|\alpha\rangle\,\langle\alpha|\,d^2\alpha, \tag{7}$$

where $|\alpha\rangle$ is a coherent state and $P(\alpha,\alpha^*)$ is the Glauber-Sudarshan P function defined as

$$P(\alpha,\alpha^*) = \frac{1}{\pi^2}\int e^{\frac{1}{2}|\beta|^2 - i\beta\alpha^* - i\beta^*\alpha}\chi(\alpha,\alpha^*)d^2\beta. \tag{8}$$

If the P function of a quantum state is positive-definite, then the state is defined as a classical state. Otherwise, it is nonclassical. There are many nonclassicality quantifications proposed for a single-mode state [18,27–30]. We first consider the nonclassicality depth [28]. For a non-positive-definite P function, a convolution of the P function

$$R(\tau,\eta,\eta^*) = \frac{1}{\pi\tau}\int e^{-1/\tau|\alpha-\eta|^2}P(\alpha,\alpha^*)d^2\alpha \tag{9}$$

may become a positive-definite function as a classical probability distribution. For a given P function, the minimum value of τ such that R function becomes positive-definite is defined as the nonclassical depth. It is a measure of nonclassicality ranging between 0 and 1. Particularly, the nonclassicality depth is a continuous measure for a Gaussian state in the range of $[0,\frac{1}{2}]$. For the covariance matrix V_1, the nonclassical depth is given by

$$\tau = \max\{0, 1/2 - \lambda_{1\min}\}, \tag{10}$$

where $\lambda_{1\min} = a - |b|$ is the minimum eigenvalue of V_1. Therefore, for any single-mode nonclassical Gaussian state, $\tau > 0$ or $a - |b| < \frac{1}{2}$.

For a single-mode Gaussian state, the degree of squeezing can also be used as a quantification of nonclassicality [11]. If a Gaussian state is squeezed, the minimum uncertainty of its phase-space quadratures, which equals the minimum eigenvalue $\lambda_{\min}$ of its covariance matrix, is smaller than $\frac{1}{2}$ [31]. Therefore, we consider the quantity $N_{\text{noncl}} = -\log_2(2\lambda_{\min})$ as the nonclassicality of a single-mode Gaussian state. For a coherent state, $N_{\text{noncl}} = 0$. For a pure squeezed state with a squeezing parameter r [31], we find $N_{\text{noncl}} = 2r$. For a thermal state with an average thermal photon number n, $N_{\text{noncl}} = -\log_2(2n+1) < 0$.

III. COMPLETELY EXTRACTING THE NONCLASSICALITY

A. Calculation of the remaining nonclassicality

The operation of the beam splitter transforms the density matrix as $\rho_{12} = \mathcal{M}_{\text{BS}}\rho_1^{(\text{in})}\otimes\rho_2^{(\text{in})}\mathcal{M}_{\text{BS}}^\dagger$, where $\mathcal{M}_{\text{BS}}$ is the BS operator and $\rho_{1,2}^{(\text{in})}$ are the density matrices for the input states of the BS. In order to study the nonclassicality of each output mode, we define $\rho_1 = \text{Tr}_2(\rho_{12})$ and $\rho_2 = \text{Tr}_1(\rho_{12})$ for each of the output modes from the output state ρ_{12}. We further define a separable output system as $\tilde{\rho}_{12} = \text{Tr}_2(\rho_{12})\otimes\text{Tr}_1(\rho_{12})$

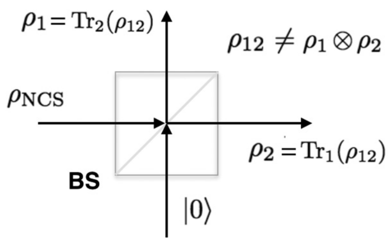

FIG. 1. A nonclassical Gaussian state ρ_{NCS} is mixed with a vacuum state at a BS, generating an output state ρ_{12}. Each output mode of the bipartite state after the BS is given by $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ and $\rho_2 = \mathrm{Tr}_1(\rho_{12})$, respectively.

(see Fig. 1), where two-mode entanglement is wiped out [32]. We show that the covariance matrix of $\tilde{\rho}_{12}$ after the tracing operation on the output state is $\tilde{V}_{\mathrm{out}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, as should be expected. Covariance matrices for single-mode states (A,B) are unaffected. The derivation is provided in Appendix A.

After this partial trace operation, we calculate the remaining nonclassical depth in a separable two-mode system, namely, $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ and $\rho_2 = \mathrm{Tr}_1(\rho_{12})$. We use the definition introduced in Refs. [12,33] to calculate the nonclassical depth for a two-mode Gaussian system. In Fig. 2, we plot the two-mode entanglement generated at the output of the BS and the remaining nonclassicality for different BS mixing angles. The nonclassicality is converted to two-mode entanglement in the BS, and weaker nonclassicality remains as the strength of the two-mode entanglement increases.

B. Depleting all nonclassicality

We observe that not all of the nonclassicality is transformed to two-mode entanglement in Fig. 2. Nonclassicality remains in the two output modes. One naturally raises the question if we can transform all of the nonclassicality into two-mode entanglement. For this reason, we put the separable state $\tilde{\rho}_{12} = \mathrm{Tr}_2(\rho_{12}) \otimes \mathrm{Tr}_1(\rho_{12})$—after recording and wiping out the generated entanglement—into the following BSs [32] to extract more of the entanglement potential. Since the two states are separable, i.e., $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ and $\rho_2 = \mathrm{Tr}_1(\rho_{12})$,

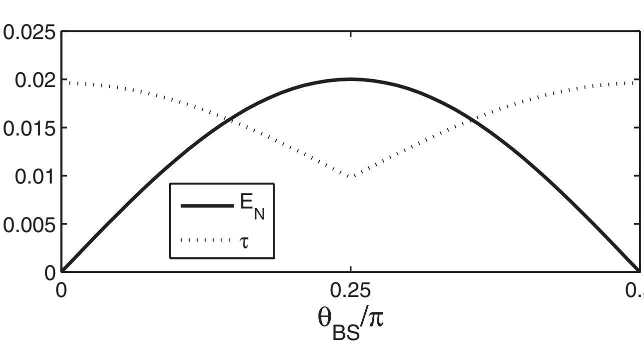

FIG. 2. Nonclassicality of the input single-mode state transforms into two-mode entanglement in a beam splitter. When the amount of extracted two-mode entanglement (E_N) increases, the nonclassicality in the output states (τ) decreases. Not all of the nonclassicality could be converted to two-mode entanglement, even for an optimum BS mixing angle.

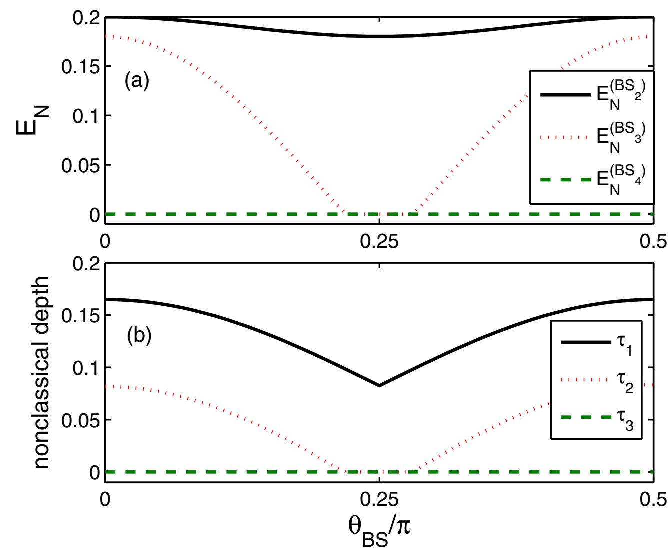

FIG. 3. (Color online) The two-mode output state of the first BS is partially traced and put in to a second BS, $\tilde{\rho}_{12}^{(\mathrm{BS}_2)} = \mathrm{Tr}_2(\rho_{12}^{(\mathrm{BS}_2)}) \otimes \mathrm{Tr}_1(\rho_{12}^{(\mathrm{BS}_2)})$, in order to extract the remaining nonclassicality as entanglement. We repeat the same procedure for four BSs. The nonclassical depth before a BS, τ_{i-1}, displays parallel behavior with the two-mode entanglement extracted from this BS, $E_N^{(\mathrm{BS}_i)}$. Nonclassicality is depleted at the fourth BS.

we mix them in the second BS as $\rho_{12}^{(\mathrm{BS}_2)} = \mathcal{M}_{\mathrm{BS}_2} \tilde{\rho}_{12} \mathcal{M}_{\mathrm{BS}_2}^{\dagger}$. In Fig. 3(a), we plot the extracted two-mode entanglement after the second BS, $E_N^{(\mathrm{BS}_2)}$. We perform a partial trace operation again, $\tilde{\rho}_{12}^{(\mathrm{BS}_2)} = \mathrm{Tr}_2(\rho_{12}^{(\mathrm{BS}_2)}) \otimes \mathrm{Tr}_1(\rho_{12}^{(\mathrm{BS}_2)})$, to obtain the remaining nonclassicality after the second BS, τ_2, as plotted in Fig. 3(b). We perform a similar procedure for two more BSs. No nonclassicality remains after the third BS, $\tau_3 = 0$. This is also confirmed by placing a fourth BS where no two-mode entanglement can be extracted, $E_N^{(\mathrm{BS}_4)} = 0$. Comparing Figs. 3(a) and 3(b), we observe that the nonclassical depth before a BS, τ_{i-1}, has parallel behavior with the two-mode entanglement extracted from this BS, $E_N^{(\mathrm{BS}_i)}$.

Even though we calculated the nonclassical depth of a two-mode state using the definition of Refs. [12,33] for a qualitative (behavior) comparison, in the preceding sections we arrive at a more useful definition for quantitative purposes.

IV. CONSERVATION RELATION OF SINGLE-MODE NONCLASSICALITY AND TWO-MODE ENTANGLEMENT

We consider, in general, two single-mode Gaussian states mixed at a BS. The nonclassicalities of the input modes are $N_{\mathrm{noncl}}^{\mathrm{in1}} = -\log_2(2\lambda_{1\mathrm{min}})$ and $N_{\mathrm{noncl}}^{\mathrm{in2}} = -\log_2(2\lambda_{2\mathrm{min}})$, where $\lambda_{\mathrm{min}i}$ is the minimum eigenvalue of V_i. After the BS, the nonclassicalities of the output modes are $N_{\mathrm{noncl}}^{\mathrm{out1}} = -\log_2(2\tilde{\lambda}_{1\mathrm{min}})$ and $N_{\mathrm{noncl}}^{\mathrm{out2}} = -\log_2(2\tilde{\lambda}_{2\mathrm{min}})$, where $\tilde{\lambda}_{1\mathrm{min}}$ ($\tilde{\lambda}_{2\mathrm{min}}$) is the minimum eigenvalue of matrix A (B).

After the BS, two-mode entanglement can be generated from input nonclassical single-mode states. Since a BS is a linear device that does not create extra nonclassicality, therefore we quantify the difference between input nonclassicality and output nonclassicality as the degree of two-mode entanglement generated via a BS. This quantity is denoted as

$$S_{\mathcal{N}} = N_{\rm noncl}^{\rm in1} + N_{\rm noncl}^{\rm in2} - N_{\rm noncl}^{\rm out1} - N_{\rm noncl}^{\rm out2} = \log_2 \frac{\tilde{\lambda}_{1\rm min}\tilde{\lambda}_{2\rm min}}{\lambda_{1\rm min}\lambda_{2\rm min}}. \quad (11)$$

We show in the following with several examples that this quantification of entanglement is equivalent to the logarithmic negativity [21]. Then we generalize this relation to a class of Gaussian states mixed at a BS.

A. A single-mode nonclassical state mixing with a vacuum

1. Conservation relation of nonclassicality depth in a BS

We first consider a simple case when a single-mode nonclassical Gaussian state is mixed with a vacuum at a BS. The covariance matrix of the two single-mode input state is given by $V_{\rm in} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, where $V_1 = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}$ and $V_2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. Here, a is real and b is complex, in general. The eigenvalues of the matrix V_1 are $\lambda_{1\rm min} = a - |b|$, $\lambda_{1\rm max} = a + |b|$, and $u \equiv \frac{1}{2\sqrt{\lambda_{1\rm min}\lambda_{1\rm max}}}$ is the purity of the state [34]. The eigenvalues of V_2 are $\lambda_{2\rm min} = \lambda_{2\rm max} = \frac{1}{2}$.

We show an interesting equation of nonclassical depth before and after the BS. Before the BS, the single-mode nonclassical state has a nonclassical depth τ. After mixing the nonclassical state with a vacuum at the BS, the nonclassical depth is calculated from matrices A and B for each output mode. The corresponding nonclassicality depths are given by $\tilde{\tau}_1 = \tau\cos^2\theta$ and $\tilde{\tau}_2 = \tau\sin^2\theta$. Therefore, we obtain

$$\tau = \tilde{\tau}_1 + \tilde{\tau}_2, \quad (12)$$

which is equivalent to

$$\lambda_{1\rm min} + \lambda_{2\rm min} = \tilde{\lambda}_{1\rm min} + \tilde{\lambda}_{2\rm min}, \quad (13)$$

where $\tilde{\lambda}_{1\rm min}$ and $\tilde{\lambda}_{2\rm min}$ are minimum eigenvalues of A and B, respectively. The conservation relation of nonclassicality depth holds for any nonclassical Gaussian state mixed with a vacuum state. Another interesting question arises: If the nonclassicality is conserved before and after the BS in such a way, where does the entanglement come from after the BS?

2. Conservation relation of nonclassicality and entanglement

In the following, we show that the quantification of two-mode entanglement in Eq. (11) is equivalent to the logarithmic negativity, and therefore the conservation relation of single-mode nonclassicality and two-mode entanglement can be obtained.

A number of separability conditions [21,26,35–37] have been proposed to test the entanglement of a bipartite system based on partial transposition [38,39]. For a two-mode Gaussian state, there are necessary and sufficient conditions [21,26,35] which can be used as measures for two-mode entanglement. Here, we consider the logarithmic negativity defined in Ref. [21]. For the output covariance matrix $V_{\rm out}$, the logarithmic negativity is given by [21]

$$E_{\mathcal{N}} = \max\left\{0, -\tfrac{1}{2}\log_2\left(S - \sqrt{S^2 - 16\,{\rm Det}[V_{\rm out}]}\right)\right\}, \quad (14)$$

where $S = 2({\rm Det}[A] + {\rm Det}[B] - 2\,{\rm Det}[C])$ and ${\rm Det}[V_{\rm out}] = {\rm Det}[V_{\rm in}] = \frac{1}{4}\lambda_{1\rm min}\lambda_{1\rm max} = \frac{1}{16u^2}$. Then we obtain the expression S in terms of the nonclassical depth τ, the purity u, and the BS angle θ as

$$S = (1-\tau)\left(\frac{1}{2u^2(1-2\tau)} + \frac{1}{2}\right) - \tau\left(\frac{1}{2u^2(1-2\tau)} - \frac{1}{2}\right)\cos(4\theta). \quad (15)$$

From Eq. (14), we find that the condition for the two-mode Gaussian state to be entangled is

$$S > \tfrac{1}{2} + 8\,{\rm Det}[V_{\rm out}]. \quad (16)$$

By rearranging the expression of S, we obtain an equivalent condition of Eq. (16) as

$$S - \left(\frac{1}{2} + 8\,{\rm Det}[V_{\rm out}]\right) = \mathcal{C}\left(\frac{(1-2\tilde{\tau}_1)(1-2\tilde{\tau}_2)}{1-2\tau} - 1\right) > 0, \quad (17)$$

where $\mathcal{C} = \frac{1}{u^2\tau} - \frac{1-2\tau}{\tau} \geqslant 2$. Therefore, we find that the quantification of two-mode entanglement in Eq. (11) is

$$S_{\mathcal{N}} = \log_2 \frac{(1-2\tilde{\tau}_1)(1-2\tilde{\tau}_2)}{1-2\tau} = \log_2\left[\frac{S - \left(\frac{1}{2} + 8\,{\rm Det}[V_{\rm out}]\right)}{\mathcal{C}} + 1\right]. \quad (18)$$

As can be seen from the above expression, $S_{\mathcal{N}} > 0$ is a necessary and sufficient condition for a two-mode Gaussian entanglement to exist, which is equivalent to the condition of logarithmic negativity. When $S \leqslant \frac{1}{2} + 8\,{\rm Det}[V_{\rm out}]$, $S_{\mathcal{N}} \leqslant 0$, which gives us a quantitative measure of how far the system is away from entanglement.

To see the validity of $S_{\mathcal{N}}$ numerically, we plot the degree of entanglement using both $S_{\mathcal{N}}$ and $E_{\mathcal{N}}$ in Fig. 4 by either varying the transmittance or the single-mode nonclassicality. We observe similar qualitative trends for both measures. We also observe that $S_{\mathcal{N}}$ is independent of the purity u of the nonclassical state, while $E_{\mathcal{N}}$ increases with u except for $\theta = 0,\pi/4,\pi/2$.

Next, we rewrite the relation Eq. (11) to obtain a conservation relation between the initial nonclassicality before the BS and the sum of the degree of entanglement and the remaining nonclassicality after BS as

$$N_{\rm noncl}^{\rm in1} + N_{\rm noncl}^{\rm in2} = N_{\rm noncl}^{\rm out1} + N_{\rm noncl}^{\rm out2} + S_{\mathcal{N}}. \quad (19)$$

This is the main result of our paper. The initial single-mode nonclassicality is equal to the sum of the output single-mode nonclassicality and output two-mode entanglement generated via the BS.

We plot the curves of $N_{\rm noncl}^{\rm in} = N_{\rm noncl}^{\rm in1} + N_{\rm noncl}^{\rm in2}$, $N_{\rm noncl}^{\rm out} = N_{\rm noncl}^{\rm out1} + N_{\rm noncl}^{\rm out2}$, and $S_{\mathcal{N}}$ in Fig. 5 to show this conservation relation using an example of a nonclassical state with $\lambda_{1\rm min} = 0.335$.

B. A single-mode nonclassical Gaussian state mixing with a thermal state

Next, we consider a pure nonclassical state with covariance matrix $V_1 = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}$ mixing with a thermal state with a

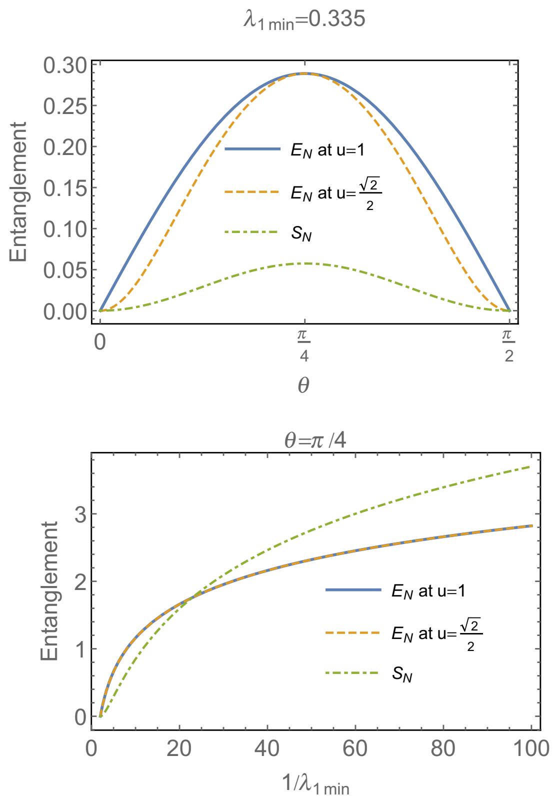

FIG. 4. (Color online) Output entanglement from mixing a nonclassical state with a vacuum at a BS quantified by two measures of the degree of entanglement. (a) For constant nonclassicality, the relation of the degree of entanglement vs θ for different purities of the input nonclassical state. (b) At an optimal BS angle, the monotonic relation of the degree of entanglement vs the inverse of the minimum eigenvalue.

covariance matrix $V_2 = \begin{pmatrix} n+\frac{1}{2} & 0 \\ 0 & n+\frac{1}{2} \end{pmatrix}$, where n is the average number of thermal photons. The input matrix is given by $V_{\rm in} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$. Here, the eigenvalues of V_1 satisfy $\lambda_{1\min}\lambda_{1\max} = a^2 - |b|^2 = \frac{1}{4}$ and the eigenvalues of V_2 are $\lambda_{2\min} = \lambda_{2\max} = n + \frac{1}{2}$. After the BS unitary transformation $U(\theta,\varphi)$, we obtain

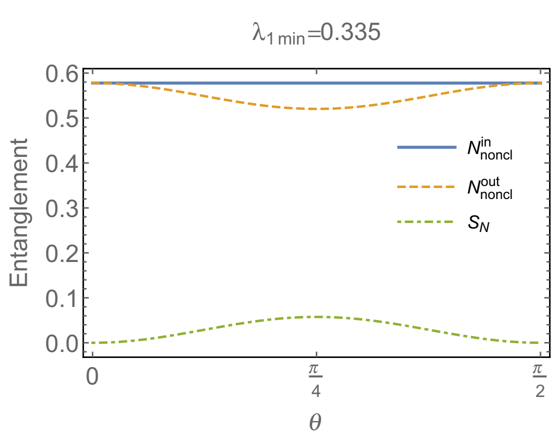

FIG. 5. (Color online) $N^{\rm in}_{\rm noncl}, N^{\rm out}_{\rm noncl}$ and $S_{\mathcal{N}}$ vs the BS angle θ for a nonclassical state mixing with a vacuum state.

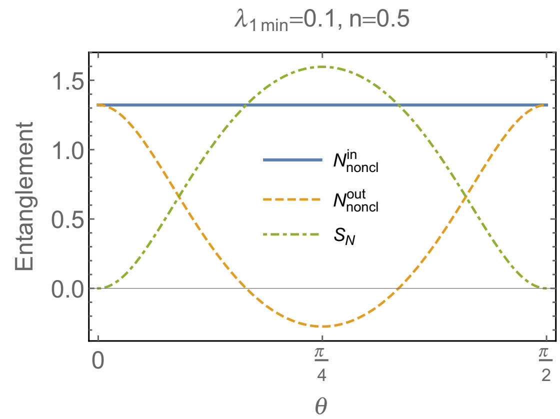

FIG. 6. (Color online) $N^{\rm in}_{\rm noncl}, N^{\rm out}_{\rm noncl}$ and $S_{\mathcal{N}}$ vs the BS angle θ for a pure nonclassical state mixing with a thermal state.

the expression of S as

$$S = \left(\lambda_{1\min} + n + \frac{1}{2}\right)\left(\frac{1}{4\lambda_{1\min}} + n + \frac{1}{2}\right) + \left(\lambda_{1\min} - n - \frac{1}{2}\right)\left(\frac{1}{4\lambda_{1\min}} - n - \frac{1}{2}\right)\cos(4\theta). \quad (20)$$

The determinant of the output matrix, ${\rm Det}[V_{\rm out}] = {\rm Det}[V_{\rm in}] = \frac{1}{4}(n+\frac{1}{2})^2$. We find the same expression as in the previous case,

$$S_{\mathcal{N}} = \log_2 \frac{\tilde{\lambda}_{1\min}\tilde{\lambda}_{2\min}}{\lambda_{1\min}\lambda_{2\min}} = \log_2\left[\frac{S - \left(\frac{1}{2} + 8\,{\rm Det}[V_{\rm out}]\right)}{\mathcal{C}} + 1\right], \quad (21)$$

where in this case $\mathcal{C} = 2(2n+1)\frac{1-2\lambda_{1\min}(2n+1)}{2n+1-2\lambda_{1\min}}$. In order to have $\mathcal{C} > 0$, we require $2\lambda_{1\min}(2n+1) < 1$, which is the condition for entanglement to appear, as discussed in Refs. [11,13].

For the input nonclassical state, $N^{\rm in1}_{\rm noncl} = \log_2(\frac{1}{2\lambda_{1\min}}) > 0$. For the input thermal state, $N^{\rm in2}_{\rm noncl} = -\log_2(1+2n) < 0$, which means antisqueezing and it is less nonclassical than a coherent state. Therefore, the total nonclassicality of the input states is $N^{\rm in}_{\rm noncl} = N^{\rm in1}_{\rm noncl} + N^{\rm in1}_{\rm noncl} = -\log_2[2\lambda_{1\min}(2n+1)] > 0$. The remaining nonclassicality in the output states is $N^{{\rm out}i}_{\rm noncl} = \log_2(\frac{1}{2\tilde{\lambda}_{i\min}})$, with $\tilde{\lambda}_{1\min(2\min)} = \lambda_{1\min(2\min)}\cos^2\theta + \lambda_{2\min(1\min)}\sin^2\theta$.

With the definitions of $S_{\mathcal{N}}$ and $N_{\rm noncl}$, we obtain the same conservation relation as in Eq. (19) as

$$N^{\rm in1}_{\rm noncl} + N^{\rm in2}_{\rm noncl} = N^{\rm out1}_{\rm noncl} + N^{\rm out2}_{\rm noncl} + S_{\mathcal{N}}. \quad (22)$$

We observe that, for optimal transfer of entanglement at the output, $\theta = \pi/4$ [11,13], $N^{\rm out1}_{\rm noncl}, N^{\rm out2}_{\rm noncl} < 0$ for $n \geqslant \frac{1}{2}$ since $\tilde{\lambda}_{1\min(2\min)} = \frac{n}{2} + \frac{1}{4} + \frac{\lambda_{1\min}}{2} > \frac{1}{2}$. Therefore, the output separable system $\tilde{\rho}_{12}$ has negative nonclassicality, which means they are antisqueezed.

We plot $N^{\rm in}_{\rm noncl}, N^{\rm out}_{\rm noncl}$ and $S_{\mathcal{N}}$ in Fig. 6, and the relation $N^{\rm out}_{\rm noncl} + S_{\mathcal{N}} = N^{\rm in}_{\rm noncl}$ can be seen quantitatively from the figure. We also observe that more degrees of entanglement than the input nonclassicality are obtained around $\theta = \pi/4$ due to the mixing with a thermal state.

C. Search for a generalized conservation relation

Next, we show that the quantification of entanglement $S_{\mathcal{N}}$ can be generalized to any Gaussian states that satisfy the following two constraints: (i) At least one of the input states is a pure state, i.e., the product of the two eigenvalues of the single-mode input state covariance matrix is equal to $\frac{1}{4}$; (ii) to make $\mathcal{C}$ positive, the eigenvalues of the input state satisfy the following conditions: $\lambda_{1\min} < \lambda_{2\min} < \lambda_{2\max} < \lambda_{1\max}$, or $\lambda_{2\min} < \lambda_{1\min} < \lambda_{1\max} < \lambda_{2\max}$, or $\lambda_{2\min} = \lambda_{1\min}$ and $\lambda_{1\max} = \lambda_{2\max}$.

We consider the case of mixing two general Gaussian states at a BS with at least one of them being a pure state. The input matrix is given by $V_{\text{in}} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, where $V_1 = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}$ and $V_2 = \begin{pmatrix} c & d \\ d^* & c \end{pmatrix}$. Without loss of generality, we require that the product of the eigenvalues of V_1 satisfies $\lambda_{1\min}\lambda_{1\max} = a^2 - |b|^2 = \frac{1}{4}$. The eigenvalues of V_2 are $\lambda_{2\min} = c - |d|$, $\lambda_{2\max} = c + |d|$, and $\lambda_{2\min}\lambda_{2\max} \geqslant \frac{1}{4}$. When $|b| = 0$ and $c - |d| < \frac{1}{2}$, this reduces to the first case of mixing a nonclassical state with a vacuum state. When $|d| = 0$, this reduces to the second case of mixing a pure nonclassical state with a thermal state. When $a - |b|, c - |d| < \frac{1}{2}$, this is the case of mixing a pure nonclassical state with another nonclassical state.

As shown in Ref. [12], the phase of the BS φ plays a role since b and d are nonzero in general. For $b, d \neq 0$, we require $\varphi = \arg(b)/2 - \arg(d)/2$ to have a conservation relation of the minimum eigenvalues before and after the BS between the initial system and the output separable system, i.e.,

$$\lambda_{1\min} + \lambda_{2\min} = \tilde{\lambda}_{1\min} + \tilde{\lambda}_{2\min}. \tag{23}$$

After the BS, the output matrix V_{out} is given in Appendix A. In general, we find

$$\begin{aligned} S = {} & (\lambda_{1\min} + \lambda_{2\min})(\lambda_{1\max} + \lambda_{2\max}) \\ & + (\lambda_{1\min} - \lambda_{2\min})(\lambda_{1\max} - \lambda_{2\max})\cos(4\theta) \end{aligned} \tag{24}$$

and

$$S - \left(\frac{1}{2} + 8\,\mathrm{Det}[V_{\text{out}}]\right) = \mathcal{C}\left(\frac{\tilde{\lambda}_{1\min}\tilde{\lambda}_{2\min}}{\lambda_{1\min}\lambda_{2\min}} - 1\right). \tag{25}$$

Here, the expression of $\mathcal{C}$ is generalized as

$$\mathcal{C} \equiv 8\lambda_{1\min}\lambda_{2\min}\frac{\lambda_{1\max} - \lambda_{2\max}}{\lambda_{2\min} - \lambda_{1\min}}. \tag{26}$$

Applying constraint (ii), we find that $\mathcal{C}$ is positive-definite. Therefore, the quantification $S_{\mathcal{N}}$ in Eq. (11) is equivalent to the logarithmic negativity for any two single-mode Gaussian states satisfying the two constraints. Then we derive the conservation relation as

$$N_{\text{noncl}}^{\text{in1}} + N_{\text{noncl}}^{\text{in2}} = N_{\text{noncl}}^{\text{out1}} + N_{\text{noncl}}^{\text{out2}} + S_{\mathcal{N}}. \tag{27}$$

A detailed derivation is provided in Appendix A. We see that the sum of single-mode nonclassicality and two-mode entanglement is conserved before and after a BS under the unitary transformation $U(\theta,\varphi)$.

As an example of mixing two nonclassical states, a quantitative conservation relation between $N_{\text{noncl}}^{\text{in}}, N_{\text{noncl}}^{\text{out}}$ and $S_{\mathcal{N}}$ is plotted in Fig. 7 for $\lambda_{1\min} = 0.1$ and $\lambda_{2\min} = 0.335$.

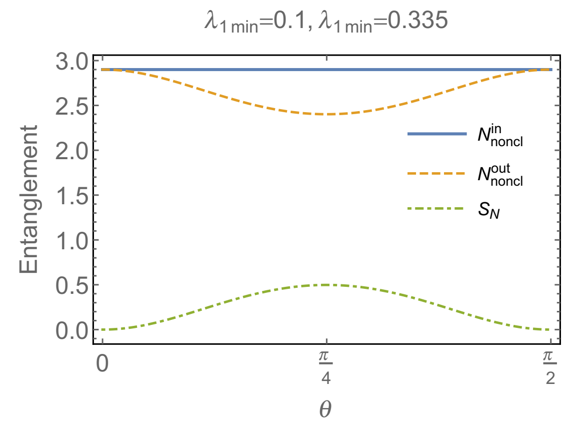

FIG. 7. (Color online) $N_{\text{noncl}}^{\text{in}}, N_{\text{noncl}}^{\text{out}}$ and $S_{\mathcal{N}}$ vs the BS angle θ for a pure nonclassical state mixing with another nonclassical state.

D. Extension to infinite number of BSs

As shown in Sec. IV A, after the first BS, we find that there is nonclassicality remaining in each of the output modes ρ_1 and ρ_2 for an initial nonclassical state mixed with a vacuum state. Therefore, we can send each output mode after the first BS, ρ_1 (ρ_2), to another BS mixing with a vacuum state to generate two sets of two-mode Gaussian entangled states, as shown in Fig. 8. The generated entanglement and the remaining nonclassicality after the second set of BSs will be equal to the input nonclassicality before the set BSs. Then we can send each output mode after the second set of BSs to the third set of BSs mixing with a vacuum state separately. In this way, the nonclassicality is split further and after each set of BSs we create a certain degree of entanglement. In each step, two conservation relations Eqs. (13) and (19) are satisfied. After an infinite number of steps, there is some nonclassicality in each of the output single-mode states, and by adding them up, we

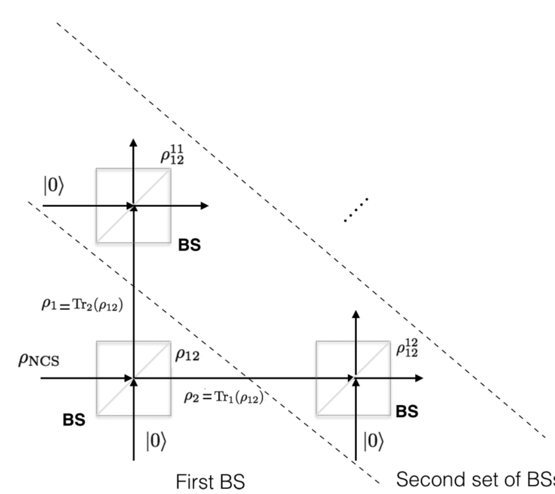

FIG. 8. A nonclassical state ρ_{NCS} is mixed with a vacuum state at a BS, generating an output state ρ_{12}. Each output mode ρ_1 or ρ_2 is mixed with a vacuum state at another BS generating two sets of output bipartite Gaussian states ρ_{12}^{11} and ρ_{12}^{12}. This process can be extended further with more BSs.

find the total remaining nonclassicality is given by

$$N^{\rm tot}_{\rm noncl} = (1 - 2\lambda_{1\rm min})\log_2 e, \tag{28}$$

which is independent on the angles of the BSs. Here, e is the Euler number. We can add all the entanglement generated in each step to obtain the total entanglement as

$$S^{\rm tot}_{\mathcal{N}} = \log_2\left(\frac{1}{2\lambda_{1\rm min}}\right) - (1 - 2\lambda_{1\rm min})\log_2 e. \tag{29}$$

A simple proof is provided in Appendix B. By extending our procedure to many sets of BSs, we find that both the quantifications of single-mode nonclassicality and two-mode entanglement are additive.

V. CONCLUSION

In this paper, we study the relation between the single-mode nonclassicality and two-mode entanglement created at a BS. We show that the input single-mode nonclassicality cannot be transferred completely into the output two-mode entanglement and nonclassicality remains in the output modes. The more the generated entanglement, the less nonclassicality remains. We use the logarithm of the minimum eigenvalue of a single-mode covariance matrix (the minimum uncertainty width) as its nonclassicality.

We also define the difference between the input nonclassicality and the output nonclassicality as a degree for two-mode entanglement, which is generated from two single-mode Gaussian states mixed at a BS. This quantification has a qualitative correspondence with the logarithmic negativity. The sum of nonclassicality and entanglement is shown to be conserved before and after a BS using these quantifications. We generalize this conservation relation to a class of two-mode Gaussian states. The extensions of many sets of BSs are discussed in the context of this conservation relation. Our work may stimulate a further interest in the unification of nonclassicality and entanglement.

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APPENDIX A: DERIVATION OF THE CONSERVATION RELATION BETWEEN ENTANGLEMENT AND NONCLASSICALITY IN A BS

For a general input two-mode Gaussian covariance, $V_{\rm in} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, with $V_1 = \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}$ and $V_2 = \begin{pmatrix} c & d \\ d^* & c \end{pmatrix}$. After the BS,

$$V_{\rm out} = U^\dagger(\theta,\varphi) V_{\rm in} U(\theta,\varphi) = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix}, \tag{A1}$$

where the matrices A, B, and C are given by

$$A = \begin{pmatrix} a\cos^2\theta + c\sin^2\theta & b\cos^2\theta + d\sin^2\theta e^{2i\varphi} \\ b^*\cos^2\theta + d^*\sin^2\theta e^{-2i\varphi} & a\cos^2\theta + c\sin^2\theta \end{pmatrix}, \tag{A2}$$

$$B = \begin{pmatrix} a\sin^2\theta + c\cos^2\theta & b\sin^2\theta e^{-2i\varphi} + d\cos^2\theta \\ b^*\sin^2\theta e^{2i\varphi} + d^*\cos^2\theta & a\sin^2\theta + c\cos^2\theta \end{pmatrix}, \tag{A3}$$

$$C = \begin{pmatrix} (a-c)e^{i\varphi} & be^{-i\varphi} - de^{i\varphi} \\ b^* e^{i\varphi} - d^* e^{-i\varphi} & (a-c)e^{-i\varphi} \end{pmatrix} \sin\theta\cos\theta. \tag{A4}$$

The elements of the covariance matrix are defined [26] as $V_{{\rm out}ij} = \frac{1}{2}{\rm Tr}[(v_i v_j + v_j v_i)\rho_{12}]$, where v_i are the position and momentum operators of the two-mode system defined as $v_1 = x_1$, $v_2 = p_1$, $v_3 = x_2$, and $v_4 = p_2$.

For the output separable system $\tilde{\rho}_{12} = {\rm Tr}_2(\rho_{12}) \otimes {\rm Tr}_1(\rho_{12})$, the elements of its covariance matrix are $\tilde{V}_{{\rm out}ij} = \frac{1}{2}{\rm Tr}[(v_i v_j + v_j v_i)\tilde{\rho}_{12}]$. For $i = 1,2$ and $j = 3,4$,

$$\begin{aligned} \tilde{V}_{{\rm out}ij} &= \tfrac{1}{2}[{\rm Tr}_1(v_i\rho_1){\rm Tr}_2(v_j\rho_2) + {\rm Tr}_2(v_j\rho_2){\rm Tr}_1(v_i\rho_1)] \\ &= 0 \end{aligned} \tag{A5}$$

for zero-mean Gaussian states. For $i,j = 1,2$,

$$\tilde{V}_{{\rm out}ij} = \tfrac{1}{2}{\rm Tr}_1[(v_i v_j + v_j v_i)\rho_1] = V_{{\rm out}ij}. \tag{A6}$$

A similar relation holds for $i,j = 3,4$. Therefore, we prove that $\tilde{V}_{\rm out} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The minimum eigenvalues of A and B are given by $\tilde{\lambda}_{1\rm min(2min)} = \lambda_{1\rm min(2min)}\sin^2\theta + \lambda_{2\rm min(1min)}\cos^2\theta$ using the phase condition $\varphi = \arg(b)/2 - \arg(d)/2$. Therefore, we obtain the first conservation relation between the minimum eigenvalues, i.e.,

$$\lambda_{1\rm min} + \lambda_{2\rm min} = \tilde{\lambda}_{1\rm min} + \tilde{\lambda}_{2\rm min}. \tag{A7}$$

S can be expressed in terms of minimum eigenvalues as

$$\begin{aligned} S &= 2({\rm Det}[A] + {\rm Det}[B] - 2\,{\rm Det}[C]) \\ &= 2(a\cos^2\theta + c\sin^2\theta)^2 - 2(|b|\cos^2\theta + |d|\sin^2\theta)^2 + 2(a\sin^2\theta + c\cos^2\theta)^2 - 2(|b|\sin^2\theta + |d|\cos^2\theta)^2 \\ &\quad -4(a-c)^2\sin^2\theta\cos^2\theta + 4(|b| - |d|)^2\sin^2\theta\cos^2\theta \\ &= 2(a^2 - |b|^2 + c^2 - |d|^2) - 8(a^2 - |b|^2 + c^2 - |d|^2 - 2ac + 2|b||d|)\sin^2\theta\cos^2\theta \\ &= 2(a^2 - |b|^2 + c^2 - |d|^2) - (a^2 - |b|^2 + c^2 - |d|^2 - 2ac + 2|b||d|)[1 - \cos(4\theta)] \\ &= (\lambda_{1\rm min} + \lambda_{2\rm min})(\lambda_{1\rm max} + \lambda_{2\rm max}) + (\lambda_{1\rm min} - \lambda_{2\rm min})(\lambda_{1\rm max} - \lambda_{2\rm max})\cos(4\theta) \\ &= (\lambda_{1\rm min} + \lambda_{2\rm min})\left(\frac{1}{4\lambda_{1\rm min}} + \lambda_{2\rm max}\right) + (\lambda_{1\rm min} - \lambda_{2\rm min})\left(\frac{1}{4\lambda_{1\rm min}} - \lambda_{2\rm max}\right)\cos(4\theta), \end{aligned} \tag{A8}$$

where we have used $\lambda_{1\min}\lambda_{1\max} = a^2 - |b|^2 = \frac{1}{4}$ and $\lambda_{2\min(2\max)} = c \mp |d|$. With $\mathrm{Det}[V_{\mathrm{out}}] = \frac{1}{4}\lambda_{2\min}\lambda_{2\max}$, we obtain

$$S - \left(\frac{1}{2} + 8\,\mathrm{Det}[V_{\mathrm{out}}]\right) = \frac{\lambda_{2\min} - \lambda_{1\min}}{2\lambda_{1\min}}(1 - 4\lambda_{1\min}\lambda_{2\max})\sin^2(2\theta). \tag{A9}$$

Using $\mathcal{C} = 2\lambda_{2\min}\frac{1-4\lambda_{1\min}\lambda_{2\max}}{\lambda_{2\min}-\lambda_{1\min}}$ for $\lambda_{1\min}\lambda_{1\max} = \frac{1}{4}$, and $\frac{\tilde{\lambda}_{1\min}\tilde{\lambda}_{2\min}}{\lambda_{1\min}\lambda_{2\min}} - 1 = \frac{(\lambda_{2\min}-\lambda_{1\min})^2}{4\lambda_{1\min}\lambda_{2\min}}\sin^2(2\theta)$, we prove the equality

$$S - \left(\frac{1}{2} + 8\,\mathrm{Det}[V_{\mathrm{out}}]\right) = \mathcal{C}\left(\frac{\tilde{\lambda}_{1\min}\tilde{\lambda}_{2\min}}{\lambda_{1\min}\lambda_{2\min}} - 1\right). \tag{A10}$$

APPENDIX B: EXTENSION OF INFINITE NUMBER OF BSs

When mixing a nonclassical state with a vacuum at a BS, we have

$$\tau = \tilde{\tau}_1 + \tilde{\tau}_2. \tag{B1}$$

At the second set of BSs, we split the nonclassical depth further by mixing each subsystem with a vacuum state. Then we have

$$\tilde{\tau}_1 = \tilde{\tau}_1^{(11)} + \tilde{\tau}_2^{(11)} \tag{B2}$$

and

$$\tilde{\tau}_2 = \tilde{\tau}_1^{(12)} + \tilde{\tau}_2^{(12)}. \tag{B3}$$

After $m+1$ steps, there are 2^{m+1} single-mode Gaussian states generated and the nonclassical depth of each state is given by $\tilde{\tau}_1^{(mj)}, \tilde{\tau}_2^{(mj)}$, where $j = 1,2,\ldots,2^m$. As $m \to +\infty$, $\tilde{\tau}_1^{(mj)}, \tilde{\tau}_2^{(mj)}$ will be infinitesimal independent of the angles of the BSs at each step. Then the nonclassicality of each state is

$$N_{\mathrm{noncl}}^{\mathrm{out}(mj)1} = -\log_2\left(1 - 2\tilde{\tau}_1^{(mj)}\right) = 2\tilde{\tau}_1^{(mj)}\log_2 e. \tag{B4}$$

The sum of all nonclassicality after $m+1$ steps is

$$\begin{aligned} N_{\mathrm{noncl}}^{\mathrm{tot}} &= \sum_{j=1}^{2^m}\left(N_{\mathrm{noncl}}^{\mathrm{out}(mj)1} + N_{\mathrm{noncl}}^{\mathrm{out}(mj)2}\right) \\ &= 2\sum_{j=1}^{2^m}\left(\tilde{\tau}_1^{(mj)} + \tilde{\tau}_2^{(mj)}\right)\log_2 e \\ &= 2\tau\log_2 e = (1 - 2\lambda_{1\min})\log_2 e, \end{aligned} \tag{B5}$$

where the conservation relation of nonclassical depth is used. Using the conservation relation between nonclassicality and entanglement before and after a BS, we obtain

$$\begin{aligned} S_{\mathcal{N}}^{\mathrm{tot}} &= N_{\mathrm{noncl}}^{\mathrm{in}} - N_{\mathrm{noncl}}^{\mathrm{tot}} \\ &= \log_2\left(\frac{1}{2\lambda_{1\min}}\right) - (1 - 2\lambda_{1\min})\log_2 e. \end{aligned} \tag{B6}$$

APPENDIX C: DISPLACED INPUT STATES

In deriving the conservation relation, we considered those input states without any displacement. Here, we show that, in general, if the input Gaussian states are displaced, the conservation relation still holds. To do this, we will show, for a single-mode input state, that its covariance matrix is invariant under any displacement on the state.

For an arbitrary input state (pure or mixed) ρ, its displaced density matrix is $\rho_\alpha = \mathcal{D}(\alpha)\rho\mathcal{D}^\dagger(\alpha)$, where $\mathcal{D}(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$. The covariance matrix for ρ is given by

$$V = \begin{pmatrix} \langle x^2\rangle - \langle x\rangle^2 & \frac{\langle xp+px\rangle}{2} - \langle x\rangle\langle p\rangle \\ \frac{\langle xp+px\rangle}{2} - \langle x\rangle\langle p\rangle & \langle p^2\rangle - \langle p\rangle^2 \end{pmatrix}, \tag{C1}$$

where $x = \frac{a+a^\dagger}{\sqrt{2}}$ and $p = \frac{a-a^\dagger}{i\sqrt{2}}$. For the displaced state ρ_α, its covariance matrix is given by

$$V_\alpha = \begin{pmatrix} \langle x^2\rangle_\alpha - \langle x\rangle_\alpha^2 & \frac{\langle xp+px\rangle_\alpha}{2} - \langle x\rangle_\alpha\langle p\rangle_\alpha \\ \frac{\langle xp+px\rangle_\alpha}{2} - \langle x\rangle_\alpha\langle p\rangle_\alpha & \langle p^2\rangle_\alpha - \langle p\rangle_\alpha^2 \end{pmatrix}, \tag{C2}$$

where $\langle O\rangle_\alpha = \mathrm{Tr}(O\rho_\alpha) = \mathrm{Tr}[O\mathcal{D}(\alpha)\rho\mathcal{D}^\dagger(\alpha)] = \mathrm{Tr}[\mathcal{D}^\dagger(\alpha)O\mathcal{D}(\alpha)\rho]$ for any operator O. By using the relation $\mathcal{D}^\dagger(\alpha)a\mathcal{D}(\alpha) = a + \alpha$ [31], we obtain $\langle x\rangle_\alpha = \langle x\rangle + \frac{\alpha+\alpha^*}{\sqrt{2}}$ and $\langle x^2\rangle_\alpha = \langle(x + \frac{\alpha+\alpha^*}{\sqrt{2}})^2\rangle$. So, $\langle x^2\rangle_\alpha - \langle x\rangle_\alpha^2 = \langle x^2\rangle - \langle x\rangle^2$. Similarly, we can show that the other matrix elements in V_α are equal to the corresponding elements in V. Therefore, for any displaced input Gaussian states, the same conservation relation applies since the covariance matrix is invariant under displacement.

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