

Quantum speed limit for a relativistic electron in a uniform magnetic field

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We analyze the influence of relativistic effects on the minimum evolution time between two orthogonal states of a quantum system. Defining the initial state as a homogeneous superposition between two Hamiltonian eigenstates of an electron in a uniform magnetic field, we obtain a relation between the minimum evolution time and the displacement of the mean radial position of the electron wave packet. The quantum speed limit time is calculated for an electron dynamics described by Dirac and Schrödinger-Pauli equations considering different parameters, such as the strength of magnetic field and the linear momentum of the electron in the axial direction. We highlight that when the electron undergoes a region with extremely strong magnetic field the relativistic and nonrelativistic dynamics differ substantially, so that the description given by the Schrödinger-Pauli equation enables the electron to travel faster than c , which is prohibited by Einstein's theory of relativity. This approach allows a connection between the abstract Hilbert space and the space-time coordinates, besides the identification of the most appropriate quantum dynamics used to describe the electron motion.

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I. INTRODUCTION

The question of how fast quantum information can be processed was tackled in 1945 by Mandelstam and Tamm (MT) [1]. They developed a criterion to find the minimum time for a closed quantum system with limited energy uncertainty ΔH to change the expectation value of a given operator by the standard of this operator. Such result was supported later by Refs. [2–4]. On the other hand, Margolus and Levitin (ML) [5] attributed the speed of a quantum evolution between two orthogonal states to the mean energy of the system $\langle \hat{H} \rangle$. In Ref. [6] it is assumed that the minimum evolution time has the expression $T_{\min} = \max\{\pi\hbar/2\Delta H, \pi\hbar/2(\langle \hat{H} \rangle - E_0)\}$, where E_0 is the lowest energy of one of the states of the superposition. A unified version of the MT and ML bounds was presented in Ref. [7]. Recent developments on this subject extended these ideas to include initial mixed states and open quantum system dynamics, obtaining realistic bounds for the speed of quantum processes [8–20]. Very recently, the non-Markovian effects on the quantum speed limit have been tested experimentally in the cavity quantum electrodynamics scenario [21].

The answer to the former question is very important for many areas of quantum physics, including quantum information and computation [22,23], quantum metrology [24], optimal control theory [25–27], and quantum thermodynamics [28,29].

Although the achievement of an exact expression for the quantum speed limit is of fundamental importance to attain precisely the minimum time of a quantum process, the correct description of the dynamics of the system of interest is as important as the former. Regarding this point we observe that for an accurate description of a system dynamics it is necessary to take into account relativistic effects. In the case of spin-1/2 particles such as electrons, the Dirac equation is able to accommodate quantum mechanics and special relativity very

well. It reproduces accurately the spectrum of the hydrogen atom, provides a natural description of the electron spin, and indicates the existence of antimatter [30]. The correction to the energy of atomic levels due to fine structure is a beautiful example of relativistic effects in low-energy quantum systems. Such correction is very small, about five orders of magnitude smaller than the energy values predicted by the nonrelativistic Schrödinger equation, but nevertheless is experimentally observable [31].

The target of this work is to encompass relativistic effects on the quantum speed limit. For this purpose we analyze the transition between two orthogonal states of an electron in a uniform magnetic field according to the Dirac equation [30] and compare it to the nonrelativistic description given by the Schrödinger-Pauli equation [32,33]. Defining the electron initial state as an homogeneous superposition of two eigenstates of the Hamiltonian, the Madelstam-Tamm and Margolus-Levitin bounds become equivalent [34]. Therefore, in some sense, our results are independent on the expression used to calculate the minimum transition time. For some states the electron mean radial position is initially different from its final one. The ratio between such average radial displacement and the minimum evolution time furnishes the average speed in which the electron travels in spacetime in the radial direction. Such speed is important for two reasons: (i) it enables us to find what kind of initial superposition state provides the greatest spatial displacement in the shortest time and (ii) for speeds higher than the speed of light in vacuum c it works as a criterion to invalidate the equation used to describe the electron dynamics. As expected, the Schrödinger-Pauli equation is the only one to violate this criterion.

This paper is organized as follows: in Sec. II we briefly describe the nonrelativistic and relativistic dynamics of an electron in a uniform magnetic field by the Schrödinger-Pauli and Dirac equations, respectively. In Sec. III we show an analysis of a particular case of an initial superposition state which gives us enough information about both quantum mechanical descriptions and used it to realize in Sec. IV a numerical calculation for looking for fastest superposition states. In Sec. V is our conclusion.

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II. MODEL AND FRAMEWORK

For didactic reasons we briefly review the nonrelativistic and relativistic dynamics of an electron in a uniform magnetic field, respectively. The Pauli Hamiltonian is

$$H = \frac{1}{2m_0}(\vec{p} + e\vec{A})^2 + \frac{e}{m_0}\vec{B} \cdot \vec{S}, \quad (1)$$

with \vec{p} being the linear mechanical momentum, e being the absolute value of the electron charge, and m_0 being the electron rest mass. The magnetic vector potential \vec{A} is expressed by the symmetric Landau gauge $\vec{A} = (\vec{B} \times \vec{r})/2$, where $\vec{B} = B\hat{z}$ is the magnetic field oriented in the z direction and \vec{r} is the vector position of the electron. The eigenstates of the Hamiltonian (1) are [32,33]

$$\psi_{n,m_l,m_s,p}(\varrho,\varphi,z) = F_{n,m_l}(\varrho,\varphi)e^{ipz/\hbar}\Gamma_{m_s}, \quad (2)$$

where the radial wave function is

$$F_{n,m_l}(\varrho,\varphi) = \frac{(-1)^{\binom{n-|m_l|}{2}} \binom{n-|m_l|}{2}!}{\sqrt{\pi \binom{n+|m_l|}{2}! \binom{n-|m_l|}{2}!}} \times \beta(\beta\varrho)^{|m_l|} L_{\binom{n-|m_l|}{2}}^{(|m_l|)}(\beta^2\varrho^2)e^{-\beta^2\varrho^2/2}e^{im_l\varphi}, \quad (3)$$

with $L_{\binom{|m_l|}{2}}^{(|m_l|)}(\beta^2\varrho^2)$ being the generalized Laguerre polynomials,

$$L_{\binom{|m_l|}{2}}^{(|m_l|)} = \sum_{j=0}^{\binom{n-|m_l|}{2}} (-1)^j \binom{\binom{n-|m_l|}{2} + |m_l|}{j} \frac{(\beta\varrho)^{2j}}{j!}. \quad (4)$$

Here $\beta \equiv \sqrt{\frac{eB}{2\hbar}}$ is the inverse of the *characteristic length* of the harmonic oscillator, the indexes $n = 0, 1, 2, \dots$ and $m_l = -n, -n+2, \dots, n-2, n$ refer to the eigenstates $F_{n,m_l}(\varrho,\varphi)$ of the 2-dimensional harmonic oscillator in the plane perpendicular to the orientation of the magnetic field and also to the coupling between the magnetic field and the orbital angular momentum. Γ_{m_s} represents the eigenstates of the spin operator S_z with eigenvalues $\hbar m_s$, so that the index $m_s = \{-1/2, +1/2\}$. p is the projection of the linear momentum in z direction. The corresponding eigenvalues of Hamiltonian (1) are

$$E_{n,m_l,m_s,p} = \frac{p^2}{2m_0} + \hbar\omega(n + m_l + 2m_s + 1), \quad \omega \equiv \frac{eB}{2m_0}. \quad (5)$$

By its turn, the relativistic dynamics of the electron is given by Dirac equation, which is expressed as

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = (c\vec{\alpha} \cdot \vec{\Pi} + \beta m_0 c^2)\psi(\vec{r},t), \quad (6)$$

where $\vec{\Pi} = \vec{p} + e\vec{A}$ is the linear canonical momentum. We are using the Bjorken-Drell convention to represent the γ matrices, here denoted by $\vec{\alpha}$ and β . The Dirac Hamiltonian eigenstates are spinors with four components, in which the two upper components have positive energy and are described by Eq. (2), while the two lower components with negative energy are

given by

$$\frac{c\vec{\sigma} \cdot \vec{\Pi}}{E + m_0 c^2} \psi_{n,m_l,m_s,p}(\varrho,\varphi,z). \quad (7)$$

The quantity E represents the eigenenergies

$$E_{n,m_l,m_s,j,p} = j\sqrt{m_0^2 c^4 + p^2 c^2 + eB\hbar c^2(n + m_l + 2m_s + 1)}, \quad (8)$$

with $j = \{+, -\}$ indicating the sign of the energy. For more details about this solution see Ref. [30].

The electron initial state is assumed to have $+1/2$ spin projection along the z direction and a Gaussian wave packet in the same spatial direction with standard deviation d and expectation value p_0 for the linear momentum operator \hat{p}_z ,

$$\psi_z(z) = \frac{1}{(\pi d^2)^{1/4}} e^{-z^2/4d^2} e^{ip_0 z/\hbar}. \quad (9)$$

Our idea is to establish a connection between the quantum speed limit and the speed in which the electron wave packet moves through the spacetime. For this purpose we consider the initial state of the system in an x - y plane as a homogeneous superposition of two radial eigenstates $F_{n,m_l}(\varrho,\varphi)$ in different Landau energy levels. After the time of evolution T_{\min} the state of the system is orthogonal to the initial one, so that the mean radial position of the electron wave packet experiences a displacement. In the next sections we analyze the relativistic effects on T_{\min} , besides the dependence of the electron's displacement on the initial superposition state and on the relativistic and nonrelativistic descriptions of quantum mechanics.

III. QUANTUM SPEED LIMIT FOR AN ELECTRON UNDER RELATIVISTIC AND NONRELATIVISTIC QUANTUM DYNAMICS

We start analyzing the nonrelativistic case of superposition between the radial eigenstates $F_{0,0}(\varrho,\varphi)$ and $F_{2,0}(\varrho,\varphi)$,

$$\psi(\vec{r},0) = \frac{1}{\sqrt{2}}[F_{0,0}(\varrho,\varphi) + F_{2,0}(\varrho,\varphi)]\psi_z(z)\Gamma_{+1/2}. \quad (10)$$

After the evolution from $\psi(\vec{r},0)$ to $\psi(\vec{r},T_{\min})$, we obtain the quantities required to evaluate the quantum speed limit criteria,

$$\Delta H = \langle H \rangle - E_0 = \frac{eB\hbar}{2m_0}. \quad (11)$$

Thus the minimum evolution time is

$$T_{\min} = \frac{\pi m_0}{eB}. \quad (12)$$

As we are considering basically the change in the radial part of the system state, we analyze the radial displacement of the electron, which enable us to set the expectation value of the linear momentum in the axial direction as $p_0 = 0$. Thus, the mean radial position at any time is given by the expression,

$$\langle \varrho \rangle_t = \frac{1}{2}[\langle 0,0|\varrho|0,0 \rangle + \langle 2,0|\varrho|2,0 \rangle + 2\langle 0,0|\varrho|2,0 \rangle \cos(\mathcal{E}t)]. \quad (13)$$

By using the Dirac notation, we nominated each eigenstate by its quantum numbers n and m_l , and $\mathcal{E} = eB/m_0$ is a constant

with dimension of frequency. Therefore, the maximum radial displacement of the electron's mean position is

$$\begin{aligned} |\langle \varrho \rangle_{T_{\min}} - \langle \varrho \rangle_0| &= |\langle 0,0|\varrho|2,0\rangle[\cos(\mathcal{E}T_{\min}) - 1]| \\ &= \sqrt{\frac{\pi\hbar}{2eB}}. \end{aligned} \quad (14)$$

The expression above shows us the relevancy of the crossed term $D_S(\varrho) = \langle 0,0|\varrho|2,0\rangle = 2\pi \int_0^\infty F_{0,0}^\dagger(\varrho,\varphi)\varrho F_{2,0}(\varrho,\varphi)d\varrho$ for the electron's displacement. From Eqs. (2), (3), and (14) we observe that for a non-null radial displacement of the electron, the initial superposition state must be built by eigenstates with the same quantum numbers of spin m_s and orbital angular momentum m_l , besides $\mathcal{E}T_{\min} \neq s\pi$, with s even. Then the average speed of the mean radial position of the electron from its initial state to the orthogonal one is

$$\bar{v} = \frac{1}{m_0} \sqrt{\frac{eB\hbar}{2\pi}}. \quad (15)$$

On the other hand, in the relativistic description with $p_0 = 0$, the minimum evolution time is

$$T_{\min} = \frac{\pi\hbar}{\sqrt{m_0^2c^4 + 4eB\hbar c^2} - \sqrt{m_0^2c^4 + 2eB\hbar c^2}}. \quad (16)$$

In what follows we write the two spinors that compose the evolved state of the system

$$U_{0,0} = N_{0,0} \begin{pmatrix} F_{0,0}(\varrho,\varphi) \\ 0 \\ \frac{cpF_{0,0}(\varrho,\varphi)}{(E_{0,0}+m_0c^2)} \\ \frac{2i\hbar c\beta F_{1,1}(\varrho,\varphi)}{(E_{0,0}+m_0c^2)} \end{pmatrix} e^{ipz/\hbar} \quad (17)$$

and

$$U_{2,0} = N_{2,0} \begin{pmatrix} F_{2,0}(\varrho,\varphi) \\ 0 \\ \frac{cpF_{2,0}(\varrho,\varphi)}{(E_{2,0}+m_0c^2)} \\ \frac{2i\hbar c\beta\sqrt{2}F_{3,1}(\varrho,\varphi)}{(E_{2,0}+m_0c^2)} \end{pmatrix} e^{ipz/\hbar}, \quad (18)$$

where $N_{0,0}$ and $N_{2,0}$ are normalization constants and $E_{0,0}$ and $E_{2,0}$ are positive eigenvalues given by Eq. (8). Therefore, the superposition state evolves in time as

$$\psi(\vec{r},t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \alpha(p)[U_{0,0}e^{-iE_{0,0}t/\hbar} + U_{2,0}e^{-iE_{2,0}t/\hbar}]dp, \quad (19)$$

with $\alpha(p)$ being the coefficient of expansion of the Gaussian wave packet defined in Eq. (9). Now we are able to calculate the radial displacement of the electron's mean position in the relativistic case, which one is made numerically [35]. In Fig. 1 we plot the average speed of the electron's wave packet when moving from the initial to final state during the time interval T_{\min} under both relativistic and nonrelativistic quantum dynamics. We noticed in the nonrelativistic case that there is a magnetic field strong enough to yield $\bar{v} \geq c$ given by $B \geq 2.77 \times 10^{10}T$, which contradicts the Einstein's theory of relativity.

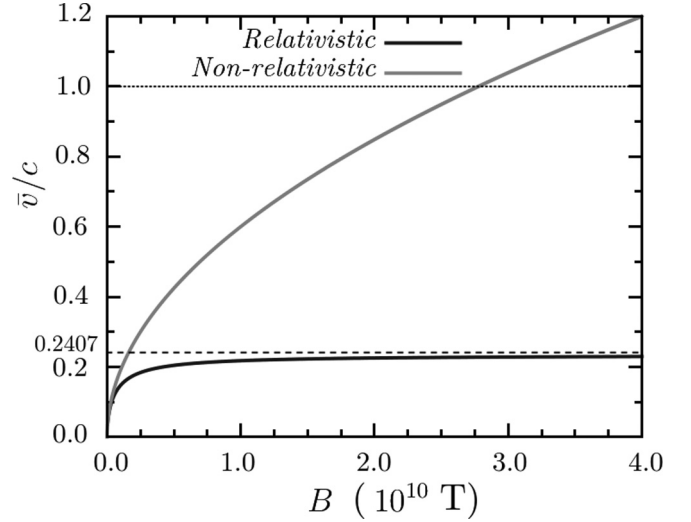


FIG. 1. Average radial speed as function of the external magnetic field according to relativistic and nonrelativistic quantum dynamics.

Naturally, it is impossible to achieve this intensity of magnetic field in a laboratory on the earth, but not in special neutron stars, called magnetars [36]. Conversely, Dirac's theory for the electron predicts the asymptotic value of $\bar{v} \simeq 0.2407c$. To attain this value we first need to evaluate the radial displacement of the electron mean position, which one depends on the crossed term $D_S(\varrho) = 2\pi U_{0,0}^\dagger \varrho U_{2,0}$ and on the minimum evolution time T_{\min} as in Eq. (16). In the limit case $B \rightarrow \infty$ the expressions for the eigenenergies can be approximated by

$$E_{0,0} \approx 2c\hbar\beta, \quad E_{2,0} \approx 2\sqrt{2}c\hbar\beta, \quad (20)$$

which renders

$$T_{\min} \approx \frac{\pi}{2c\beta(\sqrt{2}-1)}. \quad (21)$$

Inside this approximation, the spinor normalization constants become $N_{0,0} = N_{2,0} = 1/\sqrt{2}$ and the radial displacement of the electron's mean position becomes

$$\begin{aligned} |\langle \varrho \rangle_{T_{\min}} - \langle \varrho \rangle_0| &= 2 \left| \int_0^\infty \varrho D_S(\varrho) d\varrho \right| \\ &= \frac{\sqrt{\pi}}{4\beta} \left(1 + \frac{3}{2\sqrt{2}} \right). \end{aligned} \quad (22)$$

Unlike the nonrelativistic case, now the displacement in time and space have the same dependence on the magnetic field, as shown in Eqs. (21) and (22), respectively. Consequently, the maximum value of the average speed of the electron in the radial direction is

$$\bar{v} = \frac{c}{4\sqrt{2}\pi} (1 + \sqrt{2}) \approx 0.2407c. \quad (23)$$

Hence, as expected, the relativistic quantum dynamics is the most appropriate to describe the electron dynamics in the presence of high-intensity magnetic fields. Throughout the present development we observed that the relativistic theory of quantum mechanics funded by Dirac does not restrict the time interval of a quantum process of being arbitrarily small, as

shown in Eq. (16). Making a comparison between Eqs. (12) and (16) we verify that there is a quadratic dependence on the magnetic field in the nonrelativistic case in relation to the relativistic one. Such difference can turn out to be important for $B \sim m_0^2 c^2 / e\hbar \sim 5$ GT. However, our relativistic description of the electron dynamics, and consequently the quantum speed limit, applies for low-intensity magnetic fields in graphene, where the charge carriers can effectively be described by relativistic particles with zero rest mass [37,38].

IV. THE FASTEST SUPERPOSITIONS

In the preceding section we verified that the dynamics of an electron described by Schrödinger-Pauli equation violates a basic principle of Einstein's theory of relativity, which states that any object with non-null rest mass cannot travel faster than c . For that reason, we study the dependence of the quantum speed limit for an electron evolving according to the Dirac theory as function of the initial superposition state. Our main purpose here is looking for the maximum radial displacement in the shortest time interval. Since the electron's radial displacement depends strongly on the crossed term $D_S(\varrho)$, its maximum absolute value is attained when the initial and final states have the same spin orientation ($m_s = 1/2$), zero angular momentum projection $m_l = 0$, and the initial superposition state is made of two nearest neighbors eigenstates, i.e., with quantum numbers n and $n + 2$. In Fig. 1 we observe that \bar{v} increases as the intensity of the magnetic field is strengthened. Therefore, in the regime $\beta \rightarrow \infty$ the minimum evolution time between two orthogonal states, where the initial superposition state is composed by two eigenstates with positive energy (called *particle-particle* states), is

$$T_{\min} \approx \frac{\pi}{[\sqrt{n+4} - \sqrt{n+2}]\sqrt{2}c\beta} \quad (24)$$

and the crossed term turns out to be

$$D_S(\varrho) \approx \pi\varrho[F_{n,0}^\dagger F_{n+2,0} + F_{n+1,1}^\dagger F_{n+3,1}]. \quad (25)$$

After some steps we get an analytic expression for the maximum radial displacement as function of n ,

$$\begin{aligned} & |\langle \varrho \rangle_{T_{\min}} - \langle \varrho \rangle_0| \\ &= \frac{1}{\beta} \sum_{i=0}^{\left(\frac{n}{2}\right)} \sum_{j=0}^{\left(\frac{n+2}{2}\right)} \frac{(-1)^{i+j}}{i! j!} \binom{\frac{n}{2}}{\frac{n}{2} - i} \binom{\frac{n+2}{2}}{\frac{n+2}{2} - j} \\ & \quad \times \Gamma\left(i + j + 1 + \frac{1}{2}\right) \\ & \quad \times \left[1 + \frac{\sqrt{\left(\frac{n}{2} + 2\right)\left(\frac{n}{2} + 1\right)}}{(i+1)(j+1)} \left(i + j + 1 + \frac{1}{2}\right) \right]. \quad (26) \end{aligned}$$

In Fig. 2 we plot the average radial speed of the electron for different even values of n ranging in the interval $[0, 132]$. The evaluation of \bar{v} for higher values of n is very hard, provided that Eq. (26) has many factorials. The inset of such figure shows the convergence of \bar{v}/c to the asymptotic value 0.269814 found numerically. We notice that in the interval $80 \leq n \leq 132$ the value of \bar{v} changes in the fourth decimal place only, which

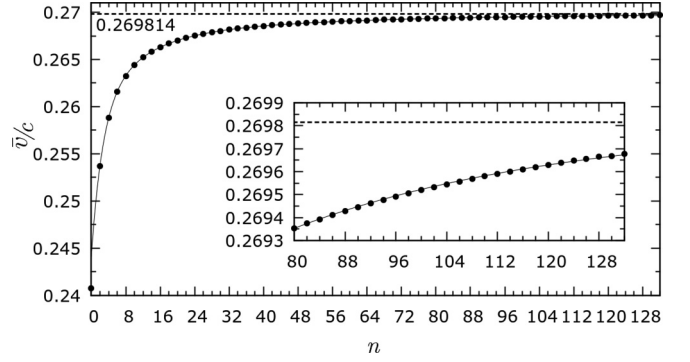


FIG. 2. Average radial speed of an electron for different initial superpositions of two positive-energy eigenstates $(U_{n,0} + U_{n+2,0})/\sqrt{2}$.

shows that the average radial speed is reaching a constant value less than c for $n \rightarrow \infty$.

Instead of considering only initial particle-particle states, we will take into account superpositions of eigenstates with negative and positive energies (called *antiparticle-particle* states). The reason we are tackling this subject only now is that it is not clear if it is fair to compare the nonrelativistic dynamics, which describes only particle states, with the relativistic antiparticle dynamics. Despite that, antiparticle-particle dynamics reveals the role played by the electron rest mass in the energy spectrum and thus imposes physical limits on the quantum speed limit [23]. Repeating the same procedure above to obtain the maximum displacement of the mean radial position of the electron, we find that the two states of the superposition must have the same spin orientation (spin up), have null angular momentum projection along the z direction, and be made of nearest neighbor eigenstates with even quantum numbers n . Assuming the negative energy eigenvalue as the lowest one in the module, according to Eq. (8) we attribute to it the quantum number n , while for the positive energy eigenvalue the quantum number $n + 2$. Thus, the minimum evolution time for the particular case $n = 0$ and $p_0 = 0$ is given by

$$T_{\min} = \frac{\pi\hbar}{\sqrt{m_0^2 c^4 + 4eB\hbar c^2} + \sqrt{m_0^2 c^4 + 2eB\hbar c^2}}. \quad (27)$$

This time is shorter than in the particle-particle case [see Eq. (16)] because the energy gap is bigger by a quantity that is at least the energy of the electron rest mass. To evaluate the mean radial displacement of the electron, we need the expression of the negative-energy spinor for a general quantum number n and null angular momentum,

$$U_{n,0}^-(\vec{r}) = N_{n,0}^- \begin{pmatrix} \frac{cp}{(E_n - m_0 c^2)} F_{n,0} \\ \frac{\sqrt{2}ic\hbar\beta\sqrt{n+2}}{(E_n - m_0 c^2)} F_{n+1,1} \\ F_{n,0} \\ 0 \end{pmatrix} e^{ipz/\hbar}, \quad (28)$$

where $N_{n,0}^-$ is the normalization constant. In addition, the radial displacement of the electron is proportional to the absolute

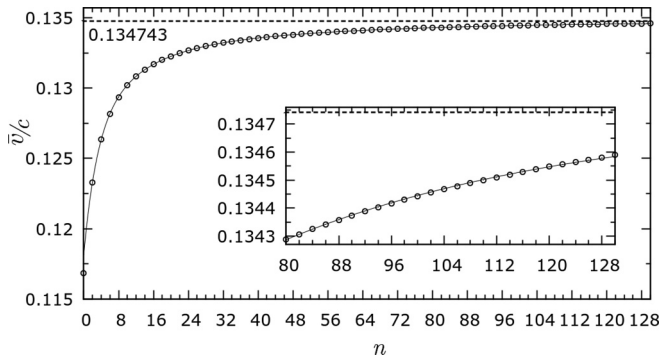


FIG. 3. Average radial speed of an electron for different initial superposition states composed by a positive- and negative-energy eigenstate $(U_{n,0}^- + U_{n+2,0})/\sqrt{2}$.

value of the crossed term

$$D_S = 2\pi q N_{n,0}^- N_{n+2,0} c p \times \left[\frac{1}{E_n - m_0 c^2} + \frac{1}{E_{n+2} + m_0 c^2} \right] F_{n,0}^\dagger F_{n+2,0}, \quad (29)$$

which one is maximized for $p_0 \approx \beta \hbar \gg m_0 c$. In Fig. 3 we plot the average radial speed of the electron to change from an n -dependent initial negative-positive state to a final one orthogonal to the former in the minimum time interval T_{\min} . The asymptotic value of \bar{v} is $0.134743c$ and lower than the speed in the positive-positive case (see Fig. 2).

Comparing Figs. 2 and 3 we observe that \bar{v} for negative-positive states is always less than \bar{v} for positive-positive states. This behavior is clarified in Fig. 4, where \bar{v}/c is plotted for both cases of initial superposition states as function of p_0 for three different values of the magnetic field. If the initial state is negative-positive, then, according to Eq. (29), the displacement of the radial mean position of the electron depends linearly on p_0 , which justifies the null value of \bar{v}/c at the origin of Fig. 4. For intermediate values of p_0 , we observe that \bar{v}/c attains a maximum value for $p_0 \approx \beta \hbar \gg m_0 c$, while for great values of $p_0 \gg \beta \hbar, m_0 c$ regardless of the initial superposition the average radial speed becomes smaller. In the latter case both positive- and negative-energy eigenstates have the same expression, and therefore the same radial displacement of the electron and T_{\min} .

In the context of Dirac's theory, this can be explained by the fact that each spinor does not describe its own particle only but also its antiparticle by the two terms in the bottom position of the spinor. One of these antiparticle terms is relevant to the whole description when the electron presents a very high linear momentum or when the particle is strongly confined in a region less than or equal to its Compton wavelength. In these cases we could say that the spinor by itself describes a superposition between its particle and its antiparticle [30,39,40].

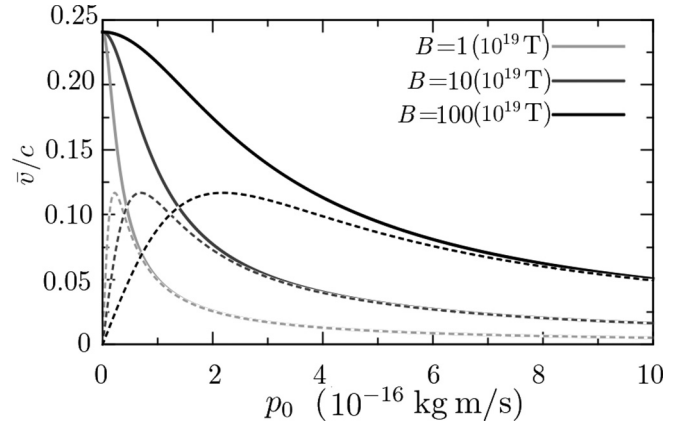


FIG. 4. Average radial speed of an electron for particle-particle (solid lines) and antiparticle-particle (dashed lines) states as function of the expectation value of the linear momentum along the z direction, p_0 .

V. CONCLUSIONS

We analyzed the role played by relativistic effects on the quantum speed limit of a system composed by an electron in a uniform magnetic field. The relativistic dynamics by itself does not restrict the minimum time of evolution of being arbitrary small, but imposes constraints on the average speed at which the electron travels along the space-time. As expected, we observed that the quantum dynamics described by Schrödinger-Pauli equation enables the electron wave packet traveling faster than c , in contradiction to Einstein's theory of relativity. Such problem is circumvented by the use of Dirac's equation. The minimum evolution time between two orthogonal states in the relativistic formulation can be significantly different from the nonrelativistic case. If the initial state of the system is a homogenous superposition of two Hamiltonian eigenstates with positive energies, then the minimum evolution time is dilated in the laboratory frame. On the other hand, if the Hamiltonian eigenstates have negative and positive energies, then the minimum evolution time is contracted in the laboratory frame. This last result can be useful for quantum computing, since it can speed up quantum gates, although a precise control over the creation of particle-antiparticle states is necessary [41].

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