# Damping of confined excitation modes of one-dimensional condensates in an optical lattice

C. Trallero-Giner,[1] Darío G. Santiago-Pérez,[2,3] Ming-Chiang Chung,[4] G. E. Marques,[5] and R. Cipolatti[6]

[1]*Facultad de Física, Universidad de La Habana, Vedado 10400, La Habana, Cuba*

[2]*Universidad de Sancti Spiritus "José Martí Pérez," Avenida de los Mártires 360, CP 62100, Sanctí Spiritus, Cuba*

[3]*Centro Latino-Americano de Física, Avenida Venceslau Braz, 71 Fundos, 22290-140 Rio de Janeiro, Rio de Janeiro, Brazil*

[4]*Department of Physics, National Chung Hsing University, Taichung 40227, Taiwan*

[5]*Departamento de Física, Universidade Federal de São Carlos, 13565-905 São Carlos, São Carlos, Brazil*

[6]*Instituto de Matemàtica, Universidade Federal do Rio de Janeiro, CP 68530, 21941-901 Rio de Janeiro, Rio de Janeiro, Brazil*

(Received 4 April 2015; published 5 October 2015)

We study the damping of the collective excitations of Bose-Einstein condensates in a harmonic trap potential loaded in an optical lattice. In the presence of a confining potential the system is inhomogeneous and the collective excitations are characterized by a set of discrete confined phononlike excitations. We derive a general convenient analytical description for the damping rate, which takes into account the trapping potential and the optical lattice for the Landau and Beliaev processes at any temperature $T$. At high temperature or weak spatial confinement, we show that both mechanisms display a linear dependence on $T$. In the quantum limit, we find that the Landau damping is exponentially suppressed at low temperatures and the total damping is independent of $T$. Our theoretical predictions for the damping rate under the thermal regime is in complete correspondence with the experimental values reported for the one-dimensional (1D) condensate of sodium atoms. We show that the laser intensity can tune the collision process, allowing a resonant effect for the condensate lifetime. Also, we study the influence of the attractive or repulsive nonlinear terms on the decay rate of the collective excitations. A general expression for the renormalized Goldstone frequency is obtained as a function of the 1D nonlinear self-interaction parameter, laser intensity, and temperature.

DOI: 10.1103/PhysRevA.92.042502 PACS number(s): 03.75.Be, 03.75.Lm, 05.45.Yv

## I. INTRODUCTION

The damping process plays a crucial role in the dynamics of Bose-Einstein condensates (BECs). Phenomena such as superfluid phase transitions [1–7], the Josephson effect [8,9], quantized vortices [10–12], and Mott insulator transitions [13], among others [14], are limited by the finite lifetime of the collective excitations of the condensed atoms, i.e., by the damping mechanisms and their dependence on temperature. After the experimental confirmation that the collective excitations are damped [2,4,15,16], the behavior of the decay rates has been of major interest in the physics of BECs. Thus, several studies and theoretical calculations of the damping of excitations have been performed in three-dimensional (3D) [1,17–23], 2D [1,22–24], and 1D systems [25–27]. For a better understanding of the damping process, it becomes necessary to consider the contribution of the parabolic confining potential. The behavior and characteristics of the decay rate and its dependence on temperature differ radically whether or not we are dealing with homogeneous systems. The main assumption for a homogeneous system is to consider the condensate density constant in the whole space.

In the present work we consider a microscopic theory for the damping rate of collective oscillations, specifically for a quasi-1D condensate confined to a parabolic harmonic trap potential and loaded into an optical lattice. The existence of a non-negligible confining external potential breaks the invariance symmetry, which leads to the damping rate showing a different qualitative behavior in comparison with the previous formalism, where the condensate is tackled as a homogeneous system (see, for example, Refs. [23,28], and references therein). The influence of both external interactions, the trap potential and laser intensity, must provide a physically richer scenario for the decay of collective excitations. The knowledge of excited states or Goldstone modes enables the characterization of the condensate dynamics in a general framework. In a spatially inhomogeneous BEC system, the label spacing of the discrete spectrum (confined phononlike modes) and also the nature and symmetry of the wave function of the collective modes are required for the calculation of the collision scattering process [21].

Cigar-shaped traps can be considered as quasi-one-dimensional systems [29]. We select such a platform of a condensate loaded simultaneously into a 1D harmonic potential and an optical lattice to characterize the phenomenon of damping and tackle the problem analytically. Within the framework of mean-field theory, the physical characteristics of a BEC in such a trapping profile are ruled by the time-dependent nonlinear Gross-Pitaevskii equation (GPE) [28,32,33] in an external potential

$$V_{\rm ext}(x) = \frac{1}{2} m\omega_0^2 x^2 - V_L \cos^2\left(\frac{2\pi}{d}x\right), \tag{1}$$

where $m$ is the alkaline-earth-metal atomic mass, $V_L$ is set by the laser intensity, $d$ is its laser wavelength, and $\omega_0$ is the frequency of the harmonic trap.

## II. THEORETICAL BACKGROUND

In the framework of the Green's-function formalism, the spectrum of the excited states is obtained by the poles

 

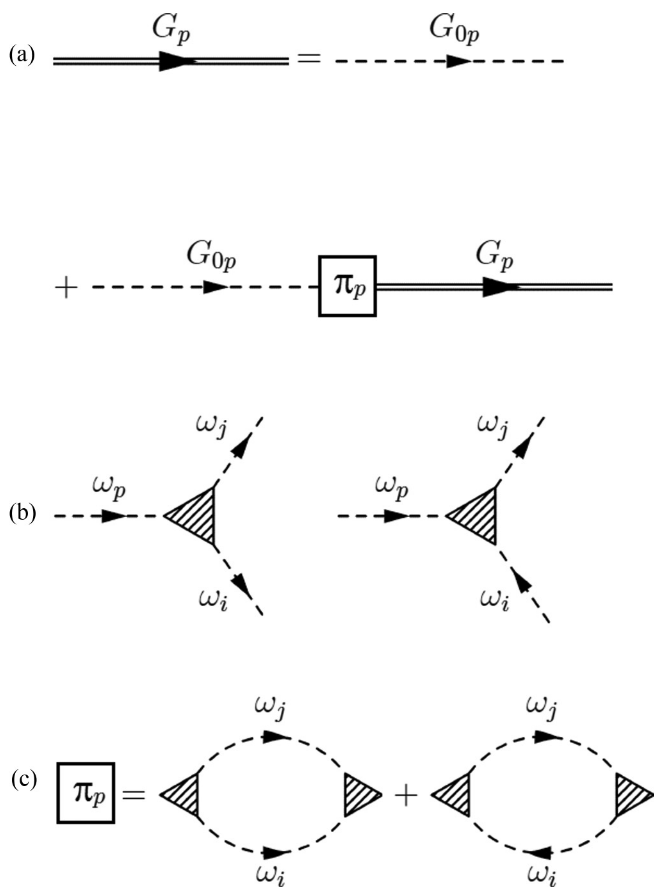


FIG. 1. (a) Diagrammatic equation for the casual Green's function $G_p$ of phonon modes in the condensate. (b) Vertices involved in the collisional phonon-phonon cubic interaction. (c) Feynman diagrams contributing to the phonon self-energy operator $\pi_p$ for Beliaev and Landau processes.

of the dressed Green's function $G_p$. The solution of the Dyson equation, shown diagrammatically in Fig. 1(a), is the renormalized Green's function $G_p$ given by

$$G_p^{-1} = G_{0p}^{-1} - \pi_p. \tag{2}$$

In the absence of the interaction $G_p \to G_{0p} = [\omega - \omega_p + i\varepsilon]^{-1}$, $\varepsilon > 0$, $\omega_p$ is the eigenfrequency of the excited state and $\pi_p$ is the self-energy contribution. The solution of Eq. (2) leads to the complex frequency $\omega = \omega_p + \pi_p$. Here $\mathrm{Re}(\pi_p)$ represents the renormalized contribution to the eigenfrequency $\omega_p$, while $\mathrm{Im}(-\pi_p)$ corresponds to the damping rate.

We assume that the decay processes are associated with the collision between confined phonon states. To first order, the collision term is described by the interaction between three interacting phonon modes, giving rise to the vertices in the self-energy diagrams shown in Fig. 1(b). It becomes clear that the diagram on the right-hand side in Fig. 1(b) does not contribute to the self-energy interaction at $T = 0$ K since a thermal excited mode $\omega_i$ must be present in the system [see Eq. (3) below]. We must recall that the thermal cloud in the present theory is assumed to be in thermal equilibrium.

Figure 1(c) presents the leading diagrams contributing to the self-energy $\pi_p$. Accordingly, the complex frequency correction can be factorized into two main processes as $\pi_p = \Delta\tilde{\omega}_L + \Delta\tilde{\omega}_B$ and in the Hartree-Fock-Bogoliubov approximation we obtain that [23]

$$\pi_p = \frac{2\pi g_1^2}{\hbar^2}\sum_{i,j}\left[\frac{2(f_i - f_j)|A_{ij}|^2}{\omega_p + \omega_i - \omega_j + i\varepsilon} + \frac{(1 + f_i + f_j)|B_{ij}|^2}{\omega_p - \omega_i - \omega_j + i\varepsilon}\right], \tag{3}$$

where $g_1$ is the coupling constant and $A_{ij}$ ($B_{ij}$) represents the matrix elements for the Landau (Beliaev) process $\omega_p + \omega_i \to \omega_j$ ($\omega_p \to \omega_i + \omega_j$). These matrix elements describe the interaction between the collective mode coupled to the thermal cloud of quasiparticles. This interaction is dictated by a three-mode-coupling matrix element and it can be treated as a perturbation of the equation of motion and retaining only three-body collisions [28]. In Eq. (3) the sum $\sum_{i,j}$ takes into account all possible virtual transitions $|\omega_i\rangle$ and $|\omega_j\rangle$ contributing to the decay rate, the term $1 + f_i + f_j$ gives us the Bose-Einstein statistical factor of the phonon $|\omega_p\rangle$ decaying into two confined phonon modes [the diagram on the left-hand side in Fig. 1(c)] $|\omega_i\rangle$ and $|\omega_j\rangle$, and $f_i - f_j$ corresponds to the thermal correction for the annihilation and creation of phonons with frequencies $\omega_i$ and $\omega_j$ [the diagram on the right-hand side in Fig. 1(c)], respectively.

### A. Bogoliubov-type excitations

Bogoliubov-type excitations can be sought by applying a small deviation from the GPE stationary solutions $|\psi_0\rangle \exp(-i\mu t/\hbar)$, i.e.,

$$|\Psi(t)\rangle = \exp(-i\mu t/\hbar)[|\psi_0\rangle + |u\rangle \times \exp(-i\omega t) + |v^*\rangle \exp(i\omega t)], \tag{4}$$

with $\mu$ the chemical potential. By linearizing the time-dependent nonlinear GPE, we obtain the Bogoliubov–de Gennes equations (BdGEs) for the eigenfrequencies $\omega$ and amplitudes $|u\rangle$ and $|v\rangle$. As stated above, if the harmonic trap potential is switched off, we have the homogeneous case, where the phonon wave vector $\mathbf{q} = q_x\mathbf{e}_x$ is a good quantum number and with excited frequency $\omega(q_x)$ depending on the phonon wave vector. Thus, the Bogoliubov low-lying excitation spectrum shows a linear phonon dispersion in $q_x$. On the other hand, when the condensate is loaded into a harmonic potential, $\omega_0 \neq 0$ in Eq. (1), the system becomes inhomogeneous and the wave vector $\mathbf{q}$ is no longer a good quantum number. In such a case the BdGE provides a set of discrete excited-state frequencies $\omega_p$ ($p = 1,2,\ldots$).

By considering the periodic potential in (1) and the nonlinear term $g_1|\phi_0|^2$ in the GPE and BdGE as a perturbation with respect to the harmonic trap potential $\frac{1}{2}m\omega_0^2x^2$, phonon frequencies $\omega_p$ up to the second order in $g_1$ and $V_0$ are given by Ref. [35]

$$\frac{\omega_p}{\omega_0} = p + \frac{\Lambda}{\sqrt{2\pi}}\left[-1 + \frac{2\Gamma(p+1/2)}{\sqrt{\pi}\,p!}\right] - \frac{V_0}{2}\exp(-\alpha^2)[L_p(2\alpha^2) - 1] - \frac{\Lambda V_0}{\sqrt{2\pi}}\exp(-\alpha^2)\left[\mathrm{Ei}\left(\frac{\alpha^2}{2}\right) - \mathcal{C} - \ln\frac{\alpha^2}{2} + \frac{\delta_p(\alpha)}{\sqrt{\pi}}\right]$$
$$+ \frac{V_0^2}{4}\exp(-2\alpha^2)[\mathrm{Chi}(2\alpha^2) - \mathcal{C} - \ln 2\alpha^2 + \rho_p(\alpha)] + \Lambda^2\left[\frac{\gamma_p}{2\pi^2} + 0.033\,106\right], \quad p = 1,2,\ldots, \tag{5}$$

with $\Lambda = g_1 N/l_0\hbar\omega_0$, $N$ the number of atoms, $l_0 = \sqrt{\hbar/m\omega_0}$ defining the characteristic unit length $\alpha = 2\pi l_0/d$, and $V_0 = V_L/\hbar\omega_0$. In addition, $L_p(z)$, $\Gamma(z)$, $\mathrm{Ei}(z)$, $\mathrm{Chi}(z)$, and $\mathcal{C}$ are the Laguerre polynomials, the Gamma function, the exponential integral, the cosine hyperbolic integral, and the Euler constant, respectively. The parameters $\gamma_p$, $\delta_p$, and $\rho_p$ are reported elsewhere [35]. Typically, the coefficient $\alpha \gg 1$, thus the linear term in $V_0$ and the cross term $V_0\Lambda$ in the above equation are negligible.

### B. Symmetry of the excited states

Owing to the inversion symmetry, the space of solutions can be decoupled into two independent subspaces $\mathcal{O}$ and $\mathcal{E}$ for $p = 1,3,\ldots$ and $p = 2,4,\ldots$ modes, respectively. Hence, the components $|u_p\rangle$ and $|v_p\rangle$ are expanded over the complete set 1D oscillator wave functions $\{\phi_{2p+1}\}$ or $\{\phi_{2p}\}$ for the odd $\mathcal{O}$ and even $\mathcal{E}$ Hilbert subspaces. The normalized eigenvectors $|\Phi_p\rangle^\dagger = [|u_p^*\rangle, |v_p\rangle]$, up to first order in $\Lambda$ and $V_0$, can be cast as [35]

$$|\Phi_p\rangle = |\phi_p\rangle + \sum_{m\neq p}\frac{4\Lambda f_{p,m} - V_0 g_{p,m}}{2(p-m)}|\phi_m\rangle - \sum_{m=0}^{\infty}\frac{\Lambda f_{p,m}}{p+m}|\phi_m\rangle, \tag{6}$$

with

$$f_{p,m} = \frac{(-1)^{(p-m)/2}}{\pi\sqrt{2m!p!}}\Gamma\left(\frac{p+m+1}{2}\right), \tag{7}$$

$$g_{p,m} = \frac{(-1)^{(p-m)/2}}{\sqrt{m!p!}}h!(2\alpha^2)^{(p-m)/2}\exp(-\alpha^2)L_h^{|p-m|}(2\alpha^2), \tag{8}$$

where $L_h^t(2\alpha^2)$ are the associated Laguerre polynomials, $h = (p+m-|p-m|)/2$, and $m+p$ is an even number.

The parity of the function $|\Phi_p\rangle$ is linked to the index $p$; if $p$ is even or odd the eigenstate $|\Phi_p\rangle$ is symmetric or antisymmetric, respectively. The decay process of a certain phonon $p$ is restricted by the symmetry property of the matrix elements in Eq. (3). The amplitudes $A_{ij}(p)$ and $B_{ij}(p)$ impose a parity selection rule for the involved states $|\Phi_p\rangle$, $|u_i\rangle$, and $|v_i\rangle$. As shown in Eqs. (9) and (11), for a symmetric (antisymmetric) state $|\Phi_p\rangle$ the amplitudes $|u_i\rangle$ and $|v_i\rangle$ must fulfill the parity condition that $i+j$ is an even (odd) number, therefore limiting the possible number of processes for Beliaev $\omega_p \to \omega_i + \omega_j$ and Landau $\omega_p + \omega_i \to \omega_j$ decay rates. Besides the symmetry of the matrix elements $A_{ij}(p)$ and $B_{ij}(p)$, for a certain eigenmode $|\Phi_p\rangle$ with frequency $\omega_p$, a key role in the damping process is ruled by the label spacing between the Bogoliubov collective oscillations $\Delta_p^{(i,j)} = (\omega_p - \omega_i - \omega_j)/\omega_0$ and $\Delta_j^{(i,p)} = (\omega_j - \omega_i - \omega_p)/\omega_0$. Fixing the frequency $\omega_p$, all allowed combinations of $\omega_i$ and $\omega_j$ cause $\Delta_p^{(i,j)}$ or $\Delta_j^{(i,p)}$ to approach zero, leading to resonant transitions for the Beliaev or Landau damping processes. This effect is shown in Fig. 2, where the influence of the intensity $V_0$ on some level spacings $\Delta_p^{(i,j)}$ [Fig. 2(a)] and $\Delta_j^{(i,p)}$ [Fig. 2(b)] is displayed. For calculations we fixed $d/l_0 = 0.25$ and $\Lambda = 2$. From the figure it can be seen that the laser intensity can be used as a external parameter to tune a particular damping process, i.e., we are able to reach certain critical values $V_0^{(p;i,j)}$ [see Fig. 2(a)], where $\Delta_p^{(i,j)}(V_0) = 0$. This is a direct consequence of the fact that the Bogoliubov-type collective excitation energies, as a function of $\Lambda$ and $V_0$, are not equidistant. Note that, for a certain state $p$, the Landau mechanism allows more combinations fulfilling the condition $\Delta_j^{(i,p)}(V_0) = 0$.

## III. DECAY RATE

In the following we consider that the damping originates by a collision process and in a first-order approximation it is described by the interaction of the three confined phonons, giving rise to a cubic interaction in the bare phonon amplitude [19,21,22,30,36]. This mechanism is represented by the vertex diagrams of the self-energy part shown in Fig. 1(b).

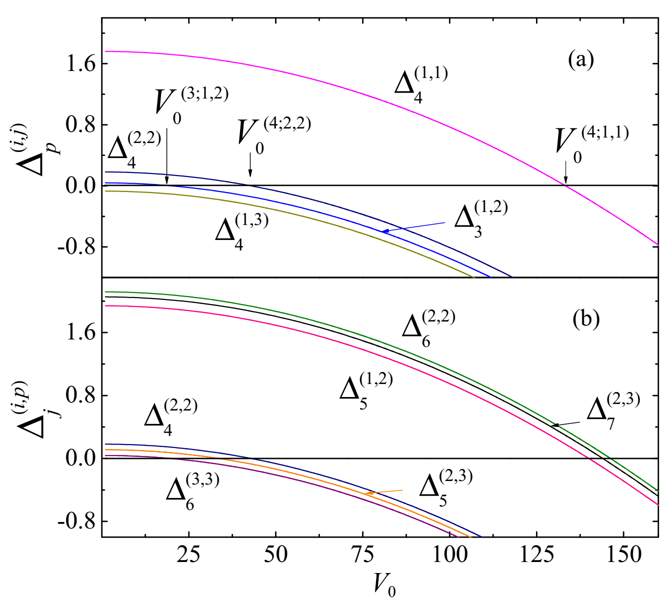


FIG. 2. (Color online) Dependence of the label spacings $\Delta_p^{(i,j)} = (\omega_p - \omega_i - \omega_j)/\omega_0$ and $\Delta_j^{(i,p)} = (\omega_j - \omega_i - \omega_p)/\omega_0$ [see Eq. (3)] on the reduced laser intensity $V_0$ for (a) Beliaev and (b) Landau damping rates, respectively. Critical values $V_0^{(p;i,j)}$, where $\Delta_p^{(i,j)}$ approaches zero, are shown by arrows.

### A. Beliaev damping

In the first-order loop approximation, the Beliaev mechanism arises from the collision of three particles where one phonon with frequency $\omega_p$ is annihilated, decaying into two confined excitations $\omega_i$ and $\omega_j$. Therefore, the allowed processes for the confined modes $\omega_p$ are those with $p = 2,3,\ldots$. Following the Feynman diagrams of Fig. 1(b) and using the eigenfunction amplitudes given in Eq. (6), we find that the decay amplitude $B_{ij}$ can be cast as [23,28]

$$B_{ij} = \int dx\, \psi_0[u_p(u_i^* u_j^* + u_i^* v_j^* + v_i^* u_j^*) + v_p(u_i^* v_j^* + v_i^* v_j^* + v_i^* u_j^*)]. \tag{9}$$

Thus, for the Beliaev damping rate we obtain

$$\gamma_B^{(p)} = \frac{\gamma^{(0)}}{2}|\Lambda|\mathcal{M}_p^{(B)}(\Lambda, V_0), \tag{10}$$

with $\gamma^{(0)} = 4\pi g_1/l_0\hbar$ and $\mathcal{M}_p^{(B)}(\Lambda, V_0)$ defined in Appendix A. Note that the Beliaev mechanism is forbidden if $\epsilon \to 0$ in Eq. (3). The energy conservation limits the real phonon transitions $\omega_p \to \omega_i + \omega_j$.

Figure 3 displays the behavior of the Beliaev damping rate $\gamma_B^{(p)}$ in units of $\gamma^{(0)}$ as a function of the reduced parameter $\Lambda$ for the first seven allowed confined modes $p = 2,\ldots,8$. In this calculation we used $T = 0$ and laser intensity $V_L = 0$. For small values of $\Lambda$ all the normalized damping seen in Fig. 3 presents a linear behavior, while for increasing values of $|\Lambda|$ the function $\gamma_B^{(p)}/\gamma^{(0)}$ behaves nonmonotonically and reaches a maximum. For a given excited state $|\Phi_p\rangle$ the position of the maximum is not symmetric with respect to the type of nonlinear interaction (repulsive $g_1 > 0$ or attractive $g_1 < 0$). In Fig. 4 we show the dependence of $\gamma_B^{(p)}$ on the dimensionless laser intensity $V_0$ and calculated $d/l_0 = 0.25$ and $T = 0$ K for

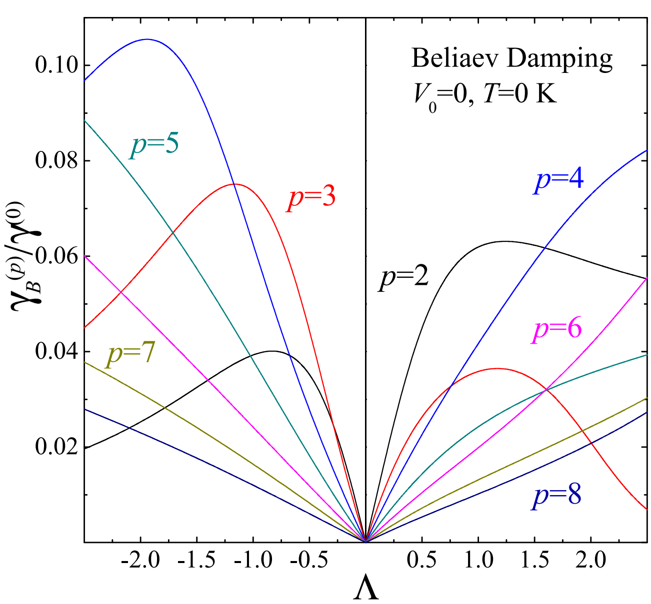


FIG. 3. (Color online) Reduced Beliaev damping $\gamma_B^{(p)}/\gamma^{(0)}$ for the Goldstone modes $p = 2,\ldots,8$ versus the dimensionless self-interaction parameter $\Lambda$ at laser intensity $V_0 = 0$ and $T = 0$ K.

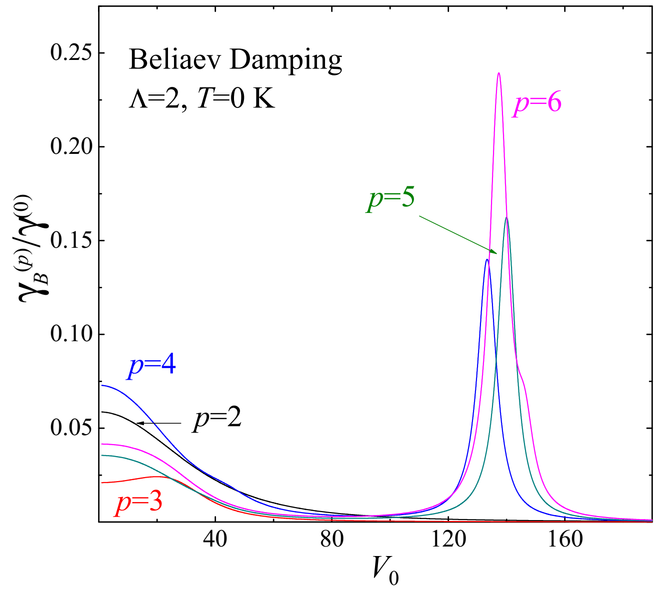


FIG. 4. (Color online) Influence of the laser intensity $V_0$ on the reduced Beliaev damping $\gamma_B^{(p)}/\gamma^{(0)}$. Resonant peaks are related to the zeros of the label spacing $\Delta_p^{(i,j)}(V_0)$ in Eq. (3).

$\Lambda = 2$. It can be observed that the excited states $p = 4$, 5, and 6 show sharp peaks at certain values of $V_0$. These features are linked to the zeros of the frequency label spacing $\Delta_p^{(i,j)}(V_0)$ as represented in Fig. 2, while the number of transitions $\omega_p \to \omega_i + \omega_j$ and the strength of the matrix elements $\overline{B_{ij}}$ dictate the relative intensity of the peaks.

### B. Landau damping

Here, in the damping process a phonon mode with frequency $\omega_p$ and a thermal excitation $\omega_i$ are annihilated and a confined phonon is created. Thus, the Landau mechanism is a thermal process at finite temperature. In the present case the vertices' phonon-phonon interaction [see Fig. 1(b)] leads to the amplitude probability [19,23]

$$A_{ij} = \int dx\, \phi_0[u_p(u_i u_j^* + v_i v_j^* + v_i u_j^*) + v_p(u_i u_j^* + v_i v_j^* + v_i u_j)]. \tag{11}$$

Thus, we have

$$\gamma_L^{(p)} = \gamma^{(0)}|\Lambda|\mathcal{M}_p^{(L)}(\Lambda, V_0), \tag{12}$$

where $\mathcal{M}_p^{(L)}(\Lambda, V_0)$ is defined in Appendix B. Hence, the total damping can be cast as

$$\gamma^{(p)} = \gamma^{(0)}|\Lambda|\left(\mathcal{M}_p^{(L)} + \tfrac{1}{2}\mathcal{M}_p^{(B)}\right). \tag{13}$$

Figure 5 presents the total damping $\gamma^{(p)}$ (solid lines) as a function of $\Lambda$ for the first five excited states. The Landau contribution $\gamma_L^{(p)}$ is represented by dashed lines. As in the case of the Beliaev process, $\gamma^{(p)}/\gamma^{(0)} \sim |\Lambda|$ for small values of the self-interaction atom-atom parameter, while for large values of $|\Lambda|$ the function $\gamma^{(p)}/\gamma^{(0)}$ has a maximum at a certain $\Lambda_p$ value. We note that $\gamma_L^{(p)}$ is smaller than $\gamma_B^{(p)}$ for all excited states $p = 2,\ldots,5$. For the Beliaev damping, the first excited

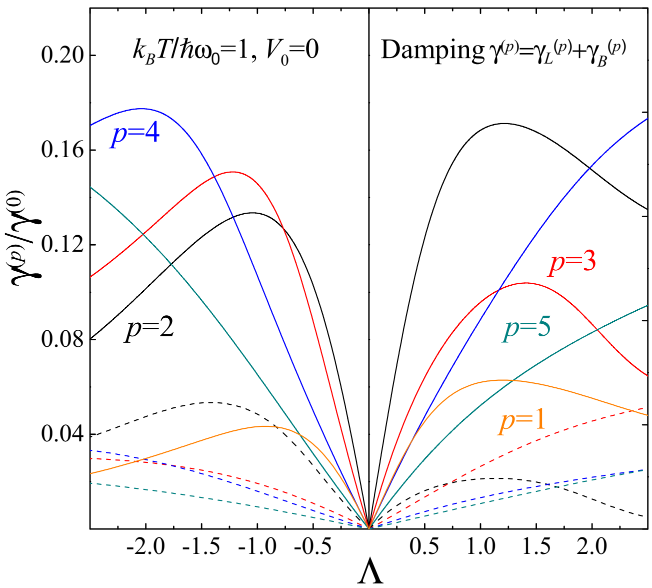


FIG. 5. (Color online) Total damping $\gamma^{(p)} = \gamma_L^{(p)} + \gamma_B^{(p)}$ in units of $\gamma^{(0)}$ for the Goldstone modes $p = 1,2,\ldots,8$ versus the dimensionless self-interaction parameter $\Lambda$. Landau damping for the modes are shown for the same color dashed lines.

state $p = 1$ is forbidden at any temperature, while for $T \neq 0$ K this mode becomes allowed for the Landau process.

The dependence of $\gamma^{(p)}$ (solid lines) for $p = 1,\ldots,5$ on the laser intensity is shown in Fig. 6. For the sake of comparison, the Landau damping contribution is represented by dashed lines. The figure shows that the total damping presents the same behavior as the Beliaev decay (see Fig. 4) and that $\gamma_L^{(p)}$ shows resonant transitions for $V_0 \sim 140$.

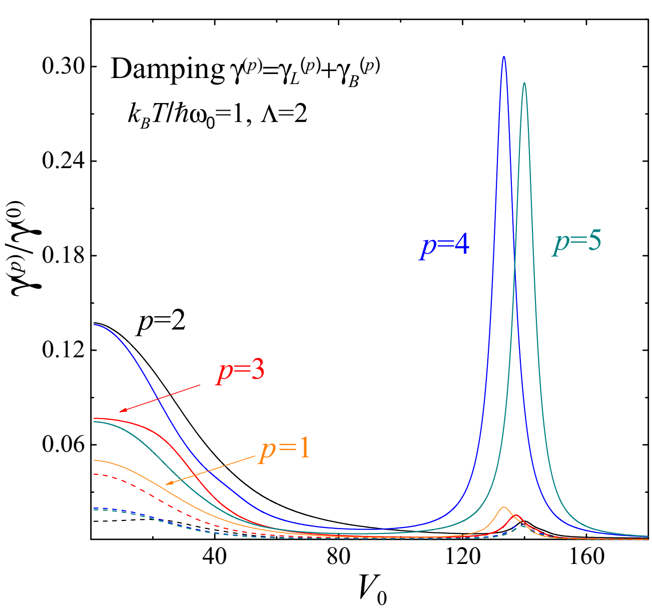


FIG. 6. (Color online) Influence of the laser intensity $V_0$ on the total damping $\gamma^{(p)} = \gamma_L^{(p)} + \gamma_B^{(p)}$ in units of $\gamma^{(0)}$ for the confined modes $p = 1,\ldots,5$. Dashed lines show the Landau damping. In the calculation $d/l_0 = 0.25$.

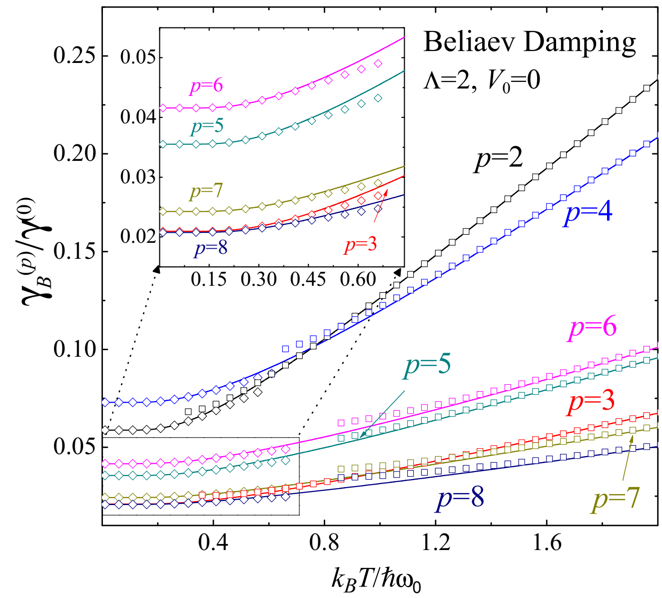


FIG. 7. (Color online) Reduced Beliaev decay rate $\gamma_B^{(p)}/\gamma^{(0)}$ versus $k_BT/\hbar\omega_0$ for the Goldstone modes $p = 2,\ldots,8$. Open squares represent the solution for the thermal regime as derived from Eq. (14). Open diamonds correspond to the low-temperature limit according to Eq. (15).

## IV. DISCUSSION OF THE RESULTS

From the reported calculations, two main results can be easily deduced: (i) the behavior of the damping rates $\gamma_B^{(p)}$, $\gamma_L^{(p)}$, and $\gamma^{(p)}$ with the temperature and (ii) the evaluation of the renormalized confined phonon frequencies. Equations (10) and (12) allow a good approach for all temperature regimes. Figures 7 and 8 depict $\gamma_B^{(p)}$ and $\gamma^{(p)}$ decays for $\Lambda = 2$ and $V_0 = 0$ as a function of the reduced temperature $k_BT/\hbar\omega_0$, respectively. From Fig. 8 we have that, in the range of temperature here considered, $\gamma_B^{(p)} > \gamma^{(p)}$ for all excited states $p > 1$.

In the thermal regime where $k_BT \gg \hbar\omega_0$, i.e., high temperature or weak quantum confinement, we obtain that $\gamma^{(p)}$, given by Eq. (13), is reduced to

$$\gamma^{(p)} = \gamma^{(0)}|\Lambda|\frac{k_BT}{\hbar\omega_0}\Bigg[\mathcal{A}_p^{(1)} + \frac{1}{2}\mathcal{B}_p^{(1)} - \frac{1}{12}\left(\frac{\hbar\omega_0}{k_BT}\right)^2\left(\mathcal{A}_p^{(0)} - \frac{1}{2}\mathcal{B}_p^{(0)}\right)\Bigg], \tag{14}$$

where the Landau $\mathcal{A}_p^{(r)}$ and Beliaev $\mathcal{B}_p^{(r)}$ coefficients ($r = 0,1$) are temperature independent [see Eqs. (A6) and (B3)]. The asymptotic behaviors for $\gamma_B^{(p)}$ and $\gamma^{(p)}$ are displayed by open squares in Figs. 7 and 8, respectively. In the case of the Beliaev damping and by comparison with the exact result in Fig. 7, we can see that the thermal regime supported by Eq. (A5) is a better approach for the lower excited states. For example, for $p = 2$ Eq. (A5) provides a good result if $k_BT > 0.5\hbar\omega_0$, while for $p = 8$ the function (A5) is valid if $k_BT > 1.3\hbar\omega_0$. In contrast, for the total damping, i.e., for higher values of $p$, the limit (14) becomes a better approach. Thus, for $p = 2$ and 4, Eq. (14) is good enough if $k_BT > \hbar\omega_0$ and $k_BT > 0.6\hbar\omega_0$, respectively.

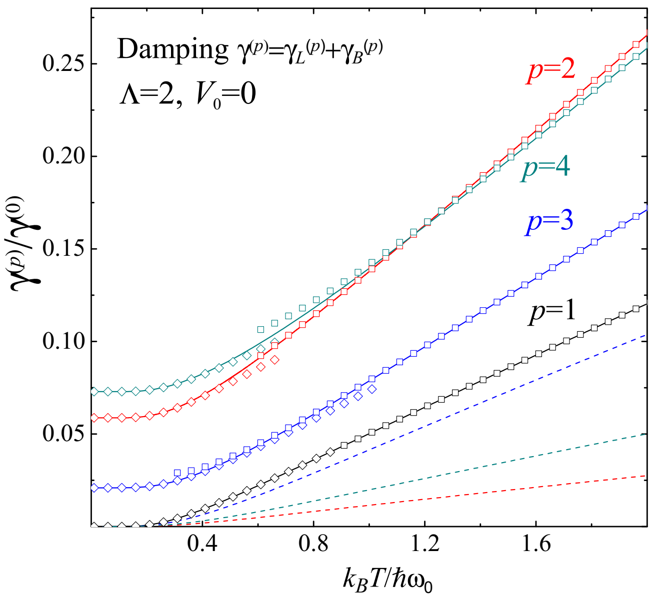


FIG. 8. (Color online) Reduced total damping $\gamma^{(p)}/\gamma^{(0)}$ for the Goldstone modes $p = 1,2,3,4$ as a function of temperature. Dashed lines: Landau damping. Weak $k_BT \gg \hbar\omega_0$ and strong $k_BT \ll \hbar\omega_0$ confinement limits are represented by open squares and open diamonds, respectively (see the text).

Note that for $p$ an odd number, the thermal regime (14) is even better. These facts are explained by the presence of the Landau matrix element $\mathcal{A}_p^{(1)}$ in the total damping calculation for $T \neq 0$ K.

The linear character of the damping rate with $T$ has been tested experimentally in the atomic gas of Na [4]. In the experiment of Ref. [4] the condensate was loaded in a trap where the transversal frequency $\omega_r \gg \omega_0$. Hence, we can argue that this is a quasi-1D condensate. The excitation frequency employed in the experiment was $\omega_{\rm ex} = 1.58\omega_0$ and $\omega_0 = 2\pi \times 19.3$ Hz. Following the Bogoliubov excitation spectrum of Eq. (5), the excitation frequency $\omega_{\rm ex}$ corresponds to the $p = 2$ confined phonon mode with a dimensional nonlinear parameter $\Lambda = 3.42$. Using the asymptotic expression for the high-temperature regime [Eq. (14)], we obtain $\gamma^{(2)} = 4.4$ s$^{-1}$ for $T = 200$ nK and 17.6 s$^{-1}$ for $T = 800$ nK, which agree quite well with the reported experimental values of 4.4 and 18 s$^{-1}$. In the evaluation a condensate of 3500 atoms is assumed and from the value of $\Lambda = 3.42$ we extract an effective 1D coupling constant $g_1 = 3.7 \times 10^{-25}$ eV m.

At very low temperature or in a strongly confined regime, i.e., $k_BT \ll \hbar\omega_0$, from Eqs. (A7) and (B4) it follows that the total damping of the exited mode $p$ can be cast as

$$\gamma^{(p)} = \left\{\mathbb{A}_p^{(1)}(T) + \tfrac{1}{2}\left[\mathbb{B}_p^{(0)} + \mathbb{B}_p^{(1)}(T)\right]\right\}. \tag{15}$$

Here the coefficients $\mathbb{A}_p^{(1)}$ and $\mathbb{B}_p^{(1)}$ decay exponentially with $T$ and $\gamma^{(p)}$ is almost constant independently of the temperature. Comparing the results of Eq. (15) with the theoretical calculations for 3D or 2D homogeneous systems, where the trap potential and confined effect are neglected, we find a different behavior for the Landau damping. Reference [23] reports the law $\gamma_L \sim T^2$, while the limit of $\gamma_L \sim T^4$ is predicted in Refs. [17,19,20]. The quantum limit or very low temperature for Beliaev damping and the total decay as a function of reduced temperature are represented by open diamonds in Figs. 7 and 8. From the figures it can be seen that, for $k_BT \ll 0.6\hbar\omega_0$, the asymptotic approach given by Eq. (15) reproduces quite well the decay processes.

An important result is the knowledge of the excited frequency shift as a function of the condensate parameters and the applied laser intensity. The real part of the self-energy in Eq. (3) leads to an analytical expression for the renormalized excited frequency $\mathrm{Re}\{\pi_p\} = \Delta\widetilde{\omega}_p$ as a function of $\Lambda$, $V_0$, and $T$. Thus,

$$\Delta\widetilde{\omega}_p = \gamma_0|\Lambda| \sum_{i,j}\left[(1 + f_i + f_j)|\overline{B_{ij}}|^2 \frac{(\omega_p - \omega_i - \omega_j)\omega_0}{(\omega_p - \omega_i - \omega_j)^2 + \varepsilon^2} + \frac{1}{2}(f_i - f_j)|\overline{A_{ij}}|^2 \frac{(\omega_j - \omega_i - \omega_p)\omega_0}{(\omega_j - \omega_i - \omega_p)^2 + \varepsilon^2}\right]. \tag{16}$$

Figure 9 shows the dependence of the dimensionless self-interaction parameter $\Lambda$ on the renormalized discrete phonon frequencies $\Delta\widetilde{\omega}_p = \omega - \omega_p$ for the reduced temperature values $k_BT/\hbar\omega_0 = 0$, 1, and 2. We conclude that for the attractive regime ($\Lambda < 0$), the renormalized shift $\Delta\widetilde{\omega}_p > 0$, while the opposite result is obtained for the repulsive interaction, i.e., $\Delta\widetilde{\omega}_p < 0$ if $\Lambda < 0$. This behavior is understood by the dependence of $\Delta\widetilde{\omega}_p$ in Eq. (16) on the label spacing $\Delta_p^{(i,j)} = (\omega_p - \omega_i - \omega_j)/\omega_0$ and $\Delta_j^{(i,p)} = (\omega_j - \omega_i - \omega_p)/\omega_0$ as a function of $\Lambda$. According to the results of Appendixes A and B, we have that in the thermal regime, $\Delta\widetilde{\omega}_p$ is proportional to $k_BT/\hbar\omega_0$. Thus a linear increase or decrease of the excited frequency with the temperature is predicted for an attractive or a repulsive interaction between atoms, respectively.

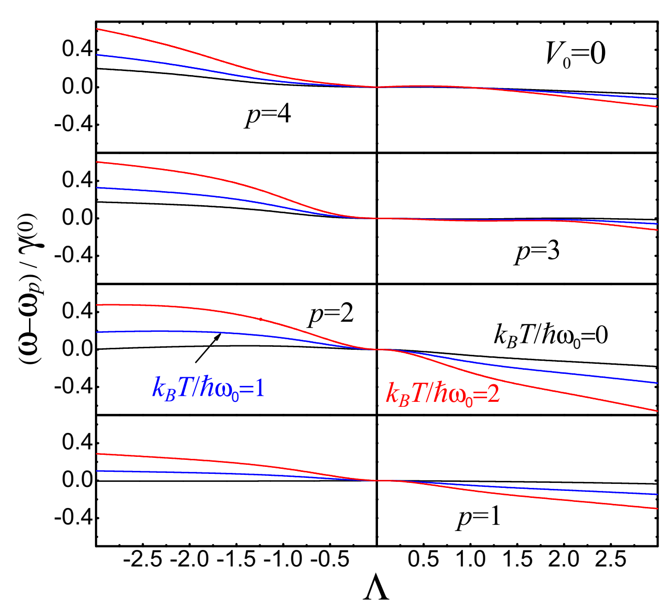


FIG. 9. (Color online) Renormalized phonon frequencies $\{\mathrm{Re}\pi_p\} = \Delta\widetilde{\omega}_p$ in units of $\gamma^{(0)}$ as a function of the reduced self-interaction parameter $\Lambda$ for three values of the temperature $k_BT/\hbar\omega_0 = 0,1,2$ in black, blue, and red lines, respectively, and phonon states $p = 1,2,3,4$.

## V. CONCLUSION

We evaluated the damping rates of confined phonon modes of 1D condensates in a harmonic trap potential loaded in an optical lattice. We remarked on the influence of the spatial confinement potential on the collective oscillations and on the damping rates as a function of the temperature. The presence of an optical lattice as an external field allows us to manipulate the decay rate of the condensate. The damping $\gamma^{(p)}$ can be turned on or off as a function of the laser intensity. Also, for a given excited state $p$ and tuning the laser intensity, it is possible to get a set of transitions $\omega_p \to \omega_j \pm \omega_i$ leading to a resonant effect for the total lifetime $1/\gamma^{(p)}$.

## ACKNOWLEDGMENTS

The authors acknowledge financial support from Brazilian agencies CNPq and FAPESP. Part of the work was supported by Alexander von Humboldt Foundation (Germany). C.T.-G. is grateful to the Max Planck Institute for Complex Systems and the Universidade Federal do Rio de Janeiro for their hospitality. M.-C.C. acknowledges MOST support under the Contract No. 102-2112-M-005-001-MY3.

## APPENDIX A: BELIAEV MATRIX ELEMENT

After substituting the perturbed wave function $|u_p\rangle$ and $|v_p\rangle$ in the matrix element (9) and neglecting terms of the second order or higher in $\Lambda$ and $V_0$, for the function $\mathcal{M}_p^{(B)}$ we get

$$\mathcal{M}_p^{(B)}(\Lambda,V_0) = \sum_{i,j}(1+f_i+f_j)|\overline{B_{ij}}|^2\mathcal{L}_p^{(+)}, \tag{A1}$$

where $\overline{B_{ij}} = T_{0pij} - \Lambda F_{pij} + V_0 H_{pij}$ [$T_{lpij}$ ($l+p+i+j$ is an even number) is reported elsewhere [35]], with

$$F_{pij} = a_{pij} + b^+_{jpi} + b^+_{ijp} + 2(b^-_{pij} + b^-_{ijp} + b^-_{jpi}), \tag{A2}$$

$$H_{pij} = c_{pij} + d_{pij} + d_{ijp} + d_{jpi}, \tag{A3}$$

$$\begin{aligned} a_{pij} &= \sum_{m\neq 0}\frac{T_{2m000}T_{2mpij}}{2m}, \quad b^{\pm}_{pij} = \sum_m{}' \frac{T_{00pm}T_{0mij}}{m\pm p}, \\ c_{pij} &= \sum_{m\neq 0}\frac{g_{0,2m}T_{0mij}}{2m}, \quad d_{pij} = \sum_{m\neq p}\frac{g_{p,m}T_{0mij}}{2(m-p)}, \end{aligned} \tag{A4}$$

where the parity condition $p+i+j$ is an even number, and the Lorenzian function

$$\mathcal{L}_p^{(\pm)} = \frac{1}{\pi}\frac{\omega_0\varepsilon}{(\omega_p \mp \omega_i - \omega_j)^2 + \varepsilon^2}.$$

In the limit of thermal regime, the probability $\mathcal{M}_p^{(B)}$ is reduced to

$$\mathcal{M}_p^{(B)}(T) = \frac{k_BT}{\hbar\omega_0}\left[\mathcal{B}_p^{(1)} + \frac{1}{12}\left(\frac{\hbar\omega_0}{k_BT}\right)^2\mathcal{B}_p^{(0)}\right], \tag{A5}$$

with

$$\mathcal{B}_p^{(r)} = \sum_{i,j}\frac{\omega_i+\omega_j}{\omega_0}\left(\frac{\omega_0^2}{\omega_i\omega_j}\right)^r|\overline{B_{ij}}|^2\mathcal{L}_p^{(+)} \quad (r=0,1). \tag{A6}$$

For low temperature we have

$$\mathcal{M}_p^{(B)}(T) = \mathbb{B}_p^{(0)} + \mathbb{B}_p^{(1)}(T), \tag{A7}$$

where

$$\begin{aligned} \mathbb{B}_p^{(r)} = &\sum_{i,j}[\exp(-\hbar\omega_i/k_BT) + \exp(-\hbar\omega_j/k_BT)]^r \\ &\times|\overline{B_{ij}}|^2\mathcal{L}_p^{(+)} \quad (r=0,1). \end{aligned} \tag{A8}$$

## APPENDIX B: LANDAU MATRIX ELEMENT

Using the wave functions $|u_p\rangle$ and $|v_p\rangle$ and Eq. (11) and neglecting terms higher than $\Lambda$ and $V_0$, we have for $\mathcal{M}_p^{(B)}$

$$\mathcal{M}_p^{(L)} = \sum_{i,j}(f_i - f_j)|\overline{A_{ij}}|^2\mathcal{L}_p^{(-)}, \tag{B1}$$

where $\overline{A_{ij}} = T_{0pij} - \Lambda D_{pij} + V_0 G_{pij}$, with

$$\begin{aligned} D_{pij} &= a_{pij} + b^+_{pij} + b^+_{ijp} + 2(b^-_{pij} + b^-_{ijp} + b^-_{jpi}), \\ G_{pij} &= c_{pij} + d_{pij} + d_{ijp} + d_{jpi}. \end{aligned}$$

If $k_BT \ll \hbar\omega_0$, the Landau probability process can approach

$$\mathcal{M}_p^{(L)}(T) = \frac{k_BT}{\hbar\omega_0}\left[\mathcal{A}_p^{(1)} - \frac{1}{12}\left(\frac{\hbar\omega_0}{k_BT}\right)^2\mathcal{A}_p^{(0)}\right] \tag{B2}$$

with

$$\mathcal{A}_p^{(r)} = \sum_{i,j}\frac{\omega_j-\omega_i}{\omega_0}\left(\frac{\omega_0^2}{\omega_i\omega_j}\right)^r|\overline{A_{ij}}|^2\mathcal{L}_p^{(-)} \quad (i=0,1), \tag{B3}$$

while for the weak confinement $k_BT \gg \hbar\omega_0$ we obtain

$$\begin{aligned} \mathcal{M}_p^{(L)}(T) &= \mathbb{A}_p^{(1)} \\ &= \sum_{i,j}[\exp(-\hbar\omega_i/k_BT) \\ &\quad - \exp(-\hbar\omega_j/k_BT)]|\overline{A_{ij}}|^2\mathcal{L}_p^{(-)}. \end{aligned} \tag{B4}$$

[1] S. T. Beliaev, Sov. Phys. JETP **34**, 299 (1958).
[2] D. S. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 420 (1996).
[3] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[4] M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 988 (1996).
[5] R. Onofrio, D. S. Durfee, C. Raman, M. Köhl, C. E. Kuklewicz, and W. Ketterle, Phys. Rev. Lett. **84**, 810 (2000).
[6] S. Burger, F. S. Cataliotti, C. Fort, F. Minardi, M. Inguscio, M. L. Chiofalo, and M. P. Tosi, Phys. Rev. Lett. **86**, 4447 (2001).
[7] K. Iigaya, S. Konabe, I. Danshita, and T. Nikuni, Phys. Rev. A **74**, 053611 (2006).

[8] F. S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trombettoni, A. Smerzi, and M. Inguscio, Science **293**, 843 (2001); **293**, 5531 (2001).
[9] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. **95**, 010402 (2005).
[10] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
[11] D. L. Feder and C. W. Clark, Phys. Rev. Lett. **87**, 190401 (2001).
[12] E. Hodby, G. Hechenblaikner, S. A. Hopkins, O. M. Maragò, and C. J. Foot, Phys. Rev. Lett. **88**, 010405 (2001).
[13] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[14] A. L. Gaunt, R. J. Fletcher, R. P. Smith, and Z. Hadzibabic, Nat. Phys. **9**, 271 (2013).
[15] D. S. Jin, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **78**, 764 (1997).
[16] D. M. Stamper-Kurn, H. J. Miesner, S. Inouye, M. R. Andrews, and W. Ketterle, Phys. Rev. Lett. **81**, 500 (1998).
[17] P. C. Hohenberg and P. C. Martin, Ann. Phys. (NY) **34**, 291 (1965).
[18] P. Szépfalusy and I. Kondor, Ann. Phys. (NY) **82**, 1 (1974).
[19] L. P. Pitaevskii and S. Stringari, Phys. Lett. A **235**, 398 (1997).
[20] W. V. Liu, Phys. Rev. Lett. **79**, 4056 (1997).
[21] P. O. Fedichev, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **80**, 2269 (1998).
[22] S. Tsuchiya and A. Griffin, Phys. Rev. A **72**, 053621 (2005).
[23] M.-C. Chung and A. B. Bhattacherjee, New J. Phys. **11**, 123012 (2009).
[24] S. S. Natu and R. M. Wilson, Phys. Rev. A **88**, 063638 (2013).
[25] F. Ferlaino, P. Maddaloni, S. Burger, F. S. Cataliotti, C. Fort, M. Modugno, and M. Inguscio, Phys. Rev. A **66**, 011604(R) (2002).
[26] S. Tsuchiya and A. Griffin, Phys. Rev. A **70**, 023611 (2004).
[27] E. Arahata and T. Nikuni, Phys. Rev. A **77**, 033610 (2008).
[28] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon, Oxford, 2003).
[29] Strictly, to fulfill the requirement of a quasi-one-dimensional system, in a certain period of time, it is necessary that the radial motion of the condensate be localized at the harmonic radial ground state, i.e., the thermal energy $k_B T \ll \hbar\omega_r$ with $\hbar\omega_r$ the transverse harmonic trap frequency. Also, the self-interaction energy between atoms has to be smaller than $\hbar\omega_r$, that is, $g_1 N/l_0 < \hbar\omega_r$, where $l_0$ is the trap length unit [30]. Note that we keep the system geometrically one dimensional, i.e., the harmonic trap in the $x$ direction $\omega_0 \gg \omega_r$, however, the interaction remains three dimensional by choosing the condition $\omega_r \gg 2\hbar/ma_3^2$ with three-dimensional scattering length $a_3$ [31].
[30] I. E. Mazets, Phys. Rev. A **83**, 043625 (2011).
[31] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
[32] C. Trallero-Giner, V. López-Richard, M.-C. Chung, and A. Buchleitner, Phys. Rev. A **79**, 063621 (2009).
[33] The 1D GPE is linked to the factorization of the 3D GPE into transverse and longitudinal motions. Due to this approach, it is possible to show that the 1D nonlinear Schrödinger equation arises with a cubic-quintic term $g_1|\psi_0|^3 - g_2|\psi_0|^5$ with $g_2 = 6\ln(4/3)|g_1|^2/\omega_r$. The term up to $g_1|\psi_0|^3$ becomes a very good approach if $g_1 > 0$. On the other hand, a non-negligible contribution is reached for the chemical potential in the case of an attractive interaction $g_1 < 0$, where the quintic nonlinear interaction plays a significant role as $g_2$ increases (for a discussion on this issue see Ref. [34]).
[34] C. Trallero-Giner, R. Cipolatti, and T. C. H. Liew, Eur. Phys. J. D **67**, 143 (2013).
[35] C. Trallero-Giner, V. López-Richard, Y. Nuñez-Fernandez, M. Oliva, G. E. Marques, and M. C. Chung, Eur. Phys. J. D **66**, 177 (2012).
[36] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, New York, 2002).