# Optical torque: Electromagnetic spin and orbital-angular-momentum conservation laws and their significance

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The physics involved in the fundamental conservation equations of the spin and orbital angular momenta leads to new laws and phenomena that are disclosed here. To this end, we analyze the scattering of an electromagnetic wave field by the canonical system constituted by a small particle, which is assumed to be dipolar in the wide sense. Specifically, under quite general conditions these laws lead to understanding the contribution and weight of each of those angular momenta to the electromagnetic torque exerted by the field on the object, which is shown to consist of an extinction and a scattering, or recoil, part. This leads to an interpretation of its effect different than that taken up until now by many theoretical and experimental works, and implies that a part of the recoil torque cancels the usually called intrinsic torque, which was often considered to be responsible for the particle spinning. In addition, we obtain the contribution of the spatial structure of the wave to this torque, unknown to this date, showing its effect in the orbiting of the object, and demonstrating that it often leads to a negative torque on a single particle, i.e., opposite to the incident helicity, producing an orbital motion contrary to its spinning. Furthermore, we establish a decomposition of the electromagnetic torque into conservative and nonconservative components in which the helicity and the spin angular momentum play a role analogous to the energy and its flux for electromagnetic forces. These phenomena are illustrated with examples of beams, also showing the difficulties of some paraxial formulations whose fields do not hold the transversality condition.

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## I. INTRODUCTION

There is a growing interest in phenomena related to the angular momenta of light and other electromagnetic fields. Advances in particle manipulation are making experiments, their assessment, interpretation with theories, and applications of increasing study and capability. This has opened a new area of research related to the twisting of both the polarization and the wave fronts of electromagnetic fields [1–20] and their effects on matter [21–36,38].

On the one hand, conservation laws for the helicity and spin of wave fields [7,12–16] were recently established based on initial extensions [11] of known conservation equations of electromagnetism. Such laws appear as fundamental as those for the energy and linear and angular momenta. In particular, [7] put forward the conservation of spin and orbital angular momenta separately. On the other hand, studies on the momenta transfer to matter and their mechanical action on bodies are enlarging the area of optical manipulation of objects. Among these, new bizarre effects like negative optical torques (i.e., opposite to the helicity of the illumination) on sets of particles have been predicted [35,39] with circularly polarized plane waves, as well experimentally observed with Gaussian beams on extended objects [36]. This phenomenon keeps an analogy with pulling forces [40–42] that have recently attracted so much attention.

However, theory and experiments on electromagnetic torques are less developed and understood than their counterparts of optical and electromagnetic forces [38,40–54]. Observations are more difficult to control and to quantitatively interpret with existing models. With few exceptions [33,35], most experimental [26,27] and theoretical [28–33,53] studies employ a static formulation (perhaps following the path of pioneering work [21]), which was shown [39] to be incomplete and not compatible with energy and angular momentum conservation. Only for extremely small (i.e., Rayleigh) particles in terms of the wavelength is the static approximation valid; this is further discussed in Sec. X, where the total electromagnetic torque is addressed.

Although several reports [22–25,27,33,38] realize the need of energy absorption by the object in order that it experiences a torque, no explicit demonstration exists of the role played in this effect by the variation of incident spin and orbital angular momenta, even though they are calculated in some cases after scattering [8,9,33,34]. Moreover, studies based on the static approximation [26–29,31,53] deal only with the so-called intrinsic torque which, as shown below, cannot account for the angular momentum transfer nor describe the resulting torque experienced by the object through energy absorption.

In a recent work [55] we established the significance of the conservation of electromagnetic helicity for scattering objects and its relevance for energy transfer between small particles and molecules, as well as for circular dichroism. Now the aim of this paper is twofold. On the one hand, we integrate the conservation laws for the spin and orbital angular momenta, extracting their physical meaning for the interaction of fields and objects, specifically in the scattering of a wave field by a small particle. On the other hand, we predict the contributions of each of these angular momenta conservation laws to the optical torque exerted by the field on the object, also taking into account that part due to the spatial structure of the field. This is illustrated with a generally magnetodielectric bi-isotropic [56] dipolar particle in the wide sense, i.e., whose scattering is fully described by the first-order partial waves (namely, Mie coefficients if it were a sphere [54,57,58]). This analysis opens a collection of new phenomena related to the electromagnetic torque, susceptible

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to further investigation from the theoretical and experimental points of view. Our discussion is classical, however, if one considers the quantum nature of light; these results may be obtained in terms of the number operators associated with left and right circular polarized modes, expressed by the product of the corresponding annihilation and creation photon operators (see [59–61]), especially since methods to identify twisted beams with different topological charge have been devised [61–63].

A Coulomb gauge is chosen dealing with fields that satisfy the transversality condition. This allows a separation of the spin and the orbital angular momenta, thus conferring sense to the existence of their respective conservation laws [7,13] separately. It is known, however, that certain formulations of optical beams have a problem with the transversality condition, and this is also noticed in this work. Hence, within this context, and within the validity of these conservation laws, we reach the following results.

(1) The conservation laws for the spin and orbital angular momenta lead to their respective contributions to the time-averaged electromagnetic torque on the object, which from each of them is composed of two parts: (a) an *extinction torque* due to the extraction of the corresponding (spin or orbital) angular momentum from the incident wave by scattering from the body, described by the interference between the incident and the scattered fields; (b) a *scattering* or *recoil torque* due to the scattered field interfering with itself.

(2) The recoil torque due to the orbital-angular-momentum conservation is equal to that yielded by the spin angular momentum conservation; both of them sum to produce the total recoil torque, which is the same as that derived from the conservation of total angular momentum [39] and coincides with its emission rate by the electric and/or magnetic dipole induced on the particle, as it should.

(3) The extinction torque obtained from the spin angular momentum conservation contains two terms: One is the so-called intrinsic torque by several authors so far, usually employed in most studies on dipolar particles based on the static theory, and which, however, we show to be canceled by a part of the total recoil torque, which is also intrinsic as it does not depend on the origin of coordinates; the other term comes from the spatial structure of the incident field which, in turn, is equal to twice the corresponding extinction torque yielded by the conservation of orbital angular momentum. If the incident wave is a plane, obviously there is no term due to the field spatial structure; hence, the orbital-angular-momentum conservation yields no extinction part, which is consistent with the absence of incident orbital angular momentum in such a wave. Also, in this latter case, the above-mentioned cancellation conveys a transfer of spin angular momentum to the particle accounted for by the remaining part of the recoil torque through absorption of the incident energy.

(4) Therefore, if the incident field is a plane wave, according to point 3 above, the existence of a recoil torque stemming from the conservation of orbital angular momentum has to come from the incident spin angular momentum, this being a manifestation of the *spin-orbit interaction* which is included in the conservation laws.

(5) For a circularly polarized incident plane wave, if the particle produces a circularly polarized scattered field, (for which it is sufficient that its electric and magnetic polarizabilities coincide), then the electromagnetic torque and force on that particle become proportional to each other.

(6) Taking into account the spatial structure of the field, the torques derived from the spin and orbital conservation laws admit a decomposition into conservative and nonconservative components, analogous to that of electromagnetic forces. There is a *gradient component* of the torque, where now the wave-field *helicity* plays a role analogous to that of the energy for the optical force and hence may be employed as *rotational optical tweezers* that aligns the torque acting on the particle along an equilibrium direction. Likewise, there is a component given by the *spin angular momentum* (which coincides with the helicity flow in the dual-symmetric formulation for quasimonochromatic fields [7] here employed), analogous to the radiation pressure, or Poynting vector part, of the force. In addition, there are other nonconservative components that describe torques due to circulation of field vortices around the dipolar object.

(7) In contrast with plane waves, incident fields with certain spatial structures, and some beams in particular, may easily produce negative electromagnetic torques on one single small particle.

## II. THE FLOW OF HELICITY

We address time-harmonic fields with electric and magnetic vectors: $\mathcal{E}(\mathbf{r},t) = \mathrm{Re}[\mathbf{E}(\mathbf{r})\exp(-i\omega t)]$ and $\mathcal{B}(\mathbf{r},t) = \mathrm{Re}[\mathbf{B}(\mathbf{r})\exp(-i\omega t)]$, a functional form that also applies to all potentials and currents. Re denotes real part. Although such fields are not source free [64,65], they are considered in regions with no sources so that they admit an angular spectrum representation of plane-wave components [65,66], either convergent, divergent, or both, satisfying at $kr \to \infty$ Sommerfeld's radiation condition. Such fields are, of course, transversal; namely, they hold Maxwell's equations: $\nabla \cdot \mathcal{E} = 0$, $\nabla \cdot \mathcal{B} = 0$.

In a nonabsorbing dielectric medium of refractive index $n = \sqrt{\epsilon\mu}$ ($\epsilon$ and $\mu$ representing the dielectric permittivity and the magnetic permeability), the helicity density $\mathscr{H}$, the flow of helicity density $\mathscr{F}$, and the tensor $\mathscr{N}_{ij}$ of density of flow of helicity flux of this light field are defined in the dual-symmetric formulation as (cf. [7,13,67])

$$\mathscr{H} = \frac{1}{2}\left(\frac{1}{\mu}\mathcal{A}\cdot\mathcal{B} - \epsilon\mathcal{C}\cdot\mathcal{E}\right), \tag{1}$$

$$\mathscr{F} = \frac{c}{2\mu}(\mathcal{E}\times\mathcal{A} + \mathcal{B}\times\mathcal{C}), \tag{2}$$

$$\mathscr{N}_{ij} = \frac{c^2}{2\mu}\left[\mathcal{E}_i\mathcal{C}_j + \mathcal{E}_j\mathcal{C}_i - \frac{1}{\epsilon\mu}(\mathcal{A}_i\mathcal{B}_j + \mathcal{A}_j\mathcal{B}_i) + \delta_{ij}\left(\frac{1}{\epsilon\mu}\mathcal{A}\cdot\mathcal{B} - \mathcal{E}\cdot\mathcal{C}\right)\right], \quad (i,j = 1,2,3). \tag{3}$$

In (1) and (2) $\mathcal{A}$ and $\mathcal{C}$ are vector potentials such that $\mathcal{B} = \nabla\times\mathcal{A}$ and $\mathcal{E} = -\nabla\times\mathcal{C}$ and are transversal in a Coulomb gauge [13]: $\nabla\cdot\mathcal{A} = \nabla\cdot\mathcal{C} = 0$. From Maxwell's equations,

$$\dot{\mathcal{A}} = -c\mathcal{E}, \quad \dot{\mathcal{C}} = -\frac{c}{\epsilon\mu}\nabla\times\mathcal{A} + \frac{4\pi}{\epsilon}\mathcal{K}, \tag{4}$$

where the overdot means $\partial_t$ and $c$ is the speed of light in vacuum. In writing $\dot{\mathcal{C}}$ in (4) we have taken into account that the electric current density, which we denote as $\mathcal{J}$, is transversal since the existence of $\mathcal{A}$ and the law $\boldsymbol{\nabla}\cdot\epsilon\mathcal{E}=4\pi\rho$ convey that the electric charge density $\rho$ and any (static) scalar potential are zero [68], and thus $\mathcal{J}$ has been expressed as $\mathcal{J}=\boldsymbol{\nabla}\times\mathcal{K}$.

The transversality of $\mathcal{A}$ and $\mathcal{C}$ also amounts through the two first Eqs. (4) to that of $\mathcal{E}$ and $\mathcal{B}$, and hence is compatible with the above argumentation based on their angular spectrum [5], fulfilled by most optical wave fields outside near-field regions. In particular, this also involves that any far-zone scattered or radiated field, which is given by the propagating part of its angular spectrum [65,66], is gauge invariant. This agrees with [5,9] and implies that since both $\mathcal{E}$ and $\mathcal{B}$ are gauge invariant, the angular momentum $\mathbf{J}$ of these fields admits a decomposition $\mathbf{J}=\mathbf{L}+\mathscr{F}$ into an orbital $\mathbf{L}$ [proportional to $\epsilon\mathcal{E}\cdot(\mathbf{r}\times\boldsymbol{\nabla})\mathcal{A}+\mu^{-1}\mathcal{B}\cdot(\mathbf{r}\times\boldsymbol{\nabla})\mathcal{C}$] and a spin $\mathscr{F}$ [proportional to $\epsilon\mathcal{E}\times\mathcal{A}+\mu^{-1}\mathcal{B}\times\mathcal{C}$; see above] angular momenta, both $\mathbf{L}$ and $\mathscr{F}$ also being gauge invariant.

Therefore, it makes sense to study the conservation laws for the spin and orbital angular momenta separately. However, as we discuss in one example, sometimes the transversality condition is not taken into account, dealing with some representations of optical beams which do not fulfill it, thus preventing their inclusion in such decomposition of the angular momentum and, in general, in any theory based on such transversality equations.

From the above conditions, the definitions $\mathscr{H}$, $\mathscr{F}$, and $\mathscr{N}_{ij}$ hold the continuity equations [7,13],

$$\dot{\mathscr{H}}+\boldsymbol{\nabla}\cdot\mathscr{F}=-\mathscr{P}, \tag{5}$$

$$\dot{\mathscr{F}}+\boldsymbol{\nabla}\cdot\mathscr{N}_{ij}=-\mathscr{R}, \tag{6}$$

where the dissipation on transmission of spin flow by interaction of the fields with matter is given by $\mathscr{P}=2\pi(\mathcal{E}\cdot\mathcal{K}-\mathcal{J}\cdot\mathcal{C})$ and $\mathscr{R}=(2\pi c/\epsilon\mu)(\mathcal{K}\times\mathcal{B}+\mathcal{J}\times\mathcal{A})$.

Equations (5) and (6) are conservation equations for the helicity and its flow, respectively, and are as fundamental as those for the energy, linear, and angular momenta. We concentrate on (6), where $\mathscr{R}$ is linked to the absorption by the particle, which gives rise, as shown later, to the mechanical action that it experiences as a consequence of the conservation of the helicity flow.

The time harmonicity of the fields and potentials converts the quantities of Eqs. (4)–(6) into

$$\mathbf{A}=-\frac{i}{k}\mathbf{E},\quad \mathbf{C}=-\frac{i}{\epsilon}\left[\frac{\mathbf{B}}{k\mu}-\frac{4\pi}{\omega}\boldsymbol{K}\right], \tag{7}$$

and the time-averaged (denoted by $\langle\cdot\rangle$):

$$\mathscr{H}=\langle\mathscr{H}\rangle=\frac{1}{2k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}(\mathbf{E}\cdot\mathbf{B}^*), \tag{8}$$

$$\mathscr{F}=\langle\mathscr{F}\rangle=\frac{c}{4nk}\mathrm{Im}\left(\epsilon\mathbf{E}^*\times\mathbf{E}+\frac{1}{\mu}\mathbf{B}^*\times\mathbf{B}\right)=\mathscr{F}_e+\mathscr{F}_m, \tag{9}$$

$\mathscr{F}_e$ and $\mathscr{F}_m$ being the electric and magnetic parts, respectively, of $\mathscr{F}$, which now coincides with the spin angular momentum density.

Also,

$$\mathscr{N}_{ij}=\langle\mathscr{N}_{ij}\rangle=\frac{c^2}{2kn\mu}\mathrm{Im}[E_i^*B_j+E_j^*B_i-\delta_{ij}\mathbf{E}^*\cdot\mathbf{B}], \tag{10}$$

which is the density of spin angular momentum flow tensor. $i,j=1,2,3$, Im denotes imaginary part, and $k=n\omega/c$, and we have written $\mathcal{A}=\mathrm{Re}[\mathbf{A}(\mathbf{r})\exp(-i\omega t)]$, $\mathcal{C}=\mathrm{Re}[\mathbf{C}(\mathbf{r})\exp(-i\omega t)]$, $\mathcal{J}=\mathrm{Re}[\boldsymbol{J}(\mathbf{r})\exp(-i\omega t)]$, and $\mathcal{K}=\mathrm{Re}[\boldsymbol{K}(\mathbf{r})\exp(-i\omega t)]$. Also $\mathscr{P}$ in (5) and $\mathscr{R}$ in (6) are replaced by the respective time-averaged quantities

$$\langle\mathscr{P}\rangle=\pi\left[\frac{2}{ck}\sqrt{\frac{\mu}{\epsilon}}\boldsymbol{\nabla}\cdot\mathrm{Im}(\boldsymbol{K}\times\mathbf{B}^*)-\frac{1}{kn}\mathrm{Im}(\boldsymbol{J}\cdot\mathbf{B}^*)+\frac{4\pi}{ck}\sqrt{\frac{\mu}{\epsilon}}\mathrm{Im}(\boldsymbol{J}\cdot\boldsymbol{K}^*)+\mathrm{Re}(\mathbf{E}\cdot\boldsymbol{K}^*)\right], \tag{11}$$

$$\langle\mathscr{R}_i\rangle=-\frac{\pi c}{\epsilon\mu}\{c\mathrm{Im}(\mathbf{E}\times\boldsymbol{J}^*)_i+\mathrm{Re}(\mathbf{B}\times\boldsymbol{K}^*)_i+\frac{2\mu}{c}\partial_j[\delta_{ij}\mathrm{Im}(\mathbf{E}\cdot\boldsymbol{K}^*)-\mathrm{Im}(E_iK_j^*+E_jK_i^*)]\}. \tag{12}$$

At this point it is pertinent to comment that in the case dealt with here of time-harmonic fields, Maxwell's equations and the above relations show that (8) and (9) are proportional to Lipkin's zilches [11], used in recent works as the field chirality $\mathscr{K}$ and flow of chirality $\mathscr{S}$ [16,17], as well as the tensor of spin flow $\mathscr{T}_{ij}$ [13]:

$$\mathscr{K}=\langle\mathscr{K}\rangle=k^2\mathscr{H}=k^2\langle\mathscr{H}\rangle, \tag{13}$$

$$\mathscr{S}=\langle\mathscr{S}\rangle=k^2\mathscr{F}=k^2\langle\mathscr{F}\rangle, \tag{14}$$

$$\mathscr{T}_{ij}=\langle\mathscr{T}_{ij}\rangle=k^2\mathscr{N}_{ij}=k^2\langle\mathscr{N}_{ij}\rangle. \tag{15}$$

The dissipative terms that appear in the continuity equations for $\mathscr{K}$, $\mathscr{S}$, and $\mathscr{T}_{ij}$, are, however, different from those of (5) and (6). Reference [13] argues that $\mathscr{F}$ and $\mathscr{N}_{ij}$ are the quantities with dimensions of angular momentum, rather than the zilches, and $\langle\mathscr{H}\rangle$, $\langle\mathscr{F}\rangle$, and $\langle\mathscr{N}_{ij}\rangle$ are those to deal with in measurements. We follow this, although for these monochromatic fields both kind of magnitudes should, as shown in (13)–(15), lead to the same interpretation of effects.

## III. SCATTERING AND THE CONSERVATION OF SPIN ANGULAR MOMENTUM

We integrate (6) in any volume $v$ that contains the object with its distribution of currents. The divergence term is transformed into the flow of an entity across the surface $\Sigma$ of this volume that will eventually be taken as a large sphere.

Let us first consider a monochromatic, elliptically polarized, plane-wave incident on a body of arbitrary form enclosed in $v$ (cf. Fig. 1). The field at any point of the surrounding medium may be represented as the sum of the incident and scattered fields; thus, its space-dependent vectors are $\mathbf{E}(\mathbf{r})=\mathbf{E}^{(i)}(\mathbf{r})+\mathbf{E}^{(s)}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})=\mathbf{B}^{(i)}(\mathbf{r})+\mathbf{B}^{(s)}(\mathbf{r})$.

The incident field is written as $\mathbf{E}^{(i)}=\mathbf{e}^{(i)}e^{ik(\mathbf{s}_i\cdot\mathbf{r})}$, $\mathbf{B}^{(i)}=\mathbf{b}^{(i)}e^{ik(\mathbf{s}_i\cdot\mathbf{r})}$, and that scattered in the far zone is written $\mathbf{E}^{(s)}=\mathbf{e}(\mathbf{s})\exp(ikr)/r$, $\mathbf{B}^{(s)}=\mathbf{b}(\mathbf{s})\exp(ikr)/r$, so that, of

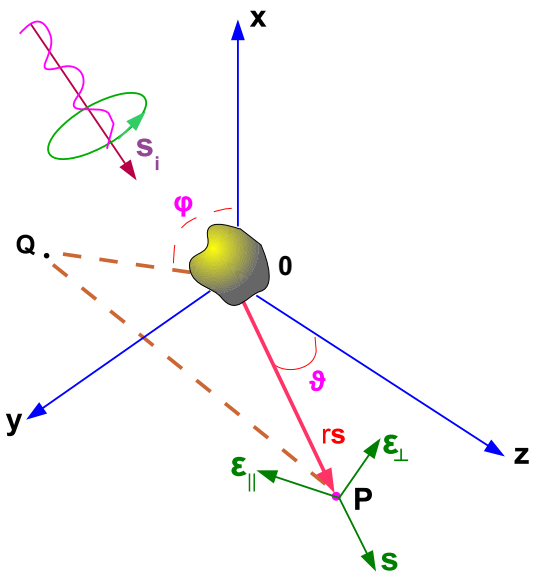


FIG. 1. (Color online) Illustration of an elliptically polarized plane-wave incident on a polarizable particle of arbitrary shape. The fields are evaluated at the point **P** of coordinates $(r,\theta,\phi)$ and position vector $\mathbf{r}=r\mathbf{s}$ (**s** being the unit vector along **r**), which eventually belongs to a sphere of integration of radius $r$, centered at some point $\mathbf{r}_0$ of the particle. $\mathbf{r}_0$ acts as the framework center **0**. The point **Q** is the projection of **P** on the plane $OXY$; the scattering plane being $OPQ$. We show the three orthonormal vectors: **s**, $\boldsymbol{\epsilon}_\parallel$ (in the plane $OPQ$ and in the sense of rotation of $\theta$), and $\boldsymbol{\epsilon}_\perp$ (normal to $OPQ$).

course, $\mathbf{b}^i=n\mathbf{s}_i\times\mathbf{e}^i$, $\mathbf{e}^i\cdot\mathbf{s}_i=\mathbf{b}^i\cdot\mathbf{s}_i=0$; $\mathbf{b}=n\mathbf{s}\times\mathbf{e}$, $\mathbf{e}\cdot\mathbf{s}=\mathbf{b}\cdot\mathbf{s}=0$.

Then the time-averaged density of spin angular momentum flow may be written $\mathscr{N}_{ij}=\mathscr{N}_{ij}^{(i)}+\mathscr{N}_{ij}^{(s)}+\mathscr{N}_{ij}'$, where

$$\mathscr{N}_{ij}^{(i)}=\frac{c^2}{2kn\mu}\mathrm{Im}\left[E_i^{(i)*}B_j^{(i)}+E_j^{(i)*}B_i^{(i)}-\delta_{ij}\mathbf{E}^{(i)*}\cdot\mathbf{B}^{(i)}\right], \quad (10a)$$

$$\mathscr{N}_{ij}^{(s)}=\frac{c^2}{2kn\mu}\mathrm{Im}\left[E_i^{(s)*}B_j^{(s)}+E_j^{(s)*}B_i^{(s)}-\delta_{ij}\mathbf{E}^{(s)*}\cdot\mathbf{B}^{(s)}\right], \quad (10b)$$

$$\mathscr{N}_{ij}'=\frac{c^2}{2kn\mu}\mathrm{Im}\left[E_i^{(i)*}B_j^{(s)}+E_i^{(s)*}B_j^{(i)}+E_j^{(i)*}B_i^{(s)}+E_j^{(s)*}B_i^{(i)}-\delta_{ij}[\mathbf{E}^{(i)*}\cdot\mathbf{B}^{(s)}+\mathbf{E}^{(s)*}\cdot\mathbf{B}^{(i)}]\right]. \quad (10c)$$

From Eq. (6) we write the rate $\mathcal{L}$ at which this density flows on interaction of the incident wave with the body, in terms of its outward flux: $\mathcal{L}=-(n^2/4\pi c^2)\int_\Sigma d\Omega r^2\mathscr{N}_{ij}\cdot\mathbf{s}$ through the surface $\Sigma$ of any large sphere of radius $r$ with center at some point $\mathbf{r}_0$ of the body. $d\Omega$ is the element of solid angle and **s** denotes the outward normal. That is, according to Eq. (6),

$$\mathcal{L}=\mathcal{L}^{(i)}+\mathcal{L}^{(s)}+\mathcal{L}', \quad (16)$$

where $\mathcal{L}^{(i)}$, $\mathcal{L}^{(s)}$, and $\mathcal{L}'$ are respectively the $\Sigma$ integrals of the projections on **s** of $-\mathscr{N}_{ij}^{(i)}$, $-\mathscr{N}_{ij}^{(s)}$, and $-\mathscr{N}_{ij}'$ on the surface of the sphere, respectively. Of course, from Eq. (6), $\mathcal{L}=(n^2/4\pi c^2)\int_\Sigma drd\Omega r^2\langle\mathscr{R}\rangle$.

From these equations, $\mathcal{L}^{(i)}=0$, so that (16) becomes

$$\mathcal{L}=\mathcal{L}^{(s)}+\mathcal{L}'. \quad (17)$$

The integrals of $-\mathscr{N}_{ij}^{s}\cdot\mathbf{s}$ and $-\mathscr{N}_{ij}'\cdot\mathbf{s}$ across $\Sigma$ leave the nonzero terms [cf. (10)]

$$\begin{aligned}\mathcal{L}^{(s)}&=-\frac{n^2}{4\pi c^2}\int_\Sigma d\Omega r^2\mathscr{N}_{ij}^{(s)}s_j\\&=\frac{n}{8\pi k\mu}\int_\Sigma d\Omega r^2\mathrm{Im}[\mathbf{E}^{(s)*}\cdot\mathbf{B}^{(s)}]\mathbf{s}\\&=-\frac{n^2}{8\pi k\mu}\int_\Sigma d\Omega\mathrm{Im}\{\mathbf{s}\cdot[\mathbf{e}^{(s)*}(\mathbf{s})\times\mathbf{e}^{(s)}(\mathbf{s})]\}\mathbf{s}\\&=-\frac{1}{8\pi k\mu}\int_\Sigma d\Omega\mathrm{Im}\{\mathbf{s}\cdot[\mathbf{b}^{(s)*}(\mathbf{s})\times\mathbf{b}^{(s)}(\mathbf{s})]\}\mathbf{s}\\&=-\frac{1}{16\pi k}\int_\Sigma d\Omega\mathrm{Im}\{\epsilon\mathbf{s}\cdot[\mathbf{e}^{(s)*}(\mathbf{s})\times\mathbf{e}^{(s)}(\mathbf{s})]\\&\quad+\frac{1}{\mu}\mathbf{s}\cdot[\mathbf{b}^{(s)*}(\mathbf{s})\times\mathbf{b}^{(s)}(\mathbf{s})]\}\mathbf{s}, \quad (18)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}'&=-\frac{n^2}{4\pi c^2}\int_\Sigma d\Omega r^2\mathscr{N}_{ij}'s_j\\&=\frac{n}{8\pi k\mu}\int_\Sigma d\Omega R^2\mathrm{Im}[\mathbf{E}^{(i)*}\cdot\mathbf{B}^{(s)}+\mathbf{E}^{(s)*}\cdot\mathbf{B}^{(i)}]\mathbf{s}\\&=-\frac{n}{2\mu k^2}\mathrm{Re}[\mathbf{B}^{(i)*}(\mathbf{r}_0)\cdot\mathbf{e}(\mathbf{s}_i)]\mathbf{s}_i\\&=\frac{n}{2\mu k^2}\mathrm{Re}[\mathbf{E}^{(i)*}(\mathbf{r}_0)\cdot\mathbf{b}(\mathbf{s}_i)]\mathbf{s}_i. \quad (19)\end{aligned}$$

In deriving (19) we have used Jones' lemma based on the principle of the stationary phase [69,70]:

$$\frac{1}{r}\int d\Omega r^2F(\mathbf{s})e^{-ik(\mathbf{s}_i\cdot\mathbf{s})r}\sim\frac{2\pi i}{k}[F(\mathbf{s}_i)e^{-ikr}-F(-\mathbf{s}_i)e^{ikr}]. \quad (20)$$

Notice from (9) and (10) that all these quantities of the conservation law (17)–(19) are also time averages. Equation (18) conveys the ellipticity of the electric or the magnetic field, as well as the dual-symmetric one containing both **e** and **b**. This electric-magnetic duality is also seen in (19).

Equation (17), together with (18) and (19), constitutes the law for the moment of the force exerted by the incident wave on the object, expressed in terms of the flux $\mathcal{L}$ of the spin flow tensor $\mathscr{N}_{ij}$ across any closed surface surrounding it. As seen, this stems from the conservation of the spin angular momentum density and, like the optical force [54], it has a contribution $\mathcal{L}^{(s)}$ of the scattered field plus one $\mathcal{L}'$ arising from the interference between this scattered field and the incident one.

This force moment $\mathcal{L}$ constitutes the contribution to the *electromagnetic torque* on the scattering object (acting the ellipticity of the scattered field as a factor in the integration) by extinction of the incident spin angular momentum and its transfer to the body at a rate proportional to a certain amplitude component of the scattered wave, projected along the polarization of the incident wave as a result of their mutual interference in the forward direction $\mathbf{s}=\mathbf{s}_i$.

## IV. CONSEQUENCES FOR A MAGNETODIELECTRIC BI-ISOTROPIC DIPOLAR PARTICLE

### A. Incident plane wave

Let us consider a magnetodielectric bi-isotropic particle, dipolar in the wide sense, i.e., characterized by its polarizabilities: electric $\alpha_e$, magnetic $\alpha_m$, and magnetoelectric $\alpha_{em}$, $\alpha_{me}$, such that, for example, if it is a sphere these quantities are given by the first-order Mie coefficients $a_1$, $b_1$, and $c_1$ as $\alpha_e = i\frac{3}{2k^3}a_1$, $\alpha_m = i\frac{3}{2k^3}b_1$, $\alpha_{em} = i\frac{3}{2k^3}c_1$, $\alpha_{me} = i\frac{3}{2k^3}d_1$, respectively [54,57,58]. Then the electric and magnetic dipole moments, $\mathbf{p}$ and $\mathbf{m}$, on the particle, induced by the incident field are

$$\mathbf{p} = \alpha_e\mathbf{E}^{(i)} + \alpha_{em}\mathbf{B}^{(i)}, \quad \mathbf{m} = \alpha_{me}\mathbf{E}^{(i)} + \alpha_m\mathbf{B}^{(i)}. \tag{21}$$

If, in addition, such sphere is chiral, then $c_1 = -d_1$ [58] and hence $\alpha_{em} = -\alpha_{me}$.

The amplitudes of the fields scattered by this particle in the far zone are

$$\mathbf{e}(\mathbf{s}) = k^2\frac{e^{ikr}}{r}\left[\epsilon^{-1}(\mathbf{s}\times\mathbf{p})\times\mathbf{s} - \sqrt{\frac{\mu}{\epsilon}}(\mathbf{s}\times\mathbf{m})\right], \tag{22}$$

$$\mathbf{b}(\mathbf{s}) = k^2\frac{e^{ikr}}{r}\left[\mu(\mathbf{s}\times\mathbf{m})\times\mathbf{s} + \sqrt{\frac{\mu}{\epsilon}}(\mathbf{s}\times\mathbf{p})\right]. \tag{23}$$

Introducing (22) and (23) into (18) and (19), evaluating the angular integrals, and substituting the results in (17), we obtain

$$\mathcal{L}^{(s)} = -\frac{k^3}{6}\mathrm{Im}\left\{\frac{1}{\epsilon}\mathbf{p}^*\times\mathbf{p} + \mu\mathbf{m}^*\times\mathbf{m}\right\} \tag{24}$$

and

$$\mathcal{L}' = -\frac{n}{2\mu}\mathrm{Re}\left[\left(\frac{1}{\epsilon}\mathbf{p}\cdot\mathbf{B}^{(i)*} - \mu\mathbf{m}\cdot\mathbf{E}^{(i)*}\right)\mathbf{s}_i\right]. \tag{25}$$

Equation (24) *is half the rate of radiated angular momentum* by the dipolar object [68], or equivalently half the *recoil optical torque* on the particle which one would obtain from the conservation of the total angular momentum by integration of Maxwell's stress tensor $T_{ij}$: $\int_\Sigma dRd\Omega r\mathbf{s}\times T_{ij}s_j$ [39].

Interestingly, since $\mathbf{p}\cdot\mathbf{s}_i = \mathbf{m}\cdot\mathbf{s}_i = 0$, (25) may also be expressed as

$$\mathcal{L}' = \langle\mathbf{\Gamma}_0\rangle. \tag{26}$$

$\langle\mathbf{\Gamma}_0\rangle$ is the *extinction electromagnetic torque*, analogous to the extinction energy in the optical theorem [35,54,70],

$$\langle\mathbf{\Gamma}_0\rangle = \tfrac{1}{2}\mathrm{Re}[\mathbf{p}\times\mathbf{E}^{(i)*} + \mathbf{m}\times\mathbf{B}^{(i)*}]. \tag{27}$$

Sometimes $\langle\mathbf{\Gamma}_0\rangle$ is called "the intrinsic torque" [31,53] exerted by the incident plane wave on the particle. However, $\mathcal{L}^{(s)}$ [Eq. (24)], is seen to be also an intrinsic torque component.

It is useful to consider a Cartesian framework (cf. Fig. 1) where the elliptically polarized incident plane wave has $\mathbf{s}_i = (0,0,1)$, its electric vector being written in a helicity basis $\boldsymbol{\epsilon}^\pm = (1/\sqrt{2})(1,\pm i,0)$ as the sum of a left-hand and a right-hand circularly polarized plane wave, so that $\mathbf{e}_i = (e_{ix},e_{iy},0) = e_i^+\boldsymbol{\epsilon}^+ + e_i^-\boldsymbol{\epsilon}^-$ and $\mathbf{b}_i = (b_{ix},b_{iy},0) = n(-e_{iy},e_{ix},0) = b_i^+\boldsymbol{\epsilon}^+ + b_i^-\boldsymbol{\epsilon}^- = -ni(e_i^+\boldsymbol{\epsilon}^+ - e_i^-\boldsymbol{\epsilon}^-)$, the sign $\pm$ standing for left-circularly polarized (LCP) (+) and right-circularly polarized (−), respectively. In such representation, the incident helicity acquires the form: $\mathscr{H}^i = (\epsilon/k)\mathrm{Im}[e_{ix}^*e_{iy}] = (\epsilon/2k)S_3 = (\epsilon/2k)[|e_i^+|^2 - |e_i^-|^2]$, which clearly expresses this magnitude as the difference between left circular and right circular intensities of the field, extensively exploited in dichroism [16,17,71] and enantiomeric molecule discrimination. $S_3 = 2\mathrm{Im}[e_{ix}^*e_{iy}] = |e_i^+|^2 - |e_i^-|^2$ represents the fourth Stokes parameter [9,70].

In the case of circular polarization of the incident wave, $e_{ix} = e$, $e$ is real, and $e_{iy} = \pm ie$ depends on whether the light is left or right circular; also $n\mathbf{e}_i = \pm i\mathbf{b}_i$ and the incident helicity is $\mathscr{H}^i = \pm e^2/k$. In addition, when the particle satisfies the first Kerker condition [72–74], thus yielding zero differential scattering cross section in the backscattering direction, $\alpha_e/\epsilon = \mu\alpha_m$, then $\mathbf{p} = \pm in\mathbf{m}$ and $\mathbf{b}(\mathbf{s}) = \mp in\mathbf{e}(\mathbf{s})$; i.e., the scattered field is circularly polarized with respect to the Cartesian system of orthonormal vectors: $(\boldsymbol{\epsilon}_\perp,\boldsymbol{\epsilon}_\parallel,\mathbf{s})$ (see Fig. 1), $\boldsymbol{\epsilon}_\perp$, and $\boldsymbol{\epsilon}_\parallel$ being, respectively, perpendicular and parallel to the scattering plane [55].

In consequence, in circular polarization one has that $\mathbf{e}(\mathbf{s}) = [\mathbf{e}(\mathbf{s})\cdot\boldsymbol{\epsilon}_\perp](1,\pm i,0)$ and $\mathbf{b}(\mathbf{s}) = [n\mathbf{e}(\mathbf{s})\cdot\boldsymbol{\epsilon}_\perp](\mp i,1,0)$. The helicity of the scattered field is proportional to its intensity—$\mathscr{H}^s = \pm\frac{\epsilon}{2k}|\mathbf{e}(\mathbf{s})|^2$—so that since then the flow of helicity becomes proportional to that of energy, one has that the torque $\mathcal{L}$ [Eq. (17)] and the flow tensor $\mathscr{N}_{ij}$ [Eq. (10)] become proportional to the time-averaged force $\langle\mathbf{F}\rangle$ and Maxwell's stress tensor $\langle T_{ij}\rangle$ [68,75]: $\mathcal{L} = \pm(1/k)\langle\mathbf{F}\rangle$, $-\mathscr{N}_{ij} = \pm(1/k)\langle T_{ij}\rangle$. This matches with a statement of [13,67] on the existence of a mapping of the helicity into the energy for circularly polarized waves.

### B. Arbitrary incident wave

The above equations are generalized for arbitrary illuminating wave fields, expressible in terms of its angular spectrum of plane-wave components [65,66],

$$\mathbf{E}^{(i)}(\mathbf{r}) = \int_{\mathcal{D}}\mathbf{e}^{(i)}(\mathbf{s})e^{ik(\mathbf{s}\cdot\mathbf{r})}d\Omega,$$
$$\mathbf{B}^{(i)}(\mathbf{r}) = \int_{\mathcal{D}}\mathbf{b}^{(i)}(\mathbf{s})e^{ik(\mathbf{s}\cdot\mathbf{r})}d\Omega, \tag{28}$$

the integration being done in the unit sphere $\mathcal{D}$ whose element of solid angle is $d\Omega$. In this regard, notice that Eqs. (22) and (23) now constitute the angular spectra of the electric and magnetic scattered vectors when they are represented in the form (28). This integration contains both propagating and evanescent waves [54,65,66], and to include them both, $\mathbf{s}_i$ in (25) must be replaced with $\mathbf{s}_i^*$, complex conjugated of $\mathbf{s}_i = (s_i^x,s_i^y,s_i^z)$, where $s_i^z = \sqrt{1-(s_i^{x2}+s_i^{y2})}$ if $s_i^{x2}+s_i^{y2} \leqslant 1$ (propagating components) and $s_i^z = i\sqrt{(s_i^{x2}+s_i^{y2})-1}$ if $s_i^{x2}+s_i^{y2} > 1$ (evanescent components).

From (28), using the same procedure as before for each plane-wave component and summing up all of them, an easy calculation shows that $\mathcal{L}^{(s)}$ remains as in (24); however, the extinction torque $\mathcal{L}'$ now becomes

$$\mathcal{L}' = \frac{n}{2k\mu}\mathrm{Im}\left\{\frac{1}{\epsilon}p_j\partial_i B_j^{(i)*} - \mu m_j\partial_i E_j^{(i)*}\right\}, \tag{29}$$

which may be also expressed as

$$\mathcal{L}' = \langle \mathbf{\Gamma}_0 \rangle + \frac{1}{2k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\nabla)\mathbf{B}^{(i)*} - \mu(\mathbf{m}\cdot\nabla)\mathbf{E}^{(i)*}\right\}. \tag{30}$$

Where $\langle \mathbf{\Gamma}_0 \rangle$ now is

$$\langle \mathbf{\Gamma}_0 \rangle = \tfrac{1}{2}\mathrm{Re}[\mathbf{p}\times\mathbf{E}^{(i)*} + \mathbf{m}\times\mathbf{B}^{(i)*}], \tag{31}$$

the fields $\mathbf{E}^{(i)}$ and $\mathbf{B}^{(i)}$ being given by Eqs. (28).

At this stage it is worth observing how taking (21) into account, Eq. (29) confers, after separating real and imaginary parts, a physical meaning in terms of the extinction torque $\mathcal{L}'$ to some quantities associated to "magnetoelectric effects" introduced in [76].

### C. On the torque involved in the conservation of the spin angular momentum

The above equations not only show that the $\mathcal{L}$ quantities have torque dimensions, as they should, but as mentioned before they manifest that $\mathcal{L}^{(s)}$ is also an intrinsic torque. Notice on comparing (30) and (31) with (26) and (27) the two additional terms in (30) due to the spatial structure of the incident field when this is not a plane wave. This term accounts for an extrinsic torque component. Also, the second and third terms of (30) have an analogy with those of the dipolar component of the electromagnetic force, whereas those contained in $\langle \mathbf{\Gamma}_0 \rangle$ keep it with those of the Lorentz's component of the optical force. This duality between the $\mathbf{E}$ and $\mathbf{B}$ vectors is also evident by comparing (29) with the expression of the electric ($e$) plus magnetic ($m$) forces on a magnetodielectric dipolar particle [54]. In this connection, the *recoil* optical torque, Eq. (24), due to interference of the scattered fields and which, as said, is also valid for an arbitrary incident field, has a formal analogy with the electric-magnetic dipole interaction electromagnetic force: $\mathbf{F}_{e-m}$ [54].

In summary, the torque on the particle $\mathcal{L} = \mathcal{L}^{(s)} + \mathcal{L}'$ [Eq. (17)] governed by the conservation of the spin angular momentum, is given by the sum of (24) plus (29) or (30). This torque has the recoil component $\mathcal{L}^{(s)}$ added to $\mathcal{L}'$, which is as necessary to describe the dynamics as the above referenced $\mathbf{F}_{e-m}$ component of the force.

However, this scattering torque (24) is only half the recoil torque exerted by the field on the particle, obtained from the conservation of total angular momentum [39]. As we shall see, the other half is obtained from the conservation of the orbital angular momentum. As expected from the premises above adopted in this formulation, this is consistent with the addition of both angular momenta to describe the total transfer to the body.

## V. DECOMPOSITION OF THE TORQUE INVOLVED IN THE CONSERVATION OF SPIN ANGULAR MOMENTUM INTO CONSERVATIVE AND NONCONSERVATIVE COMPONENTS

Like the electromagnetic force [51,52,54], the optical torque is amenable of a decomposition into conservative and nonconservative components, as shown next.

At this point we emphasize again that the vectors $\mathbf{E}^{(i)}$ and $\mathbf{B}^{(i)}$ satisfy the transversality condition $\nabla\cdot\mathbf{E}^{(i)} = 0$ and $\nabla\cdot\mathbf{B}^{(i)} = 0$. From now on we omit the $(i)$ superindex. Taking in (21) the real and imaginary parts of the polarizabilities and using the vector identities

$$\begin{aligned}
&\mathrm{Re}[(\mathbf{B}^*\cdot\nabla)\mathbf{E}] = \tfrac{1}{2}(Y+X),\ \ \mathrm{Re}[(\mathbf{E}\cdot\nabla)\mathbf{B}^*] = \tfrac{1}{2}(Y-X),\\
&\mathrm{Im}[(\mathbf{B}^*\cdot\nabla)\mathbf{E}] = \tfrac{1}{2}(S+R),\quad \mathrm{Im}[(\mathbf{E}\cdot\nabla)\mathbf{B}^*] = \tfrac{1}{2}(S-R),\\
&X = \mathrm{Re}[\nabla\times(\mathbf{E}\times\mathbf{B}^*)],\\
&Y = \mathrm{Re}[\nabla(\mathbf{E}\cdot\mathbf{B}^*)] - kn\mathrm{Im}[\mathbf{E}^*\times\mathbf{E}]\\
&\qquad + k_0\mathrm{Im}[\mathbf{B}^*\times\mathbf{B}],\\
&R = \mathrm{Im}[\nabla\times(\mathbf{E}\times\mathbf{B}^*)],\\
&S = \mathrm{Im}[\nabla(\mathbf{B}^*\cdot\mathbf{E})];\ k_0 = \omega/c,
\end{aligned} \tag{32}$$

we obtain the following expressions for the parts of $\mathcal{L} = \mathcal{L}' + \mathcal{L}^{(s)}$ [Eq. (17)]. First, $\mathcal{L}'$ [Eqs. (29) or (30)] becomes

$$\begin{aligned}
\mathcal{L}' = \frac{n}{2k\mu}\Bigg\{&\alpha_e^R\left[k_0\frac{\mu}{\epsilon}\nabla\mathscr{H} - \frac{1}{2\epsilon}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right]\\
&+\alpha_m^R\left[k_0\mu^2\nabla\mathscr{H} + \frac{\mu}{2}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right]\\
&+\frac{\alpha_e^I}{2\epsilon}\left[\frac{4\mu k^2}{c}\mathscr{F} - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle + \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right]\\
&+\frac{\mu\alpha_m^I}{2}\left[\frac{4\mu k^2}{c}\mathscr{F} - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle - \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right]\\
&-8\pi\frac{\mu}{\epsilon}\left(\alpha_{me}^I\nabla\langle w_e\rangle - \alpha_{em}^I\nabla\langle w_m\rangle\right)\\
&-\frac{2nk}{c}\frac{\mu}{\epsilon}\left(\alpha_{me}^R\nabla\times\mathscr{F}_e - \alpha_{em}^R\nabla\times\mathscr{F}_m\right)\\
&+\frac{8\pi k\mu}{c}\sqrt{\frac{\mu}{\epsilon}}\left(\alpha_{me}^R - \alpha_{em}^R\right)\langle\mathbf{S}\rangle\Bigg\}.
\end{aligned} \tag{33}$$

The helicity $\mathscr{H}$ and the spin angular momentum density that appear in (33) were defined in Eqs. (8) and (9), respectively. Moreover, $\langle w_e\rangle = (\epsilon/16\pi)|\mathbf{E}|^2$ and $\langle w_m\rangle = (1/16\pi\mu)|\mathbf{B}|^2$ are the time-averaged electric and magnetic energy densities, respectively, and $\langle\mathbf{S}\rangle = (c/8\pi\mu)\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*)$ represents the time-averaged Poynting vector. The $R$ and $I$ superindices of the polarizabilities denote their real and imaginary parts, respectively.

Notice that there are conservative terms containing $\alpha_e^R k_0\nabla\mathscr{H}$ and $\alpha_m^R k_0\nabla\mathscr{H}$, which represent a *gradient optical torque* where now the helicity $\mathscr{H}$ plays a role analogous to that of the field energy for the optical force. This may constitute the basis of what we shall call *rotational optical tweezers*, positioning the particle in equilibrium points where there is no lateral torque. This is addressed in some of the examples below. Also the imaginary parts $\alpha_{me}^I$ and $\alpha_{em}^I$ yield gradient terms of the energies, like in the gradient force components. These latter terms of the cross polarizabilities change sign if the chirality of the particle, characterized by $\alpha_{em}$ and $\alpha_{me}$, is reversed, a fact that suggests its use for enantiomeric separation [31]. Likewise, there are conservative terms $\nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)$, this time with the imaginary parts $\alpha_e^I$ and $\alpha_m^I$.

The polarizabilities $\alpha_e^R$ and $\alpha_m^R$ also appear in curl nonconservative parts that contain the vortex circulation of the reactive alternating flow of energy $\nabla \times \mathrm{Im}(\mathbf{E} \times \mathbf{B}^*)$ [68]. These latter terms are sometimes zero, like for a highly collimated circularly polarized Gaussian beam. On the other hand, the $\alpha_e^I$ and $\alpha_m^I$ nonconservative terms proportional to $\mathscr{F}$, come from $\langle \mathbf{\Gamma}_0 \rangle$ in Eq. (30). They are the analogous to the *radiation pressure* terms containing the field linear momentum in the time-averaged force. It is remarkable the analogy according to which while in this latter case the conservation of the wave linear momentum leads to the radiation pressure part of the optical force, in the optical torque the conservation of the spin angular momentum $\mathscr{F}$ leads to terms that similarly contain this quantity and play the role of a *spinning torque*.

There are also energy flows in the cross polarizabilities $\alpha_{me}^R$ and $\alpha_{em}^R$ that contain the curl of both the Poynting vector and the flow of spin angular momentum and thus represent their respective vortices surrounding the particle.

Likewise, the recoil torque, Eq. (24), is written as

$$\begin{aligned}\mathcal{L}^{(s)} = -\frac{k^3}{6}\Bigg\{ & \frac{4nk}{c}\Bigg[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e + \mu^2|\alpha_m|^2\mathscr{F}_m \\ & + \frac{1}{\epsilon^2}|\alpha_{em}|^2\mathscr{F}_m + \mu^2|\alpha_{me}|^2\mathscr{F}_e\Bigg] \\ & + \frac{16\pi\mu}{c}\left[\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{em}) - \mu\mathrm{Im}(\alpha_m^*\alpha_{me})\right]\langle \mathbf{S}\rangle \\ & - 2\left[\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Re}(\alpha_m^*\alpha_{me})\right]\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\Bigg\}, \quad (34)\end{aligned}$$

which, in addition to nonconservative terms with electric and magnetic spins and with the energy flux, has conservative terms if the particle is bi-isotropic, containing the reactive or stored energy and its alternating flow [68], $\mathrm{Im}(\mathbf{E} \times \mathbf{B}^*)$, which taking into account the identity, $k\mathrm{Im}\{\mathbf{E} \times \mathbf{B}^*\} = (1/2)\nabla|\mathbf{E}|^2 - \mathrm{Re}\{(\mathbf{E}^* \cdot \nabla)\mathbf{E}$, yields gradient-force-like terms, proportional to $\nabla|\mathbf{E}|^2$, which change sign if the particle chirality varies from left handed to right handed.

Thus, in summary, we see that the helicity and the spin play a role in the optical torque somewhat analogous to the energy and the Poynting vector in the optical force.

In the particular case in which the particle is chiral, then [16] $\alpha_{em} = -\alpha_{me}$ and (33)–(36) reduce to

$$\begin{aligned}\mathcal{L}' = \frac{n}{2k\mu}\Bigg\{ & \alpha_e^R\left[k_0\frac{\mu}{\epsilon}\nabla\mathscr{H} - \frac{1}{2\epsilon}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right] \\ & + \alpha_m^R\left[k_0\mu^2\nabla\mathscr{H} + \frac{\mu}{2}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right] \\ & + \frac{\alpha_e^I}{2\epsilon}\left[\frac{4\mu k^2}{c}\mathscr{F} - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle + \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right] \\ & + \frac{\mu\alpha_m^I}{2}\left[\frac{4\mu k^2}{c}\mathscr{F} - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle - \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right] \\ & - 8\pi\frac{\mu}{\epsilon}\alpha_{me}^I\nabla\langle w\rangle - \frac{2nk}{c}\frac{\mu}{\epsilon}\alpha_{me}^R\nabla\times\mathscr{F} \\ & + \frac{16\pi k\mu}{c}\sqrt{\frac{\mu}{\epsilon}}\alpha_{me}^R\langle\mathbf{S}\rangle\Bigg\}, \quad (35)\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{(s)} = -\frac{k^3}{6}\Bigg\{ & \frac{4nk}{c}\Bigg[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e + \mu^2|\alpha_m|^2\mathscr{F}_m \\ & + |\alpha_{me}|^2\left(\frac{1}{\epsilon^2}\mathscr{F}_m + \mu^2\mathscr{F}_e\right)\Bigg] \\ & - \frac{16\pi\mu}{c}\left[\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{me}) + \mu\mathrm{Im}(\alpha_m^*\alpha_{me})\right]\langle\mathbf{S}\rangle \\ & + 2\left[\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{me}) - \mu\mathrm{Re}(\alpha_m^*\alpha_{me})\right]\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\Bigg\}, \quad (36)\end{aligned}$$

where $\langle w \rangle$ denotes the density of electromagnetic energy: $\langle w \rangle = \langle w_e \rangle + \langle w_m \rangle$.

It is interesting that if the particle, in addition to being chiral, is dual [72–74], $\epsilon^{-1}\alpha_e = \mu\alpha_m$, then (35) and (36) become

$$\begin{aligned}\mathcal{L}' = \frac{n}{k}\Bigg\{ & \frac{\alpha_e^R}{\epsilon}k_0\nabla\mathscr{H} + \frac{2\alpha_e^I}{\epsilon}\left[\frac{k^2}{c}\mathscr{F} - \frac{2\pi}{c}\nabla\times\langle\mathbf{S}\rangle\right] \\ & - 4\pi\frac{1}{\epsilon}\alpha_{me}^I\nabla\langle w\rangle - \frac{nk}{c}\frac{1}{\epsilon}\alpha_{me}^R\nabla\times\mathscr{F} + \frac{8\pi k}{c}\sqrt{\frac{\mu}{\epsilon}}\alpha_{me}^R\langle\mathbf{S}\rangle\Bigg\}, \quad (37)\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{(s)} = -\frac{2k^3}{3c}\Bigg\{ & nk\left[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F} + |\alpha_{me}|^2\left(\frac{1}{\epsilon^2}\mathscr{F}_m + \mu^2\mathscr{F}_e\right)\right] \\ & - 8\pi\frac{\mu}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{me})\langle\mathbf{S}\rangle\Bigg\}. \quad (38)\end{aligned}$$

Thus, the reactive parts disappear. We obtain two kind of terms in (37) and (38): those conservative in $\mathcal{L}'$ linked to the gradient of the helicity and energy and those nonconservative in $\mathcal{L}'$ and $\mathcal{L}^{(s)}$ containing the flows of helicity and energy, as well as their curls. Once again, this has a suggestive analogy with the gradient, radiation pressure, and curl components of the optical force [54].

## VI. THE ORBITAL ANGULAR MOMENTUM

In the nonabsorbing dielectric medium of refractive index $n = \sqrt{\epsilon\mu}$ the field density of orbital angular momentum $\mathbf{L}$ and of its flow $\mathbf{\Lambda}_{ij}, (i,j = 1,2,3)$, are defined in the dual-symmetric formulation here employed, as [7]

$$\mathbf{L} = \frac{c}{2\mu}[\mathcal{E}\cdot\mathscr{L}\mathcal{A} + \mathcal{B}\cdot\mathscr{L}\mathcal{C}], \quad (39)$$

$$\mathscr{L} = \mathbf{r}\times\nabla, \quad (40)$$

$$\begin{aligned}\mathbf{\Lambda}_{ij} = \frac{c^2}{2\mu}[ & \epsilon_{ikl}\epsilon_{jmn}r_k(\mathcal{B}_n\partial_l\mathcal{A}_m - \mathcal{E}_n\partial_l\mathcal{C}_m) \\ & + \mathcal{B}_j\mathcal{A}_i - \mathcal{E}_j\mathcal{C}_i], \quad (i,j,k,l,m,n = 1,2,3), \quad (41)\end{aligned}$$

where $\epsilon_{ikl}$ is the antisymmetric Levi-Civita tensor. In (39) and (41) $\mathcal{A}$ and $\mathcal{C}$ are the vector potentials introduced in Sec. II.

Under the premises established in Sec. II, $\mathbf{L}$ and $\mathbf{\Lambda}_{ij}$ fulfill the continuity equation [7]

$$\dot{\mathbf{L}} + \nabla\cdot\mathbf{\Lambda}_{ij} = -\mathscr{M}, \quad (42)$$

where $-\mathscr{M}$ represents the dissipation by transfer of orbital angular momentum on interaction of the fields with matter.

In addition to (7), the time harmonicity of the fields leads to the time averages [7]

$$\mathbf{L} = \langle \mathbf{L} \rangle = \frac{c}{4nk} \mathrm{Im} \left\{ \epsilon \mathbf{E}^* \cdot \mathscr{L} \mathbf{E} + \frac{1}{\mu} \mathbf{B}^* \cdot \mathscr{L} \mathbf{B} \right\}, \quad (43)$$

$$\mathbf{\Lambda}_{ij} = \langle \mathbf{\Lambda}_{ij} \rangle = \frac{c^2}{4kn\mu} \mathrm{Im} \{ \epsilon_{jkl} (B_l^* \mathscr{L} E_k + E_l \mathscr{L} B_k^*) + B_i^* E_j + B_j^* E_i \}, \quad (i,j = 1,2,3), \quad (44)$$

where $\mathscr{M}$ in (42) is replaced by its average,

$$\langle \mathscr{M} \rangle = \frac{\pi c}{nk} \{ \mathrm{Im}[\mathbf{J}^* \cdot \mathscr{L} \mathbf{E}] - \mathrm{Re}[\mathbf{J}^* \cdot \mathscr{L} \mathbf{E}] - \partial_j \mathrm{Im}[\epsilon_{ikl} \epsilon_{jmn} x_k (\partial_l K_m) E_n^* + E_j^* K_i] \}, \quad (45)$$

$x_k$ being the $k$th-Cartesian component of the position vector $\mathbf{r}$.

## VII. CONSERVATION OF THE ORBITAL ANGULAR MOMENTUM IN SCATTERING

Equation (42) expresses the conservation of orbital-angular-momentum flow and, again, conveys a density of force moment, or torque. We next find what this means. As before, we integrate it on any volume surrounding the scattering object. The divergence term is then transformed into a flow across the volume surface $\Sigma$ that will eventually be taken as a sphere of large radius $r$.

With reference to Fig. 1, let us first consider a monochromatic, elliptically polarized, plane-wave incident on an arbitrary body. The field at any point of the surrounding medium is again represented as the sum of the incident and the scattered field. Then the flow (or time-averaged flow) density of orbital angular momentum may be written by means of $\mathbf{\Lambda}_{ij}$ decomposed as $\mathbf{\Lambda}_{ij} = \mathbf{\Lambda}_{ij}^{(i)} + \mathbf{\Lambda}_{ij}^{(s)} + \mathbf{\Lambda}'_{ij}$, where

$$\mathbf{\Lambda}_{ij}^{(i)} = \frac{c^2}{4kn\mu} \mathrm{Im} \{ \epsilon_{jkl} ( B_l^{(i)*} \mathscr{L} E_k^{(i)} + E_l^{(i)} \mathscr{L} B_k^{(i)*} ) + B_i^{(i)*} E_j^{(i)} + B_j^{(i)*} E_i^{(i)} \}, \quad (46)$$

$$\mathbf{\Lambda}_{ij}^{(s)} = \frac{c^2}{4kn\mu} \mathrm{Im} \{ \epsilon_{jkl} ( B_l^{(s)*} \mathscr{L} E_k^{(s)} + E_l^{(s)} \mathscr{L} B_k^{(s)*} ) + B_i^{(s)*} E_j^{(s)} + B_j^{(s)*} E_i^{(s)} \}, \quad (47)$$

$$\mathbf{\Lambda}'_{ij} = \frac{c^2}{4kn\mu} \mathrm{Im} \{ \epsilon_{jkl} ( B_l^{(i)*} \mathscr{L} E_k^{(s)} + E_l^{(i)} \mathscr{L} B_k^{(s)*} + B_l^{(s)*} \mathscr{L} E_k^{(i)} + E_l^{(s)} \mathscr{L} + B_k^{(i)*} ) + B_i^{i*} E_j^{(s)} + B_j^{(i)*} E_i^{(s)} + B_i^{(s)*} E_j^{(i)} + B_j^{(s)*} E_i^{(i)} \}. \quad (48)$$

From Eq. (42) the rate $\Phi_{\mathscr{L}}$ at which the total orbital flow varies on interaction of the wave field with the body is given as before by the outward flux: $-(n^2/4\pi c^2) \int_\Sigma d\Omega r^2 \mathbf{\Lambda}_{ij} \cdot \mathbf{s}$ through the surface of any large sphere $\Sigma$ of radius $r$ with center at some point $\mathbf{r}_0$ of the region occupied by the object. That is, according to Eq. (42),

$$\Phi_{\mathscr{L}} = \Phi_{\mathscr{L}}^{(i)} + \Phi_{\mathscr{L}}^{(s)} + \Phi'_{\mathscr{L}}, \quad (49)$$

where $\Phi_{\mathscr{L}}^{(i)}$, $\Phi_{\mathscr{L}}^{(s)}$, and $\Phi'_{\mathscr{L}}$ are the $\Sigma$ integrals of the projections on $\mathbf{s}$ of $-\mathbf{\Lambda}_{ij}^{(i)}$, $-\mathbf{\Lambda}_{ij}^{(s)}$, and $-\mathbf{\Lambda}'_{ij}$, respectively. $\Phi_{\mathscr{L}} = (n^2/4\pi c^2) \int_\Sigma dr d\Omega r^2 \langle \mathscr{M} \rangle$.

From these equations we have that $\Phi_{\mathscr{L}}^{(i)} = 0$, so that (49) becomes

$$\Phi_{\mathscr{L}} = \Phi_{\mathscr{L}}^{(s)} + \Phi'_{\mathscr{L}}. \quad (50)$$

Operating with (46)–(48) and using the identity $\epsilon_{jlk}\epsilon_{kmn} = \delta_{jm}\delta_{ln} - \delta_{jn}\delta_{lm}$, the integrals of $-\mathbf{\Lambda}_{ij}^{(s)} \cdot \mathbf{s}$ and $-\mathbf{\Lambda}'_{ij} \cdot \mathbf{s}$ across $\Sigma$ lead to

$$\begin{aligned} \Phi^s_{\mathscr{L}} &= -\frac{n^2}{4\pi c^2} \int_\Sigma d\Omega r^2 \mathbf{\Lambda}_{ij}^{(s)} s_j \\ &= -\frac{1}{8\pi k} \int_\Sigma d\Omega r^2 \mathrm{Im}[\epsilon \mathbf{e}^*(\mathbf{s}) \cdot \mathscr{L} \mathbf{e}(\mathbf{s})] \\ &= -\frac{1}{8\pi k} \int_\Sigma d\Omega r^2 \mathrm{Im} \left[ \frac{1}{\mu} \mathbf{b}^*(\mathbf{s}) \cdot \mathscr{L} \mathbf{b}(\mathbf{s}) \right] \\ &= -\frac{1}{16\pi k} \int_\Sigma d\Omega r^2 \mathrm{Im} \left[ \epsilon \mathbf{e}^*(\mathbf{s}) \cdot \mathscr{L} \mathbf{e}(\mathbf{s}) + \frac{1}{\mu} \mathbf{b}^*(\mathbf{s}) \cdot \mathscr{L} \mathbf{b}(\mathbf{s}) \right]. \end{aligned} \quad (51)$$

Using again Jones' lemma, Eq. (20), we get

$$\begin{aligned} \Phi'_{\mathscr{L}} &= -\frac{n^2}{4\pi c^2} \int_\Sigma d\Omega r^2 \mathbf{\Lambda}'_{ij} s_j \\ &= -\frac{1}{4k^2} \mathrm{Re} \{ \epsilon \mathbf{e}^{(i)*} \cdot [\mathscr{L} \mathbf{e}(\mathbf{s})]_{\mathbf{s}=\mathbf{s}_i} \} \\ &= -\frac{1}{4k^2} \mathrm{Re} \left\{ \frac{1}{\mu} \mathbf{b}^{(i)*} \cdot [\mathscr{L} \mathbf{b}(\mathbf{s})]_{\mathbf{s}=\mathbf{s}_i} \right\}. \end{aligned} \quad (52)$$

Equations (51) and (52) constitute the conservation law for the orbital angular momentum of either the electric or the magnetic field, as well as of the dual-symmetric one with both $\mathbf{e}$ and $\mathbf{b}$. Equation (50), together with (51) and (52), defines the force moment, or torque, by transfer of this orbital angular momentum from the incident field to the object, expressed in terms of the flow of the tensor $\mathbf{\Lambda}_{ij}$ across any closed surface surrounding this body. It is very interesting that the rate at which this incident orbital angular momentum $\Phi'_{\mathscr{L}}$ [Eq. (52)] is extinguished by transference to the body is proportional to a sort of cross-orbital angular momentum given by the interaction of the incident field with the scattered field in the forward direction $s = s_i$.

Moreover, recalling that the incident field is a plane wave, and since $\mathbf{r} = r\mathbf{s}$, one has for the incident electric orbital angular momentum

$$\begin{aligned} &\mathrm{Im}[\mathbf{E}^{(i)*} \cdot \mathscr{L} \mathbf{E}^{(i)}] \\ &= \mathrm{Im}\{\mathbf{E}^{(i)*} ikr \mathscr{L} [\mathbf{s}_i \cdot \mathbf{s}] \mathbf{E}^{(i)}\} \\ &= \mathrm{Im}\{\mathbf{e}^{(i)*} ikr [\mathbf{s} \times \mathbf{s}_i] \mathbf{e}^{(i)}\} = |\mathbf{e}^{(i)}|^2 kr (\mathbf{s} \times \mathbf{s}_i), \end{aligned} \quad (53)$$

with an analogous expression for the magnetic momentum. Then the total incident orbital angular momentum enclosed in

a sphere of large radius $R$ is s given by

$$\int_0^R dr r^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \times \mathrm{Im}[\mathbf{E}^{(i)*}\cdot \mathscr{L}\mathbf{E}^{(i)} + \mathbf{B}^{(i)*}\cdot \mathscr{L}\mathbf{B}^{(i)}] = 0,$$

as expected. Therefore, the extinction torque from this orbital angular momentum [Eq. (52)] is zero if the incident wave is a plane. (However, as we see next, in general, it is not zero for any arbitrary field.)

On the other hand, the physical meaning of the flow of $-\mathbf{\Lambda}^s_{ij}$ [Eq. (51)] corresponds to a force moment of the scattered field on the particle. This is the consequence for this recoil torque of the conservation of orbital angular momentum. However, since the incident plane wave has no orbital momentum, this recoil contribution has to come from a transfer of the incident spin angular momentum into the scattered orbital angular momentum. Hence, this effect constitutes the essence of the *spin-orbit* interaction [7,8] as a consequence of the scattering of the incident field by the object.

## VIII. EFFECTS ON A MAGNETODIELECTRIC BI-ISOTROPIC DIPOLAR PARTICLE: THE ELECTROMAGNETIC TORQUE INVOLVED IN THE CONSERVATION OF THE ORBITAL ANGULAR MOMENTUM

### A. Incident plane wave

Let us now address a magnetodielectric bi-isotropic dipolar particle. Using Eqs. (21)–(23), evaluating the angular integrals of (51), and substituting the results in (50), we obtain the extinction torque from the orbital-angular-momentum transfer to this scattering body:

$$\Phi'_{\mathscr{L}} = -\frac{n}{4\mu}\mathrm{Re}\left[\frac{1}{\epsilon}(\mathbf{p}\cdot\mathbf{s}_i)\mathbf{B}^{(i)*} - \mu(\mathbf{m}\cdot\mathbf{s}_i)\mathbf{E}^{(i)*}\right]. \quad (54)$$

Likewise, we get the scattering, or recoil, torque from the orbital angular momentum:

$$\Phi^s_{\mathscr{L}} = -\frac{k^3}{6}\mathrm{Im}\left\{\frac{1}{\epsilon}\mathbf{p}^*\times\mathbf{p} + \mu\mathbf{m}^*\times\mathbf{m}\right\}. \quad (55)$$

Equation (55) is identical to (24) for the scattering torque from the spin. Both recoil torques sum to yield the resultant scattering torque, which coincides with that derived from the conservation of the total angular momentum $\mathbf{J}$ through the integration $\int_\Sigma r\mathbf{s}\times T_{ij}s_j ds$, [39].

On the other hand, Eqs. (21) and the transversality of the incident field show that $\Phi'_{\mathscr{L}}$ given by (54) is zero, as expected from the above discussion. However, Eq.(55) exhibits the nonzero scattering contribution $\Phi^s_{\mathscr{L}}$ (recoil torque) to $\Phi_{\mathscr{L}}$ [Eq. (50)], which, as mentioned before, expresses the *spin-orbit interaction* involving the conversion and transfer of the spin into orbital-angular-momentum torque by scattering with the object.

### B. Generalization to an arbitrary incident field

The above equations are generalized to an arbitrary illuminating wave field expressible by its angular spectrum (28), which, introduced in (54), yields

$$\Phi'_{\mathscr{L}} = \frac{1}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\nabla)\mathbf{B}^* - \mu(\mathbf{m}\cdot\nabla)\mathbf{E}^*\right\}. \quad (56)$$

This form is equal to half the contribution from the spatial structure of the incident field to the extinction torque derived above from the spin conservation, as seen by comparing (56) with the second term of (30).

On the other hand, Eq. (55) for $\Phi^s_{\mathscr{L}}$ remains valid for an arbitrary incident field. Thus, we obtain from (50) using (55) and (56) the following torque involved in the conservation of orbital angular momentum:

$$\Phi_{\mathscr{L}} = \frac{1}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\nabla)\mathbf{B}^* - \mu(\mathbf{m}\cdot\nabla)\mathbf{E}^*\right\} - \frac{k^3}{6}\mathrm{Im}\left\{\frac{1}{\epsilon}\mathbf{p}^*\times\mathbf{p} + \mu\mathbf{m}^*\times\mathbf{m}\right\}. \quad (57)$$

## IX. DECOMPOSITION OF THE ANGULAR MOMENTUM TORQUE INTO CONSERVATIVE AND NONCONSERVATIVE COMPONENTS

Using (21) and taking into account the vector identities (32), we express the two components $\Phi'_{\mathscr{L}}$ [Eq. (56)] and $\Phi^s_{\mathscr{L}}$ [Eq. (55)] of the orbital conservation torque [Eq. (57)] of the transversal fields, again dropping the superindex $(i)$, as

$$\Phi'_{\mathscr{L}} = \frac{n}{2k\mu}\left\{\alpha_e^R\left[k_0\frac{\mu}{\epsilon}\nabla\mathscr{H} - \frac{1}{2\epsilon}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right] + \alpha_m^R\left[k_0\mu^2\nabla\mathscr{H} + \frac{\mu}{2}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right] + \frac{\alpha_e^I}{2\epsilon}\left[\frac{4\mu k^2}{c}(\mathscr{F}_m - \mathscr{F}_e) - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle + \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right] + \frac{\mu\alpha_m^I}{2}\left[\frac{4\mu k^2}{c}(\mathscr{F}_m - \mathscr{F}_e) - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle - \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right] - 8\pi\frac{\mu}{\epsilon}\left(\alpha_{me}^I\nabla\langle w_e\rangle - \alpha_{em}^I\nabla\langle w_m\rangle\right) - \frac{2nk}{c}\frac{\mu}{\epsilon}\left(\alpha_{me}^R\nabla\times\mathscr{F}_e - \alpha_{em}^R\nabla\times\mathscr{F}_m\right) + \frac{k\mu}{n}\left(\alpha_{em}^I + \alpha_{me}^I\right)\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right\}, \quad (58)$$

and

$$\Phi^s_{\mathscr{L}} = -\frac{k^3}{6}\left\{\frac{4nk}{c}\left[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e + \mu^2|\alpha_m|^2\mathscr{F}_m + \frac{1}{\epsilon^2}|\alpha_{em}|^2\mathscr{F}_m + \mu^2|\alpha_{me}|^2\mathscr{F}_e\right] + \frac{16\pi\mu}{c}\left[\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{em}) - \mu\mathrm{Im}(\alpha_m^*\alpha_{me})\right]\langle\mathbf{S}\rangle - 2\left[\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Re}(\alpha_m^*\alpha_{me})\right]\right\}. \quad (59)$$

If the particle is chiral, $\alpha_{em} = -\alpha_{me}$, (58) and (59) reduce to

$$\Phi'_{\mathscr{L}} = \frac{n}{k\mu}\left\{\alpha_e^R\left[k_0\frac{\mu}{\epsilon}\nabla\mathscr{H} - \frac{1}{2\epsilon}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right]\right.$$
$$+\alpha_m^R\left[k_0\mu^2\nabla\mathscr{H} + \frac{\mu}{2}\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right] + \frac{\alpha_e^I}{2\epsilon}\left[\frac{4\mu k^2}{c}(\mathscr{F}_m - \mathscr{F}_e) - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle + \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right]$$
$$\left. + \frac{\mu\alpha_m^I}{2}\left[\frac{4\mu k^2}{c}(\mathscr{F}_m - \mathscr{F}_e) - \frac{8\pi\mu}{c}\nabla\times\langle\mathbf{S}\rangle - \nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*)\right] - 8\pi\frac{\mu}{\epsilon}\alpha_{me}^I\nabla\langle w\rangle - \frac{2nk}{c}\frac{\mu}{\epsilon}\alpha_{me}^R\nabla\times\mathscr{F}\right\}, \tag{60}$$

$$\Phi^s_{\mathscr{L}} = -\frac{k^3}{6}\left\{\frac{4nk}{c}\left[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e + \mu^2|\alpha_m|^2\mathscr{F}_m + |\alpha_{me}|^2\left(\frac{1}{\epsilon^2}\mathscr{F}_m + \mu^2\mathscr{F}_e\right)\right] - \frac{16\pi\mu}{c}\left[\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{me}) + \mu\mathrm{Im}(\alpha_m^*\alpha_{me})\right]\langle\mathbf{S}\rangle\right.$$
$$\left. + 2\left[\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{me}) - \mu\mathrm{Re}(\alpha_m^*\alpha_{me})\right]\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*)\right\}. \tag{61}$$

When the particle is dual ($\alpha_e/\epsilon = \mu\alpha_m$), Eqs. (60) and (61) become

$$\Phi'_{\mathscr{L}} = \frac{n}{k}\left\{\frac{\alpha_e^R}{\epsilon}k_0\nabla\mathscr{H} + \frac{2\alpha_e^I}{\epsilon}\left[\frac{k^2}{c}(\mathscr{F}_m - \mathscr{F}_e) - \frac{2\pi}{c}\nabla\times\langle\mathbf{S}\rangle\right] - 4\pi\frac{1}{\epsilon}\alpha_{me}^I\nabla\langle w\rangle - \frac{nk}{c}\frac{1}{\epsilon}\alpha_{me}^R\nabla\times\mathscr{F}\right\}. \tag{62}$$

$$\Phi^s_{\mathscr{L}} = -\frac{2k^3}{3c}\left\{nk\left[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F} + |\alpha_{me}|^2\left(\frac{1}{\epsilon^2}\mathscr{F}_m + \mu^2\mathscr{F}_e\right)\right] - 8\pi\frac{\mu}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{me})\langle\mathbf{S}\rangle\right\}. \tag{63}$$

Similar remarks as for the analogous decomposition of the torque from the spin, above, apply to these latter equations.

## X. THE TOTAL ELECTROMAGNETIC TORQUE

From all the previous analysis we conclude that the *total electromagnetic torque* on the particle, $\langle\mathbf{\Gamma}\rangle$, stemming from the conservation laws of both the spin and the orbital angular momenta, is given by the sum of $\mathcal{L}$ [Eq. (17)] [cf. also (24) and (29) or (30)] and $\Phi_{\mathscr{L}}$ [Eq. (57)], which we express as

$$\langle\mathbf{\Gamma}\rangle = \langle\mathbf{\Gamma}'\rangle + \langle\mathbf{\Gamma}^s\rangle, \tag{64}$$

where the electromagnetic *extinction torque* is

$$\langle\mathbf{\Gamma}'\rangle = \mathcal{L}' + \Phi'_{\mathscr{L}} = \langle\mathbf{\Gamma}_0\rangle + \frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\nabla)\mathbf{B}^{(i)*} - \mu(\mathbf{m}\cdot\nabla)\mathbf{E}^{(i)*}\right\}$$
$$= -\frac{\langle\mathbf{\Gamma}_0\rangle}{2} + \frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}p_j\partial_i B_j^{(i)*} - \mu m_j\partial_i E_j^{(i)*}\right\}, \tag{65}$$

and the electromagnetic *recoil* or *scattering torque* reads

$$\langle\mathbf{\Gamma}^s\rangle = \mathcal{L}^{(s)} + \Phi^s_{\mathscr{L}} = -\frac{k^3}{3}\mathrm{Im}\left\{\frac{1}{\epsilon}\mathbf{p}^*\times\mathbf{p} + \mu\mathbf{m}^*\times\mathbf{m}\right\}. \tag{66}$$

As mentioned above, we emphasize that this total torque is also obtained from the conservation of the total angular momentum $\langle\mathbf{J}\rangle$ [39]. The analysis given here, however, elucidates the contribution of both the spin and the orbital angular momenta through their respective conservation laws and shows that both torques are added, like the angular momenta.

We notice that, making use of (21), one has [dropping the superindex $(i)$ in all following equations, understanding that we deal with incident fields]

$$\langle\mathbf{\Gamma}_0\rangle + \langle\mathbf{\Gamma}^s\rangle = \left[\frac{\alpha_e^I}{2} - \frac{k^3}{3}\left(\frac{|\alpha_e|^2}{\epsilon} + \mu|\alpha_{me}|^2\right)\right]\mathrm{Im}(\mathbf{E}^*\times\mathbf{E}) + \left[\frac{\alpha_m^I}{2} - \frac{k^3}{3}\left(\mu|\alpha_m|^2 + \frac{|\alpha_{em}|^2}{\epsilon}\right)\right]\mathrm{Im}(\mathbf{B}^*\times\mathbf{B})$$
$$+\left[\frac{\alpha_{me}^R - \alpha_{em}^R}{2} - \frac{2k^3}{3}\left(\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Im}(\alpha_m\alpha_{me}^*)\right)\right]\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*)$$
$$-\left[\frac{\alpha_{me}^I + \alpha_{em}^I}{2} - \frac{2k^3}{3}\left(\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Re}(\alpha_m\alpha_{me}^*)\right)\right]\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*). \tag{67}$$

However, introducing (21) in the optical theorem that represents the conservation of energy [54],

$$\mathcal{W}^a + \frac{ck^4}{3n}[\epsilon^{-1}|\mathbf{p}|^2 + \mu|\mathbf{m}|^2] = \frac{\omega}{2}\mathrm{Im}[\mathbf{p}\cdot\mathbf{e}_i^* + \mathbf{m}\cdot\mathbf{b}_i^*]; \tag{68}$$

with $\mathcal{W}^a$ standing for the rate of energy absorption by the particle, one has

$$\frac{\mathcal{W}^a}{\omega} = \left[\frac{\alpha_e^I}{2} - \frac{k^3}{3}\left(\frac{|\alpha_e|^2}{\epsilon} + \mu|\alpha_{me}|^2\right)\right]|\mathbf{E}|^2 + \left[\frac{\alpha_m^I}{2} - \frac{k^3}{3}\left(\mu|\alpha_m|^2 + \frac{|\alpha_{em}|^2}{\epsilon}\right)\right]|\mathbf{B}|^2 + \left[\frac{\alpha_{me}^R - \alpha_{em}^R}{2} - \frac{2k^3}{3}\left(\frac{1}{\epsilon}\mathrm{Im}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Im}(\alpha_m\alpha_{me}^*)\right)\right]\mathrm{Im}(\mathbf{E}\cdot\mathbf{B}^*) + \left[\frac{\alpha_{me}^I + \alpha_{em}^I}{2} - \frac{2k^3}{3}\left(\frac{1}{\epsilon}\mathrm{Re}(\alpha_e^*\alpha_{em}) + \mu\mathrm{Re}(\alpha_m\alpha_{me}^*)\right)\right]\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*). \tag{69}$$

It is interesting that for fields such that $\mathbf{B} = \mp\mathbf{E}$, (e.g., the circularly polarized field scattered from a dual particle, mentioned in Sec. IV), one has from (67) and (69) that

$$\langle\mathbf{\Gamma}_0\rangle + \langle\mathbf{\Gamma}^s\rangle = \frac{\epsilon\sigma^{(a)}}{8\pi k}\mathrm{Im}(\mathbf{E}^{(*)}\times\mathbf{E}) = \frac{n\sigma^{(a)}}{2\pi c}\mathscr{F}_e, \tag{70}$$

where we have defined the absorption cross section as $\sigma^{(a)} = (8\pi k/\omega|\mathbf{E}|^2)\mathcal{W}^{(a)}$. This occurs, in particular, for an incident circularly polarized plane wave and for some beams, as well as for a pure electric, or a pure magnetic, dipolar particle [in this latter case $\mathscr{F}_e$ should be replaced with $\mathscr{F}_m$ in (70)]. In such situations we see in (70) that the component $\langle\mathbf{\Gamma}_0\rangle$ of the extinction torque $\langle\mathbf{\Gamma}'\rangle$ [cf. (65)] is canceled by $\langle\mathbf{\Gamma}^s\rangle$, thus only remaining in the sum (70) that part of $\langle\mathbf{\Gamma}^s\rangle$ which contains $\sigma^{(a)}$.

More generally, for any wave field, if the particle is magnetodielectric, although not bi-isotropic ($\alpha_{em} = \alpha_{me} = 0$), separating in (69) the electric and magnetic parts by writing $\mathcal{W}^a = \mathcal{W}_e^a + \mathcal{W}_m^a$ and introducing the electric and magnetic cross sections, $\sigma_e^{(a)} = (8\pi k/\omega|\mathbf{E}|^2)\mathcal{W}_e^{(a)}$ and $\sigma_m^{(a)} = (8\pi k/\omega|\mathbf{B}|^2)\mathcal{W}_m^{(a)}$, one has on writing the optical theorem (69) separately for the electric and the magnetic parts

$$\langle\mathbf{\Gamma}_0\rangle + \langle\mathbf{\Gamma}^s\rangle = \frac{\epsilon}{8\pi k}\sigma_e^{(a)}\mathrm{Im}(\mathbf{E}^{(*)}\times\mathbf{E}) + \frac{\epsilon}{8\pi k}\sigma_m^{(a)}\mathrm{Im}(\mathbf{B}^{(*)}\times\mathbf{B}) = \frac{n}{2\pi c}\left(\sigma_e^{(a)}\mathscr{F}_e + \sigma_m^{(a)}\mathscr{F}_m\right). \tag{71}$$

These equations are seen to be compatible with (33), (36), (58), and (59).

Hence, in contrast with some previous work [26–32] and in agreement with some experiments and calculations [22–25,33,35,36], these equations show that this particle does not experience a torque and spins due to the so-far-called intrinsic torque: $\langle\mathbf{\Gamma}_0\rangle$ [which, as we have just seen, is canceled by a part of the recoil torque on this kind of particles through the optical theorem relating imaginary parts and moduli of the permittivities, Eq. (69)], but because it absorbs the incident energy and, as shown by (70) and (71), receives the spin $\mathscr{F}_e$ and/or $\mathscr{F}_m$ through the remaining part of the recoil torque that contains the absorption cross section. On the other hand, as we see below, the orbiting of the particle is due to the beam shape carried out in the torque by the second term of (65).

In this regard, it is thus pertinent to remark that, in contrast with previous torque theories that do not comply with the optical theorem, the proportionality factors between the optical torque and the electric and magnetic spin angular momenta is given by the absorption cross sections, not by the imaginary part of the corresponding electric and magnetic polarizabilities. This is a consequence of the static starting point and hence of the lack of consideration of the recoil torque in such theories.

It is well known that the static regime applies for Rayleigh particles (for which $ka \ll 1$, $a$ being their linear dimension, i.e., for example their radius if they are spheres) [54,57,70]. In this quasistatic limit the optical torque obtained in this paper reduces to the one from such theories.

If, for example, the dipolar particle is purely dielectric ($\alpha_m = \alpha_{em} = \alpha_{me} = 0$, $\sigma_e^{(a)}$ is just denoted as $\sigma^{(a)}$) Eq. (69), which becomes $-(2/3\epsilon)k^3|\alpha_e|^2 = -\alpha_e^I + \epsilon\sigma^{(a)}/4\pi k$, introduced in the recoil torque $\langle\mathbf{\Gamma}^s\rangle = (1/3\epsilon)|\alpha_e|^2\mathrm{Im}[\mathbf{E}^*\times\mathbf{E}]$ [cf. Eq. (67)], leads to cancellation of $\langle\mathbf{\Gamma}_0\rangle$ by the corresponding $\alpha_e^I$ term of $\langle\mathbf{\Gamma}^s\rangle$. Hence, Eqs. (64), (65), and (67) lead to a torque on this particle:

$$\langle\mathbf{\Gamma}\rangle = \frac{\epsilon\sigma^{(a)}}{8\pi k}\mathrm{Im}[\mathbf{E}^*\times\mathbf{E}] + \frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\nabla)\mathbf{B}^*\right\}. \tag{72}$$

We see at once that the first term of (72), which we extend to the right side of (71), and that represents the *spin torque* that we denote as $\langle\mathbf{\Gamma}_\sigma\rangle$, *characterizes the rotation of the particle on its axis due to the spin transfer by absorption of the incident energy* through $\sigma^{(a)}$.

In connection with the aforementioned static approximation, notice again that the above relationship, $-(2/3\epsilon)k^3|\alpha_e|^2 = -\alpha_e^I + \epsilon\sigma^{(a)}/4\pi k$, becomes $\alpha_e^I = \epsilon\sigma^{(a)}/4\pi k$ only in the Rayleigh limit [54,57,70].

Concerning the orbital movement due to the field structure, generally involved in the second term of (65), it is useful to consider the vast kind of wave fields that posses a vortex [1–4]. Expressing $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, as well as the position vector $\mathbf{r}$, in cylindrical coordinates $(R,\phi,z)$ and extracting the vortex phase $e^{il\phi}$, we write such incident fields as

$$\mathbf{E}(\mathbf{r}) = e^{il\phi}\tilde{\mathbf{E}}(\mathbf{r}), \qquad \mathbf{B}(\mathbf{r}) = e^{il\phi}\tilde{\mathbf{B}}(\mathbf{r}), \tag{73}$$

with $l$ being the topological charge. So that from (73) and after Eq. (21), we express this term of (65) as

$$\frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}^* - \mu(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{E}^*\right\}$$
$$= \frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\left\{\left(\frac{1}{\epsilon}\alpha_e^R - \mu\alpha_{me}^R\right)\left[\mathrm{Im}(\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* - \frac{l}{R}\mathrm{Re}(\tilde{E}_\phi\tilde{\mathbf{B}}^*)\right] + \left(\frac{1}{\epsilon}\alpha_{em}^R - \mu\alpha_m^R\right)\left[\mathrm{Im}(\tilde{\mathbf{B}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* - \frac{l}{R}\mathrm{Re}(\tilde{B}_\phi\tilde{\mathbf{B}}^*)\right]\right.$$
$$\left. + \left(\frac{1}{\epsilon}\alpha_e^I - \mu\alpha_{me}^I\right)\left[\mathrm{Re}(\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* + \frac{l}{R}\mathrm{Im}(\tilde{E}_\phi\tilde{\mathbf{B}}^*)\right] + \left(\frac{1}{\epsilon}\alpha_{em}^I - \mu\alpha_m^I\right)\left[\mathrm{Re}(\tilde{\mathbf{B}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* + \frac{l}{R}\mathrm{Im}(\tilde{B}_\phi\tilde{\mathbf{B}}^*)\right]\right\}. \tag{74}$$

For example, if the particle is purely dielectric and isotropic (only $\alpha_e \neq 0$), then the right side of (74) reduces to

$$\frac{3}{4k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}^*\right\} = \frac{3}{4kn}\left\{\alpha_e^R\left[\mathrm{Im}(\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* - \frac{l}{R}\mathrm{Re}(\tilde{E}_\phi\tilde{\mathbf{B}}^*)\right] + \alpha_e^I\left[\mathrm{Re}(\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^* + \frac{l}{R}\mathrm{Im}(\tilde{E}_\phi\tilde{\mathbf{B}}^*)\right]\right\}, \tag{75}$$

which accounts for *the particle orbital movement*, and expressing again $\alpha_e^I$ in terms of $\sigma^{(a)}$ through the aforementioned optical theorem, $\alpha_e^I = \epsilon\sigma^{(a)}/4\pi k + (2/3\epsilon)k^3|\alpha_e|^2$, shows that there is an $l$-number torque component, proportional to $l\sigma^{(a)}/R$, making the particle orbit around the vortex axis via the transfer of orbital angular momentum by absorption; there is also another orbital component by transfer through the particle scattering cross section involved in $l|\alpha_e|^2/R$ [54,73]. Both components are amenable to observation [2].

For fields with a well-defined incidence direction, like optical beams, the term $(3/4kn)\alpha_e^I\mathrm{Re}(\tilde{\mathbf{E}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{B}}^*$ of (75), which plays for the torque a role analogous to the $(1/2)\alpha_e^I\mathrm{Im}\{(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}^*\}$ component of the orbital momentum in the optical force [52,54], may make negative the resultant longitudinal torque (i.e., opposite to the incident helicity). This is illustrated below with incident Bessel and Gaussian beams.

There are more components of this extrinsic torque [Eq. (75)] in the transversal $\boldsymbol{\phi}$ and $\mathbf{R}$ directions; but for the purely dielectric particle the $\hat{\mathbf{z}}$ component of the two terms of $\alpha_e^R$ in (75) is zero for a variety of beams, like those discussed in the following examples. More complex effects of this sort remain to be studied in Eq. (74) when $\alpha_m$, $\alpha_{em}$, and $\alpha_{me}$ are nonzero.

## XI. EXAMPLE 1: INCIDENT BESSEL BEAM

As a first illustration we consider a transversal electric (TE) Bessel beam [41,77] propagating along $OZ$ in air, with a vortex of topological charge $l$, whose electric and magnetic vectors are

$$\mathbf{E}^{(i)}(\mathbf{r}) = \frac{e}{k}e^{il\phi}e^{ik_z z}\left[-\frac{lx}{R^2}J_l(KR) + i\frac{Ky}{R}J_l'(KR) - \frac{ly}{R^2}J_l(KR) - i\frac{Kx}{R}J_l'(KR), 0\right],$$
$$\mathbf{B}^{(i)}(\mathbf{r}) = \frac{ek_z}{k^2}e^{il\phi}e^{ik_z z}\left[\frac{ly}{R^2}J_l(KR) + i\frac{Kx}{R}J_l'(KR) - \frac{lx}{R^2}J_l(KR) + i\frac{Ky}{R}J_l'(KR), \frac{K^2}{k_z}J_l(KR)\right],$$
$$\mathbf{k} = (k_x, k_y, k_z), \quad \mathbf{K} = (k_x, k_y, 0), \quad k = |\mathbf{k}| = \frac{\omega}{c}, \tag{76}$$
$$\epsilon = \mu = n = 1, \quad \mathbf{r} = (x, y, z), \quad \mathbf{R} = (x, y, 0),$$

incident on a dielectric particle whose induced dipole moment is $\mathbf{p} = \alpha_e\mathbf{E}^{(i)}$.

This beam is elliptically polarized. First, we note that the helicity of this field is

$$\mathscr{H} = e^2\frac{lk_z K}{k^4 R}J_l(KR)J_l'(KR). \tag{77}$$

Then, according to (29) or (30),

$$\mathcal{L}' = \frac{1}{2k}\mathrm{Im}\,p_j\partial_i B_j^{(i)*} = \langle\boldsymbol{\Gamma}_0\rangle + \frac{1}{2k}\mathrm{Im}\{(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}^{(i)*}\}, \tag{78}$$

with

$$\langle\boldsymbol{\Gamma}_0\rangle = \alpha_e^I\frac{e^2}{k^2}\frac{lK}{R}J_l(KR)J_l'(KR)(0,0,1) = \frac{k^2}{k_z}\alpha_e^I\mathscr{H}(0,0,1), \tag{79}$$

which shows that this component of the torque follows the helicity. Also, Eq. (79) manifests a contribution of the vortex charge $l$ to this part of the torque.

On the other hand,

$$\mathcal{L}' = 2\Phi'_{\mathscr{L}} = \frac{1}{2k}\mathrm{Im}\{(\mathbf{p}\cdot\nabla)\mathbf{B}^{(i)*}\} = \frac{e^2k_z}{2k^4}\left\{\alpha_e^R\frac{lK}{R^2}[KJ_l(KR)J_l''(KR) + KJ_l'^2(KR) - \frac{1}{R}J_l(KR)J_l'(KR)](x,y,0)\right.$$
$$\left. + \alpha_e^I\left[\frac{l^2}{R^4}J_l^2(KR) - 2\frac{l^2K}{R^3}J_l(KR)J_l'(KR) + \frac{K^2}{R^2}J_l'^2(KR)\right](y,-x,0) - 2\frac{lK^3}{k_zR}J_l(KR)J_l'(KR)(0,0,1)\right]\Bigg\}, \tag{80}$$

where $J_l'(KR) = dJ_l(KR)/d(KR)$.

Equation (80) exhibits an azimuthal dependence $(y,-x) = -R\hat{\boldsymbol{\phi}}$ of the $\alpha_e^I$ terms. ($R$ and $\phi$ are the cylindrical coordinates in the transversal plane of the beam.) Hence, although $\langle\mathbf{\Gamma}_0\rangle$ is longitudinal, i.e., along $OZ$, the beam structure creates and modulates a transversal azimuthal component of $\mathcal{L}'$. Moreover, there is a radial torque $(x,y) = R\hat{\boldsymbol{r}}$ in the $\alpha_e^R$ terms due to the helicity gradient. This is extensive to the torque part: $\frac{1}{4k}\mathrm{Im}\{(\mathbf{p}\cdot\nabla)\mathbf{B}^{(i)*}\}$ [cf. Eq. (57)] coming from the orbital-angular-momentum conservation law.

It is interesting to analyze the different parts in Eqs. (78)–(80) from the point of view of the decompositions (33) and (58). There are the conservative components:

$$\frac{1}{2}\alpha_e^R\nabla\mathscr{H} = \frac{e^2}{2k^4}\frac{lk_zK}{R^2}\alpha_e^R[KJ_l(KR)J_l''(KR) + KJ_l'^2(KR) - \frac{1}{R}J_l(KR)J_l'(KR)](x,y,0), \tag{81}$$

$$\frac{1}{4k}\alpha_e^I\nabla\mathrm{Re}(\mathbf{E}\cdot\mathbf{B}^*) = 0. \tag{82}$$

Equation (81) shows the aforementioned radial gradient torque whose sign oscillates with the distance $R$ to the beam axis.

The nonconservative components,

$$\frac{k}{c}\alpha_e^I\mathscr{F}_e = \frac{1}{2}\langle\mathbf{\Gamma}_0\rangle \tag{83}$$

and

$$\frac{k}{c}\alpha_e^I\mathscr{F}_m - \frac{2\pi}{kc}\alpha_e^I\nabla\times\langle\mathbf{S}\rangle$$
$$= \frac{e^2k_z}{2k^4}\alpha_e^I\left\{\left[\frac{l^2}{R^4}J_l^2(KR) - 2\frac{l^2K}{R^3}J_l(KR)J_l'(KR) + \frac{K^2}{R^2}J_l'^2(KR)\right](y,-x,0) - 2\frac{lK^3}{k_zR}J_l(KR)J_l'(KR)(0,0,1)\right\}, \tag{84}$$

having used $x^2J_l''(x) + xJ_l'(x) + (x^2-l^2)J_l(x) = 0$.

Also,

$$-\frac{1}{4k}\alpha_e^I\nabla\times\mathrm{Im}(\mathbf{E}\times\mathbf{B}^*) = 0. \tag{85}$$

On the other hand,

$$\mathcal{L}^s = \Phi^s_{\mathscr{L}} = -\frac{k}{3}e^2|\alpha_e|^2\frac{lK}{R}J_l(KR)J_l'(KR)(0,0,1), \tag{86}$$

which is contributed by the electric helicity flow term $-\frac{k^3}{6}\frac{4nk}{c}[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e$ of Eqs. (36) and (59).

Using, as shown above, the optical theorem $\alpha_e^I - (2/3\epsilon)k^3|\alpha_e|^2 = \epsilon\sigma^{(a)}/4\pi k$ and since $K^2 + k_z^2 = k^2$, the resulting torque becomes

$$\langle\mathbf{\Gamma}(\mathbf{r})\rangle = \frac{e^2}{4\pi k^3}\sigma^{(a)}\frac{lK}{R}J_l(KR)J_l'(KR)(0,0,1) + \frac{3e^2k_z}{4k^4}\left\{\alpha_e^R\frac{lK}{R^2}[KJ_l(KR)J_l''(KR) + KJ_l'^2(KR)\right.$$
$$- \frac{1}{R}J_l(KR)J_l'(KR)](x,y,0) + \alpha_e^I\left[\left[\frac{l^2}{R^4}J_l^2(KR) - 2\frac{l^2K}{R^3}J_l(KR)J_l'(KR) + \frac{K^2}{R^2}J_l'^2(KR)\right](y,-x,0)\right.$$
$$\left.\left. - 2\frac{lK^3}{k_zR}J_l(KR)J_l'(KR)(0,0,1)\right]\right\}. \tag{87}$$

The spin torque $\langle\mathbf{\Gamma}_\sigma\rangle$ [see Eq. (72)] that results from the sum $\langle\mathbf{\Gamma}_0\rangle + \mathcal{L}^s + \Phi^s_{\mathscr{L}}$ [cf. Eqs. (83) and (86)] is the first term of (87) and, as said above, describes the particle spinning following the incident helicity by transfer of spin angular momentum through energy absorption. Equation (87) also shows that, in particular if the wave were plane, this term would be the only contribution to the torque, there being no orbiting of the particle, as expected. Otherwise, there is a radial helicity-gradient term of $\alpha_e^R$ and an azimuthal vortex term of $\alpha_e^I$ in (87) due to both the beam transversal structure and the interaction of the transversal $\mathbf{E}$ and $\mathbf{B}$ with the longitudinal $\mathbf{B}$. The latter manifested through the particle scattering and

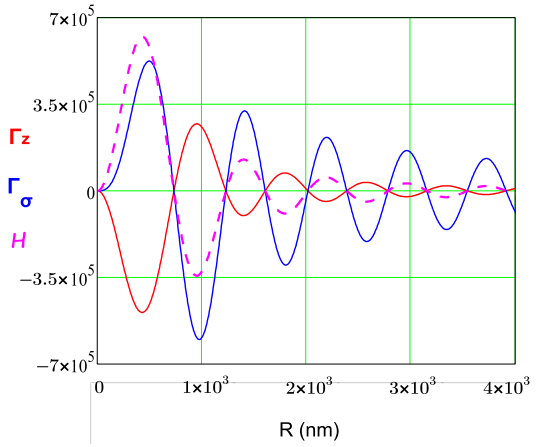


FIG. 2. (Color online) Doped Si sphere of radius 230 nm illuminated at $\lambda = 1350$ nm, where its electric polarizability dominates by a TE Bessel beam; $K = 0.9k$, $l = 2$, $\alpha_e^R = 1.095 \times 10^7\,\mathrm{nm}^3$, $\alpha_e^I = 6.083 \times 10^6\,\mathrm{nm}^3$, $\sigma^{(a)} = 0.1$. Red line, longitudinal component of the time-averaged optical torque $\langle \mathbf{\Gamma} \rangle_z$; blue line, longitudinal spin torque $\langle \mathbf{\Gamma}_\sigma \rangle$ transferred by absorption (its magnitude appears multiplied by $10^4$); dashed pink line, helicity $\mathcal{H}$ of the incident beam (its magnitude appears multiplied by $10^5$). The three quantities have circular symmetry with annular spatial distribution in the transversal $XY$ plane, and thus in this representation they are even functions if the axis of the radial coordinate $R$ is extended to the left of 0. The torques and helicity are plotted in $\mathrm{nm}^3$ and $\mathrm{nm}^{-1}$, respectively, since they are normalized to half the incident intensity density: $e^2$.

absorption of the incident beam as seen from the optical theorem and producing an orbiting of the particle around the vortex axis $OZ$ at maximum intensity points, which are stable due to the gradient force.

Figure 2 shows three quantities in the transversal plane, where they have an annular spatial distribution, for a Si sphere (refractive index $n_p = 3.5$), illuminated at a wavelength near its dipolar resonance [57,78] and $K = 0.9k$. The particle is assumed to have been slightly doped to confer it a low absorption, so that its spin torque is observable through $\sigma^{(a)}$. The optical torque $\langle \mathbf{\Gamma} \rangle_z$ is dominated by the last term of (87), which is several orders of magnitude larger than the spin torque $\langle \mathbf{\Gamma}_\sigma \rangle$. Also, while the latter follows the incident field helicity, the former opposes it, thus giving rise to a longitudinal $\mathbf{z}$-negative optical torque on the particle, which consequently orbits around the vortex axis in the opposite sense to its spinning. Diminishing $K$ results in a weaker torque, but does not change its negative sign. It occurs similarly on varying $l$, which only changes the oscillation points along $R$, slowly diminishing their amplitudes as $|l|$ grows.

It is also interesting in connection with the gradient torque (81) and the corresponding behavior of the helicity in Fig. 2 that this optical torque transversal component is maximum and zero at those points where $\mathscr{H}$ changes sign and is extreme, respectively. These positions coincide with those where the gradient force magnitude on the particle is maximum and zero, respectively. Hence, while this latter force tends to place the object at an equilibrium position of minimum energy, the gradient torque tends to a zero transversal value, both in its spinning and orbital movement [cf. Eqs. (33) and (58) for the spin and angular gradient optical torques]. As anticipated in Sec. V, this effect can be exploited as an additional degree of freedom in what we may call *rotational optical tweezer*s.

## XII. EXAMPLE 2: INCIDENT GAUSSIAN BEAM

### A. TE and TM Gaussian beam

We shall now consider a wave field commonly used in several works and illustrations. This is a Gaussian beam in air, both TE and TM, circularly polarized, with spot size $2\sigma$ and a vortex of topological charge $l$,

$$\begin{aligned}
&\mathbf{E}^{(i)}(\mathbf{r}) = e(1, \pm i, 0) e^{il\phi} e^{ikz} e^{-\frac{R^2}{\sigma^2}},\\
&\mathbf{B}^{(i)}(\mathbf{r}) = \frac{1}{ik}\nabla \times \mathbf{E}^{(i)}(\mathbf{r}) \simeq \frac{1}{ik}\partial_z \times \mathbf{E}^{(i)}(\mathbf{r}) = \hat{\mathbf{z}} \times \mathbf{E}^{(i)}(\mathbf{r})\\
&\qquad = \mp i \mathbf{E}^{(i)}(\mathbf{r}) = e(\mp i, 1, 0) e^{il\phi} e^{ikz} e^{-\frac{R^2}{\sigma^2}},\\
&\mathbf{k} = (k_x, k_y, k_z), \quad \mathbf{K} = (k_x, k_y, 0), \quad k = |\mathbf{k}| = \frac{\omega}{c},\\
&\epsilon = \mu = n = 1, \quad \mathbf{r} = (x, y, z), \quad \mathbf{R} = (x, y, 0),
\end{aligned} \tag{88}$$

illuminating a dielectric particle whose induced dipole moment is $\mathbf{p} = \alpha_e \mathbf{E}^{(i)}$.

This is an interesting case because, as we see below, although some aspects of the forces stemming from the above paraxial expressions of the $\mathbf{E}$ and $\mathbf{B}$ vectors are easily obtained, these fields are not divergenceless and, as we see next, are not appropriate for any theory based on the free-space Maxwell equations, $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$, like the torque formulation developed here. Neither the total angular momentum density nor its $\mathbf{z}$ flux across a transversal $XY$ plane can be expressed as the sum of the corresponding spin plus orbital densities and fluxes, as expected from the remarks of Sec. II.

First, we note that the helicity of this field is

$$\mathscr{H} = \pm \frac{1}{k} e^2 e^{-\frac{2R^2}{\sigma^2}}. \tag{89}$$

It is convenient to analyze the dynamics exerted by this beam on a dielectric particle from the point of view of the mean optical force, which is [48,51,52,54]

$$\langle \mathbf{F} \rangle = \frac{\alpha_e^R}{4} \nabla |\mathbf{E}|^2 + \alpha_e^I \frac{4\pi k}{c} \mathbf{P}_0. \tag{90}$$

We omit the superindex $(i)$, understanding that in all following equations we are dealing with the incident field.

The gradient, pulling the particle to the beam axis, is

$$\nabla |\mathbf{E}|^2 = \frac{4}{\sigma^2} e^2 e^{-\frac{2R^2}{\sigma^2}} (-x, -y, 0), \quad (-x, -y, 0) = -R\hat{\mathbf{r}}, \tag{91}$$

the $i$th component of the orbital momentum $\mathbf{P}_0$ being [52]

$$\begin{aligned}
&\mathbf{P}_0 = \frac{c}{8\pi k} \mathrm{Im}\{E_j^* \partial_i E_j\} = \langle \mathbf{S} \rangle + \frac{c}{8\pi k} \mathrm{Im}\{(\mathbf{E}^* \cdot \nabla)\mathbf{E}\},\\
&i, j = 1, 2, 3,
\end{aligned} \tag{92}$$

with the mean energy flow

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} e^2 e^{-\frac{2R^2}{\sigma^2}} (0, 0, 1) \tag{93}$$

and

$$\mathrm{Im}\{(\mathbf{E}^*\cdot\boldsymbol{\nabla})\mathbf{E}\} = e^2 e^{-\frac{2R^2}{\sigma^2}}\left(\pm\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(y,-x,0). \quad (94)$$

Now observe that since for the paraxial TE and TM field (88), $\boldsymbol{\nabla}\cdot\mathbf{E}\neq 0$, one has

$$\mathrm{Im}\{(\mathbf{E}^*\cdot\boldsymbol{\nabla})\mathbf{E}\} = \mathrm{Im}\{(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}^*\} - \tfrac{1}{2}\boldsymbol{\nabla}\times\mathrm{Im}\{\mathbf{E}^*\times\mathbf{E}\}. \quad (95)$$

The existence of the nonzero first term in the right side of (95) is a fact that should be recalled when expressing the orbital momentum in terms of the electric spin $\langle\mathscr{F}_e\rangle$ curl: $\boldsymbol{\nabla}\times\mathrm{Im}\{\mathbf{E}^*\times\mathbf{E}\}$ for fields that are not divergenceless.

Notice that the orbital momentum $\mathbf{P}_0$, described by (92)–(94), contains the term $\pm\frac{2}{\sigma^2}$ [cf. Eq. (94)], which is essential to describing the spin curl component of the force [cf. Eq. (95)],

$$\frac{1}{2}\boldsymbol{\nabla}\times\mathrm{Im}\{\mathbf{E}^*\times\mathbf{E}\} = \mp e^2\frac{4}{\sigma^2}e^{-\frac{2R^2}{\sigma^2}}(y,-x,0),$$
$$(y,-x,0) = -R\hat{\boldsymbol{\phi}}, \quad (96)$$

which, together with

$$\mathrm{Im}\{(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{E}^*\} = e^2 e^{-\frac{2R^2}{\sigma^2}}\left(\mp\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(y,-x,0), \quad (97)$$

yield the expression (94) when they both are introduced in (95). In fact, the omission of the $\pm\frac{2}{\sigma^2}$ leads to the well-known fundamental paradox of having a zero spin momentum [the last term of Eq. (95)], while the spin angular momentum $\mathscr{F}_e$ is not zero [79].

Equations (90) and (92)–(94), characterize the force due to $\mathbf{P}_0$, pushing the particle along $z$ with the radiation pressure characterized by $\langle\mathbf{S}\rangle$ and in the azimuthal direction due to $\mathrm{Im}\{(\mathbf{E}^*\cdot\boldsymbol{\nabla})\mathbf{E}$ given by the divergence and spin curl terms of (95). Also, from the above equations one has that the total angular momentum $\mathbf{J}$ does not hold $\mathbf{J}=\mathbf{L}+\mathscr{F}$, where $\mathbf{J}=\mathbf{r}\times\langle\mathbf{S}\rangle$ and $\mathbf{L}=\mathbf{r}\times\mathbf{P_0}$. Neither their corresponding flows across the transversal $XY$ plane fulfill such decomposition.

One would write the torque $\mathbf{R}\times\alpha_e^I\langle\mathbf{S}\rangle$, given by the moment of the radiation pressure with respect to the beam axis, as azimuthal, $-\frac{c}{8\pi}e^2e^{-\frac{2R^2}{\sigma^2}}\alpha_e^I\hat{\boldsymbol{\phi}}$, whereas the torque $\mathbf{R}\times\frac{\alpha_e^I}{2}\mathrm{Im}\{(\mathbf{E}^*\cdot\boldsymbol{\nabla})\mathbf{E}$ from the moment of the remaining orbital part of the force would be longitudinal, $-e^2e^{-\frac{2R^2}{\sigma^2}}\alpha_e^I R^2(\pm\frac{2}{\sigma^2}-\frac{l}{R^2})\hat{\mathbf{z}}$ (notice this orbital $\hat{\mathbf{z}}$ component being proportional to the vortex index $l$ and independent of $R$), to which the moment of the spin contribution $\mathbf{R}\times\frac{\alpha_e^I}{2}(\boldsymbol{\nabla}\times\mathrm{Im}\{\mathbf{E}^*\times\mathbf{E}\})$ is $\pm e^2\frac{4}{\sigma^2}e^{-\frac{2R^2}{\sigma^2}}\alpha_e^I R^2\hat{\mathbf{z}}$. However, as we have already seen, the torque by transfer of spin to the particle actually comes from its absorption cross section. If there is no absorption (or birefringence [22]), there is no spinning of the particle due to the spin angular momentum.

On the other hand, according to (29) or (30),

$$\mathcal{L}' = \frac{n}{2k\mu}\mathrm{Im}\frac{1}{\epsilon}p_j\partial_i B_j^{(i)*} = \langle\boldsymbol{\Gamma}_0\rangle$$
$$+\frac{1}{2k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}^{(i)*}\right\} = \langle\boldsymbol{\Gamma}_0\rangle + 2\Phi'_{\mathscr{L}}, \quad (98)$$

with $\epsilon=\mu=n=1$, $k=|\mathbf{k}|=\frac{\omega}{c}$, and

$$\langle\boldsymbol{\Gamma}_0\rangle = \pm\alpha_e^I e^2 e^{-\frac{2R^2}{\sigma^2}}(0,0,1) = k\alpha_e^I\mathscr{H}(0,0,1), \quad (99)$$

$$\frac{1}{2k}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Im}\left\{\frac{1}{\epsilon}(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}^{(i)*}\right\}$$
$$= \frac{e^2}{2k}e^{-\frac{2R^2}{\sigma^2}}\left\{\alpha_e^R\left(\pm\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(-x,-y,0)\right.$$
$$\left.\pm\alpha_e^I\left(\pm\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(y,-x,0)\right\}.$$
$$\times\left(\epsilon=\mu=n=1,\ k=|\mathbf{k}|=\frac{\omega}{c}\right). \quad (100)$$

As expected, Eqs. (98)–(100) show features similar to those seen above for the Bessel beam concerning the influence of the structure of the beam, in both the longitudinal and the transversal components of these torque parts.

On the other hand, the recoil torques are

$$\mathcal{L}^s = \Phi^s_{\mathscr{L}} = \mp\frac{k^3}{3\epsilon}e^2|\alpha_e|^2e^{-\frac{2R^2}{\sigma^2}}(0,0,1), \quad (101)$$

which are contributed by the electric helicity flow term, $-\frac{k^3}{6}\frac{4nk}{c}[\frac{1}{\epsilon^2}|\alpha_e|^2\mathscr{F}_e$, of Eqs. (36) and (59).

When the resultant torque [Eq. (64)] $\langle\boldsymbol{\Gamma}(\mathbf{r})\rangle = \mathcal{L}+\Phi_{\mathscr{L}} = \mathcal{L}'+\mathcal{L}^s+\Phi'_{\mathscr{L}}+\Phi^s_{\mathscr{L}}$ exerted by this beam on the particle is evaluated, one obtains

$$\langle\boldsymbol{\Gamma}(\mathbf{r})\rangle = e^2e^{-\frac{2R^2}{\sigma^2}}\left\{\pm\frac{\epsilon}{4\pi k}\sigma^{(a)}(0,0,1)\right.$$
$$+\frac{3}{4k}\left[\alpha_e^R\left(\pm\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(-x,-y,0)\right.$$
$$\left.\left.\pm\alpha_e^I\left(\pm\frac{2}{\sigma^2}-\frac{l}{R^2}\right)(y,-x,0)\right]\right\}. \quad (102)$$

Once again, we obtain in the first term of (102) the spin torque, with its $\pm$ sign, along $OZ$, proportional to the absorption cross section, accounting for the particle spinning. We also see in (102) the helicity gradient radial term, as well as an azimuthal part proportional to $\mathrm{Im}\{(\mathbf{E}^*\cdot\boldsymbol{\nabla})\mathbf{E}\}$ according to Eq. (94).

However, no $z$ component due to orbital angular momentum transfer is exhibited by (102). This is a consequence of obviating the fundamental conditions—$\boldsymbol{\nabla}\cdot\mathbf{E}=\mathbf{0}$, $\boldsymbol{\nabla}\cdot\mathbf{B}=\mathbf{0}$—in the paraxial representation (88) of the Gaussian beam. Of course, not all terms of the decompositions (33) and (58) are valid for this beam. Hence, although some aspects of the force exerted by a TE and TM beam, like that of (88), are easily described with such representation of its $\mathbf{E}$ and $\mathbf{B}$ vectors, other effects of the dynamics as discussed in Eq. (102) are not adequately described.

### B. Gaussian beams satisfying the transversality condition

Paraxial expressions of vortex Gaussian beams that, being $\partial_z \simeq ik_z$, satisfy the transversality condition should have $E_z^{(i)} = (i/k_z)\nabla_\perp \cdot \mathbf{E}_\perp^{(i)}$ [52] and $\mathbf{E}_\perp^{(i)} = (E_x^{(i)}, E_y^{(i)})$. A first example, which we denote as PG1, has the annular spatial field distribution in the $XY$ plane:

$$\mathbf{E}^{(i)}(\mathbf{r}) = ek^l R^l \left\{ 1, \pm i, \frac{i}{k_z} e^{\pm i\phi} \left[ \frac{l}{R}(1 \mp 1) - \frac{2R}{\sigma^2} \right] \right\} \times e^{il\phi} e^{ik_z z} e^{-\frac{R^2}{\sigma^2}}. \tag{103}$$

Alternatively, a second instance, which we denote as PG2, is similar to an hypergeometric Gaussian beam in the pupil plane:

$$\mathbf{E}^{(i)}(\mathbf{r}) = e\left[ 1, \pm i, \left( \pm \frac{2}{k_z \sigma^2} + \frac{l}{k_z R^2} \right)(y \mp ix) \right] e^{ik_z z} e^{-\frac{R^2}{\sigma^2}}, \tag{104}$$

$$\mathbf{B}^{(i)}(\mathbf{r}) = -i\frac{1}{k}\nabla \times \mathbf{E}^{(i)}, \tag{105}$$

$$\mathbf{r} = (x, y, z), \mathbf{R} = (x, y, 0).$$

We choose $l = 1$ and the upper signs in (103) for the PG1 beam. According to the first term of (72), the spin torque $\langle \mathbf{\Gamma}_\sigma \rangle$ that it induces on the above dielectric dipolar particle is

$$\langle \mathbf{\Gamma} \rangle_\sigma = e^2 k^2 R^2 e^{-\frac{2R^2}{\sigma^2}} \frac{\sigma^{(a)}}{2\pi k} \left\{ \frac{1}{k_z \sigma^2}(-y, x, 0) + \frac{1}{2}(0, 0, 1) \right\}. \tag{106}$$

Whereas, by the same token, the PG2 beam gives rise to

$$\langle \mathbf{\Gamma} \rangle_\sigma = \pm e^2 e^{-\frac{2R^2}{\sigma^2}} \frac{\sigma^{(a)}}{4\pi k} \times \left\{ \left( \pm \frac{2}{k_z \sigma^2} + \frac{l}{k_z R^2} \right)(-y, x, 0) + (0, 0, 1) \right\}. \tag{107}$$

For brevity, we concentrate on these longitudinal $z$ torques, as this component was the source of the discrepancy encountered with (88). They are

$$\langle \mathbf{\Gamma}(\mathbf{r}) \rangle_z = e^{-\frac{2R^2}{\sigma^2}} R^2 \left[ \frac{k\sigma^{(a)}}{4\pi} + 6\frac{e^2}{\sigma^2}\alpha_e^l \left( \frac{R^2}{\sigma^2} - 1 \right) \right](0, 0, 1) \tag{108}$$

for the PG1 beam and

$$\langle \mathbf{\Gamma}(\mathbf{r}) \rangle_z = e^2 e^{-\frac{2R^2}{\sigma^2}} \left\{ \pm \frac{\sigma^{(a)}}{4\pi k} + \frac{3}{k^2 \sigma^2} \left( l \mp 1 \pm \frac{2R^2}{\sigma^2} \right) \alpha_e^l \right\}(0, 0, 1) \tag{109}$$

for the PG2 beam.

Equations (108) and (109) contain the topological charge, or orbital number, $l$, coming from the $\langle \mathbf{S} \rangle$ component of the orbital momentum $\mathbf{P_0}$ [Eq. (92)], which describes the trajectory of the particle around the vortex following the sign of $l$. There is no contribution to this $l$ term of $\langle \mathbf{\Gamma}(\mathbf{r}) \rangle_z$ from $\mathrm{Im}[(\mathbf{E}^* \cdot \nabla)\mathbf{E}]$ since, for example for PG2, the latter yields $4e^2 e^{-\frac{2R^2}{\sigma^2}}(\pm 4/\sigma^2)(y, -x, 0)$, whose momentum $\mathbf{R} \times \mathrm{Im}[(\mathbf{E}^* \cdot \nabla)\mathbf{E}]$ contributes to the $\mp(3/k^2\sigma^2)$ factor of $\alpha_e^l$ in (109). Of course, again the sign of the incident spin angular momentum is followed by the spinning particle as it absorbs incident energy through $\sigma^{(a)}$.

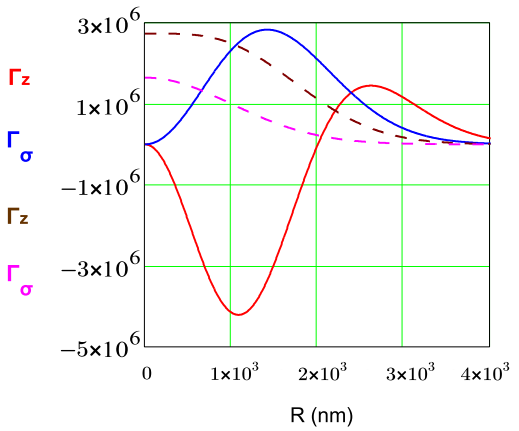


FIG. 3. (Color online) Doped Si sphere of radius 230 nm illuminated at $\lambda = 1350$ nm by either of the Gaussian beams PG1 [Eq. (103)] or PG2 [Eq. (104)] with LCP transversal components. $l = 1$ for PG1, and $l = 2$ for PG2. $\sigma = 2; \mu$m. $\alpha_e^R = 1.095 \times 10^7$ nm$^3$, $\alpha_e^l = 6.083 \times 10^6$ nm$^3$. Solid red line, longitudinal component of $\langle \Gamma \rangle_z$ due to PG1; solid blue line, longitudinal spin torque $\langle \Gamma_\sigma \rangle$ from PG1 (its magnitude appears multiplied by $10^5$); dashed brown line, longitudinal component of $\langle \Gamma \rangle_z$ due to PG2 (its magnitude appears multiplied by 14); dashed pink line, longitudinal spin torque $\langle \Gamma_\sigma \rangle$ from PG2 (its magnitude appears multiplied by $10^6$). The four quantities have circular symmetry with an annular spatial distribution in the transversal $XY$ plane, and thus in this representation they are even functions if the axis of the radial coordinate $R$ is extended to the left of 0. They are plotted in nm$^3$ since they are normalized to half the incident intensity density: $e^2$.

Figure 3 shows the annular spatial distribution of the spin and longitudinal total torques on the Si sphere addressed above, again illuminated at $\lambda = 1350$ nm by either the beam PG1 (with $l = 1$) or PG2 (with $l = 2$), choosing the upper sign in (103) and (104). The longitudinal $OZ$ torque is dominated by the $\alpha_e^l$ term of Eqs. (108) and (109), which is several orders of magnitude larger than the spin torque $\langle \mathbf{\Gamma}_\sigma \rangle$. The latter clearly displays the beam transversal Gaussian shape at whose peak the particle is stably placed by the gradient force, while it orbits around the vortex and spins on its axis. Increasing $|l|$ enlarges the strength of $\langle \mathbf{\Gamma} \rangle_z$.

It is remarkable that while the PG2 beam induces a resultant $z$ torque that follows its helicity like the spin torque, the interaction of the transversal and longitudinal components of the PG1 wave field induces a negative electromagnetic torque on the particle, which consequently orbits around the vortex axis in opposite sense to its spinning. This example, as well as the one above concerning a TE Bessel beam, illustrates that, in contrast with negative radiation pressure and tractor beams [40–42], the production of negative torques is quite common and exists beyond illumination with special oscillating spatial profiles.

We should remark that in the examples here shown, the torque is enhanced by illumination in the electric dipole resonance region of this kind of high $n_p$ particles. By contrast, we obtained torques one order of magnitude smaller on latex

spheres ($n_p = 1.5$) of the same size in the equivalent region ($\lambda = 700$ nm).

## XIII. CONCLUSIONS

We have studied the significance and consequences of the conservation laws of the spin and orbital angular momenta for the scattering of arbitrary wave fields, satisfying the transversality condition with an object, which we have illustrated by a magnetodielectric bi-isotropic dipolar particle in the wide sense. We have shown that these laws describe the respective contributions of these angular momenta to the different parts of the torque exerted by the field on the particle, which is seen to consist of an extinction and a scattering, or recoil, component. The latter is shown to cancel the often-called intrinsic torque, contained in the extinction component and modeled in some previous works, without compatibility with energy conservation, to account for the optical torque felt by the body, and that as shown here plays no role in the resultant optical torque, which is actually felt by the particle through its absorption cross section. This latter quantity is what remains from the recoil torque component after the above-mentioned cancellation.

Each of these two laws describe a half of the recoil torque acting on the object. The transfer of spin and orbital angular momenta, on the other hand, arises while they are extinguished from the incident field. This is described by the torque extinction parts due to the interference of the incident and scattered fields. If the incident wave is a plane, and thus it has no orbital angular momentum, its transfer characterized by the recoil torque is a manifestation of the spin-orbit interaction.

The role of the spatial structure of the incident field in the torque has been established, showing that the contribution from the spin angular momentum conservation is twice that from the orbital-angular-momentum continuity law. While the spinning of the particle through absorption of the incident energy always follows the incident helicity, its orbiting may result opposite to the incident spin, this giving rise to a resultant negative optical torque on a single particle. I have illustrated this with Bessel and Gaussian beams.

The electromagnetic torque admits a decomposition into conservative and nonconservative components in which the helicity and its flow play a role analogous to that of the energy and the Poynting vector in the optical force. In particular, the gradient torque has the potential to add a new variable to optical tweezers setups.

In addition to contributing to an understanding of the physical mechanisms involved in the transfer of angular momentum from the field to a body, ruled by these conservation laws, and their several consequences, this study opens new avenues of research both from the experimental and theoretical points of view that, with the improvement of particle manipulation techniques, adds new degrees of freedom in the analysis and control of possible fundamental phenomena and their applications.

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