Universal form of the carrier frequency of scalar and vector paraxial X waves with orbital angular momentum and arbitrary frequency spectrum

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We report on the dependence of the carrier frequency of a nondiffracting optical pulse on the amount of orbital angular momentum it carries. We provide a unified universal form of such a dependence for the cases of both scalar and vector pulses with arbitrary frequency spectra. For the case of paraxial optical pulses we consider two different examples, namely, pulses with exponentially decaying spectra and Gaussian spectra.

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I. INTRODUCTION

Optical pulses, i.e., superpositions of plane waves traveling at different frequencies, are nowadays a fundamental tool in many fields of basic and applied research such as atomic physics, spectroscopy, communications, material processing, and medicine [1,2]. In particular, nondiffracting pulses, i.e., light fields that do not spread during propagation in both time and space, have attracted a great deal of interest in the past decade for their interesting and peculiar properties [3]. Among this class of solutions of Maxwell's equations, X waves are surely the most well known. First introduced in acoustics [4,5], they rapidly found interesting applications in many areas of physics, such as nonlinear optics [6,7], condensed matter [8], quantum optics [9], waveguide arrays [10,11], and optical communications [12].

A very interesting characteristic of nondiffracting waves is that they are essentially polychromatic superpositions of Bessel beams. It is well known that Bessel beams are eigenstates of the Helmholtz equation carrying m units of orbital angular momentum (OAM) [13]. Having at our disposal optical pulses with OAM will surely open new horizons and possibility for the applications of optical pulses. Despite this fact, however, most of the literature concerning nondiffracting waves only deals with pulses generated by zeroth-order Bessel beams, i.e., by beams carrying no OAM at all.

Since the seminal works of Berry and Nye [14] and Allen *et al.* [15], however, the concept of OAM of light was extensively studied from both fundamental [16–19] and experimental points of view, leading to striking applications such as optical tweezers [20] and spanners [21]. Having the possibility of bringing the concept of OAM into the domain of optical pulses will therefore represent a very important step forward towards new horizons and applications for optical pulses. Even though some recent works deal with the angular momentum of optical pulses [22–26], the generation of femtosecond vortex beams [27,28], and vortex supercontinua [29], OAM is still seldom associated with optical pulses.

Very recently, we have proposed a simple method for introducing OAM into the domain of optical pulses, namely, by generalizing the well-known fundamental X-wave solution (valid for $m=0$) to the case $m\neq 0$ [30]. We found that for the particular case of fundamental X waves, the OAM content of an optical pulse affects its temporal properties by self-compressing the pulse as the OAM parameter m grows and by making the carrier frequency of the pulse grow with m [30]. In this work we intend to generalize those findings, with particular attention on the role of m in the determination of the carrier frequency of an optical pulse.

To do that we consider a nondiffracting optical pulse with arbitrary frequency spectrum and we derive the general expression of the carrier frequency as a function of the OAM parameter m. Within the paraxial approximation, moreover, we show that such a dependence can be written in a universal manner, encompassing the cases of both scalar and vector pulses.

This work is organized as follows. In Sec. II we introduce the concept of a nondiffracting optical pulse with OAM. Section III is devoted to the calculation of the carrier frequency of such a pulse and its relation to the OAM parameter m for the case of a scalar and a vector pulse. Two explicit examples, namely, the cases of an exponentially decaying spectrum and a Gaussian spectrum, are carried out in Sec. IV. A summary is given in Sec. V. Throughout this work natural units $c=\hbar=1$ will be implicitly assumed. This means, in particular, that the dispersion relation of an electromagnetic field propagating in vacuum is given by $\omega=k$.

II. NONDIFFRACTING OPTICAL PULSES WITH OAM

We start our analysis by considering a solution $\psi(\mathbf{r};\omega)$ of the scalar Helmholtz equation, i.e., $(\nabla^2+\omega^2)\psi(\mathbf{r};\omega)=0$. From this solution it is possible to construct an exact solution of the scalar wave equation $(\nabla^2-\partial_t^2)\phi(\mathbf{r},t)=0$ in the following way:

$$\phi(\mathbf{r},t)=\int d\omega\, f(\omega)e^{-i\omega t}\psi(\mathbf{r};\omega), \tag{1}$$

where $f(\omega)$ is an arbitrary spectral function. Let us now consider a Bessel beam [31] as a solution of the Helmholtz equation, i.e.,

$$\psi(\mathbf{r};\omega)=J_m(\omega\sin\vartheta_0 R)e^{im\theta}e^{i\omega z\cos\vartheta_0}, \tag{2}$$

where $R=\sqrt{x^2+y^2}$, $\theta=\arctan(y/x)$, $J_m(x)$ is the Bessel function of the first kind of order m [32], and ϑ_0 is the Bessel cone angle, i.e., the beam's characteristic parameter.

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Nondiffracting optical pulses, i.e., solutions of the wave equation that are localized in space and time, can be then defined, according to Eq. (1), as a polychromatic superposition of Bessel beams [3] as follows:

$$\phi(\mathbf{r},t) = \int d\omega\, f(\omega) J_m(\rho\omega) e^{i(\omega\zeta + m\theta)}, \tag{3}$$

where $\zeta = z\cos\vartheta_0 - t$ is the comoving coordinate attached to the nondiffracting pulse itself and $\rho = R\sin\vartheta_0$ has been introduced for later convenience. Equation (3) with $m = 0$ describes the well-known fundamental X waves [3]. If $m \neq 0$ instead, Eq. (3) describes a scalar nondiffracting optical pulse carrying m units of OAM [30].

An exact vectorial solution to Maxwell's equation can be built from the scalar field described by Eq. (3) by means of the so-called Hertz potential method [33]. By choosing $\mathbf{\Pi}(\mathbf{r},t) = \phi(\mathbf{r},t)\hat{\mathbf{f}}$ (where $\hat{\mathbf{f}}$ is an arbitrary unit vector) as the Hertz potential, in fact, the electric and magnetic fields can be written as follows [33]:

$$\mathbf{E}(\mathbf{r},t) = \nabla \times \nabla \times \mathbf{\Pi}(\mathbf{r},t), \tag{4a}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\partial}{\partial t}[\nabla \times \mathbf{\Pi}(\mathbf{r},t)]. \tag{4b}$$

These vector electric and magnetic fields can be thought of as polychromatic superpositions of the corresponding monochromatic electric and magnetic fields generated by the scalar field $\psi(\mathbf{r};\omega)\exp(-i\omega t)$ in Eq. (1). If we in fact define the monochromatic Hertz potential as $P(\mathbf{r};\omega) = \psi(\mathbf{r};\omega)\exp(-i\omega t)\hat{\mathbf{f}}$, this will, according to Eq. (4), generate the monochromatic vector electric and magnetic fields $\mathcal{E}(\mathbf{r},t;\omega)$ and $\mathcal{B}(\mathbf{r},t;\omega)$, respectively. Once these fields are known, the electric and magnetic fields for a nondiffracting optical pulse can be simply calculated as follows:

$$\mathbf{E}(\mathbf{r},t) = \int d\omega\, f(\omega)\mathcal{E}(\mathbf{r},t;\omega), \tag{5a}$$

$$\mathbf{B}(\mathbf{r},t) = \int d\omega\, f(\omega)\mathcal{B}(\mathbf{r},t;\omega). \tag{5b}$$

The explicit expressions of the monochromatic vector electric and magnetic fields are reported in Appendix A.

III. CALCULATION OF THE CARRIER FREQUENCY

The aim of this section is to give an explicit expression of the dependence of the carrier frequency of an optical nondiffracting pulse from the OAM parameter m. We begin by considering the case of a scalar optical pulse and then generalize our results to vector pulses. We will conclude the section with a unified description of the two cases.

A. Scalar case

To begin, let us rewrite the scalar pulse given by Eq. (3) in the form

$$\phi(\mathbf{r},t) = A_m(\rho,\theta,\zeta)e^{i\psi_m(\rho,\theta,\zeta)}, \tag{6}$$

where $A_m(\rho,\theta,\zeta)$ is the pulse amplitude and $\psi_m(\rho,\theta,\zeta)$ is the pulse phase. These quantities can be calculated by isolating the real and imaginary parts in Eq. (3) and calculating the modulus and phase as usual. In particular, for the pulse phase we obtain the following result:

$$\psi_m(\rho,\theta,\zeta) = \arctan\left[\frac{\mathcal{S}_m^{(0)}(\rho,\theta,\zeta)}{\mathcal{C}_m^{(0)}(\rho,\theta,\zeta)}\right], \tag{7}$$

where

$$\mathcal{S}_m^{(n)}(\rho,\theta,\zeta) = \int d\omega\, \omega^n f(\omega) J_m(\rho\omega)\sin(\omega\zeta + m\theta), \tag{8a}$$

$$\mathcal{C}_m^{(n)}(\rho,\theta,\zeta) = \int d\omega\, \omega^n f(\omega) J_m(\rho\omega)\cos(\omega\zeta + m\theta). \tag{8b}$$

To obtain the carrier frequency associated with the scalar pulse in Eq. (6), let us recall that a general optical pulse $E(\mathbf{r},t)$ can always be written as the product of an envelope function $\mathcal{A}(\mathbf{r},t)$ (generally assumed to slowly oscillate with respect to the underlying field) times an exponential factor oscillating with the carrier frequency ω_c, i.e.,

$$E(\mathbf{r},t) = \mathcal{A}(\mathbf{r},t)e^{i\omega_c t}. \tag{9}$$

A comparison between this expression and the scalar pulse given by Eq. (6) reveals that in order to write the field $\phi(\mathbf{r},t)$ in the form of Eq. (9), one should define the carrier frequency associated with $\phi(\mathbf{r},t)$ as follows [30]:

$$\omega_c = \left.\frac{\partial\psi_m(\rho,\theta,\zeta)}{\partial\zeta}\right|_{\zeta=0}. \tag{10}$$

Substituting Eq. (7) into this equation gives the following result:

$$\omega_c = \frac{\int d\omega\, \omega\, f(\omega)J_m(\rho\omega)}{\int d\omega\, f(\omega)J_m(\rho\omega)}. \tag{11}$$

This is the first result of our work. The carrier frequency of a nondiffracting scalar optical field carrying m units of OAM is given by the (normalized) center of mass of the spectrum $f(\omega)$ of the pulse itself, weighted with the Bessel function $J_m(\rho\omega)$. The m dependence of the carrier frequency is therefore a direct consequence of the presence of the Bessel function. Equation (11) is a very general result. Without any further assumption, however, Eq. (10) is of little help in understanding how OAM influences the carrier frequency of an optical pulse. Moreover, there are few analytical solutions of integrals involving Bessel functions and most of the time only numerical solutions to Eq. (10) would be available, thus reducing the possibility of analyzing Eq. (11) for general spectra.

If we assume the pulse to be paraxial, however, Eq. (11) admits an insightful analytical solution, even for the case of an arbitrary spectrum $f(\omega)$. To do that, let us recall that the argument of the Bessel function appearing in Eq. (11) is $\rho\omega = R\omega\sin\vartheta_0$. In the paraxial case, $\vartheta_0 \ll 1$ (which means $\rho \ll 1$) and the Bessel functions appearing in Eq. (11) can be expanded in a Taylor series with second-order accuracy with respect to ρ, namely,

$$J_m(\rho\omega) = \rho^m\left(\frac{\omega^m}{2^m m!} - \frac{\rho^2\omega^{m+2}}{2^{m+2}(m+1)!}\right) + O(\rho^4). \tag{12}$$

Using this expansion and defining the mth moment $g(m)$ of the spectral function $f(\omega)$ as

$$g(m) = \int d\omega\, \omega^m f(\omega), \tag{13}$$

Eq. (11) can be written in the following elegant form:

$$\omega_c = \frac{g(m+1)}{g(m)} + \rho^2 \frac{g(m+3)}{4(m+1)g(m)} + O(\rho^4). \tag{14}$$

From this equation one can deduce that, at the leading order in ρ (i.e., at the leading order in ϑ_0), the dependence of the carrier frequency of the scalar pulse (3) from the OAM parameter m is given as the ratio between the $(m+1)$th and mth moments of the frequency spectrum $f(\omega)$, namely,

$$\omega_c = \frac{g(m+1)}{g(m)}. \tag{15}$$

B. Extension to vector pulses

The results obtained in the previous section are valid for a scalar pulse. In many situations, however, the scalar description of an optical pulse is no longer adequate and a full vector theory must be employed. As the properties of scalar and vector pulses are ostensibly different, it is very likely that Eq. (14) will be affected as well by the vector nature of the pulse. To prove this, let us consider the case of a vector pulse generated from the Hertz potential $\mathbf{\Pi}(\mathbf{r},t) = \phi(\mathbf{r},t)\hat{\mathbf{z}}$, with $\phi(\mathbf{r},t)$ given by Eq. (3). Using Eqs. (4) and defining the quantity

$$\mathcal{I}_m^{(n)}(\rho,\theta,\zeta) = \mathcal{C}_m^{(n)}(\rho,\theta,\zeta) + i\mathcal{S}_m^{(n)}(\rho,\theta,\zeta), \tag{16}$$

the components of the electric field can be written as follows:

$$E_x(\mathbf{r},t) = \frac{i\cos\vartheta_0}{R}\left[\rho\cos\theta\mathcal{I}_{m-1}^{(2)} - m\mathcal{I}_m(1)\right], \tag{17}$$

$$E_y(\mathbf{r},t) = \frac{i\cos\vartheta_0}{R}\left[\rho\sin\theta\mathcal{I}_{m-1}^{(2)} + im\mathcal{I}_m(1)\right], \tag{18}$$

$$E_z(\mathbf{r},t) = \sin^2\vartheta_0\mathcal{I}_m^{(2)}. \tag{19}$$

From these expressions it appears clear that in the paraxial limit, either the x or the y component of the electric field can be taken as representative of the field itself, while the z component can be neglected, as it is $O(\vartheta_0^2)$. Without loss of generality, we then concentrate on the x component. By rewriting $E_x(\mathbf{r},t)$ as in Eq. (9), the phase $\Psi_m(\rho,\theta,\zeta)$ assumes the following expression:

$$\Psi_m(\rho,\theta,\zeta) = -\arctan\left[\frac{\sin\vartheta_0\cos\theta\mathcal{C}_{m-1}^{(2)} - m\mathcal{C}_m^{(1)}}{\sin\vartheta_0\cos\theta\mathcal{S}_{m-1}^{(2)} - m\mathcal{S}_m^{(1)}}\right], \tag{20}$$

where the arguments of $\mathcal{C}_m^{(n)}$ and $\mathcal{S}_m^{(n)}$ have been dropped for the sake of simplicity. Substituting the expression (20) into Eq. (10) then gives the following result for the carrier frequency of a vector pulse:

$$\omega_c = \frac{A_m^{(n)}(\rho)\rho^2\cos^2\theta + m^2 B_m^{(n)}(\rho)}{C_m^{(n)}(\rho)\rho^2\cos^2\theta + m^2 D_m^{(n)}(\rho)}, \tag{21}$$

where

$$A_m^{(n)}(\rho) = \mathcal{G}_{m-1}^{(2)}\left[\mathcal{G}_m^{(3)} - m\mathcal{G}_m^{(2)}\right] - m\mathcal{G}_m^{(1)}\mathcal{G}_{m-1}^{(3)}, \tag{22}$$

$$B_m^{(n)}(\rho) = m^2\mathcal{G}_m^{(1)}\mathcal{G}_m^{(2)}, \tag{23}$$

$$C_m^{(n)}(\rho) = \mathcal{G}_{m-1}^{(2)}\left[\mathcal{G}_{m-1}^{(2)} - m\mathcal{G}_m^{(2)}\right], \tag{24}$$

$$D_m^{(n)}(\rho) = m^2\left[\mathcal{G}_m^{(2)}\right]^2, \tag{25}$$

and $\mathcal{G}_m^{(n)} \equiv \mathcal{G}_m^{(n)}(\rho) = \mathcal{I}_m^{(n)}(\rho,0,0)$.

Equation (21) is the vector counterpart of Eq. (11). As in the scalar case discussed above, this expression is very general and holds for any light field, despite its paraxiality. Again, in its present form Eq. (21) is of little use, as it does not admit further simplifications, which would allow us to gain more insight into the role of m in the definition of the carrier frequency ω_c of a vector pulse. In the paraxial approximation $\vartheta_0 \ll 1$ and substituting Eq. (12) into Eq. (21) gives the following result:

$$\omega_c = \frac{g(m+2)}{g(m+1)} - \rho^2\left[a_m(\theta)\frac{g(m+2)g(m+3)}{g^2(m+1)} + b_m(\theta)\frac{g(m+4)}{g(m+1)}\right] + O(\rho^4), \tag{26}$$

with

$$a_m(\theta) = \frac{1}{8m(m+1)}\{3\cos 2m\theta[(m+1)\cos\theta + 1] + \sin\theta[3(m+1)\sin 2m\theta - \sin\theta] + (m+1) - \cos^2\theta\}, \tag{27}$$

$$b_m(\theta) = \frac{1}{4m(m+1)}[(m+2)\cos\theta\sin m\theta - m\sin\theta\cos m\theta]. \tag{28}$$

In particular, at the leading order in ϑ_0, we obtain

$$\omega_c = \frac{g(m+2)}{g(m+1)}. \tag{29}$$

This result should be compared with the same result obtained for the scalar case, namely, with Eq. (15).

C. Unified expression for ω_c

A direct comparison of Eqs. (26) with (14) allows us to write the following unified expression of the OAM dependence of the carrier frequency ω_c of an optical pulse in the paraxial regime:

$$\omega_c = \frac{g(m+s)}{g(m+s-1)} + \vartheta_0^2 F_m^{(s)}(R,\theta) + O\left(\vartheta_0^4\right). \tag{30}$$

Notice that for $s = 1$ this equation reduces to Eq. (11), while for $s = 2$ it reduces to Eq. (26). Therefore, for $s = 1$ Eq. (30) describes a scalar pulse, while for $s = 2$ it describes a vector pulse. The explicit expression of $F_m^{(s)}(R,\theta)$ for the scalar and vector cases can be obtained by comparing Eq. (30) with Eqs. (14) and (26) and it is given as follows:

$$F_m^{(s)}(R,\theta) = \begin{cases} \frac{g(m+3)}{4(m+1)g(m)}R^2, & s = 1 \\ \frac{a_m(\theta)\gamma_1(m) + b_m(\theta)\gamma_2(m)}{g^2(m+1)}R^2, & s = 2, \end{cases} \tag{31}$$

where $\gamma_1(m) = g(m+2)g(m+3)$ and $\gamma_2(m) = g(m+1)g(m+4)$.

At the leading order in ϑ_0, therefore, we have the following result:

$$\omega_c = \frac{g(m+s)}{g(m+s-1)}. \tag{32}$$

This is the main result of our work. The dependence of ω_c on the OAM parameter m can be written in a unified universal form as given above. Moreover, since the carrier frequency is given as the ratio between the $(m+s)$th and $(m+s-1)$th moments of the pulse spectrum $f(\omega)$, the dependence of ω_c on the OAM parameter m can be suitably controlled by carefully engineering the pulse spectrum $f(\omega)$.

IV. EXAMPLES

In this section we apply the results obtained above to two explicit cases, namely, optical pulses with exponentially decaying spectra, which can be taken as a model for single-cycle pulses [34], and optical pulses with Gaussian spectra. For the sake of simplicity, we consider well-collimated pulses, so we can neglect the ϑ_0^2 term in the definition of ω_c.

A. Exponentially decaying spectrum

If we choose the following expression for $f(\omega)$:

$$f(\omega) = \Theta(\omega)e^{-\alpha\omega}, \tag{33}$$

where α is a constant with the dimensions of a time that accounts for the width of the spectrum and $\Theta(\omega)$ is the Heaviside step function [32], then, according to Eq. (30), we obtain the result

$$\alpha\omega_c = m + s. \tag{34}$$

In this case, therefore, the carrier frequency varies linearly with the amount of OAM carried by the pulse, as shown in Fig. 1 for the case of both scalar ($s = 1$) and vector ($s = 2$) pulses. This result is in accord with the one presented in Ref. [30]. The explicit expressions of the electric and magnetic fields corresponding to this choice of spectrum are reported in Appendix B.

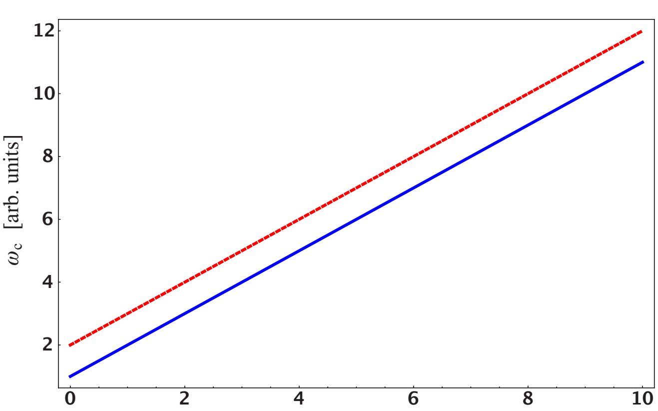

FIG. 1. (Color online) Dependence of the carrier frequency ω_c of a scalar (blue line) and a vector (red dashed line) pulse with an exponentially decaying spectrum from the OAM parameter m. As can be seen, ω_c varies linearly with m in both cases. The scalar (or vector) nature of the pulse only accounts for a different value of ω_c at $m = 0$. For this graph, $\alpha = 1$ is assumed, meaning that ω_c is a dimensionless quantity.

B. Gaussian spectrum

As a second, and actually experimentally realizable, example, we choose a Gaussian-like spectrum, i.e.,

$$f(\omega) = \frac{\gamma}{\sqrt{2\pi}}\sqrt{\frac{\omega}{\omega_0}}e^{-\gamma^2(\omega-\omega_0)^2}\Theta(\omega), \tag{35}$$

where ω_0 is the central frequency of the spectrum and $1/\gamma$ accounts for the spectrum width. Note that the above spectrum defines the so-called Bessel X pulses [35]. The explicit expressions of the electric- and magnetic-field components corresponding to the choice of spectrum given by Eq. (35) are given in Appendix C.

The carrier frequency of a Bessel X pulse carrying m units of OAM is then given by the following relation:

$$\gamma\omega_c = \frac{\mathcal{H}^{(1)}_{m+s}(3/4,-1/4,-\Omega^2)+2\Omega\mathcal{H}^{(2)}_{m+s}(5/4,-1/4,-\Omega^2)}{\mathcal{H}^{(1)}_{m+s}(1/4,1/4,-\Omega^2)+2\Omega\mathcal{H}^{(2)}_{m+s}(3/4,3/4,-\Omega^2)}, \tag{36}$$

where $\Omega = \gamma\omega_0$ and

$$\mathcal{H}^{(1)}_k(a,b,x) = \Gamma\left(a+\frac{k}{2}\right){}_1F_1\left(b-\frac{k}{2};\frac{1}{2};x\right), \tag{37}$$

$$\mathcal{H}^{(2)}_k(a,b,x) = \Gamma\left(a+\frac{k}{2}\right){}_1F_1\left(b-\frac{k}{2};\frac{3}{2};x\right), \tag{38}$$

with ${}_1F_1(a;b;x)$ the confluent hypergeometric function [32]. In this case, as can be seen from Fig. 2, the dependence of ω_c on m is more complicated than in the previous case and in particular for small m, ω_c grows nonlinearly with the OAM parameter.

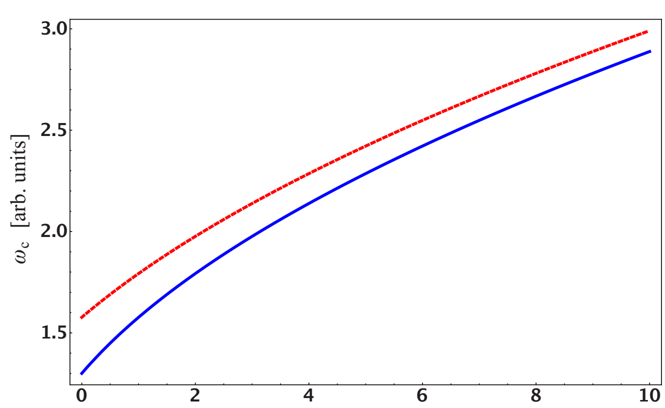

FIG. 2. (Color online) Dependence of the carrier frequency ω_c of a scalar (blue line) and a vector (red dashed line) pulse with a Gaussian spectrum from the OAM parameter m. As can be seen, the dependence of ω_c on m is more complicated that the case shown in Fig. 1, as in this case a Gaussian spectrum is assumed. The scalar (or vector) nature of the pulse only accounts for a different value of ω_c at $m = 0$. For this graph, $\alpha = 1$ is assumed, meaning that ω_c is a dimensionless quantity.

It is interesting, however, in this case, to consider explicitly two limiting cases. First, we consider the case when $\gamma \to 0$, corresponding to a pulse with a very broad spectrum. Taking the limit of Eq. (36) for $\gamma \to 0$ then gives the following result:

$$\gamma\omega_c \simeq \frac{\Gamma(3/4+(m+s)/2)}{\Gamma(1/4+(m+s)/2)}, \tag{39}$$

which for small m gives

$$\gamma\omega_c \simeq a_s + b_s m, \tag{40}$$

where a_s and b_s assume different numerical values for the scalar ($s=1$) and vector ($s=2$) cases, respectively. Their explicit expression is not reported. The interested reader can derive it by explicitly calculating the coefficient of the first-order Taylor expansion of Eq. (39). For small values of m, therefore, the carrier frequency varies linearly with the OAM content of the pulse, in accordance with the results for a very broad (exponentially decaying) spectrum given by Eq. (34).

As a second case, let us consider the opposite limit, i.e., when $\gamma \to \infty$, corresponding to a quasimonochromatic pulse. The expansion of Eq. (36) around $\gamma = \infty$ thus gives the following result:

$$\omega_c \simeq \omega_0. \tag{41}$$

This is in accord with the fact that for a monochromatic beam oscillating at a frequency ω_0, there is no dependence on the OAM parameter m for the carrier frequency.

V. CONCLUSION

In this work we have explicitly calculated the expression of the carrier frequency for a scalar and a vector nondiffracting optical pulse carrying m units of OAM. We have shown that, in general, the carrier frequency of an OAM-carrying pulse is influenced by the amount of OAM carried by the pulse itself. For the simple case of paraxial pulses, we proved that this relation can be written as the (normalized) $(m+s)$th moment of the spectrum of the pulse, where s distinguishes between the scalar and the vector nature of the pulse. Moreover, we have also shown that the functional form of this relation is universally valid and can be applied to both the scalar and the vector case.

To validate our results, we considered explicitly the case of an optical pulse with an exponentially decaying spectrum and a Gaussian spectrum and we calculated explicitly the dependence of the carrier frequency of such pulses on the OAM parameter m. Our calculations show that in both cases the carrier frequency of an optical pulse grows as the amount of OAM carried by the pulse itself grows. While for the case of an exponentially decaying spectrum ω_c depends linearly on m, for the more realistic (and experimentally realizable) case of Bessel X pulses, ω_c has a complicated dependence on m. In this latter case, we have shown that for Bessel X pulses with a broad spectrum, the linear growth of ω_c with m is a very good approximation.

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APPENDIX A: MONOCHROMATIC FIELDS

In this appendix we give an explicit expression for the monochromatic fields $\mathcal{E}(\mathbf{r},t;\omega)$ and $\mathcal{B}(\mathbf{r},t;\omega)$ generated by the scalar field (3). We assume, for the sake of simplicity, that the monochromatic Hertz potential $P(\mathbf{r},t;\omega)$ is oriented along the z axis, i.e.,

$$P(\mathbf{r},t;\omega) = J_m(\omega \sin\vartheta_0 R)e^{im\theta}e^{i\omega(z\cos\vartheta_0 - t)}\hat{\mathbf{z}}. \tag{A1}$$

Using Eqs. (4), the monochromatic electric and magnetic fields generated by the above Hertz potential are given by

$$\begin{aligned}
\mathcal{E}_x(\mathbf{r},t;\omega) &= A_m(\mathbf{r};\omega)\big[f_{m-1}^{(1)}(\mathbf{r};\omega)\cos\theta - f_m^{(2)}(\mathbf{r};\omega)\big],\\
\mathcal{E}_y(\mathbf{r},t;\omega) &= A_m(\mathbf{r};\omega)\big[f_{m-1}^{(1)}(\mathbf{r};\omega)\sin\theta + i f_m^{(2)}(\mathbf{r};\omega)\big],\\
\mathcal{E}_z(\mathbf{r},t;\omega) &= \omega^2\sin^2\vartheta_0 e^{i(m\theta-\zeta\omega)}J_m(R\omega\sin\vartheta_0)
\end{aligned}$$

for the monochromatic electric field and

$$\begin{aligned}
\mathcal{B}_x(\mathbf{r},t;\omega) &= -iA_m(\mathbf{r};\omega)\big[f_{m-1}^{(1)}(\mathbf{r};\omega)\sin\theta + i f_m^{(2)}(\mathbf{r};\omega)\big],\\
\mathcal{B}_y(\mathbf{r},t;\omega) &= iA_m(\mathbf{r};\omega)\big[f_{m-1}^{(1)}(\mathbf{r};\omega)\cos\theta - f_m^{(2)}(\mathbf{r};\omega)\big],\\
\mathcal{B}_z(\mathbf{r},t;\omega) &= 0
\end{aligned}$$

for the monochromatic magnetic field, where $R=\sqrt{x^2+y^2}$, $\tan\theta = y/x$, $\zeta = z\cos\vartheta_0 - t$, and

$$\begin{aligned}
A_m(\mathbf{r};\omega) &= \frac{i\omega\cos\vartheta_0}{R}e^{i(m\theta-\zeta\omega)},\\
f_m^{(1)}(\mathbf{r};\omega) &= R\omega\sin\vartheta_0 J_m(R\omega\sin\vartheta_0),\\
f_m^{(2)}(\mathbf{r};\omega) &= mJ_m(R\omega\sin\vartheta_0)e^{i\theta}.
\end{aligned}$$

APPENDIX B: PULSES WITH AN EXPONENTIALLY DECAYING SPECTRUM: ELECTRIC AND MAGNETIC FIELDS

Let us now consider the exponentially decaying spectrum introduced in Sec. IV A, i.e.,

$$f(\omega) = e^{-\alpha\omega}\Theta(\omega). \tag{B1}$$

Substituting this expression into Eqs. (5), using the expressions of the monochromatic field components derived in Appendix A, and using the integral [36]

$$\int_0^\infty dx\, x^{\mu-1}J_\nu(\beta x)e^{-\gamma x} = \beta^\nu\Theta_{\mu,\nu}\,{}_2F_1(\tilde{a},\tilde{b};\tilde{c};\xi), \tag{B2}$$

where $\Theta_{\mu,\nu} = \Gamma(\mu+\nu)/[2^\nu\gamma^{\mu+\nu}\Gamma(\nu+1)]$, $\tilde{a} = (\mu+\nu)/2$, $\tilde{b} = \tilde{a}+1/2$, $\tilde{c} = \nu+1$ and $\xi = -R^2\sin^2\vartheta_0/(\alpha+i\zeta)^2$, the electric and magnetic fields for a nondiffracting optical pulse with an exponentially decaying spectrum can be written as

$$\begin{aligned}
E_x(\mathbf{r},t) = {}& C_m(\mathbf{r},t,\vartheta_0)e^{im\theta}[e^{i\theta}\,{}_2F_1(a,b;m+1;\xi)\\
&- 2\cos\theta\,{}_2F_1(a,b;m;\xi)],
\end{aligned} \tag{B3a}$$

$$\begin{aligned}
E_y(\mathbf{r},t) = {}& C_m(\mathbf{r},t,\vartheta_0)e^{im\theta}[-ie^{i\theta}\,{}_2F_1(a,b;m+1;\xi)\\
&- 2\sin\theta\,{}_2F_1(a,b;m;\xi)],
\end{aligned} \tag{B3b}$$

$$\begin{aligned}
E_z(\mathbf{r},t) = {}& C_m(\mathbf{r},t,\vartheta_0)e^{im\theta}[-ie^{i\theta}\,{}_2F_1(a,b;m+1;\xi)\\
&- 2\sin\theta\,{}_2F_1(a,b;m;\xi)]
\end{aligned} \tag{B3c}$$

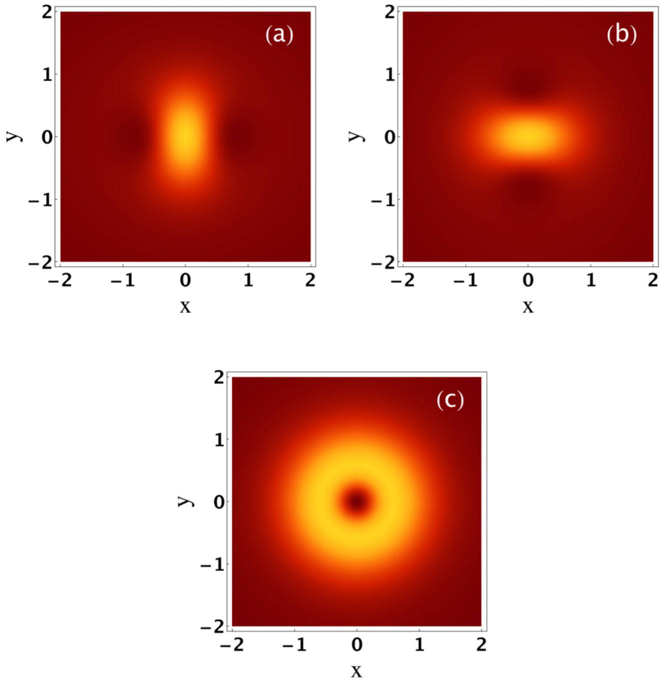

FIG. 3. (Color online) Plot of the intensity distribution of the components of the electric field as given by Eqs. (B3) in the plane $\zeta = 0$: (a) $|E_x(\mathbf{r},t)|^2$, (b) $|E_y(\mathbf{r},t)|^2$, and (c) $|E_z(\mathbf{r},t)|^2$. The x and y axes are expressed in units of ϑ_0. For these plots, the following parameters were used: $m = 1$, $\vartheta_0 = 1$, and $\alpha = 1$.

for the electric field, where $a = (m+2)/2$, $b = (m+3)/2$,

$$C_m(\mathbf{r},t,\vartheta_0) = \left[\frac{m(m+1)}{i2^m}\right]\frac{\cos\vartheta_0(R\sin\vartheta_0)^m}{R(\alpha+i\zeta)^{m+2}}, \tag{B4}$$

and ${}_2F_1(a,b;m;x)$ is the Gaussian hypergeometric function [32]. For the magnetic field, we instead obtain the following result:

$$\begin{aligned} B_x(\mathbf{r},t) &= D_m(\mathbf{r},t,\vartheta_0)e^{im\theta}[e^{i\theta}\,{}_2F_1(a,b;m+1;\xi) \\ &\quad + 2\sin\theta\,{}_2F_1(a,b;m;\xi)], && \text{(B5a)} \\ B_y(\mathbf{r},t) &= D_m(\mathbf{r},t,\vartheta_0)e^{im\theta}[e^{i\theta}\,{}_2F_1(a,b;m+1;\xi) \\ &\quad - 2\cos\theta\,{}_2F_1(a,b;m;\xi)], && \text{(B5b)} \\ B_z(\mathbf{r},t) &= 0, && \text{(B5c)} \end{aligned}$$

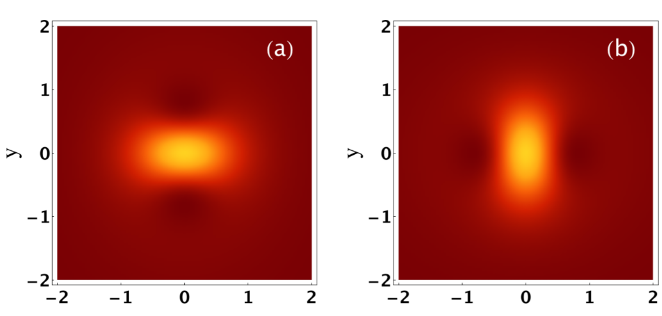

FIG. 4. (Color online) Plot of the intensity distribution of the components of the magnetic field as given by Eqs. (B5) in the plane $\zeta = 0$: (a) $|B_x(\mathbf{r},t)|^2$ and (b) $|B_y(\mathbf{r},t)|^2$. The x and y axes are expressed in units of ϑ_0. For these plots, the following parameters were used: $m = 1$, $\vartheta_0 = 1$, and $\alpha = 1$.

where

$$D_m(\mathbf{r},t,\vartheta_0) = \frac{C_m(\mathbf{r},t,\vartheta_0)}{\cos\vartheta_0}. \tag{B6}$$

Plots of the components of the electric and magnetic fields defined above are given in Figs. 3 and 4, respectively.

APPENDIX C: PULSES WITH A GAUSSIAN SPECTRUM: ELECTRIC AND MAGNETIC FIELDS

We now consider the Gaussian spectrum introduced in Sec. IV B, i.e.,

$$f(\omega) = \frac{\gamma}{\sqrt{2\pi}}\sqrt{\frac{\omega}{\omega_0}}e^{-\gamma^2(\omega-\omega_0)^2}\Theta(\omega). \tag{C1}$$

To construct vector solutions of the wave equation from this spectrum, we have to deal with integrals of the following form:

$$\int_0^\infty d\omega\sqrt{\frac{\gamma^2\omega}{\omega_0}}e^{-\gamma^2(\omega^2+\Omega^2)/2}e^{\gamma^2\omega\Omega}J_m(\rho\omega). \tag{C2}$$

An exact analytical solution to the above integral does not exist in general, mainly because of both the square root factor $\sqrt{\omega}$ and the exponential factor $\exp(\gamma^2\Omega\omega)$ appearing in the integral. We can however give an approximate solution to the integral in Eq. (C2) if we are able to find a suitable approximation for such terms. To do that we note that the asymptotic expansion of the modified Bessel function of the first kind $I_m(x)$ for $x \to \infty$ reads

$$I_m(x) \simeq \frac{e^x}{\sqrt{2\pi x}}[1 + e^{-2x}e^{i(m+1/2)\pi}]. \tag{C3}$$

Knowing this, we can try to replace the exponentially diverging term in Eq. (C2) with the modified Bessel function of the first kind as

$$e^{\gamma^2\Omega\omega} \simeq \sqrt{2\pi\gamma^2\Omega\omega}\,I_m(\gamma^2\Omega\omega), \tag{C4}$$

thus obtaining an analytically solvable integral, whose solution is given by [36]

$$\begin{aligned} \mathcal{I}_m(\alpha,\beta,\gamma) &= \int_0^\infty d\omega\,\omega\,e^{-\alpha\omega^2}I_m(\beta\omega)J_m(\gamma\omega) \\ &= \frac{1}{2\alpha}e^{(\beta^2-\gamma^2)/4\alpha}J_m\left(\frac{\beta\gamma}{2\alpha}\right). \end{aligned} \tag{C5}$$

To validate such an approximation, we make use of the results presented in Ref. [35] for the case $m = 0$. There, in fact, it is said that this approximation holds for all the practical (i.e., experimentally accessible) cases and for optical pulses as short as 1 fs as well. Although in Ref. [35] this approximation was used only to describe Bessel X pulses generated by zeroth-order Bessel beams, i.e., only for the case $m = 0$, its validity can be straightforwardly extended to the case $m \neq 0$. As can be seen from the expansion (C3), in fact, the dependence on m does not play any significant role, as the m is contained only in the second term of the expansion, which vanishes for $x \to \infty$. This then allows us to take as valid the approximation given by Eq. (C4) also for the case $m \neq 0$.

Substituting the spectrum (C1) into Eqs. (5) and using the above results and the expressions for the monochromatic fields

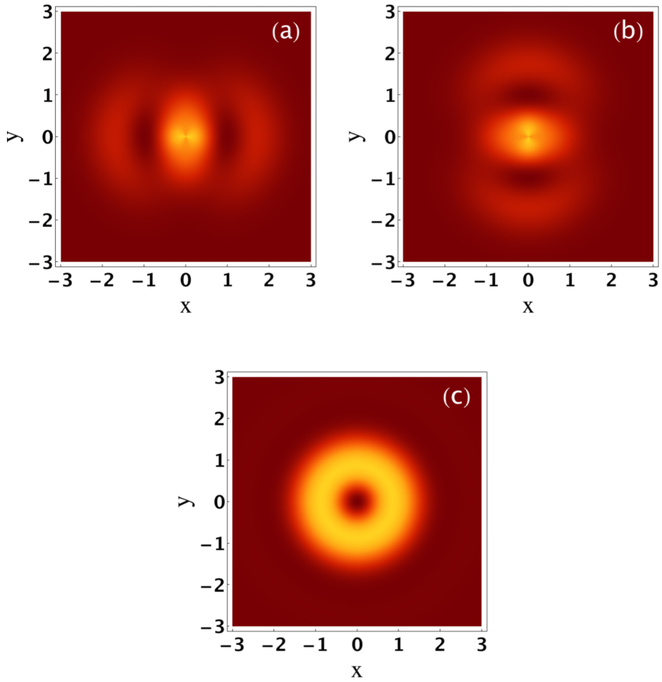

FIG. 5. (Color online) Plot of the intensity distribution of the components of the electric field as given by Eqs. (C6) in the plane $\zeta = 0$: (a) $|E_x(\mathbf{r},t)|^2$, (b) $|E_y(\mathbf{r},t)|^2$, and (c) $|E_z(\mathbf{r},t)|^2$. The x and y axes are expressed in units of ϑ_0. For these plots, the following parameters were used: $m = 1$, $\vartheta_0 = 1$, and $\alpha = 1$.

given in Appendix A, the electric and magnetic fields for a nondiffracting optical pulse with Gaussian spectrum can be written as

$$E_x(\mathbf{r},t) = -\frac{m\cos\vartheta_0 e^{i(m+1)\theta}}{R}\frac{\partial F_m(\varrho,\zeta)}{\partial\zeta} - \frac{i\sin 2\vartheta_0\cos\theta e^{im\theta}}{R}\frac{\partial^2 F_{m-1}(\varrho,\zeta)}{\partial\zeta^2}, \tag{C6a}$$

$$E_y(\mathbf{r},t) = \frac{im\cos\vartheta_0 e^{i(m+1)\theta}}{R}\frac{\partial F_m(\varrho,\zeta)}{\partial\zeta} - \frac{i\sin 2\vartheta_0\sin\theta e^{im\theta}}{R}\frac{\partial^2 F_{m-1}(\varrho,\zeta)}{\partial\zeta^2}, \tag{C6b}$$

$$E_z(\mathbf{r},t) = -\sin^2\vartheta_0 e^{im\theta}\frac{\partial^2 F_m(\varrho,\zeta)}{\partial\zeta^2} \tag{C6c}$$

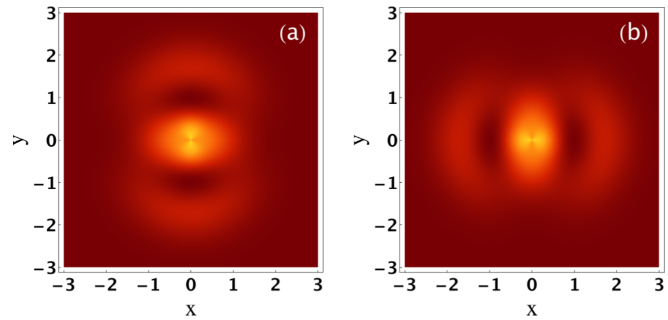

FIG. 6. (Color online) Plot of the intensity distribution of the components of the magnetic field as given by Eqs. (C9) in the plane $\zeta = 0$: (a) $|B_x(\mathbf{r},t)|^2$ and (b) $|B_y(\mathbf{r},t)|^2$. The x and y axes are expressed in units of ϑ_0. For these plots, the following parameters were used: $m = 1$, $\vartheta_0 = 1$, and $\alpha = 1$.

for the electric field, where $\varrho = R\sin\vartheta_0$,

$$F_m(\varrho,\zeta) = \sqrt{Z(\zeta)}e^{-(\varrho^2+\zeta^2)/2\gamma^2}J_m[Z(\zeta)\omega_0\varrho]e^{i\omega_0\zeta}, \tag{C7}$$

and

$$Z(\zeta) = 1 + i\frac{\zeta}{\omega_0\gamma^2}. \tag{C8}$$

Equation (C7) is the result of the integral (C2) for the case of the scalar field in Eq. (3). Therefore, $Z(\zeta)$ can be interpreted, in analogy with Gaussian beams, as an effective complex q parameter for the pulse. This means that the quantity $\zeta/(\omega_0\gamma^2)$ can be interpreted as the equivalent of the pulse's Rayleigh range. This is in accord with the fact that, essentially, Eq. (C5) is the polychromatic counterpart of a Bessel Gaussian beam.

For the magnetic field, we obtain instead the following result:

$$B_x(\mathbf{r},t) = -\frac{ime^{i(m+1)\theta}}{R}\frac{\partial F_m(\varrho,\zeta)}{\partial\zeta} + i\sin\vartheta_0\sin\theta e^{im\theta}\frac{\partial^2 F_{m-1}(\varrho,\zeta)}{\partial\zeta^2}, \tag{C9a}$$

$$B_y(\mathbf{r},t) = -\frac{me^{i(m+1)\theta}}{R}\frac{\partial F_m(\varrho,\zeta)}{\partial\zeta} - i\sin\vartheta_0\cos\theta e^{im\theta}\frac{\partial^2 F_{m-1}(\varrho,\zeta)}{\partial\zeta^2}, \tag{C9b}$$

$$B_z(\mathbf{r},t) = 0. \tag{C9c}$$

Plots of the components of the electric and magnetic fields defined above are given in Figs. 5 and 6, respectively.

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