

**Bell inequalities violated using detectors of low efficiency**

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(Received 14 August 2015; published 9 November 2015)

We define a family of binary-outcome  $n$ -party  $m \leq n$  settings per party Bell inequalities whose members require the least detection efficiency for their violation among all known inequalities of the same type. This gives upper bounds for the minimum value of the critical efficiency—below which no violation is possible—achievable for such inequalities. For  $m = 2$ , our family reduces to the one given by Larsson and Semitecolos in 2001 [*Phys. Rev. A* **63**, 022117 (2001)]. For  $m > 2$ , a gap remains between these bounds and the best lower bounds. The violating state near the threshold efficiency always approaches a product state of  $n$  qubits.

DOI: [10.1103/PhysRevA.92.052104](https://doi.org/10.1103/PhysRevA.92.052104)

PACS number(s): 03.65.Ud, 03.67.—a

**I. INTRODUCTION**

In a Bell experiment distant parties perform measurements on a shared physical system. From very reasonable assumptions it follows that the correlations between their results must satisfy certain inequalities. Quantum mechanics predicts that often these Bell inequalities will not be respected [1,2]. Their violation, which excludes all local realistic explanations for the quantum world and which is often referred to as nonlocality, is one of the most surprising and counterintuitive features of quantum physics. One of the earliest experimental demonstrations of the Bell violation was done by Aspect *et al.* [3], and many experiments on various systems have been performed since (see, e.g., Refs. [4,5]). Unfortunately, technical imperfections open various loopholes, which in principle allow local realistic explanation for the experimental results. Notably, a very recent experiment conducted by Hensen *et al.* [6] managed to close the two main loopholes, the so-called detection and locality loopholes, simultaneously.

The detection loophole appears in experiments performed on systems of photons [7]. Some photons get lost during transmission, and the efficiency of the present-day photon detectors is also limited. If the proportion of photons detected falls below a certain critical value  $\eta_{\text{crit}}$ , which depends both on the Bell inequality and on the system, then the results become compatible with local realism. As an important property characterizing the inequality and the system, Vallone *et al.* [8] introduced the notion of the robustness of nonlocality  $R$ . They defined this quantity as the maximum fraction of detection events that can be lost such that the remaining ones still do not admit a local model. It is easy to see that  $R = 1 - \eta_{\text{crit}}$ . In the present paper it will be assumed that the detection efficiency is the same for all parties. There are experiments, for example, those involving entangled photon-ion systems, where some of the parties (the ones working with the ions) have virtually no loss of events. Then the experiment tolerates higher losses for the remaining parties than in the symmetric case considered here [9,10].

Another important loophole is the locality loophole, whose closure requires spacelike separation between the parties. This can realistically be ensured only in Bell tests performed on photon systems. Each loophole has been closed by some experiment, in particular, both the locality loophole and the detection loophole were closed by photons. (See the respective Refs. [5] and [11] and a comprehensive review [12] on this

topic.) Moreover, Hensen *et al.* [6] reports a recent experiment which closes both loopholes simultaneously using entangled electron spins. Such an experimental verification is very interesting (however, a loophole-free Bell experiment using only photons is still needed) and also fairly important from a philosophical point of view, although, as so many predictions of quantum mechanics have proven correct, at present virtually everyone is quite convinced that the Bell violations observed so far are genuine and not the results of some conspiracy plotted by nature, using different loopholes in different situations to mislead us.

However, loophole-free Bell violation is important from a practical point of view as well, since it is the basis of the so-called device-independent (DI) protocols of quantum information technology (see, for example, Ref. [13] for a recent review). Such protocols are based on the fact that Bell violation is impossible to fake (modulo technical loopholes). DI protocols make it possible for anyone to check the proper functioning of an apparatus without knowing much about the details of its internal structure. Therefore construction faults would be easier to notice, and it would be harder for a malicious manufacturer to build back doors into the device. This approach promises to perform cryptographic tasks with an unprecedented security [14], produce genuinely random numbers [15], and carry out black-box state tomography [16].

The violation of the simplest Bell inequality, the two-party ( $n = 2$ ) two-setting per party ( $m = 2$ ), binary-outcome Clauser-Horne-Shimony-Holt (CHSH) inequality [17], requires at least  $\eta_{\text{crit}} = 2/3 \approx 0.667$  detection efficiency [18]. This value may be reached with a partially entangled two-qubit state, which approaches a product state near the threshold efficiency. In this case the robustness of nonlocality is  $R = 1/3$ . As far as we know, with two parties performing an arbitrary number  $m$  of measurements on a pair of qubits, no better value has been achieved. In Ref. [10] a bipartite four-setting inequality was found with a somewhat lower critical efficiency of  $\eta_{\text{crit}} = (\sqrt{5} - 1)/2 \approx 0.618$ . To get this value a pair of ququarts had to be used. Larsson and Semitecolos [19] have introduced a family of  $n \geq 2$  party binary-outcome Bell inequalities with  $m = 2$  settings per party with  $\eta_{\text{crit}} = n/(2n - 1)$ . Later, this threshold value was proven to be optimal by Massar and Pironio [20]. That is, no two-setting  $n$ -party Bell inequality may be violated with any lower detection efficiency. Note that this threshold holds true for measurements with any number of outcomes and any

dimensionality of the local Hilbert space used. This result also means that for  $m = 2$  an efficiency of less than  $1/2$  is never enough.

Let  $\eta_*$  be the smallest of the critical detection efficiencies belonging to the Bell inequalities characterized by a certain number of parties and numbers of settings and measurement outcomes for each party. Above  $\eta_*$  there are quantum measurements exhibiting correlations that cannot be explained by any local hidden-variables model. The value for  $\eta_{\text{crit}}$  given by Ref. [19] and cited above is just  $\eta_*$  for the set of binary-outcome  $n$ -party two-setting per party Bell inequalities. For  $n$ -party and more than two-setting per party ( $m > 2$ ) inequalities,  $\eta_*$  is not known. Massar and Pironio [20] have given a lower bound for this quantity by constructing explicit local-hidden-variables models reproducing the correlations for efficiencies below this bound. For  $m = 2$  the bound is the same as that of Ref. [19]. Based on combinatorial considerations, Buhrman *et al.* [21] have derived an upper bound for  $\eta_*$ , when  $m$  is some power of 2. Reference [22] also provided upper bounds for many  $(n, m)$  combinations ( $m$  prime) by giving explicit Bell inequalities corresponding to those efficiency values. The inequalities of that work may become useful in practice as well, because they require the Greenberger-Horne-Zeilinger (GHZ) state [23], well distinguishable measurement settings even near the threshold efficiency, and the noise tolerance is also quite good.

Very recently tripartite Bell inequalities have been constructed [24] requiring low detection efficiencies for their violation with the  $W$  state [25]. With three measurement settings for one of the parties and two settings for the other two parties, a critical efficiency of  $\eta_{\text{crit}} = 0.6$  has been achieved. The same efficiency has been necessary for an inequality symmetric for the permutations of the parties with  $m = 3$  settings per party. For  $m = 4, 6,$  and  $8$  this value has been improved to  $\eta_{\text{crit}} = 0.509\,036, 0.502\,417,$  and  $0.501\,338,$  respectively. For larger  $m$  the construction would have required too many computational resources, but the tendency indicates that the result would have remained above  $0.5$ . However, if a small admixture of a product state to the  $W$  state has been allowed,  $\eta_{\text{crit}} = 0.5$  has been found for an  $m = 4$  inequality. This value did not improve when  $m$  was increased to  $6$ .

In this work we have used an iterative procedure alternating a linear programming step and a semidefinite programming step to get the  $m = 3, n = 3,$  and  $n = 4$  binary-outcome Bell inequalities with the smallest possible  $\eta_{\text{crit}}$ . In this procedure we made assumptions neither about the symmetry of the inequality nor about the properties of the state, not even about the dimensionality of the Hilbert space. We have produced tripartite inequalities reaching  $\eta_{\text{crit}} = 0.5$  already with three binary-outcome settings per party, and for  $m = 3, n = 4$  we have obtained  $\eta_{\text{crit}} = 6/13 \approx 0.461\,54$ . We could get the same values with confining ourselves to permutationally symmetric inequalities. Numerical calculations showed that the state giving the maximum violation near the threshold efficiency is a state of  $n$  qubits, namely, a mixture of the  $W$  state and a product state that approaches the product state near  $\eta_{\text{crit}}$ , the same behavior as that observed for the two-setting inequalities of Larsson and Semitecolos [19]. The measurement settings could be chosen the same for all parties. Each measurement operator could be characterized by a single

angle that approached zero at  $\eta_{\text{crit}}$  but at different paces for the different settings, again, similarly to what is seen in Ref. [19]. These observations helped us to make the generalization and to define a whole family of Bell inequalities with any  $m$  and  $n \geq m$ . The members of the family require the least detection efficiency for their violation among all known inequalities of the same type, giving upper bounds for  $\eta_*$ . For  $m = 2$  the inequalities are the same (apart from swapping of the two measurement settings) as those of Ref. [19].

The structure of this paper is as follows. In Sec. II, we give explicitly the family of multipartite Bell expressions. Then, in Sec. III, we give the optimal quantum violation of these Bell inequalities with qubit systems accounting for finite detection efficiencies. Our main result is presented in formula (8), providing (to the best of our knowledge) the best upper bound on  $\eta_*$  for any  $n \geq m$  parties and  $m$  settings per party. To get this formula, we have used two claims: one regarding the equality of certain quantum conditional probabilities and the negligibility of others, the other concerning the classical bound. The first claim is proven in Sec. IV and the second one in Sec. V. To demonstrate the behavior of the relevant quantities, we highlight the example  $n = m = 3$  in Sec. VI, where we give the explicit Bell expression and also show the optimal measurement settings (angles) and Bell violation departing from the value of the critical detection efficiency. Finally, we summarize our conclusions.

## II. THE INEQUALITIES

We consider the Bell scenario with  $n$  distant observers. Each observer may freely choose between a set of  $m$  binary measurement settings with possible outcomes zero and one. Let  $A_{ij}$  denote the  $j$ th measurement of the  $i$ th party. A Bell inequality corresponding to this setup may be written as

$$\sum_{j_1=0}^m \sum_{j_2=0}^m \dots \sum_{j_n=0}^m B_{j_1 j_2 \dots j_n} P(A_{1j_1} A_{2j_2} \dots A_{nj_n}) \leq L, \quad (1)$$

where  $L$  is the classical bound, and  $P(A_{1j_1} A_{2j_2} \dots A_{nj_n}) \equiv P(11 \dots 1 | A_{1j_1} A_{2j_2} \dots A_{nj_n})$  denotes the conditional probability of all parties getting outcome 1, given they have measured  $A_{1j_1}, A_{2j_2},$  and so on. For the sake of brevity of the notation, for terms involving only a subset of observers, we have introduced a zeroth measurement whose outcome is always 1 and which is always supposed to be performed. This measurement appears in these terms for the observers, not members of the subset considered. Also note that the value of  $B_{00 \dots 0}$  in Eq. (1) is chosen to be zero to make the value of  $L$  unambiguous.

The inequalities we introduce in the present paper are invariant with respect to the permutations of the parties, and  $n \geq m$  holds. Besides terms involving all  $n$  observers, they contain only terms with  $n - 1$  parties (that is, terms containing equal or less than  $n - 2$  parties are zero). Due to the permutational symmetry, Bell coefficients having the same set of indices in different orders have the same value. Therefore we only give explicit values for coefficients with indices  $j_1 \leq j_2 \leq j_3 \dots \leq j_n$ .

The nonzero  $(n - 1)$ -party Bell coefficients have the value of  $-1$ . Besides the index zero for the party not involved, they have indices from 2 to  $m$  once, plus index  $m$  another

$n - m$  times if  $n > m$ . For example, if  $m = 2$  and  $n > 2$ , then  $B_{02\dots 2} = -1$ , or if  $m = 3$  and  $n > 3$ , then  $B_{023\dots 3} = -1$ . For larger values of  $m$  the nonzero coefficients are

$$B_{0234\dots mm\dots m} = -1, \quad (2)$$

and the ones we get by permuting the indices of the coefficients above. Thus there are  $n!/(n - m + 1)!$  nonzero  $(n - 1)$ -party coefficients.

Furthermore, we have  $n$ -party Bell coefficients whose value is positive. They are the ones that have the same set of indices as the nonzero  $(n - 1)$ -party coefficients above, but index zero is replaced by anything from 1 to  $m$ . If this index is  $m$ , the value of the coefficient is  $(n - m + 1)$ ; otherwise it is 1. Therefore, we may write (again for larger  $m$ )

$$\begin{aligned} B_{1234\dots mm\dots m} &= 1 \\ B_{2234\dots mm\dots m} &= 1 \\ B_{2334\dots mm\dots m} &= 1 \\ B_{2344\dots mm\dots m} &= 1 \\ &\vdots \\ B_{234\dots mmm\dots m} &= n - m + 1. \end{aligned} \quad (3)$$

Again, all coefficients we get by permuting the indices have the same value. Thus the first and the last line represent  $n!/(n - m + 1)!$  and  $n!/(n - m + 2)!$  coefficients, respectively, while the lines in between represent  $n!/2(n - m + 1)!$  coefficients.

There are also some  $n$ -party Bell coefficients having negative values,

$$B_{k_1 k_2 k_3 k_4 \dots k_n} < 0, \quad (4)$$

where the indices are such that the quantum values of the conditional probabilities multiplying these coefficients calculated with the quantum state and the measurement operators we will give later are vanishingly small compared to the quantum conditional probabilities multiplying the coefficients given by Eqs. (2) and (3). Then the actual values of these coefficients will not influence the quantum value of the Bell expression [the left-hand side of Eq. (1)]; therefore we may choose them freely. As we will show later, this freedom makes it possible to achieve that the classical bound  $L$  appearing in Eq. (1) is zero, which is appropriate for our purpose. There are many solutions, and they all give the same value for the threshold detector efficiency. The maximum quantum violation above the threshold efficiency does depend on the actual choice, but we have made no attempt to find the best one.

All other Bell coefficients are zero.

### III. QUANTUM VIOLATION WITH DETECTORS OF LIMITED EFFICIENCY

Let us consider the quantum violation of our Bell inequality. If we confine ourselves to von Neumann measurements on a pure  $n$ -qubit state  $|\psi\rangle$ , the quantum value of the conditional probability appearing in Eq. (1) may be written as

$$P_Q(A_{1j_1} A_{2j_2} \dots A_{nj_n}) = \langle \psi | \bigotimes_{i=1}^n \hat{A}_{ij_i} | \psi \rangle, \quad (5)$$

where  $\hat{A}_{ij}$  for  $1 \leq j \leq m$  is the measurement operator corresponding to measurement  $A_{ij}$ ,  $\hat{A}_{i0} = \hat{I}$  is the identity operator in the subspace of the  $i$ th observer, and  $\bigotimes_{i=1}^n \hat{A}_{ij_i}$  denotes the tensor product of the operators of the parties. The state  $|\psi\rangle$  we consider in this paper is invariant with respect to the permutations of the parties, and all parties have the same set of measurement operators, that is,  $\hat{A}_{ij} = \hat{A}_j$  is independent of  $i$ . In our case, the Bell inequality of Eq. (1) is permutationally invariant. If the classical bound is zero, the condition for its quantum violation can be written as

$$\begin{aligned} &\sum_{j_1=0}^m \sum_{j_2=j_1}^m \dots \sum_{j_n=j_{n-1}}^m \pi(j_1, j_2, \dots, j_n) B_{j_1 j_2 \dots j_n} \\ &\times \langle \psi | \bigotimes_{i=1}^n \hat{A}_{j_i} | \psi \rangle > 0, \end{aligned} \quad (6)$$

where  $\pi(j_1, j_2, \dots, j_n)$  denotes the number of independent permutations of its arguments. Let us suppose that the observers have detectors of limited efficiency  $\eta < 1$ . Let them agree that each of them signal outcome zero whenever the particle is not detected. Note that in principle the observers may agree on more general local strategies in the event of no detection (e.g., in case of no detection some of the parties may signal outcome 1). In fact, we have no general proof that the present one is the optimal strategy.

In this case each term in the left-hand side (l.h.s.) of Eq. (6) must be multiplied by the probability of detecting all particles concerned to get the condition for the detection of the violation. This probability is  $\eta^l$  for an  $l$ -party term, where  $l$  is the number of nonzero indices. At the threshold efficiency  $\eta_{\text{crit}}$  the resulting expression is zero. This condition gives an  $n$ th-order equation for  $\eta_{\text{crit}}$ . In our case the solution is simple, because we have restricted ourselves to  $n$ -party and  $(n - 1)$ -party terms. The condition for being able to detect the violation is

$$\begin{aligned} &\eta^{n-1} \sum_{j_2=1}^m \sum_{j_3=j_2}^m \dots \sum_{j_n=j_{n-1}}^m \pi(0, j_2, \dots, j_n) B_{0j_2 \dots j_n} \\ &\times \langle \psi | \hat{I} \otimes \bigotimes_{i=2}^n \hat{A}_{j_i} | \psi \rangle + \eta^n \sum_{j_1=1}^m \sum_{j_2=j_1}^m \dots \\ &\times \sum_{j_n=j_{n-1}}^m \pi(j_1, j_2, \dots, j_n) B_{j_1 j_2 \dots j_n} \\ &\times \langle \psi | \bigotimes_{i=1}^n \hat{A}_{j_i} | \psi \rangle > 0. \end{aligned} \quad (7)$$

Then  $\eta_{\text{crit}}$  is  $-1$  times the ratio of the  $(n - 1)$ -party and  $n$ -party contributions in the expression above. The former must be negative while the latter positive such that the l.h.s. is positive above  $\eta_{\text{crit}}$ .

In the next section we will show that with an appropriate choice of the state vector and the measurement settings, the quantum conditional probabilities associated with each Bell coefficient given in Eqs. (2) and (3) are equal. As the same number appears both in the numerator and the denominator of the ratio giving  $\eta_{\text{crit}}$ , we can simplify with it. The factor  $\pi$  giving the number of independent permutations of the indices

is  $n!/(n-m+1)!$  for the  $(n-1)$ -party coefficient and for the  $n$ -party coefficient given in the first line of Eq. (3). The factor is one-half of this for the next  $m-2$  coefficients, and it is  $n!/(n-m+2)!$  for the last one. Taking into account the values of the coefficients we can give the expression for  $\eta_{\text{crit}}$  as

$$\eta_{\text{crit}} = \frac{1}{1 + \frac{m-2}{2} + \frac{n-m+1}{n-m+2}} = \frac{2}{2 + m - \frac{2}{n-m+2}}. \quad (8)$$

Here we have simplified the fraction by the common factor  $n!/(n-m+1)!$ . This expression is our main result. Our Bell inequalities are such that their violation, at least in principle, can be detected by using detectors of as low efficiency as given above. Therefore this is an upper bound for the threshold detector efficiency which may be achieved for  $n$ -party  $m \leq n$  setting binary-outcome Bell inequalities. The expression gives  $2/(m+1)$  for  $n=m$  and  $2/(m+2)$  in the limit of large  $n$ . However, if we increase  $n$ , we can achieve less than by increasing both  $m$  and  $n$  by just 1.

To prove Eq. (8) we must show that the relevant quantum conditional probabilities are really equal as we have claimed and that the local bound is zero. We will do this in the next two sections.

#### IV. THE QUANTUM CONDITIONAL PROBABILITIES

Let all measurements be real ones, that is, performed in the  $X-Z$  plane. Then each operator  $\hat{A}_j$  can be characterized by a single real variable  $\phi_j$ :

$$\begin{aligned} \hat{A}_j|0\rangle &= \frac{1 - \cos \phi_j}{2}|0\rangle - \frac{\sin \phi_j}{2}|1\rangle \equiv c_j^-|0\rangle + s_j|1\rangle \\ \hat{A}_j|1\rangle &= -\frac{\sin \phi_j}{2}|0\rangle + \frac{1 + \cos \phi_j}{2}|1\rangle \equiv s_j|0\rangle + c_j^+|1\rangle. \end{aligned} \quad (9)$$

A zero angle corresponds to the measurement giving outcome 1 with probability 1 for the  $|1\rangle$  state. Let all measurement angles be small, and let them obey the following hierarchy:

$$0 \leq |\phi_1| \ll |\phi_2| \ll |\phi_3| \ll \dots \ll |\phi_m|. \quad (10)$$

We will show later that above  $\eta_{\text{crit}}$  if  $\phi_m$  is small enough and the differences between the orders of magnitudes of the consecutive angles are large enough the Bell inequality will always be violated with the appropriate state. Numerical results for the optimum choice in the case of  $n=m=3$  will be shown in Sec. VI.

Let the state be

$$|\psi\rangle = \cos \alpha|0\rangle - \sin \alpha|W\rangle, \quad (11)$$

where

$$|0\rangle \equiv |00\dots 0\rangle \quad (12)$$

$$|W\rangle \equiv \frac{1}{\sqrt{n}}(|10\dots 0\rangle + |01\dots 0\rangle + \dots + |00\dots 1\rangle). \quad (13)$$

This state is permutationally symmetric. The angle  $\alpha$  will be small, and therefore the state is predominantly the separable  $|0\rangle$  state with a very small amount of  $|W\rangle$  state added.

First we calculate the quantum conditional probability by multiplying the  $(n-1)$ -party Bell coefficient given by Eq. (2).

From Eqs. (5) and (11) we get

$$\begin{aligned} P_Q(A_{10}A_{22}A_{33}\dots A_{mm}A_{(m+1)m}\dots A_{nm}) \\ = \cos^2 \alpha S_{00} - 2 \cos \alpha \sin \alpha S_{W0} + \sin^2 \alpha S_{WW}, \end{aligned} \quad (14)$$

where

$$S_{\sigma\tau} \equiv \langle \sigma | \hat{I} \otimes \bigotimes_{i=2}^n \hat{A}_i | \tau \rangle, \quad (15)$$

with  $\hat{A}_i \equiv \hat{A}_m$  when  $i \geq m$ . As the first step to calculate these matrix elements, from Eqs. (9) and (12) we get:

$$\hat{I} \otimes \bigotimes_{i=2}^n \hat{A}_i |0\rangle = |0\rangle \otimes \bigotimes_{i=2}^n (c_i^-|0\rangle + s_i|1\rangle). \quad (16)$$

Then, by using Eqs. (12) and (13) it is easy to calculate  $S_{00}$  and  $S_{W0}$ . We may also get  $S_{WW}$  from a similar, but somewhat lengthier calculation. The result is:

$$\begin{aligned} S_{00} &= \prod_{i=2}^n c_i^- \\ S_{W0} &= \frac{1}{\sqrt{n}} \sum_{k=2}^n \frac{s_k}{c_k} \prod_{i=2}^n c_i^- \equiv \frac{S_{00}}{\xi} \\ S_{WW} &= S_{00} \left( \frac{1}{n} + \frac{1}{\xi^2} \right). \end{aligned} \quad (17)$$

As  $|c_i^-| \ll |s_i|$ ,  $\xi$  is a small number. By substituting Eqs. (17) into Eq. (14) we get

$$\begin{aligned} P_Q(A_{10}A_{22}A_{33}\dots A_{mm}A_{(m+1)m}\dots A_{nm}) \\ = S_{00} \left[ \left( \cos \alpha - \frac{\sin \alpha}{\xi} \right)^2 + \frac{\sin^2 \alpha}{n} \right]. \end{aligned} \quad (18)$$

The first term in the square bracket dominates over the second one. However, as we will see later, the contribution from the  $n$ -party terms is of the same order as the second term. The threshold efficiency may only be finite if the first term vanishes. We can achieve that by choosing the mixing angle  $\alpha$  characterizing the quantum state such that  $\text{tg} \alpha = \xi$ . As the role of this choice is to eliminate the leading-order term, to achieve this in practice would require an extremely precise preparation of the state. Then we arrive at

$$P_Q(A_{10}A_{22}A_{33}\dots A_{mm}A_{(m+1)m}\dots A_{nm}) = \frac{S_{00} \sin^2 \alpha}{n}. \quad (19)$$

To get the factors multiplying the Bell coefficients of Eqs. (3) and (4), let us calculate the general  $n$ -party conditional probability. Analogously to Eqs. (14) and (15) we may write

$$\begin{aligned} P_Q(A_{1j_1}A_{2j_2}\dots A_{nj_n}) \\ = \cos^2 \alpha F_{00} - 2 \cos \alpha \sin \alpha F_{W0} + \sin^2 \alpha F_{WW}, \end{aligned} \quad (20)$$

where

$$F_{\sigma\tau} \equiv \langle \sigma | \bigotimes_{i=1}^n \hat{A}_{i j_i} | \tau \rangle. \quad (21)$$

We note that  $F_{\sigma\tau}$  depends on the set of indices  $j_i$  ( $i=1, \dots, n$ ) characterizing the particular  $n$ -party coefficient. We will not

explicitly mark this dependence. At the same time,  $S_{\sigma\tau}$  defined in Eq. (15) has no such dependence, as it corresponds to the well-defined indices of the single nonzero  $(n-1)$ -party coefficient. We can calculate the matrix elements above similarly to the ones given in Eq. (17), and we get:

$$\begin{aligned} F_{00} &= \prod_{i=1}^n c_{j_i}^- \\ F_{W0} &= \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{s_{jk}}{c_{jk}^-} \prod_{i=1}^n c_{j_i}^- \equiv \frac{F_{00}}{\chi} \\ F_{WW} &= \frac{F_{00}}{\chi^2}. \end{aligned} \quad (22)$$

By substituting Eqs. (22) into Eq. (20) we get

$$\begin{aligned} P_Q(A_{1j_1} A_{2j_2} \dots A_{nj_n}) &= F_{00} \left( \cos \alpha - \frac{\sin \alpha}{\chi} \right)^2 \\ &= F_{00} \sin^2 \alpha \left( \frac{1}{\xi} - \frac{1}{\chi} \right)^2. \end{aligned} \quad (23)$$

Here we used the relation  $\operatorname{tg} \alpha = \xi$ .

Let us consider the quantum conditional probabilities multiplying the  $n$ -party coefficients whose values are given explicitly in Eq. (3). The product giving  $F_{00}$  includes exactly the same elements as the one giving  $S_{00}$ , plus one more [see Eqs. (17) and (22)]. The same is true for the sums defining  $\chi$  and  $\xi$ . The extra element is the one of index  $l$  for the  $l$ th line of Eq. (3). Then  $F_{00} = S_{00} c_l^-$  and  $1/\chi = 1/\xi + s_l/(c_l^- \sqrt{n})$ . By substituting these values into Eq. (23) we get that the result is  $s_l^2/c_l^-$  times the value given in Eq. (19) for the  $(n-1)$ -party conditional probability. But this extra factor is one in the limit of small angles. Therefore, for our state and measurement settings, the quantum conditional probabilities multiplying each of the Bell coefficients of Eqs. (2) and (3) are equal indeed and have the value given in Eq. (19).

Now, let us consider which are the Bell coefficients Eq. (4) whose value may freely be chosen. From Eqs. (19) and (23) it follows that the quantum probability characterized by indices  $k_1 \leq k_2 \leq \dots \leq k_n$  is negligible compared to the significant ones if

$$\frac{\prod_{i=1}^n c_{k_i}^-}{\prod_{i=2}^n c_i^-} \left( \sum_{i=2}^n \frac{s_i}{c_i^-} - \sum_{i=1}^n \frac{s_{k_i}}{c_{k_i}^-} \right)^2 \ll 1. \quad (24)$$

Here we substituted the values of  $S_{00}$ ,  $F_{00}$ ,  $\xi$ , and  $\chi$  from Eqs. (17) and (22). However,  $c_i^- = (1 - \cos \phi_i)/2 \approx \phi_i^2/4$  and  $s_i = -\sin \phi_i/2 \approx -\phi_i/2$ , if the angles are small; therefore the above condition may be written as

$$\left[ \frac{\prod_{i=1}^n \phi_{k_i}}{\prod_{i=2}^n \phi_i} \left( \sum_{i=1}^n \frac{1}{\phi_{k_i}} - \sum_{i=2}^n \frac{1}{\phi_i} \right) \right]^2 \ll 1. \quad (25)$$

If  $k_1 = 1$ , due to the hierarchy imposed by Eq. (10) the dominant term in the parentheses is  $1/\phi_1$ . Then the condition above may be written as  $(\prod_{i=2}^n \phi_{k_i} / \prod_{i=2}^n \phi_i)^2 \ll 1$ . This relation holds if  $\phi_{k_i} \leq \phi_i$  for  $2 \leq i \leq n$ , and for at least one index  $w$  the inequality is strict, in which case  $\phi_{k_w} \ll \phi_w$ . When deriving the formulas we have implicitly supposed that  $\phi_1 \neq 0$ . However, the result is true for  $\phi_1 = 0$  as well.

If  $k_i = i+1$  for  $i < l$ , while  $k_l = k_{l-1} = l$ , the dominant term in the parentheses is  $1/\phi_l$  and the condition becomes  $(\prod_{i=l+1}^n \phi_{k_i} / \prod_{i=l+1}^n \phi_i)^2 \ll 1$ . Then if the same relations hold between  $\phi_i$  and  $\phi_{k_i}$  as for the  $k_1 = 1$  case, but now for  $l+1 \leq i \leq n$ , Eq. (25) will hold. To ensure that the classical bound is zero, it is sufficient to allow just those Bell coefficients to have negative values that correspond to  $k_i$  obeying the conditions above.

Now, let us show that above  $\eta_{\text{crit}}$  [see Eq. (8)] the measurement operators may be chosen such that the l.h.s. of Eq. (7) is positive, that is, the Bell inequality is violated, indeed. Let us underestimate the l.h.s. of Eq. (7). First, let us underestimate the factors of the positive Bell coefficients of Eq. (3) by replacing all of them by the value of the smallest one of them, namely, by  $s_m^2/c_m^- \equiv 1 - \varepsilon_1$  times the factor of the  $(n-1)$ -party coefficient of Eq. (19). It is easy to see that  $\varepsilon_1 > 0$ , and its value can be made as small as necessary by choosing  $\phi_m$ , the largest one of the measurement angles small enough. At the same time, let us overestimate the factors of the negative Bell coefficients of Eq. (4) by replacing them all by the largest one of them, which is nothing else than the factor of the  $(n-1)$ -party coefficient times  $\varepsilon_2$ , where  $\varepsilon_2$  is the largest value the l.h.s. of Eq. (24) may take with  $k_i$  corresponding to one of the appropriate Bell coefficients. If the measurement angles are small, the l.h.s. of Eq. (24) may be approximated well by some  $(\phi_p/\phi_q)^2$ , where  $p < q$ , or by products of such factors. Then, taking into account the hierarchy of Eq. (10),  $\varepsilon_2$  is approximately the largest of  $(\phi_{q-1}/\phi_q)^2$ . Therefore  $\varepsilon_2$  can be arbitrarily small if all the ratios of the absolute values of the consecutive angles in Eq. (10) are small enough.

After some straightforward calculations, Eq. (7) with its l.h.s. underestimated as above may be written as

$$R \left( \eta - \eta_{\text{crit}} - \varepsilon_1 \frac{\eta}{\eta_{\text{crit}}} - \varepsilon_2 \eta Z \right) > 0, \quad (26)$$

where

$$R \equiv \eta^{n-1} \frac{n!}{(n-m+1)!} \frac{S_{00} \sin^2 \alpha}{n} \quad (27)$$

and

$$\begin{aligned} Z &\equiv - \frac{(n-m+1)!}{n!} \\ &\times \sum_{k_1 < k_2 < \dots < k_n} \pi(k_1, k_2, \dots, k_n) B_{k_1 k_2 \dots k_n}. \end{aligned} \quad (28)$$

The sum above goes through all combinations of  $k_i$ , corresponding to a negative full correlation Bell coefficient. When deriving the formulas above, we used Eqs. (8) and (19). The factor  $Z$  is a finite positive number for any  $m$  and  $n \geq m$ . Then, for  $\eta > \eta_{\text{crit}}$  whenever  $\varepsilon_1$  and  $\varepsilon_2$  are small enough, which they can be, if the largest angle is small enough and the order-of-magnitude differences between the consecutive measurement angles are large enough, Eq. (26) is satisfied, which means that the Bell inequality is violated. The strategy above to choose the measurement settings is far from optimal. Unfortunately, as we show in Sec. VI, even the optimally chosen measurement settings get very close to each other, becoming more and more indistinguishable as we approach  $\eta_{\text{crit}}$ , even for  $n = m = 3$ . Therefore not only the preparation

of the state but the setting of the measurements would require extreme precision.

## V. THE CLASSICAL CASE

For an  $n$ -party Bell inequality with  $m$  measurement settings per party the classical bound is zero if

$$\sum_{j_1=0}^m \sum_{j_2=0}^m \cdots \sum_{j_n=0}^m B_{j_1 j_2 \dots j_n} a_{1j_1} a_{2j_2} \dots a_{nj_n} \leq 0 \quad (29)$$

for all deterministic local strategies, which are defined by the actual choices of  $a_{ij}$  ( $i = 1, \dots, n$ ;  $j = 1, \dots, m$ ), where each of them may take the value of zero or 1, while  $a_{i0} = 1$  for all  $i$ . The  $a_{i0}$  values appear in the subcorrelation terms. A deterministic strategy means that each party has a definite outcome for each measurement setting with probability 1. This outcome for the  $j$ th measurement of the  $i$ th party is given by  $a_{ij}$ . Let us denote the matrix with elements  $a_{ij}$  by  $\bar{a}$ . Then for permutationally symmetric inequalities Eq. (29) may be rewritten as

$$\sum_{j_1=0}^{j_2} \sum_{j_2=0}^{j_3} \cdots \sum_{j_n=0}^m B_{j_1 j_2 \dots j_n} C(\bar{a}, j_1, j_2, \dots, j_n) \leq 0, \quad (30)$$

where

$$\begin{aligned} C(\bar{a}, j_1, j_2, \dots, j_n) &= \frac{\pi(j_1, j_2, \dots, j_n)}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i j_{\sigma(i)}} \\ &= \frac{\pi(j_1, j_2, \dots, j_n)}{n!} \text{perm}[\bar{\alpha}(\bar{a}, j_1, j_2, \dots, j_n)]. \end{aligned} \quad (31)$$

The sum in the first line extends over all elements  $\sigma$  of the symmetric group  $S_n$ , that is, over all permutations of the numbers  $1, 2, \dots, n$ . We should only have summed over the independent permutations of the numbers  $j_1, j_2, \dots, j_n$ . The difference is taken care of by the factor in front of the summation, with  $\pi(j_1, j_2, \dots, j_n)$  denoting the number of independent permutations of its arguments. The sum itself is nothing else but the permanent of the  $n \times n$  square matrix  $\bar{\alpha}(\bar{a}, j_1, j_2, \dots, j_n)$  consisting of the  $j_1$ th,  $j_2$ th, ...,  $j_n$ th columns of the  $\bar{a}$  strategy matrix.

For the Bell inequality given by Eqs. (2)–(4), the condition (30) that the classical bound is zero may be written as

$$\begin{aligned} &2\text{perm}[\bar{\alpha}(\bar{a}, 1234 \dots mm \dots m)] \\ &+ \text{perm}[\bar{\alpha}(\bar{a}, 2234 \dots mm \dots m)] \\ &+ \text{perm}[\bar{\alpha}(\bar{a}, 2334 \dots mm \dots m)] \\ &+ \text{perm}[\bar{\alpha}(\bar{a}, 2344 \dots mm \dots m)] + \dots \\ &+ 2\frac{n-m+1}{n-m+2} \text{perm}[\bar{\alpha}(\bar{a}, 234 \dots mmm \dots m)] \\ &- 2\text{perm}[\bar{\alpha}(\bar{a}, 0234 \dots mm \dots m)] - Q(\bar{a}) \leq 0. \end{aligned} \quad (32)$$

Here we have used Eq. (31); we have substituted the actual values of  $\pi(j_1, j_2, \dots, j_n)$  and simplified the inequality by a factor of  $2(n-m+1)!$ . The  $Q(\bar{a}) \geq 0$  denotes the contribution of terms due to the negative  $n$ -party Bell coefficients of Eq. (4), multiplied by minus  $2(n-m+1)!$ .

A permanent is independent of the order of the rows and columns of the matrix. Therefore, we may rearrange the columns of the matrices appearing in Eq. (32) such that their  $j$ th column is the  $(j+1)$ th column of  $\bar{a}$  if  $j \leq m-1$ , the  $m$ th column if  $m-1 \leq j \leq n-1$ , and only their last column differs from each other. However, the sum of two permanents whose matrices differ from each other only in one single column (row) is the permanent of the matrix having the sum of those columns (rows) in that position. Furthermore, multiplying a permanent with a factor has the same effect as multiplying one of the columns (rows) of the matrix. Therefore, Eq. (32) may be rewritten as

$$\text{perm}[\bar{\beta}(\bar{a})] - Q(\bar{a}) \leq 0, \quad (33)$$

where

$$\begin{aligned} \beta_{ij} &= a_{i(j+1)} \quad j \leq m-1 \\ \beta_{ij} &= a_{im} \quad m-1 \leq j \leq n-1 \\ \beta_{in} &= 2a_{i1} + \sum_{k=2}^{m-1} a_{ik} + 2\frac{n-m+1}{n-m+2} a_{im} - 2. \end{aligned} \quad (34)$$

Our Bell inequality is appropriate if Eq. (32) holds for all deterministic strategies. The permanents appearing in the expression cannot be negative, as all the elements of the matrices involved are 0 or 1. Therefore, strategies giving zero for all terms with a positive sign trivially satisfy Eq. (32). We also do not have to consider explicitly any strategy leading to a nonzero factor for a Bell coefficient whose value does not affect the threshold efficiency [Eq. (4)]. The  $Q(\bar{a})$  depends linearly on such a coefficient; if we choose its absolute value large enough, we can always satisfy Eq. (32).

First let us consider strategies giving a positive value for the first term of Eq. (32). This is true if and only if there is at least one party having outcome 1 for the first measurement setting, at least one other party having outcome 1 for the second measurement setting, and so on, and finally, there are at least  $n-m+1$  other parties having outcome 1 for their  $m$ th measurement setting. For a permutationally invariant inequality, all strategies that differ from each other only in the order of parties give the same constraints for the Bell coefficients. Therefore, without sacrificing generality, we may assume that the first party's first measurement outcome is 1, the second party's second measurement outcome is 1, and so on, and finally, the last  $n-m$  party's  $m$ th measurement outcome is 1, that is,  $a_{ik} = 1$ , where  $k = \min(i, m)$ . We only have to consider strategies whose matrices contain zeros to the left of these elements in each row, that is,  $a_{ik'} = 0$  if  $0 < k' < k$ . If  $a_{ik'}$  were 1, then the factor of  $B_{12 \dots (i-1)k'(i+1) \dots mmm \dots}$  would not be zero. However, the quantum conditional probability associated with this coefficient is  $(\phi_{k'}/\phi_k)^2 \ll 1$  times that associated with the significant coefficients [this follows from Eq. (25)]; therefore we need not consider such a strategy explicitly indeed. The matrix of the remaining strategies for

$m = 6$  and  $n = 9$ , without the zeroth column, is as follows:

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 1 & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 1 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & 1 & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (35)$$

With such a strategy, matrix  $Q(\bar{a})$  appearing in Eqs. (32) and (33) is zero; terms contributing to it have zero factors. It is also clear that increasing the number of parties beyond  $m$  does not allow any more free parameters.

For this example Eq. (33) may be written as follows:

$$\text{perm} \begin{pmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{16} & a_{16} & a_{16} & \beta_{1n} \\ 1 & a_{23} & a_{24} & a_{25} & a_{26} & a_{26} & a_{26} & a_{26} & \beta_{2n} \\ 0 & 1 & a_{34} & a_{35} & a_{36} & a_{36} & a_{36} & a_{36} & \beta_{3n} \\ 0 & 0 & 1 & a_{45} & a_{46} & a_{46} & a_{46} & a_{46} & \beta_{4n} \\ 0 & 0 & 0 & 1 & a_{56} & a_{56} & a_{56} & a_{56} & \beta_{5n} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \beta_{mn} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \beta_{mn} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \beta_{mn} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \beta_{mn} \end{pmatrix} \leq 0. \quad (36)$$

Let us expand the permanent along its last row. There are  $n - m + 1$  ones in that row, all in columns equal to each other. These terms of the expansion give  $n - m + 1$  times the permanent of the matrix with that column and the last row deleted. The contribution from the last element will be  $\beta_{mn}$  times the permanent of the matrix with its last row and last column deleted. The sum of the contributions may be expressed as the permanent of a single matrix whose  $i$ th element in its last column is  $(n - m + 1)\beta_{in} + \beta_{mn}a_{im}$ , which we can calculate from Eqs. (34) and (35). The value we get is

$$(n - m + 1) \left( 2a_{i1} + \sum_{k=2}^{m-1} a_{ik} + 2 \frac{n - m}{n - m + 1} a_{im} - 2 \right), \quad (37)$$

which is just  $n - m + 1$  times the last column of the matrix with  $n - 1$  parties. Therefore the value of the permanent for  $n$  parties is  $n - m + 1$  times the value for  $n - 1$  parties. If we repeat this  $n - m$  times we get  $(n - m + 1)!$  times the permanent for  $n = m$ .

Let us consider  $n = m$ . For  $m = 6$  Eq. (33) may be written as

$$\text{perm} \begin{pmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \sum_{j=2}^6 a_{1j} \\ 1 & a_{23} & a_{24} & a_{25} & a_{26} & \sum_{j=3}^6 a_{2j} - 1 \\ 0 & 1 & a_{34} & a_{35} & a_{36} & \sum_{j=4}^6 a_{3j} - 1 \\ 0 & 0 & 1 & a_{45} & a_{46} & a_{45} + a_{46} - 1 \\ 0 & 0 & 0 & 1 & a_{56} & a_{56} - 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \leq 0. \quad (38)$$

The values in the last column follow from Eq. (34). If we expand the permanent along its last row we get the

permanent of a matrix of the same form corresponding to  $m - 1$  measurement settings. If we repeat this  $m - 1$  times, we finally get zero for the permanent. Therefore Eq. (33), and equivalently, Eq. (32) is satisfied as an equality. Although we demonstrated this result for the example of  $m = 6$ , it is obvious that it is true for any  $m$ .

So far we have dealt only with strategies that lead to a nonzero factor for the first term of Eq. (32). Now let us consider strategies giving zero for the first  $l - 1$  terms of Eq. (32) and nonzero for the  $l$ th term. If  $n = m = 8$  and  $l = 5$  the strategy matrix to be considered without the zeroth column is the following:

$$\begin{pmatrix} a_{11} & 1 & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & 1 & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & 1 & a_{35} & a_{36} & a_{37} & a_{38} \\ 0 & 0 & 0 & 0 & 1 & a_{46} & a_{47} & a_{48} \\ 0 & 0 & 0 & 0 & 1 & a_{56} & a_{57} & a_{58} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{67} & a_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (39)$$

Elements  $a_{k(l-1)} = 0$  and  $a_{kl} = 0$  for  $k < l$ ; otherwise the  $k$ th term of Eq. (32) would be positive. Furthermore,  $a_{ij} = 0$  if  $j < i$  for  $i > l$ . For example,  $a_{73}$  above is zero. If it were not, the factor multiplying  $B_{23345568}$  would be positive. The quantum conditional probability associated to this coefficient is small by a factor of  $(\phi_5/\phi_7)^2 \ll 1$  [see Eq. (25)]. For the interesting strategies a larger number of parties would mean an additional  $n - m$  copies of the last line, which contains a single 1 in the last position. Only this property of the matrix was used when we showed that increasing  $n$  beyond  $m$  introduces only a positive factor, which does not affect the validity of the inequality. Therefore it is enough to deal with  $n = m$ .

For the present example Eq. (33) may be written as

$$\text{perm} \begin{pmatrix} 1 & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & \beta_{1m} \\ a_{22} & 1 & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & \beta_{2m} \\ a_{32} & a_{33} & 1 & a_{35} & a_{36} & a_{37} & a_{38} & \beta_{3m} \\ 0 & 0 & 0 & 1 & a_{46} & a_{47} & a_{48} & \beta_{4m} \\ 0 & 0 & 0 & 1 & a_{56} & a_{57} & a_{58} & \beta_{5m} \\ 0 & 0 & 0 & 0 & 1 & a_{67} & a_{68} & \beta_{6m} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{78} & \beta_{7m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \beta_{8m} \end{pmatrix} \leq 0. \quad (40)$$

This permanent is equal to the product of the permanents of two blocks along the diagonal of sizes  $(l - 2) \times (l - 2)$  and  $(n - l + 2) \times (n - l + 2)$ . In the present case,

$$\text{perm} \begin{pmatrix} 1 & a_{13} & a_{14} \\ a_{22} & 1 & a_{24} \\ a_{32} & a_{33} & 1 \end{pmatrix} \times \text{perm} \begin{pmatrix} 1 & a_{46} & a_{47} & a_{48} & \beta_{4m} \\ 1 & a_{56} & a_{57} & a_{58} & \beta_{4m} \\ 0 & 1 & a_{67} & a_{68} & \beta_{4m} \\ 0 & 0 & 1 & a_{78} & \beta_{4m} \\ 0 & 0 & 0 & 1 & \beta_{4m} \end{pmatrix}. \quad (41)$$

If we substitute the values of the last column of the second matrix from Eq. (34), we get a matrix whose structure is very similar to the one in Eq. (38). There is one important difference. Now the last elements of each row always equal the sum of the preceding elements minus 2. In the case of Eq. (38), this is not true for the first row, the minus 2 is missing. Due to this difference, if we follow the steps we have taken in the case of Eq. (34), we will not get zero but  $-2$ . Therefore, Eq. (32) is satisfied now as a strict inequality.

To determine the values of the negative  $n$ -party Bell coefficients, it is necessary to calculate the values of more general permanents, which do not have zero entries at the right positions, making the calculation possible even analytically. There is no efficient way to evaluate a general permanent; therefore for larger  $m$  and  $n$  the determination of the explicit form of the Bell inequality may become infeasible. For small  $m$  and  $n$  it is not difficult; in the next section we show an appropriate inequality for  $n = m = 3$ .

## VI. THE 333 CASE

For  $m = n = 3$  a permutationally invariant inequality with the properties required may be defined by the following coefficients:

$$\begin{aligned} B_{023} &= -1 & B_{112} &= -1 & B_{113} &= -1 & B_{122} &= -2 \\ B_{123} &= +1 & B_{223} &= +1 & B_{233} &= +1. \end{aligned} \quad (42)$$

Bell coefficients having the same set of indices in different orders as any of the coefficients above have the same value. All other coefficients are zero.

We got this inequality by iterating a linear programming step and a semidefinite programming step. The former provides the Bell inequality of zero classical bound with no single-party marginals having the lowest critical detector efficiency from known matrix elements of the measurement operators. The procedure is the same as the one we used in Ref. [24]. In the semidefinite programming step we applied the method of Navascués *et al.* [26] at level 3 (that is, the maximum length of all tuples of operators is 3 in the sequence) to get an approximation for the matrix elements and a quite tight upper bound for the violation. As starting values, we have chosen three random settings on the Bloch sphere (the same for each party) along with a random symmetric three-qubit state. Note the initial qubit measurements generally do not confine the state to a three-qubit state during the iteration procedure. We have repeated the above iterative procedure many times with different (randomly generated) starting values. The best inequality we obtained is the one given above. Note that the only assumption we have made is that we allowed no single-party marginals, we actually allowed nonsymmetric inequalities, and the procedure constrained neither the state, nor the measurement operators, not even the dimensionality of the Hilbert space.

We applied the seesaw algorithm [27] to the inequality using three-qubit states. From this calculation it turned out that this space was enough to get the violation and  $\eta_{\text{crit}} = 1/2$ . From this calculation we could also determine the structure of the state and the behavior of the measurement operators. We also derived a Bell inequality with  $m = 3$  and  $n = 4$ ,

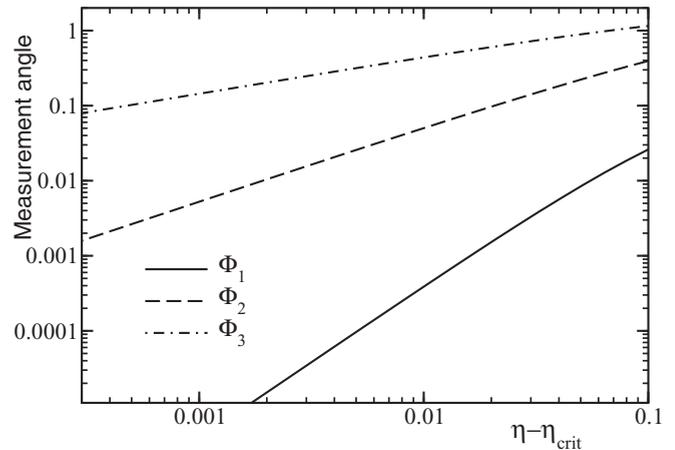


FIG. 1. The optimum measurement angles for the  $m = n = 3$  inequality as functions of the detector efficiency.

and we have studied its behavior with the seesaw algorithms. These results, and those for  $m = 2$  and  $n > 2$  by Larsson and Semitecolos [19], made it possible for us to make the generalization and to get the family of inequalities presented in this paper. Near  $\eta_{\text{crit}}$  all these inequalities can be violated by the  $|00\dots 0\rangle$  state, with a small admixture of the  $n$ -particle  $W$  state. For  $m = 2$  and any  $n$  these are provably the best inequalities as far as  $\eta_{\text{crit}}$  is concerned. For  $m = 3$  and  $n = 3$  or  $n = 4$ , we could not find any better with our numerical method. For larger values of  $n$  and  $m$  we cannot rule out that there are some other families of inequalities requiring some other types of states allowing even lower detection efficiencies.

We have calculated numerically the optimum measurement angles and the maximum violation as functions of the detector efficiency above the threshold efficiency  $\eta_{\text{crit}} = 0.5$  for the inequality given in Eq. (42). To get more accurate numbers, we did not use the seesaw algorithm to get these results, only the knowledge we had gained from it about the form of the solution. We used the analytical expressions given by Eqs. (17) and (22) for the matrix elements and optimized numerically the measurement angles and the mixing angle  $\alpha$ .

The results are shown in Figs. 1 and 2. The first measurement angle tends to zero at the fastest pace; it behaves as the square of  $\eta - \eta_{\text{crit}}$ . If we choose its value to be exactly zero, the violation changes very little. The second measurement angle is proportional with  $\eta - \eta_{\text{crit}}$ , while the third one behaves as the square root of  $\eta - \eta_{\text{crit}}$ . The violation is proportional with the sixth power of  $\eta - \eta_{\text{crit}}$ .

## VII. SUMMARY

In the present paper we have defined a family of binary-outcome  $n$ -party Bell inequalities with  $m \leq n$  measurement settings for each party, whose  $m > 2$  members can be violated by less detection efficiency than any other inequalities known so far. This gives upper bounds for the minimum value  $\eta_*$  of the critical efficiency achievable for such inequalities. The family is the generalization of the one given by Larsson and Semitecolos [19] for  $m = 2$ . Unfortunately, for  $m > 2$  there is still a gap between our upper bounds and the best lower bounds  $n/[(n-1)m+1]$  given by Massar and Pironio [20].

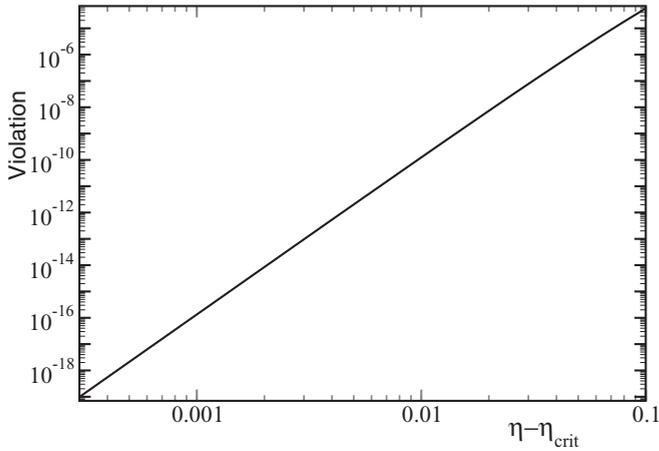


FIG. 2. The maximum violation of the  $m = n = 3$  inequality as a function of the detector efficiency.

For  $m = n = 3$  our upper bound is  $1/2$ , while the lower bound is  $3/7$ . As we have found no better Bell inequality while making assumptions neither about the symmetry nor about the violating state,  $\eta_*$  may well agree with our bound for both this case and also for  $m = 3, n = 4$ . For  $n = m$  our bound Eq. (8) gives  $\eta_* \leq 2/(m + 1)$ , while from Ref. [20]  $\eta_* \geq 1/(m - 1 + 1/m)$  follows, a factor of 2 difference for large  $m$ . Buhrman *et al.* [21] gives a worse upper bound of  $\eta_* \leq 8/m$ . When  $n \rightarrow \infty, \eta_* \leq 2/(m + 2)$ , as it follows from Eq. (8). In this case the lower bound is  $\eta_* \geq 1/m$ , so there is still a factor of 2 difference for large  $m$ . The upper bound of Ref. [21] behaves very similarly to ours; it is  $\eta_* \geq 1/m$  [see Eq. (4) in their paper]. It is interesting to note that  $\eta_*$  does not approach zero if  $n$  alone goes to infinity, it is proportional to  $1/m$ . However, if  $m \rightarrow \infty$  while  $n$  remains finite, the lower bound of Ref. [20] goes to zero. Reference [21] does not give a useful upper bound for this situation, while in the case of Ref. [22] the critical efficiency remains finite, proportional to  $1/n$ . Unfortunately, the present work tells nothing about the  $m > n$  case, so even the qualitative behavior of  $\eta_*$  in this case remains open.

For the present family of Bell inequalities the violating state approaches a product state near the threshold efficiency. The same behavior has been observed by Vallone *et al.* [8] for several other Bell inequalities, e.g., for the two-qubit chained inequalities with any number of measurement settings. They even concluded that the complementary behavior of entanglement and nonlocality might be general. In the multipartite case this cannot be so, as the inequalities given in Refs. [22] and [24]

are violated by the maximally entangled GHZ and  $W$  states near  $\eta_{\text{crit}}$ , respectively. In the bipartite case, a counterexample is shown in the Appendix. However, it is still possible that the conjecture by Vallone *et al.* is correct for inequalities with the smallest critical efficiency, the ones with  $\eta_{\text{crit}} = \eta_*$ .

Our multipartite Bell inequalities involve high-order correlators among the particles, which are challenging to access in the case of many particles. However, recently experimentally more friendly binary-outcome Bell inequalities have been constructed involving only two-body correlators [28]. We pose it as an interesting problem to find such low-order Bell inequalities which are suited to Bell violation using detectors of low efficiencies.

## ACKNOWLEDGMENTS

We acknowledge financial support from the Hungarian National Research Fund OTKA (No. K111734) and a János Bolyai grant of the Hungarian Academy of Sciences.

## APPENDIX: TWO-PARTY BELL INEQUALITY WHERE THE MAXIMAL ROBUSTNESS OF NONLOCALITY IS ATTAINED WITH A MAXIMALLY ENTANGLED STATE

To prove the title of the Appendix, let us consider the following Bell inequality:

$$I_2 \equiv P(A_1, B_1) + P(A_1, B_2) + P(A_2, B_1) - P(A_2, B_2) - cP(A_1) - cP(B_1) \leq 2 - 2c, \quad (\text{A1})$$

where  $c = \frac{2(2^{1/4}-1)}{\sqrt{2}-1} \simeq 0.9136$ . We choose that both Alice and Bob always output “0” in case of nondetection, and hence all measurements have binary outputs (it turns out that other choices will not lead to lower detection efficiency thresholds). Then the measurement probabilities are modified according to  $P(A_x, B_y) \rightarrow \eta^2 P(A_x, B_y)$ ,  $P(A_x) \rightarrow \eta P(A_x)$  and similarly for Bob. By introducing these expressions in Eq. (A1) and dividing by  $\eta^2$ , we obtain the detection-efficiency-dependent inequality:

$$I_2(\eta) = P(A_1, B_1) + P(A_1, B_2) + P(A_2, B_1) - P(A_2, B_2) - c \frac{P(A_1)}{\eta} - c \frac{P(B_1)}{\eta} \leq \frac{2 - 2c}{\eta^2}. \quad (\text{A2})$$

By setting  $\eta = c$ , we get the Clauser-Horne inequality on the l.h.s., whereas the r.h.s. becomes  $(\sqrt{2} - 1)/2$ , which is just the maximal quantum violation of the Clauser-Horne inequality attainable with a pair of maximally entangled qubits [29,30]. From this it follows that as  $\eta$  approaches  $c \simeq 0.9136$  from above, the maximal violation of the Bell inequality (A1) is given by a state converging to the maximally entangled two-qubit state.

- [1] J. S. Bell, On the Einstein Podolsky Rosen Paradox, *Physics* **1**, 195 (1964).  
 [2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, *Rev. Mod. Phys.* **86**, 419 (2014).  
 [3] A. Aspect, J. Dalibard, and G. Roger, Experimental test of Bell's inequalities using time-varying analyzers, *Phys. Rev. Lett.* **49**,

1804 (1982); A. Aspect, Bell's inequality test: More ideal than ever, *Nature (London)* **398**, 189 (1999).

- [4] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Experimental violation of a Bell's inequality with efficient detection, *Nature (London)* **409**, 791 (2001); M. Ansmann *et al.*, Violation of Bell's inequality in

- Josephson phase qubits, *ibid.* **461**, 504 (2009); J. Hofmann, M. Krug, N. Ortegel, L. Gérard, M. Weber, W. Rosenfeld, and H. Weinfurter, Heralded entanglement between widely separated atoms, *Science* **337**, 72 (2012); G. Carvacho *et al.*, Postselection-Loophole-Free Bell Test Over an Installed Optical Fiber Network, *Phys. Rev. Lett.* **115**, 030503 (2015).
- [5] B. G. Christensen *et al.*, Detection-Loophole-Free Test of Quantum Nonlocality, and Applications, *Phys. Rev. Lett.* **111**, 130406 (2013); M. Giustina *et al.*, Bell violation using entangled photons without the fair-sampling assumption, *Nature (London)* **497**, 227 (2013).
- [6] B. Hensen *et al.*, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, *Nature (London)* **526**, 682 (2015).
- [7] P. M. Pearle, Hidden-variable example based upon data rejection, *Phys. Rev. D* **2**, 1418 (1970); C. Branciard, Detection loophole in Bell experiments: How postselection modifies the requirements to observe nonlocality, *Phys. Rev. A* **83**, 032123 (2011).
- [8] G. Vallone, G. Lima, E. S. Gómez, G. Cañas, J. Å. Larsson, P. Mataloni, and A. Cabello, Bell scenarios in which nonlocality and entanglement are inversely related, *Phys. Rev. A* **89**, 012102 (2014).
- [9] A. Cabello and J. A. Larsson, Minimum Detection Efficiency for a Loophole-Free Atom-Photon Bell Experiment, *Phys. Rev. Lett.* **98**, 220402 (2007); N. Brunner, N. Gisin, V. Scarani, and C. Simon, Detection Loophole in Asymmetric Bell Experiments, *ibid.* **98**, 220403 (2007).
- [10] T. Vértesi, S. Pironio, and N. Brunner, Closing the detection loophole in Bell experiments using qudits, *Phys. Rev. Lett.* **104**, 060401 (2010).
- [11] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Violation of Bell's Inequality under Strict Einstein Locality Conditions, *Phys. Rev. Lett.* **81**, 5039 (1998); W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Violation of Bell Inequalities by Photons More Than 10 km Apart, *ibid.* **81**, 3563 (1998).
- [12] J. Å. Larsson, Loopholes in Bell inequality tests of local realism, *J. Phys. A* **47**, 424003 (2014).
- [13] V. Scarani, The device-independent outlook on quantum physics, *Acta Physica Slovaca* **62**, 347 (2012).
- [14] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography against Collective Attacks, *Phys. Rev. Lett.* **98**, 230501 (2007); U. Vazirani and T. Vidick, Fully Device-Independent Quantum Key Distribution, *ibid.* **113**, 140501 (2014).
- [15] S. Pironio *et al.*, Random numbers certified by Bell's theorem, *Nature (London)* **464**, 1021 (2010); R. Colbeck, Quantum and relativistic protocols for secure multi-party computation, Ph.D. thesis, University of Cambridge, 2006; R. Colbeck and A. Kent, Private randomness expansion with untrusted devices, *J. Phys. A* **44**, 095305 (2011).
- [16] D. Mayers and A. Yao, Self testing quantum apparatus, *Quantum Inf. Comput.* **4**, 273 (2004); T. H. Yang, T. Vértesi, J.-D. Bancal, V. Scarani, and M. Navascués, Robust and Versatile Black-Box Certification of Quantum Devices, *Phys. Rev. Lett.* **113**, 040401 (2014).
- [17] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, *Phys. Rev. Lett.* **23**, 880 (1969).
- [18] P. H. Eberhard, Background level and counter efficiencies required for a loophole-free Einstein-Podolsky-Rosen experiment, *Phys. Rev. A* **47**, R747 (1993); G. Lima, E. B. Inostroza, R. O. Vianna, J. Å. Larsson, and C. Saavedra, Optimal measurement bases for Bell tests based on the Clauser-Horne inequality, *ibid.* **85**, 012105 (2012).
- [19] J.-Å. Larsson and J. Semitecolos, Strict detector-efficiency bounds for  $n$ -site Clauser-Horne inequalities, *Phys. Rev. A* **63**, 022117 (2001).
- [20] S. Massar and S. Pironio, Violation of local realism versus detection efficiency, *Phys. Rev. A* **68**, 062109 (2003).
- [21] H. Buhman, P. Høyer, S. Massar, and H. Röhrig, Combinatorics and quantum nonlocality, *Phys. Rev. Lett.* **91**, 047903 (2003).
- [22] K. F. Pál, T. Vértesi, and N. Brunner, Closing the detection loophole in multipartite Bell tests using Greenberger-Horne-Zeilinger states, *Phys. Rev. A* **86**, 062111 (2012).
- [23] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bells Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, Holland, 1989), pp. 69–72.
- [24] K. F. Pál and T. Vértesi, Bell inequalities violated using detectors of low efficiency, *Phys. Rev. A* **92**, 022103 (2015).
- [25] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, *Phys. Rev. A* **62**, 062314 (2000).
- [26] M. Navascués, S. Pironio, and A. Acín, Bounding the set of quantum correlations, *Phys. Rev. Lett.* **98**, 010401 (2007); A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, *New J. Phys.* **10**, 073013 (2008).
- [27] R. F. Werner and M. M. Wolf, Bell inequalities and entanglement, *Quantum Inf. Comput.* **1**, 1 (2001); K. F. Pál and T. Vértesi, Maximal violation of a bipartite three-setting, two-outcome Bell inequality using infinite-dimensional quantum systems, *Phys. Rev. A* **82**, 022116 (2010).
- [28] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, and A. Acín, Detecting nonlocality in many-body quantum states, *Science* **344**, 1256 (2014); J. Tura, A. B. Sainz, T. Vértesi, A. Acín, M. Lewenstein, and R. Augusiak, Translationally invariant multipartite Bell inequalities involving only two-body correlators, *J. Phys. A* **47**, 424024 (2014); J. Tura, A. B. Sainz, T. Grass, R. Augusiak, A. Acín, and M. Lewenstein, Entanglement and nonlocality in many-body systems: A primer, [arXiv:1501.02733](https://arxiv.org/abs/1501.02733); J. Tura, R. Augusiak, A. B. Sainz, B. Lücke, C. Klempt, M. Lewenstein, and A. Acín, Nonlocality in many-body quantum systems detected with two-body correlators, *Ann. Phys.* **362**, 370 (2015).
- [29] B. S. Cirel'son, Quantum generalizations of Bell's inequality, *Lett. Math. Phys.* **4**, 93 (1980).
- [30] J. F. Clauser and M. A. Horne, Experimental consequences of objective local theories, *Phys. Rev. D* **10**, 526 (1974).