Fluctuating hydrodynamics for a discrete Gross-Pitaevskii equation: Mapping onto the Kardar-Parisi-Zhang universality class

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We show that several aspects of the low-temperature hydrodynamics of a discrete Gross-Pitaevskii equation (GPE) can be understood by mapping it to a nonlinear version of fluctuating hydrodynamics. This is achieved by first writing the GPE in a hydrodynamic form of a continuity and a Euler equation. Respecting conservation laws, dissipation and noise due to the system's chaos are added, thus giving us a nonlinear stochastic field theory in general and the Kardar-Parisi-Zhang (KPZ) equation in our particular case. This mapping to KPZ is benchmarked against exact Hamiltonian numerics on discrete GPE by investigating the nonzero temperature dynamical structure factor and its scaling form and exponent. Given the ubiquity of the Gross-Pitaevskii equation (also known as the nonlinear Schrödinger equation), ranging from nonlinear optics to cold gases, we expect this remarkable mapping to the KPZ equation to be of paramount importance and far reaching consequences.

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I. INTRODUCTION

Low-dimensional classical and quantum systems are often very counterintuitive and exhibit properties different from their higher dimensional counterparts [1,2]. One such example is the width of the line shape of the phonon peaks in the dynamical structure factor. Contrary to the expected k^2 behavior in higher dimensions [3], the power is anomalous in low dimensions. It is well known that linearized hydrodynamics, which predicts a diffusive broadening, fails in one dimension (1D). This immediately creates a need for a nonlinear hydrodynamics that could describe low-dimensional systems. Such a theory beyond the conventional Luttinger liquid would describe the superdiffusive broadening in low-dimensional systems.

A system that provides a remarkable platform for probing low-dimensional fluids is the system of a 1D weakly interacting Bose gas at nonzero temperature [4]. Using a variant of Bragg spectroscopy [5,6] one could probe the dynamical structure factor of the Bose gas, thereby unraveling the nonlinear phenomenon in low-dimensional fluids.

It is to be noted that in the Lieb-Liniger model, i.e., gas of bosons in one dimension with contact interaction (of which the GPE is a semiclassical approximation), the dynamical structure factor at zero temperature has a width that scales with k^2. This is, in fact, a general feature at zero temperature for one-dimensional systems with nonlinear dispersion [1]. However, here we study the finite-temperature regime where one would expect an anomalous exponent in low dimensions ($k^{3/2}$). The problem of describing within a single theoretical framework both the finite-temperature phenomena and the zero-temperature results [1] remains to be an open unsolved question.

The underlying theory that describes [7] this cold atomic system, namely, the Gross-Pitaevskii equation (GPE) or the nonlinear Schrödinger (NLS) equation is ubiquitous in areas such as optics, cold gases, and mathematical physics. Although the strictly continuum GPE is integrable, the experimental realizations break integrability in one or more ways, such as, the presence of a lattice or trapping potential, energy loss, and escape of unwanted evaporation of particles. Here, we focus on the discrete (lattice [8]) version of GPE which is the generic nonintegrable case. Such a discrete GPE has been realized in experiments on waveguide lattices [9].

The ubiquity of such a class of equations and cutting edge technologies available to probe statistical properties of such systems enhances an urgent need for writing down a stochastic nonlinear theory that makes transparent the role of various components that result in a complex nonlinear-driven-dissipative phenomenology. Establishing this strong connection between GPE and stochastic nonlinear differential equations (which turns out to be a two-component KPZ equation in our case) helps in using the tools available in the literature to make far-reaching predictions about statistical mechanics of systems such as a 1D Bose gas or optical waveguides. In the converse, one could also use such systems as an experimental test bed for KPZ phenomena, providing much needed additional experimental realizations [10–12] of KPZ physics.

In this article, we analyze the low-temperature hydrodynamics of GPE, which is known to be a valid description for systems such as 1D weakly interacting Bose gas or optical waveguides. We present a discrete GPE that governs the dynamics of such complex fields (which are atomic fields in the case of cold atoms or optical fields in waveguides). We write continuity-like and Euler-like equations for the macroscopic density and velocity fields and derive the nonlinear fluctuating hydrodynamics. The coefficients of the resulting nonlinear fluctuating theory are expressed in terms of underlying parameters of the system (such as coupling strength and background density). We then present results for the dynamical structure factor $S(k,\omega)$ (i.e., Fourier transform of correlation function of fields obeying nonlinear fluctuating hydrodynamic theory), namely, its scaling function and the underlying anomalous exponent. This effective nonlinear hydrodynamic theory is finally benchmarked against exact Hamiltonian numerics. Our results also support a recent remarkable conjecture that the long-wavelength dynamics of a classical 1D fluid at finite temperature is in the Kardar-Parisi-Zhang (KPZ) universality class [13]. In addition to confirmation of the 3/2 exponent, we have taken a big step forward in showing agreement with the Prahofer-Spohn scaling function [14]. Therefore, the notoriously difficult problem of computing the dynamical structure factor (density-density correlations) can now be connected to correlation functions of familiar stochastic equations.

II. NONLINEAR FLUCTUATING HYDRODYNAMICS AND GPE

The semiclassical Hamiltonian describing a strictly one-dimensional gas of bosons of mass m and contact interaction strength g is given by

$$H = \int dx \left[\frac{|\partial_x \psi|^2}{2m} + \frac{g}{2}|\psi|^4 \right], \tag{1}$$

which in conjugation with Poisson brackets $\{\psi^*(x),\psi(y)\} = i\delta(x-y)$ gives the time-dependent GPE,

$$i\partial_t \psi = -\frac{1}{2m}\partial_x^2 \psi + g|\psi|^2\psi. \tag{2}$$

This is a continuum integrable model. However, physical realizations are not in this ideal limit. The generic case is nonintegrable due to several possibilities, such as the presence of a lattice rather than continuum (breaking translational invariance), interactions being nonzero range, existence of external potential, and dissipation. Here, we will assume that we are not in the ideal integrable limit. In other words, integrability is destroyed and this nonlinear classical system is chaotic at nonzero temperature. The specific integrability breaking we consider is the discrete GPE (NLS) on a 1D lattice [Eq. (12)] but our results are generic. From the perspective of optical applications, g is called a Kerr nonlinearity and $|\psi|^2$ is the intensity of light field.

We examine the hydrodynamics of the equilibrium steady state that this chaotic system approaches at long times. We are interested in hydrodynamic scaling of the density-density correlation $S(x,t) = \langle |\psi(x,t)|^2 |\psi(0,0)|^2\rangle - \langle |\psi(0,0)|^2\rangle^2$ with $\langle\cdot\rangle$ denoting the average over the statistical steady state. $\psi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ defines the density $\rho(x,t)$ and phase θ. The velocity is $v(x,t) = \frac{1}{m}\frac{\partial\theta(x,t)}{\partial x}$. We work at low enough temperature that the rate at which phase slips occur at equilibrium is negligible. Hence, velocity is a conserved quantity, as is density. The continuity and Euler equations are

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \quad \partial_t v + \partial_x\left(\frac{v^2}{2} + \frac{g}{m}\rho\right) = 0\,. \tag{3}$$

The equilibrium state has average density $\rho_0 = \langle |\psi(x)|^2\rangle$ and we consider the case of zero average velocity.

In the regime we are considering, the equation for $S(x,t)$ above refers to small deviations from the average density. Hence if we linearize Eq. (3), taking $\rho \to \rho_0 + \varrho$ and $v \to 0 + v$, we obtain $\partial_t \vec{u} + \partial_x[A\vec{u}] = 0$ with $\vec{u} = \binom{\varrho}{v}$, $A = \begin{pmatrix} 0 & \rho_0 \\ \frac{g}{m} & 0 \end{pmatrix}$. This gives us the right and left moving sound modes with speed $c = \sqrt{g\rho_0/m}$. In other words, it is the dynamics of a linearized Luttinger liquid whose dynamical structure factor $S(k,\omega)$ consists only of a pair of δ function peaks at $\omega = \pm c|k|$, corresponding to undamped phonons. We need to add to this the scattering between phonons due to nonlinearities. In linear fluctuating hydrodynamics one adds damping and noise, which broadens the sound peaks in $S(k,\omega)$, giving them a line width that scales "diffusively" as $\Gamma(k) \sim k^2$. This works fine in dimension $d \geqslant 3$ [15], but fails in 1D [16]. An example showing this anomaly involves simulations of Fermi-Pasta-Ulam (FPU) chains which report superdiffusive broadening of the sound peaks [2,17–20]. To capture such behavior, it has been proposed recently to use a nonlinear extension of fluctuating hydrodynamics [21]. We will follow this strategy to obtain the hydrodynamic scaling of $S(k,\omega)$, which we then compare to exact Hamiltonian numerics. The prescription of nonlinear fluctuating hydrodynamics [21] consists of adding diffusion and noise matrices in Eq. (3) giving

$$\partial_t \vec{u} + \partial_x \left[A\vec{u} + \frac{1}{2}\sum_{\alpha,\beta=1}^{2} \vec{H}_{\alpha,\beta} u_\alpha u_\beta - \partial_x(D\vec{u}) + B\vec{\xi} \right] = 0, \tag{4}$$

where D and B are diffusion and noise matrices. Above, the Hessian matrix $H^\gamma_{\alpha,\beta} = \partial_{u_\alpha}\partial_{u_\beta} j^\gamma$ along with $\vec{j} = (\varrho v, \frac{1}{2}v^2)$ captures the nonlinear terms in the underlying GPE. Here, $\vec{\xi}$ is a Gaussian white noise with mean 0 and covariance given by $\langle \xi_\alpha(x,t)\xi_{\alpha'}(x',t')\rangle = \delta_{\alpha\alpha'}\delta(x-x')\delta(t-t')$.

Dropping quadratic terms would correspond to linear fluctuating hydrodynamics which yields diffusive sound peaks. For our application we are interested in the stationary, mean zero process governed by Eq. (4), again denoted by $\varrho(x,t)$ and $v(x,t)$. The equal time, static correlations are expected to have short-range correlations. Hence, we define the susceptibilities as follows:

$$\begin{aligned} c_1 &= \int dx \left[\langle \varrho(x,0)\varrho(0,0)\rangle - \rho_0^2\right], \\ c_2 &= \int dx \langle v(x,0)v(0,0)\rangle, \end{aligned} \tag{5}$$

where cross terms vanish because $\varrho(x,t)$ and $v(x,t)$ have different parity. The fluctuation dissipation relation is given by $DC + CD = BB^\dagger$ [C is a diagonal matrix containing Eq. (5)]. In addition, space-time stationarity enforces in general the relation $AC = CA^{\mathrm{T}}$, which implies the relation $c_2 = \frac{c^2}{\rho_0^2}c_1$ for GPE. In Eq. (3) the linear terms dominate and to obtain better insight into the solution one has to transform to normal modes which have a definite propagation velocity (speed of sound). We therefore introduce a linear transformation in component space, by setting $\binom{\phi_-}{\phi_+} = R\binom{\varrho}{v}$ such that R satisfies $RAR^{-1} = \mathrm{diag}(-c,c)$. In addition we require that the ϕ susceptibilities are normalized to unity, which means $R\,\mathrm{diag}(c_1, \frac{c^2}{\rho_0^2}c_1)R^{\mathrm{T}} = 1$. Up to an overall sign, R is uniquely determined and given by $R = \frac{1}{c\sqrt{2c_1}}\begin{pmatrix} -c & \rho_0 \\ c & \rho_0 \end{pmatrix}$. Then the equation for the normal modes (i.e, the left and right chiral sectors, also known as eigenmodes) reads

$$\partial_t \phi_\sigma + \partial_x[\sigma c\phi_\sigma + \langle \vec{\phi}, G^\sigma \vec{\phi}\rangle - \partial_x(D_{\mathrm{rot}}\phi)_\sigma + (B_{\mathrm{rot}}\xi)_\sigma] = 0, \tag{6}$$

where $\sigma = \pm$ refers to right and left modes and "rot" indicates the matrices rotated by R matrix, $D_{\rm rot} = RDR$ and $B_{\rm rot} = RB$. The coupling matrix is given by

$$G^- = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}, \quad G^+ = \frac{c}{2\rho_0}\sqrt{\frac{c_1}{2}}\begin{pmatrix}-1 & 1\\ 1 & 3\end{pmatrix}. \tag{7}$$

Since Eq. (6) is nonlinear, it is still difficult to compute the covariance $\langle\phi_\sigma(x,t)\phi_{\sigma'}(0,0)\rangle$ for the mean zero, stationary process. One central observation is that in leading order the two peaks in $S(x,t)$ separate linearly in time. Hence, in Eq. (6) for ϕ_σ, the terms $\phi_\sigma\phi_{-\sigma}$ and $(\phi_{-\sigma})^2$ turn out to be irrelevant [20] compared to $(\phi_\sigma)^2$. Albeit they may effectively renormalize nonuniversal coefficients (in front of all terms), they do not impact the universal properties. Therefore, preserving universality we can decouple Eq. (6) into two components giving stochastic Burgers equation (KPZ in "height function" h_σ where $\phi_\sigma = \partial_x h_\sigma$) and, for this, the exact scaling function is available [14] and tabulated [22],

$$\langle\phi_\sigma(x,t)\phi_\sigma(0,0)\rangle = (\lambda t)^{-2/3} f_{\rm KPZ}((\lambda t)^{-2/3}(x-\sigma ct)) \tag{8}$$

valid for large x,t. λ is a nonuniversal coefficient, which here is explicitly calculated to be $\lambda = 2\sqrt{2}|G^\sigma_{\sigma\sigma}| = \frac{3c}{\rho_0}\sqrt{c_1}$. The value of λ derived above will get renormalized [20] due to the discarded nonlinearities as explained above. Note that λ does not depend on $D_{\rm rot}$ or $B_{\rm rot}$. This says that while some dissipation and noise are needed to maintain stationarity, the asymptotic form of correlation is dominated by $G^\sigma_{\sigma\sigma}$. In several molecular dynamics-type simulations, one computes the correlation $S(x,t)$ [23] (where x is the space coordinate) directly. But more conventionally, as also relevant in this paper, we study the structure function, which is defined as the space-time Fourier transform of $S(x,t)$. We define $\hat{S}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}S(x,t)dx$. As argued before, the asymptotic scaling form is expected to be of the form $\hat{S}(k,t) = \frac{1}{2}(e^{ikct}+e^{-ikct})c_1\hat{f}_{\rm KPZ}(k(\lambda|t|)^{2/3})$. The two sound peaks are symmetric reflections of each other. Considering only the right mover and setting $\omega_k = ck$, we get

$$\begin{aligned}\hat{S}(k,\omega+\omega_{\rm k}) &= \int dt e^{i\omega t}\frac{1}{2}c_1\hat{f}_{\rm KPZ}(k(\lambda|t|)^{2/3})\\ &= \int dt e^{i(\omega/\lambda|k|^{3/2})t}(\lambda|k|^{3/2})^{-1}\frac{1}{2}c_1\hat{f}_{\rm KPZ}(|t|^{2/3}).\end{aligned} \tag{9}$$

By defining $h(\omega) = \int dt e^{i\omega t}\hat{f}_{\rm KPZ}(|t|^{2/3})$ we arrive at

$$\hat{S}(k,\omega+\omega_{\rm k}) = \tfrac{1}{2}c_1(\lambda|k|^{3/2})^{-1}h(\omega/\lambda|k|^{3/2}). \tag{10}$$

If the maximum of $\hat{S}$ is normalized to 1, then the prefactor in Eq. (10) is set to 1 and h is replaced by $h/h(0)$.

The hypothesis about the decoupling of chiral fields is a subtle issue. The $\phi_\sigma(x,t)$ fields fluctuate without any spatial decay. In fact, only the correlations are peaked near $\pm ct$. However, the decoupling of the components can be seen directly on the level of mode coupling in the one-loop approximation. As supported by numerical solutions [23], it is safe to use the diagonal approximation $\langle\phi_\sigma(x,t)\phi_{\sigma'}(0,0)\rangle = \delta_{\sigma\sigma'}f_\sigma(x,t)$. In the one-loop approximation, one has (ν is the phenomenologically added dissipation)

$$\begin{aligned}\partial_t f_\sigma(x,t) &= \left(-\sigma c\partial_x + \tfrac{1}{2}\nu\partial_x^2\right)f_\sigma(x,t)\\ &\quad + \int_0^t ds\int dy f_\sigma(x-y,s)\partial_y^2 M(y,s),\end{aligned} \tag{11}$$

with the memory kernel given by $M(x,t) = 2\sum_{\sigma\sigma'=\pm}(G^\sigma_{\sigma\sigma'})^2 f_\sigma(x,t)f_{\sigma'}(x,t)$. The terms with $\sigma \neq \sigma'$ have a very small overlap. But diagonal terms proportional to $(G^\sigma_{\sigma\sigma})^2$ do contribute to the long time behavior. By explicit computation we can check that the self-interaction term dominates the mutual one. Equation (11) can be studied numerically by an iteration scheme. The asymptotic shape of the sound peak and the true scaling function $f_{\rm KPZ}$ have a relative error of about 4% [23]. It is of utmost importance to have such a deterministic expression [Eq. (11)] for the correlators of 1D Bose gas that captures physics beyond a Luttinger liquid. All the above needs to be benchmarked against brute-force Hamiltonian numerics of the underlying GPE Hamiltonian.

III. HAMILTONIAN NUMERICS OF DISCRETE GPE

We now go to the discrete version of above time-dependent GPE [Eq. (2)] that now governs the dynamics of a complex-valued $\psi(n,t)$, with integer $n = 1,\ldots,N$ and periodic boundary conditions. Discretization is achieved by substituting $x \to na$ where a is the lattice spacing and Na is the system size L. The discrete version of time-dependent GPE reads

$$i\frac{d}{dt}\psi(n,t) = \mathcal{F}_{\rm inv}\left[\frac{k_q^2}{2m}\tilde{\psi}(k_q,t)\right] + g|\psi(n,t)|^2\psi(n,t), \tag{12}$$

where $k_q = \frac{2\pi}{Na}q$ for integer $q = (-\frac{N}{2}+1),\ldots,\frac{N}{2}$ and $\mathcal{F}_{\rm inv}$ denotes our inverse-Fourier transform, $\mathcal{F}_{\rm inv}\{G(k_q)\} = \frac{1}{a}\frac{1}{N}\sum_{n=1}^N G(k_q)e^{-\frac{2\pi i}{N}nq}$ (slightly unconventional due to the explicit presence of lattice spacing a). The local energy and the local number density, $|\psi(n)|^2$, are conserved. According to standard classifications, Eq. (12) is listed as nonintegrable [8]. Hence one would expect that H and $\mathsf{N} = a\sum_{n=1}^N|\psi(n)|^2$ are the only conserved fields and that the set of equilibrium states is of the form $Z^{-1}e^{-\beta(H-\mu\mathsf{N})}$, $\beta > 0$, $\mu \in \mathbb{R}$, in the limit of large N. Therefore, the above discretization scheme for the integrable continuum GPE breaks the underlying integrability. In fact, we find that in order to make connection to fluctuating hydrodynamics and subsequently KPZ, we require broken integrability and the resulting chaos.

In this section, we describe the Hamiltonian exact numerics [24] starting from Eq. (12). The time evolution is obtained by the well-known leap-frog splitting technique where the system is evolved alternatively (setting $g = m = 1$, and choosing $\rho_0 = 1$) by kinetic, $\tilde{\psi}(k_q,t) \to e^{-ik_q^2\tau/2}\tilde{\psi}(k_q,t)$, and potential, $\psi(n,t) \to e^{-i\tau|\psi(n,t)|^2}\psi(n,t)$, terms in sequence $\mathcal{V}_{\frac{\tau}{2}}\cdot\mathcal{T}_\tau\cdot\mathcal{V}_{\frac{\tau}{2}}$, with time step τ.

In the simulation [24] we measure the structure function $\tilde{S}(k_q,\omega)$: At each time step we obtain the time evolved

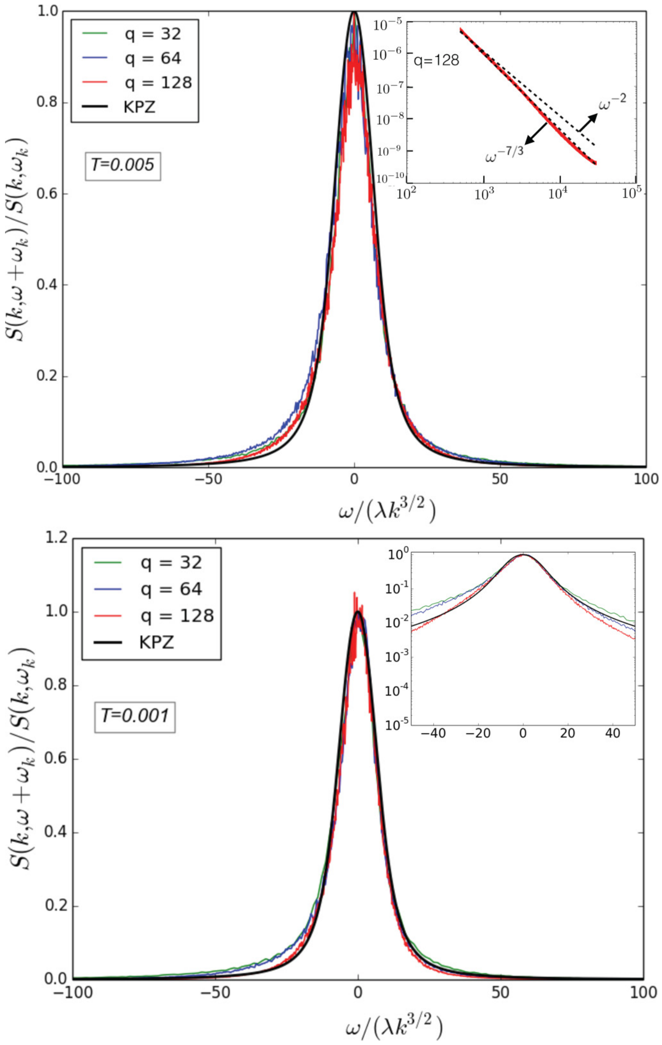

FIG. 1. (Color online) (Top) Comparison between exact numerics of discrete GPE with nonlinear fluctuating hydrodynamics. Here the temperature $T = 0.005$, $L = 5 \times 2^{14}$. The best fit to the scaling function $h(\omega)/h(0)$ is given by $\lambda_{\rm opt} \sim 0.005$ (and theoretical $\lambda \sim 0.045$) with universal tail (shown by log-log inset) close to $\omega^{-\frac{7}{3}}$ at significantly large frequencies [14]. (Bottom) Similar comparison for a different set of parameters. Here the temperature $T = 0.001$, $L = 10 \times 2^{13}$. The best fit to the scaling function $h(\omega)/h(0)$ is given by $\lambda_{\rm opt} \sim 0.0028$ (and theoretical $\lambda \sim 0.0041$). The wave vectors are given by $k = \frac{2\pi}{L} q$ where the values of the integers q are given in the legends. The inset shows the dynamical structure factor on a logarithm scale. All quantities plotted are in dimensionless units.

density $\rho(n,t) = |\psi(n,t)|^2$, which we then space-time Fourier transform to $\tilde{\rho}(k_q,\omega)$. Then the dynamical structure factor is the ensemble average: $\tilde{S}(k_q,\omega) = \langle|\tilde{\rho}(k_q,\omega)|^2\rangle$. These results are expected to depend only on the total energy and particle number of the initial conditions because the dynamics are chaotic and we expect ergodic behavior. The chaos for our parameters $a = 5$ and $\tau = 2$ has been confirmed by observing positive Lyapunov exponents.

For random initial conditions we assume that the Fourier coefficients ϱ_k,θ_k are independent Gaussian random variables with mean 0 and covariance given by

$$\langle|\varrho_{k_q}^2|\rangle = \frac{\rho_0}{2L}\frac{\alpha_{k_q} T}{\xi_{k_q}}, \quad \langle|\theta_{k_q}^2|\rangle = \frac{1}{2\rho_0 L}\frac{T}{\alpha_{k_q}\xi_{k_q}}, \tag{13}$$

where $\xi_{k_q} = \sqrt{\frac{k_q^2}{2}(\frac{k_q^2}{2} + 2\rho_0)}$ and $\alpha_{k_q} = \sqrt{\frac{k_q^2}{\frac{k_q^2}{4}+\rho_0}}$.

In Ref. [24] the low-temperature dynamical structure factor was simulated numerically and the KPZ scaling exponent was observed, i.e., phonon line width $\Gamma_k \sim |k|^z$ with $z = 1.510 \pm 0.018$ was found. Here we provide a more quantitative comparison with the full scaling function (Fig. 1). Importantly, we have also presented the mapping to nonlinear fluctuating hydrodynamics, which tells us that the structure factor should be of the form shown in Eq. (10). This means that the structure factor we obtain from our brute-force simulations must have the KPZ scaling exponent and follow the scaling function in the hydrodynamic limit. In Fig. 1 we show the remarkable quantitative agreement between exact Hamiltonian numerics and the expectations of a nonlinear hydrodynamic theory with fluctuations. Our results are in a regime where the system is not near integrability, due to a chosen large lattice spacing a. We notice that, on approaching integrability [25] by reducing a we find strong deviations from KPZ, both in terms of scaling form and exponent. Since the generic nonintegrable case is the central agenda of our paper, our findings in the crossover regime will be discussed elsewhere [26]. The discrepancy between the optimally chosen value of λ ($\lambda_{\rm opt}$) and the one expected from KPZ correspondence ($\lambda = 2\sqrt{2}|G^{\sigma}_{\sigma\sigma}|$) probably arises due the fact that higher-order nonlinearities and the different chiral sectors effectively renormalize the first relevant nonlinearity of the specific chiral sector under consideration. Such a disagreement has also been seen recently in case of the FPU problem [27]. One therefore requires an effective renormalization scheme to make more precise connections between λ and $\lambda_{\rm opt}$.

IV. CONCLUSION

We have demonstrated a strong connection between the statistical mechanics of a discrete NLS/GPE and a nonlinear hydrodynamic theory with fluctuations. This was done by first formulating the GPE in terms of hydrodynamic variables (conjugate classical fields) and then adapting a recent procedure in formulating a fluctuating version of nonlinear hydrodynamic theory [21]. In our case, the resulting theory is shown to be of the KPZ universality class. This immediately enables us to use the rich physics of KPZ class and a well-established one-loop approximation to make predictions for GPE. This was then benchmarked by exact Hamiltonian numerics. Given the wide range of phenomena described by these equations, our results have implications in fields ranging from cold gases to nonlinear optics. Moreover, extending this mapping to coupled nonlinear Schrödinger equations (also an experimentally realized situation in both cold atoms [28,29] and nonlinear optics [30]) is shown to give a variety [31] of interesting dynamical critical phenomena arising due to coupled stochastic differential equations [26].

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