Multiscale quantum simulation of quantum field theory using wavelets

Gavin K. Brennen,[1] Peter Rohde,[2] Barry C. Sanders,[3,4,1] and Sukhwinder Singh[1]

[1]*Centre for Engineered Quantum Systems, Department of Physics and Astronomy, Macquarie University, North Ryde, NSW 2109, Australia*

[2]*Centre for Quantum Computation and Intelligent Systems (QCIS), Faculty of Engineering and Information Technology, University of Technology, Sydney, NSW 2007, Australia*

[3]*Institute for Quantum Science and Technology, University of Calgary, Alberta, Canada T2N 1N4*

[4]*Program in Quantum Information Science, Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada*

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A successful approach to understand field theories is to resolve the physics into different length or energy scales using the renormalization group framework. We propose a quantum simulation of quantum field theory which encodes field degrees of freedom in a wavelet basis—a multiscale description of the theory. Since wavelet families can be constructed to have compact support at all resolutions, this encoding allows for quantum simulations to create particle excitations which are local at some chosen scale and provides a natural way to associate observables in the theory to finite-resolution detectors.

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I. INTRODUCTION

Wavelets are a versatile basis to represent functions which are neither localized in position nor momentum. They are best known for their use [1] in signal processing such as in the Joint Photographic Experts Group (JPEG) compression where they can represent and compress data at multiple spatial scales with low loss of fidelity. They are also being adopted to speed up calculations for a plethora of problems in science, including quantum molecular dynamics [2], density functional theory [3], and Monte Carlo simulations on lattice [4], which are of enormous importance for quantum chemistry, solid state, and statistical physics. Further, there are potential applications to high-energy physics where a wavelet basis been proposed as a way to regularize quantum field theories [5].

At the same time that these advances have been made in classical computations, algorithms have been developed to attack difficult problems in quantum mechanics by using quantum simulators [6]. However, most quantum algorithms for simulation of dynamics in real space use some version of bases which are localized in position and/or momentum and mapped into each other by Fourier transforms. While the quantum Fourier transform is efficient, more efficient evolutions may be possible for quantum states which are not localized in either basis.

In Ref. [7], the authors provide a quantum algorithm to simulate scalar bosonic field theories which achieves accurate estimation of scattering matrix probabilities in a time exponentially faster than known classical algorithms. Here we present a wavelet-based quantum simulation. A key feature of this basis choice is that we need not discretize space; rather, we choose a representative scale to capture features of the wave function and can add smaller scale features in a controlled manner. There are several advantages to using wavelets in the context of quantum simulation algorithms for quantum field theory. First, wavelets have a built-in scaling structure which could be used to compute expectation values of operators such as energy density and two-point correlation functions at different length scales. This information could then be used to compute fixed points of renormalization flows [8]. Second, the wavelet basis has a well-defined procedure to include local gauge invariance via covariant derivatives at every length scale [9,10]. Third, in the spirit of quantum information, a wavelet basis is a natural one to describe quantum fields by the scale of a measurement. This can obviate issues with divergences of Greens functions that arise in calculations using pointlike operators [11].

We first briefly introduce in Sec. II the essential features of wavelets focusing on a particular family, the Daubechies wavelets, which are related to each other by dyadic scaling and discrete translations. In Sec. III we represent the Hamiltonian for a scalar bosonic field theory in $d = 1$ spatial dimension in a wavelet basis where the mode operators have support over the finite spatial extent of wavelets. This construction has a straightforward extension to higher d. We show how to encode the ground state of the free field theory in a register of qubits or bosonic modes and how to create single-particle excitations and turn on quartic interactions. The complexity of this simulation is shown in Sec. IV to be similar, up to a multiplicative factor logarithmic in the number of modes, to the algorithm of Ref. [7] that uses the discretized position basis. A summary of our results and outlook are given in Sec. V.

II. BASIC PROPERTIES OF DAUBECHIES WAVELETS

Wavelets constitute an orthonomal basis for the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the line and we briefly review some of their properties here. For a comprehensive survey see Ref. [1]. Generically, wavelets are defined in terms of a mother wavelet function $w(x)$ and a father scaling function $s(x)$ by taking linear combinations of shifts and rescalings thereof. For the remainder we focus on one family known as Daubechies $\mathcal{K}$ wavelets where the role of $\mathcal{K} \in \mathbb{Z}^+$ will be described below. First, we introduce two unitary operators on $L^2(\mathbb{R})$: $\mathcal{T}$ for discrete translation and $\mathcal{D}$ for scaling defined by the action on a function $f \in L^2(\mathbb{R})$:

$$\mathcal{D}f(x) = \sqrt{2}f(2x); \quad \mathcal{T}f(x) = f(x-1). \tag{1}$$

The father scaling function $s(x)$ is a solution to the linear renormalization group equation

$$s(x) = \mathcal{D}\left[\sum_{n=0}^{2\mathcal{K}-1} h_n \mathcal{T}^n s(x)\right], \tag{2}$$

reading *first block average and then rescale*. The $2\mathcal{K}$ real coefficients $\{h_n\}$ are computed analytically for $\mathcal{K} < 4$ and are solved for numerically otherwise. Given the solution to $s(x)$, scale 2^{-k} scaling functions are defined by applying n unit translations followed by k scaling transformations on the father:

$$s_n^k(x) = D^k T^n s(x). \tag{3}$$

The scaling functions are normalized so

$$\int dx\ s_n^k(x) = 1. \tag{4}$$

The mother wavelet $w(x)$ and the father $s(x)$ have the property that they are neither localized in position or momentum. The wavelets take the following form:

$$w(x) = \sum_{n=0}^{2\mathcal{K}-1} g_n \mathcal{D}\mathcal{T}^n s(x) = \sum_{n=0}^{2\mathcal{K}-1} g_n s_n^1(x), \tag{5}$$

where the set of coefficients $\{g_n\}$ are obtained from $\{h_n\}$ by reversing the order and alternating signs: $g_n = (-1)^n h_{2\mathcal{K}-1-n}$. Scale 2^{-k} wavelets are obtained by translating and scaling the mother:

$$w_n^k(x) = \mathcal{D}^k \mathcal{T}^n w(x). \tag{6}$$

The index $\mathcal{K}$ specifies the number of vanishing moments of the wavelets, i.e.,

$$\int dx\ w(x)x^p = 0 \quad p = 0, \ldots, \mathcal{K}.$$

The vanishing of the zeroth moment is synonymous with the admissibility condition which guarantees that the wavelet basis is square integrable [1]. Choosing larger $\mathcal{K}$ means more features can be captured at a given scale, however, at the expense of additional computational cost since more translations are needed during block averaging. Daubechies wavelets are optimal in the sense that they have the smallest size support for a given number of vanishing moments [1]. The basis functions $s_n^k(x)$ and $w_n^k(x)$ have support on $[2^{-k}n, 2^{-k}(n + 2\mathcal{K} - 1)]$ and satisfy the following orthonormality relations:

$$\begin{aligned} \int dx\ s_n^k(x) s_m^k(x) &= \delta_{m,n}, \\ \int dx\ s_n^k(x) w_m^{k+l}(x) &= 0 \quad (l \geqslant 0), \\ \int dx\ w_n^k(x) w_m^l(x) &= \delta_{m,n}\delta_{k,l}. \end{aligned} \tag{7}$$

By the last relation, the wavelets constitute normalized wave functions. The scaling functions at scale 2^{-k} are complete in that

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2^k}} s_n^k(x) = 1. \tag{8}$$

A final important property of the Daubechies $\mathcal{K}$ wavelets is that they are $\mathcal{K} - 2$ times differentiable.

Linear superpositions of functions $\{s_n^k(x)\}_{n=-\infty}^{\infty}$ (with square summable coefficients) span a subspace $\mathcal{H}_k$ of $L^2(\mathbb{R})$, which is the scale 2^{-k} subspace and which is a proper subspace of a smaller scale space $\mathcal{H}_k \subset \mathcal{H}_{k+m}$ $(m > 0)$. Linear combinations of the scale 2^{-k} wavelet functions $\{w_n^k(x)\}_{n=-\infty}^{\infty}$ span the orthocomplement $\mathcal{W}_k$ of $\mathcal{H}_k$ in $\mathcal{H}_{k+1}$: $\mathcal{H}_{k+1} = \mathcal{H}_k \oplus \mathcal{W}_k$. We can use a set of scaling functions $\{s_n^k(x)\}_{n=-\infty}^{\infty}$ to represent features down to scale 2^{-k} and a set of wavelets $\{w_n^k(x)\}_{n=-\infty}^{\infty}$ to represent features down to scale $2^{-(k+1)}$ that cannot be represented at scale 2^{-k}. The whole space has the following decomposition satisfied for *any* finite k:

$$L^2(\mathbb{R}) = \mathcal{H}_k \bigoplus_{l=k}^{\infty} \mathcal{W}_l, \tag{9}$$

meaning that for a fixed scale 2^{-k} the set

$$\left\{s_n^k(x)\right\}_{n=-\infty}^{\infty} \bigcup \left\{w_n^l(x)\right\}_{n=-\infty, l=k}^{\infty, \infty}$$

span a basis for $L^2(\mathbb{R})$.

III. A WAVELET REPRESENTATION OF QUANTUM FIELDS

A. Free field ground state represented in the wavelet basis

The class of theories we address are the massive scalar bosonic $\hat{\Phi}^4$ theory in $d \in \mathbb{N}$ spatial dimensions. These are given by the Hamiltonian:

$$\hat{H} = \hat{H}^{(0)} + \hat{H}^{(I)}, \tag{10}$$

where the free field contribution is

$$\hat{H}^{(0)} = \int d^d x \frac{1}{2}\left\{\hat{\Pi}^2(\boldsymbol{x},t) + [\nabla\hat{\Phi}^2(\boldsymbol{x},t)] + m_0^2\hat{\Phi}^2(\boldsymbol{x},t)\right\} \tag{11}$$

and the interaction term is

$$\hat{H}^{(I)} = \int d^d x\ \frac{\lambda_0}{4!}\hat{\Phi}^4(\boldsymbol{x},t). \tag{12}$$

The canonical momentum is

$$\hat{\Pi}(\boldsymbol{x},t) = \frac{\partial\hat{\Phi}(\boldsymbol{x},t)}{\partial t}, \tag{13}$$

which, together with the field, are normalized to satisfy the equal time commutation relation $[\hat{\Phi}(\boldsymbol{x},t), \hat{\Pi}(\boldsymbol{y},t)] = i\delta^d(\boldsymbol{x} - \boldsymbol{y})$ $(\hbar \equiv 1)$. Here the phase velocity of waves in this theory is set so the speed of light is 1, the bare mass is m_0, and the strength of the interactions is dictated by λ_0.

To apply wavelets to the field theory we follow the prescription given in Ref. [10]. Because the Hamiltonian involves terms with no higher than first derivatives, it suffices to choose the Daubechies $\mathcal{K} = 3$ wavelet family which have continuous first derivatives for the scale and wavelet functions (Fig. 1). This will guarantee that we have analytic forms for the coupling matrix elements in the wavelet basis while also providing for a minimal size support for the functions, a feature which reduces the number of nonzero coupling terms that appear in

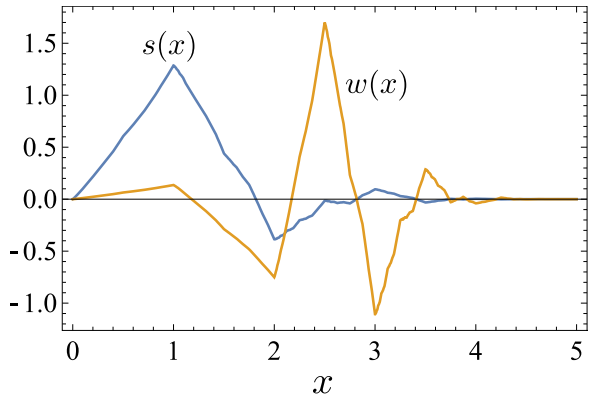

FIG. 1. (Color online) The father scaling function $s(x)$ and mother wavelet $w(x)$ with support on [0,5] for the Daubechies $\mathcal{K}=3$ wavelet family. The functions have continuous first derivatives.

the Hamiltonian. We present the $d=1$ case as it makes the notation considerably simpler and captures the salient features of the algorithm. The wavelet representation can easily be extended to higher dimensions (see Appendix B) using a Cartesian product of wavelets and scale functions. First we decompose the field and its conjugate in the wavelet basis as:

$$\begin{aligned}&\hat{\Phi}(x,t)\\&\quad=\sum_{n\in\mathbb{Z}}\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n,t)s_n^{l_{\min}}(x)+\sum_{n\in\mathbb{Z}}\sum_{l=l_{\min}}^{\infty}\hat{\Phi}^{[\mathbf{w}]l}(n,t)w_n^l(x)\\&\hat{\Pi}(x,t)\\&\quad=\sum_{n\in\mathbb{Z}}\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n,t)s_n^{l_{\min}}(x)+\sum_{n\in\mathbb{Z}}\sum_{l=l_{\min}}^{\infty}\hat{\Pi}^{[\mathbf{w}]l}(n,t)w_n^l(x),\end{aligned}\tag{14}$$

where the coarsest scale in the theory corresponds to $2^{-l_{\min}}$. Henceforth, we drop the dependence of the fields and their conjugates on time. The discrete field operators are projections of the field operators onto the scaling and wavelet functions (here $l\geqslant l_{\min}$):

$$\begin{aligned}\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n)&=\int dx\ \hat{\Phi}(x)s_n^{l_{\min}}(x),\\\hat{\Phi}^{[\mathbf{w}]l}(n)&=\int dx\ \hat{\Phi}(x,t)w_n^l(x)\\\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n)&=\int dx\ \hat{\Pi}(x)s_n^{l_{\min}}(x),\\\hat{\Pi}^{[\mathbf{w}]l}(n)&=\int dx\ \hat{\Pi}(x)w_n^l(x),\end{aligned}\tag{15}$$

and they satisfy the following equal time commutation relations (assuming here that $l_{\min}\leqslant r,s$):

$$\begin{aligned}&[\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n),\hat{\Phi}^{[\mathbf{s}]l_{\min}}(m)]=0,\quad[\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n),\hat{\Pi}^{[\mathbf{s}]l_{\min}}(m)]=0\\&[\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n),\hat{\Pi}^{[\mathbf{s}]l_{\min}}(m)]=i\delta_{n,m}\\&[\hat{\Phi}^{[\mathbf{w}]r}(n),\hat{\Phi}^{[\mathbf{w}]s}(m)]=0,\quad[\hat{\Pi}^{[\mathbf{w}]r}(n),\hat{\Pi}^{[\mathbf{w}]s}(m)]=0\\&[\hat{\Phi}^{[\mathbf{w}]r}(n),\hat{\Pi}^{[\mathbf{w}]s}(m)]=i\delta_{r,s}\delta_{n,m}\\&[\hat{\Phi}^{[\mathbf{w}]r}(n),\hat{\Phi}^{[\mathbf{w}]s}(m)]=0,\quad[\hat{\Pi}^{[\mathbf{w}]r}(n),\hat{\Pi}^{[\mathbf{w}]s}(m)]=0\\&[\hat{\Phi}^{[\mathbf{w}]r}(n),\hat{\Pi}^{[\mathbf{w}]s}(m)]=0,\quad[\hat{\Pi}^{[\mathbf{w}]r}(n),\hat{\Phi}^{[\mathbf{w}]s}(m)]=0.\end{aligned}\tag{16}$$

The discrete annihilation operators, for the scaling and wavelet fields, respectively, are

$$\begin{aligned}\hat{a}^{l_{\min}}(n)&=\frac{1}{\sqrt{2}}\left[\sqrt{\gamma(l_{\min})}\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n)+i\frac{1}{\sqrt{\gamma(l_{\min})}}\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n)\right]\\\hat{b}^{r}(n)&=\frac{1}{\sqrt{2}}\left[\sqrt{\gamma(r)}\hat{\Phi}^{[\mathbf{w}]r}(n)+\frac{i}{\sqrt{\gamma(r)}}\hat{\Pi}^{[\mathbf{w}]r}(n)\right],\end{aligned}\tag{17}$$

and the inverse relations are

$$\begin{aligned}\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n)&=\frac{1}{\sqrt{2\gamma^{[\mathbf{s}]}(l_{\min})}}[\hat{a}^{l_{\min}\dagger}(n)+\hat{a}^{l_{\min}}(n)]\\\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n)&=i\sqrt{\frac{\gamma^{[\mathbf{s}]}(l_{\min})}{2}}[\hat{a}^{l_{\min}\dagger}(n)-\hat{a}^{l_{\min}}(n)]\\\hat{\Phi}^{[\mathbf{w}]r}(n)&=\frac{1}{\sqrt{2\gamma^{[\mathbf{w}]}(r)}}[\hat{b}^{r\dagger}(n)+\hat{b}^{r}(n)]\\\hat{\Pi}^{[\mathbf{w}]r}(n)&=i\sqrt{\frac{\gamma^{[\mathbf{w}]}(r)}{2}}[\hat{b}^{r\dagger}(n)-\hat{b}^{r}(n)].\end{aligned}\tag{18}$$

Each annihilates the free field vacuum and together with the set of adjoint creation operators they satisfy the bosonic commutation relations:

$$\begin{aligned}[\hat{a}^{l_{\min}}(n),\hat{a}^{l_{\min}\dagger}(m)]&=\delta_{m,n}\\[\hat{b}^{l}(n),\hat{b}^{j\dagger}(m)]&=\delta_{m,n}\delta_{j,l},\end{aligned}\tag{19}$$

with all others commutators vanishing. The Hilbert space for the free field theory is spanned by linear combinations of products of the creation operators from the set $a^{l_{\min}\dagger}(m),b^{l\dagger}(m)$ applied to $|G\rangle$.

The coefficients γ depend on the scale $2^{-l_{\min}}$ and the mass m_0 as follows:

$$\begin{aligned}\gamma^{[\mathbf{s}]}(l_{\min})&=\frac{1\pm\sqrt{1-4\nu^{[\mathbf{s}]}(l_{\min})\eta^{[\mathbf{s}]}(l_{\min})}}{2\nu^{[\mathbf{s}]}(l_{\min})}\\\gamma^{[\mathbf{w}]}(r)&=\frac{1\pm\sqrt{1-4\nu^{[\mathbf{w}]}(r)\eta^{[\mathbf{w}]}(r)}}{2\nu^{[\mathbf{w}]}(r)},\end{aligned}\tag{20}$$

where $|G\rangle$ is the free field vacuum state,

$$\begin{aligned}\nu^{[\mathbf{s}]}(l_{\min})&=\langle G|\hat{\Phi}^{[\mathbf{s}]l_{\min}}(0)\hat{\Phi}^{[\mathbf{s}]l_{\min}}(0)|G\rangle\\\nu^{[\mathbf{w}]}(r)&=\langle G|\hat{\Phi}^{[\mathbf{w}]r}(0)\hat{\Phi}^{[\mathbf{w}]r}(0)|G\rangle\\\eta^{[\mathbf{s}]}(l_{\min})&=\langle G|\hat{\Pi}^{[\mathbf{s}]l_{\min}}(0)\hat{\Pi}^{[\mathbf{s}]l_{\min}}(0)|G\rangle\\\eta^{[\mathbf{w}]}(r)&=\langle G|\hat{\Pi}^{[\mathbf{w}]r}(0)\hat{\Pi}^{[\mathbf{w}]r}(0)|G\rangle,\end{aligned}\tag{21}$$

and the $\pm$ sign is chosen according to the case that makes the expression positive.

Following Ref. [10] we decompose the free field Hamiltonian into three pieces

$$\hat{H}^{(0)}=\hat{H}_{\mathrm{ss}}+\hat{H}_{\mathrm{ww}}+\hat{H}_{\mathrm{sw}}.\tag{22}$$

We fix a scale $2^{-l_{\min}}$ so the Hilbert space is decomposed as in Eq. (9). Then the constituent terms of the Hamiltonian

are

$$\begin{aligned}\hat{H}_{\rm ss} &= \frac{1}{2}\Bigg(\sum_{n\in\mathbb{Z}} :\hat{\Pi}^{[\mathbf{s}]l_{\rm min}}(n)\hat{\Pi}^{[\mathbf{s}]l_{\rm min}}(n): \\ &\quad + m_0^2\sum_{n\in\mathbb{Z}} :\hat{\Phi}^{[\mathbf{s}]l_{\rm min}}(n)\hat{\Phi}^{[\mathbf{s}]l_{\rm min}}(n): \\ &\quad + \sum_{m,n\in\mathbb{Z}} :\hat{\Phi}^{[\mathbf{s}]l_{\rm min}}(m)D^{k}_{m,n}\hat{\Phi}^{[\mathbf{s}]l_{\rm min}}(n):\Bigg) \\ \hat{H}_{\rm ww} &= \frac{1}{2}\Bigg(\sum_{n\in\mathbb{Z}}\sum_{l\geqslant l_{\rm min}} :\hat{\Pi}^{[\mathbf{w}]l}(n)\hat{\Pi}^{[\mathbf{w}]l}(n): \\ &\quad + m_0^2\sum_{n\in\mathbb{Z}}\sum_{l\geqslant l_{\rm min}} :\hat{\Phi}^{[\mathbf{w}]l}(n)\hat{\Phi}^{[\mathbf{w}]l}(n): \\ &\quad + \sum_{m,n\in\mathbb{Z}}\sum_{l,j\geqslant l_{\rm min}} :\hat{\Phi}^{[\mathbf{w}]l}(m)D^{l,j}_{m,n}\hat{\Phi}^{[\mathbf{w}]j}(n):\Bigg) \\ \hat{H}_{\rm sw} &= \frac{1}{2}\sum_{m,n\in\mathbb{Z}}\sum_{l\geqslant l_{\rm min}} :\hat{\Phi}^{[\mathbf{w}]l}(m)D^{l,l_{\rm min}}_{m,n}\hat{\Phi}^{[\mathbf{s}]l_{\rm min}}(n):,\end{aligned} \tag{23}$$

where $:\hat{O}:$ indicates normal ordering of the operator $\hat{O}$ is taken. The operator $\hat{H}_{\rm ss}$ describes physics at a scale $2^{-l_{\rm min}}$ involving interactions between scale field degrees of freedom, $\hat{H}_{\rm ww}$ describes physics at finer scales involving interactions between wavelet degrees of freedom, and $\hat{H}_{\rm sw}$ describes coupling between scale fields at resolution $2^{-l_{\rm min}}$ and finer wavelet degrees of freedom. While there are an infinite number of finer scale degrees of freedom we truncate to $l_{\rm max}$ consistent with momentum cutoffs in physical theories. Specifically, the maximum momentum for a single-particle excitation is $p_{\rm max} \simeq 2^{l_{\rm max}}$ as described in Sec. III C. The coupling coefficients are

$$\begin{aligned}D^{l_{\rm min}}_{m,n} &= \int dx\ \partial_x s^{l_{\rm min}}_m(x)\cdot\partial_x s^{l_{\rm min}}_n(x) \\ D^{l,j}_{m,n} &= \int dx\ \partial_x w^{l}_m(x)\cdot\partial_x w^{j}_n(x) \\ D^{l,l_{\rm min}}_{m,n} &= 2\int dx\ \partial_x w^{l}_m(x)\cdot\partial_x s^{l_{\rm min}}_n(x).\end{aligned} \tag{24}$$

Many of these coefficients are computed in Ref. [10] for the Daubuchies $\mathcal{K}=3$ wavelets. The choice of $\mathcal{K}=3$ ensures a continuous first derivative of the scaling functions which allows for computing these overlaps exactly. Because the functions have compact support, the coefficients vanish unless $|n-m|\leqslant 4$.

Let the physical one-dimensional volume be La where $L\in\mathbb{N}$ and a is the unit of length at the base scale, and assume periodic boundaries. The size of L will be determined by the long-wavelength physics that one wishes to capture. At smaller scales, 2^{-l}, the unit of length is $a2^{-l}$. We will work in normalized length units such that $a=1$, and we choose our base scale so $l_{\rm min}=0$ such that the support of the scaling function $s^0_0(x)=s(x)$ is the interval [0,5]. A plot of these functions is shown in Fig. 2.

Now let us introduce notation for basis vectors in the wavelet basis. Basis vectors $|r\rangle_{w^j_m}$ denote field amplitude r in the wavelet mode w^j_m such that $\hat{\Phi}^{[\mathbf{w}]j}(m)|r\rangle_{w^j_m} = r|r\rangle_{w^j_m}$ and, similarly, $\hat{\Phi}^{[\mathbf{s}]0}(m)|r\rangle_{s^0_m} = r|r\rangle_{s^0_m}$. We adopt a simplified notation for states in the tensor product space of the

$$V = L2^{l_{\rm max}+1}, \tag{25}$$

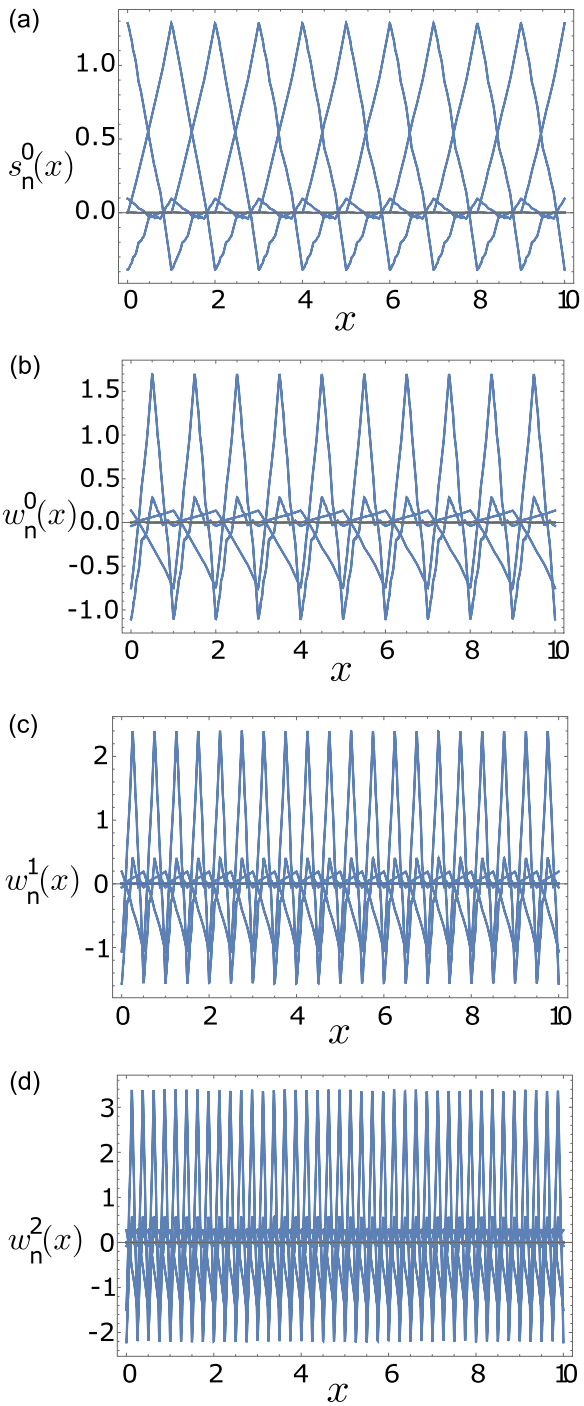

FIG. 2. (Color online) One-dimensional Daubechies $\mathcal{K}=3$ scale functions and wavelets plotted as a function of position x at three scales for a system of size $L=10$ with periodic boundaries. (a) Scale functions $\{s^0_n(x)\}_{n=0}^{L-1}$; (b) wavelets $\{w^0_n(x)\}_{n=0}^{L-1}$; (c) $\{w^1_n(x)\}_{n=0}^{2L-1}$; (d) wavelets $\{w^2_n(x)\}_{n=0}^{4L-1}$. Results in the main text are for a system with periodic boundaries.

modes utilizing the vector $\boldsymbol{r} = (r_0, \ldots r_{V-1})^T$ with

$$\begin{aligned}|\mathbf{r}\rangle = {} & |r_0\rangle_{s_0^0} \otimes \cdots \otimes |r_{L-1}\rangle_{s_{L-1}^0} \otimes |r_L\rangle_{w_0^0} \\ & \otimes \cdots \otimes |r_{2L-1}\rangle_{w_{L-1}^0} \\ & \otimes |r_{2L}\rangle_{w_0^1} \otimes \cdots \otimes |r_{4L-1}\rangle_{w_{2L-1}^1} \\ & \otimes |r_{4L}\rangle_{w_0^2} \cdots \otimes |r_{V-1}\rangle_{w_{V/2-1}^{l_{\max}}}. \end{aligned} \tag{26}$$

The ground state of the free field theory $\hat{H}^{(0)}$ is then approximated by

$$|G\rangle \approx \mathcal{N}^{-1} \int dr_0 \ldots \int dr_{V-1}\, e^{-\frac{1}{2}\boldsymbol{r}^T K^{1/2} \boldsymbol{r}} |\mathbf{r}\rangle, \tag{27}$$

where the normalization is $\mathcal{N}^{-1} = \det(K^{1/2})^{1/4}/\pi^{V/4}$. Here the coupling matrix is

$$K = \begin{bmatrix} [K_{\rm ss}] & [K_{\rm sw}(0)] & \cdots & [K_{\rm sw}(l_{\max})] \\ [K_{\rm sw}(0)]^T & [K_{\rm ww}(0,0)] & \ldots & [K_{\rm ww}(0,l_{\max})] \\ \vdots & & \ddots & \\ [K_{\rm sw}(l_{\max})]^T & [K_{\rm ww}(0,l_{\max})]^T & \cdots & [K_{\rm ww}(l_{\max},l_{\max})] \end{bmatrix}. \tag{28}$$

The scale-scale mode couplings are encoded in $K_{\rm ss}$, the scale-wavelet couplings in $K_{\rm sw}$, and the wavelet-wavelet couplings in $K_{\rm ww}$. These matrices are

$$\begin{aligned} [K_{\rm ss}]_{a,b} &= \left(m_0^2 - D^0_{0,0}\right)\delta_{a,b} + \tfrac{1}{2}\left(D^0_{0,(a-b) \bmod L} + D^0_{0,(b-a) \bmod L}\right) \quad (0 \leqslant a,b < L) \\ [K_{\rm sw}(l)]_{a,b} &= D^{0,l}_{a,b} \quad (0 \leqslant a < L, 0 \leqslant b < L2^l, 0 \leqslant l \leqslant l_{\max}) \\ [K_{\rm ww}(l,j)]_{a,b} &= m_0^2 \delta_{a,b}\delta_{j,l} + D^{l,j}_{a,b} (0 \leqslant a < L2^l, 0 \leqslant b < L2^j, 0 \leqslant j \leqslant l \leqslant l_{\max}). \end{aligned} \tag{29}$$

The values of these coupling overlap integrals for Daubechies $\mathcal{K} = 3$ wavelets are obtained from the following relations. First we use the scaling function components defined in Eq. (2)

$$\begin{aligned} h_0 &= \frac{1}{16\sqrt{2}}(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}) \\ h_1 &= \frac{1}{16\sqrt{2}}(5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}}) \\ h_2 &= \frac{1}{16\sqrt{2}}(10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}}) \\ h_3 &= \frac{1}{16\sqrt{2}}(10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}}) \\ h_4 &= \frac{1}{16\sqrt{2}}(5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}}) \\ h_5 &= \frac{1}{16\sqrt{2}}(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}). \end{aligned} \tag{30}$$

The coefficients $g_n = (-1)^n h_{5-n}$. The coefficients $D^0_{m,n} = D^0_{n,m}$ with

$$\begin{aligned} D^0_{0,0} &= 5.2576013450, \\ D^0_{0,1} &= -3.3828986455 \\ D^0_{0,2} &= 0.87333354692, \\ D^0_{0,3} &= -0.11139112377 \\ D^0_{0,4} &= -5.3243362257 \times 10^{-3}, \end{aligned} \tag{31}$$

and $D^0_{m,n} = 0$ for $|m - n| > 4$. Because the derivatives of translations of the father functions form a partition of unity [1],

$$\sum_n n \partial_x s^0_n(x) = 1,$$

the coefficients satisfy the following constraint:

$$\sum_n n D^0_{m,n} = 0.$$

The other coefficients are

$$\begin{aligned} D^{0,l}_{a,b} = {} & 2^{2(l+1)}(\langle a|[H(l)]^{l+1} D(l) G^T(l)|b\rangle \\ & + \langle a + L|[H(l)]^{l+1} D(l) G^T(l)|b\rangle \\ & + \langle a|[H(l)]^{l+1} D(l) G^T(l)|b + 2^l L\rangle) \\ D^{l,j}_{a,b} = {} & 2^{2(l+1)}(\langle a|G(l,j)[H(l,j)]^{l-j} D(l,j) G^T(l,j)|b\rangle \\ & + \langle a + 2^j L|G(l,j)[H(l,j)]^{l-j} D(l,j) G^T(l,j)|b\rangle \\ & + \langle a|G(l,j)[H(l,j)]^{l-j} D(l,j) G^T(l,j)|b + 2^l L\rangle), \end{aligned} \tag{32}$$

where the scale-dependent matrices are

$$\begin{aligned} H(l) &= \sum_{m,n=0}^{2^{(l+2)}(L+4)-5} h_{n-2m}|m\rangle\langle n| \\ H(l,j) &= \sum_{m,n=0}^{2^{(l-j+1)}(2(2^j L-4))-5} h_{n-2m}|m\rangle\langle n| \\ D(l) &= \sum_{m,n=0}^{2^{(l+2)}(L+4)-5} D^0_{m,n}|m\rangle\langle n| \\ D(l,j) &= \sum_{m,n=0}^{2^{(l-j+1)}(2(2^j L-4))-5} D^0_{m,n}|m\rangle\langle n| \end{aligned}$$

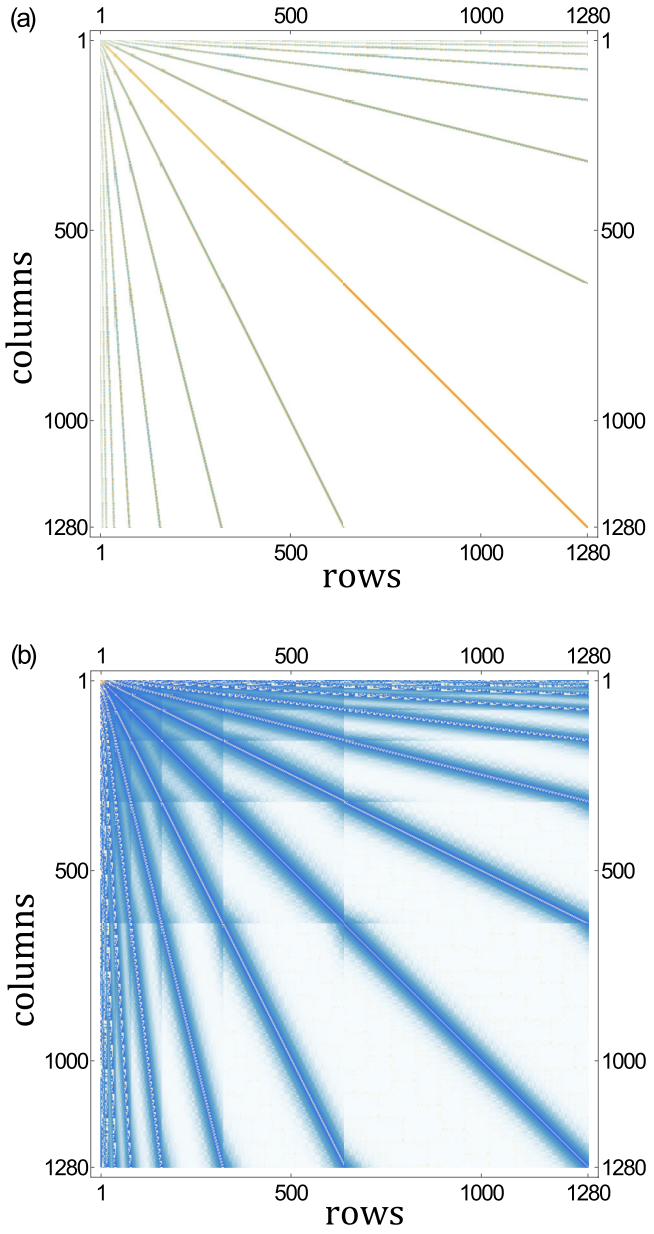

FIG. 3. (Color online) Visualization of a 1D free scalar bosonic field (mass $m_0 = 1$) in the wavelet basis. The system size is $L = 10$, and $l_{\min} = 0, l_{\max} = 6$ so the total number of modes is $V = 1280$. The modes are ordered according to Eq. (26) with coarsest scale modes in the upper left and finest wavelet modes in the lower right. White indicates zero values. (a) The coupling matrix K. Diagonal stripes show couplings within a given scale while off diagonal stripes represent couplings between scales. (b) Field correlation matrix $\Gamma^{\Phi,\Phi} = \frac{1}{2}K^{-1/2}$ as defined in Eq. (38).

$$G(l) = \sum_{m,n=0}^{2^{(l+2)}(L+4)-5} g_{n-2m}|m\rangle\langle n|$$
$$G(l,j) = \sum_{m,n=0}^{2^{(l-j+1)}(2(2^j L-4)-5} g_{n-2m}|m\rangle\langle n|. \quad (33)$$

An example of a K matrix is plotted in Fig. 3(a). Because the wavelets have compact support, the coupling matrix is sparse, having $\simeq 10V\log_2(V)$ nonzero elements, with the factor of 10 arising from the fact that Daubechies $\mathcal{K}$ wavelets have overlap with $2(2\mathcal{K}-1)$ translates within any given scale.

B. Constructing the ground state of the free field theory

We encode the vacuum state $|G\rangle$ in Eq. (27) into a qubit register. As described in Appendix A, let the values r_j be discretized via an m bit string $x_j = x_{j,0}x_{j,1}\ldots x_{j,m-1}$ according to $r_j(x_j) = \delta_\Phi(-1)^{x_{j,0}}\sum_{r=1}^{k-1}2^{x_{j,r}}$ with δ_Φ the field amplitude resolution. The field resolution scales like $\delta_\Phi = O(\sqrt{\frac{\epsilon}{VE}})$ (see Sec. IV), where E is a bound on the expectation value of the energy during the simulation and ϵ quantifies the distance between the truncated many body state and the true ground state of the theory.

The ground state is then represented as a state of $m \times V$ qubits:

$$|G\rangle \approx \mathcal{N}^{-1} \sum_{\{x_{j,r}\in\{0,1\}\}_{j=0,r=0}^{V-1,m-1}} e^{-\frac{1}{2}\mathbf{r}(\{x_j\})^T K^{1/2}\mathbf{r}(\{x_j\})} \times |x_{0,0}\ldots x_{0,m-1}\rangle_{s_0^0}\ldots|x_{V-1,0}\ldots x_{V-1,m-1}\rangle_{w_{V/2-1}^{l_{\max}}}. \quad (34)$$

To construct this ground state using quantum gates one can use the Kitaev-Webb circuit [12]. The cost of that construction is dominated by the $O(V^{2.376})$ time complexity associated with the LDL matrix decomposition of the matrix $K^{1/2}$.

The field operators expressed in the qubit basis are

$$\hat{\Phi}^{[s]0}(n) = \delta_\Phi\sigma_{n,0}^z\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{n,v}$$
$$\hat{\Phi}^{[w]j}(n) = \delta_\Phi\sigma_{L2^j+n,0}^z\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{L2^j+n,v}. \quad (35)$$

The momentum operators are not diagonal in the qubit basis so we need to first transform the state to a basis which is diagonal via the $m-1$ qubit quantum Fourier transform (QFT) $\mathcal{F}$ (which acts on all but the sign bit),

$$\hat{\Pi}^{[s]0}(n) = \delta_\Phi\sigma_{n,0}^z\mathcal{F}^\dagger\left[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{n,v}\right]\mathcal{F}$$
$$\hat{\Pi}^{[w]j}(n) = \delta_\Phi\sigma_{L2^j+n,0}^z\mathcal{F}^\dagger\left[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{L2^j+n,v}\right]\mathcal{F}. \quad (36)$$

For the free field theory, the ground state is a Gaussian which is completely characterized by the covariance matrix Γ defined as

$$\Gamma_{j,k} = \mathrm{Re}[\mathrm{tr}[\rho(\hat{\mathbf{r}}_j - \langle\hat{\mathbf{r}}_j\rangle)(\hat{\mathbf{r}}_k - \langle\hat{\mathbf{r}}_k\rangle)]], \quad (37)$$

where $\langle\hat{\mathbf{r}}_j\rangle$ is the expectation value of j-th component of the $2V$-dimensional vector of operators:

$$\hat{\mathbf{r}} = [\hat{\Phi}^{[s]0}(0),\ldots,\hat{\Phi}^{[s]0}(L-1),\hat{\Phi}^{[w]0}(0)\ldots,\hat{\Phi}^{[w]0}(L-1),$$
$$\ldots,\hat{\Phi}^{[w]l_{\max}}(0),\ldots,\hat{\Phi}^{[w]l_{\max}}(2^{l_{\max}}L-1),$$
$$\hat{\Pi}^{[s]0}(0),\ldots,\hat{\Pi}^{[s]0}(L-1),\hat{\Pi}^{[w]0}(0)\ldots,\hat{\Pi}^{[w]0}(L-1),$$
$$\ldots,\hat{\Pi}^{[w]l_{\max}}(0),\ldots,\hat{\Pi}^{[w]l_{\max}}(2^{l_{\max}}L-1)]^T.$$

The covariance matrix is block diagonal in the field modes and their conjugates:

$$\Gamma = \begin{pmatrix} \Gamma^{\Phi,\Phi} & \Gamma^{\Phi,\Pi} \\ \Gamma^{\Pi,\Phi} & \Gamma^{\Pi,\Pi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} K^{-1/2} & 0 \\ 0 & K^{1/2} \end{pmatrix}. \tag{38}$$

A visualization of the correlation matrix $\Gamma^{\Phi,\Phi}$ for a massive free field theory is given in Fig. 2(b). We note that the matrix $K^{1/2}$ that appears in the exponent of the of the ground-state amplitudes has many negligible elements. Numerics on systems of size $V = 2560$ modes (size $L = 10$ and $l_{\max} = 7$) and for a wide range of masses, show that $\sim$3/4 of the matrix elements have magnitude $<10^{-8}$. Hence, in practice, the $O(V^{2.376})$ scaling for the construction of the ground state might be improved making use algorithms that exploit the matrix sparsity [13].

For completeness, in Appendix C we also describe how to construct the ground state using an encoding with a bosonic network instead of qubits. Because the state is Gaussian, the preparation procedure requires only Gaussian operations on single modes or pairs of modes.

C. Particle creation in the free field theory

Let us consider the steps in a quantum algorithm to create a particle excitation above the vacuum ground state of the free field theory. A simple choice here is to choose the particle's wave function to be the wavelet $\psi(x) = w_n^r(x)$. That is, we want to construct the state

$$\hat{b}^{r\dagger}(n)|G\rangle. \tag{39}$$

The momentum operator in the wavelet basis is [10]

$$\begin{aligned}\hat{p} = -\Bigg[&\sum_{m,n} :\hat{\Pi}^{[\mathbf{s}]l_{\min}}(m) P^{k}_{m,n} \hat{\Phi}^{[\mathbf{s}]l_{\min}}(n): \\ &+ \sum_{(m,l),(n,j)} :\hat{\Pi}^{[\mathbf{w}]l}(m) P^{l,j}_{m,n} \hat{\Phi}^{j}(n): \\ &+ \sum_{m,l,n} :\hat{\Pi}^{[\mathbf{w}]l}(m) P^{l}_{m,n} \hat{\Phi}^{[\mathbf{s}]l_{\min}}(n): \Bigg],\end{aligned} \tag{40}$$

where

$$\begin{aligned} P^{k}_{m,n} &= \int dx\, s_m^{l_{\min}}(x)\partial_x s_n^{l_{\min}}(x) \\ P^{l,j}_{m,n} &= \int dx\, w_m^{l}(x)\partial_x w_n^{j}(x) \\ P^{l}_{m,n} &= \int dx \left[w_m^{l}(x)\partial_x s_n^{l_{\min}}(x) + s_n^{l_{\min}}(x)\partial_x w_m^{l}(x)\right]. \end{aligned} \tag{41}$$

TABLE I. Values of overlap integrals used to determine the momentum of excited state wave packets for Daubechies $\mathcal{K} = 3$ wavelets. Note $P^{0}_{0,m} = -P^{0}_{0,-m}$ and $P^{0,0}_{0,m} = -P^{0,0}_{0,-m}$, and for $|m| > 4$ the values are zero.

m	$P^{0}_{0,m}$	$P^{0,0}_{0,m}$
0	0	0
1	0.745203	−1.32599
2	−0.145203	0.146573
3	0.014612	−0.014612
4	0.000342	−0.000342

See, for example, Table I. Note that translational shifts in the wave function do not change the momentum. Furthermore, from the scaling properties of the scaling functions and wavelets, $P^{k}_{m,n} = -P^{k}_{n,m} = 2^k P^{0}_{0,n-m}$ and $P^{l,l}_{m,n} = -P^{l,l}_{n,m} = 2^l P^{0,0}_{0,n-m}$.

For the excited state in Eq. (39), the expectation value of the momentum (assuming $r \geqslant k$) is

$$\langle G|\hat{b}^{r}(n)\hat{p}\hat{b}^{r\dagger}(n)|G\rangle = -\frac{i}{2}2^r P^{0,0}_{0,0} = 0. \tag{42}$$

Finite momentum excited states can be created from a superposition of wavelets. Consider the state

$$|E\rangle = [\alpha_{r,n}\hat{b}^{r\dagger}(n) + \beta_{r,m}\hat{b}^{r\dagger}(m)]|G\rangle, \tag{43}$$

with $r \geqslant k$ and $|\alpha_{r,n}|^2 + |\beta_{r,m}|^2 = 1$. We find

$$\langle E|\hat{p}|E\rangle = 2^r P^{0,0}_{0,n-m}\mathrm{Im}[\alpha_{r,n}\beta^*_{r,m}].$$

For a given scale, the maximum magnitude momentum eigenstate is obtained for $n - m = -1$, $\alpha_{r,n} = 1/\sqrt{2}$, $\beta_{r,m} = \mp i/\sqrt{2}$ in which case $\langle E|\hat{p}|E\rangle = \pm 2^{r-1}P^{0,0}_{0,-1} = \pm 0.663 \times 2^r$. Hence the maximum momentum of a single-particle state is

$$p_{\max} = 0.663 \times 2^{l_{\max}}. \tag{44}$$

A generalized single-particle excitation at scale 2^{-r} is defined by $f^\dagger|G\rangle$, where $\hat{f}^\dagger = \sum_{r,n}\alpha_{r,n}b^{r\dagger}(n)$ and $\sum_{r,n}|\alpha_{r,n}|^2 = 1$. Following the approach in Ref. [7] we introduce an ancillary qubit a interacting with the register qubits via

$$\hat{H}_\psi = \hat{f}^\dagger \otimes (|1\rangle\langle 0|)_a + \hat{f} \otimes (|0\rangle\langle 1|)_a. \tag{45}$$

If we can simulate the evolution by $\hat{H}_\psi$, then $e^{-i\hat{H}_\psi \pi/2}|G\rangle|0\rangle_a = -i\,\hat{f}^\dagger|G\rangle|1\rangle_a$ and we have the excited state up to a phase with no entanglement left between the ancilla and the register. The Hamiltonian written out explicitly in the qubit representation is

$$\begin{aligned}\hat{H}_\psi = \frac{\delta_\Phi}{\sqrt{2}}\sum_{r,n}\Bigg[&\Bigg(\mathrm{Re}[\alpha_{r,n}]\sqrt{\gamma^{[\mathbf{w}]}(r)}\sigma^z_{L2^r+n,0}\Bigg[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{L2^r+n,v}\Bigg] \\ &+ \mathrm{Im}[\alpha_{r,n}]\frac{1}{\sqrt{\gamma^{[\mathbf{w}]}(r)}}\sigma^z_{L2^r+n,0}\mathcal{F}^\dagger\Bigg[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{L2^r+n,v}\Bigg]\mathcal{F}\Bigg)\otimes\sigma^x_a \\ &+ \Bigg(\mathrm{Im}[\alpha_{r,n}]\sqrt{\gamma^{[\mathbf{w}]}(r)}\sigma^z_{r,0}\Bigg[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{r,v}\Bigg] - \mathrm{Re}[\alpha_{r,n}]\frac{1}{\sqrt{\gamma^{[\mathbf{w}]}(r)}}\sigma^z_{r,0}\mathcal{F}^\dagger\Bigg[\sum_{v=1}^{m-1}2^v(|1\rangle\langle 1|)_{r,v}\Bigg]\mathcal{F}\Bigg)\otimes\sigma^y_a\Bigg],\end{aligned} \tag{46}$$

where the factors $\gamma^{[w]}(r)$ is defined according to Eq. (20) and can be computed either ahead of time on a classical machine or by measurements of correlation functions in the free field ground state [see Eq. (21)]. The evolution generated by the noncommuting terms in $\hat{H}_\Psi$ then can be simulated efficiently by Trotter decomposition [14]. Note that the overhead cost to implement the QFT is $O(m^2)$.

A method to prepare single-particle excitation in a bosonic network encoding is given in Appendix C.

D. Interacting field theory

The Hamiltonian including interactions in the wavelet basis is

$$\hat{H} = \hat{H}_{\rm ss} + \hat{H}_{\rm ww} + \hat{H}_{\rm sw} + \hat{H}^{(I)}, \tag{47}$$

where the interaction term is

$$\begin{aligned}\hat{H}^{(I)} = \frac{\lambda_0}{4!} \sum_{z's\in\{w,s\}} \sum_{j's} \sum_{n's} \int dx\ & f_{n_1}^{z_1,j_1}(x) f_{n_2}^{z_2,j_2}(x) \\ & \times f_{n_3}^{z_3,j_3}(x) f_{n_4}^{z_4,j_4}(x) \\ & : \hat{\Phi}^{[z_1]j_1}(n_1)\hat{\Phi}^{[z_2]j_2}(n_2)\hat{\Phi}^{[z_3]j_3}(n_3)\hat{\Phi}^{[z_4]j_4}(n_4) : ,\end{aligned} \tag{48}$$

where

$$f_n^{z,j} = \begin{cases} s_n^{l_{\rm min}}(x) & j = l_{\rm min} \text{ and } z = s \\ w_n^j(x) & j \geqslant l_{\rm min} \text{ and } z = w. \end{cases} \tag{49}$$

Because the scale functions and wavelets have compact support, the number of nonzero summands in the interaction scales like $O[V \log_2(V)]$.

IV. RESOURCE SCALING OF THE QUANTUM WAVELET SIMULATION

The overall efficiency of the bosonic field theory simulator in the wavelet basis can be obtained by comparing it with the discretized position basis algorithm which was carefully analyzed in Ref. [7]. In the latter algorithm the real valued field, which is a function of the continuous position degree of freedom, is discretized by treating the volume as finite and composed of $N \in \mathbb{N}$ points equally spaced by physical length a' in each dimension. The longest wavelength physics that can be captured is Na' and the highest momentum that can be represented is $1/a'$. Furthermore, the amplitude for the field at each of the $V' = N^d$ discrete points in space is discretized to values in the set $\delta_\Phi[-2^{b-1}, \ldots, 2^{b-1}]$ where $b = \log_2(\Phi_{\rm max}/\delta_\Phi)$ [15].

The efficiency of the algorithm is quantified in terms of two important quantities: the total energy bound E such that the evolved state satisfies $\langle\psi(t)|\hat{H}|\psi(t)\rangle \leqslant E$ for all times in the simulation and the error ϵ which is defined in terms of fidelity of the truncated and discretized many-body state with the true state: $|\langle\Psi|\Psi_{\rm cut}\rangle| \geqslant 1 - \varepsilon$. A cutoff in the maximum field amplitude $\Phi_{\rm max} = O(\sqrt{\frac{V'E}{m_0^2\varepsilon}})$ ensures the above fidelity. By the Fourier relation between conjugate variables, the momentum cutoff is $\Pi_{\rm max} = \delta_\Phi^{-1}$, and upper bounding the expectation values of $\hat{\Pi}(\boldsymbol{x})$ and $\hat{\Pi}^2(\boldsymbol{x})$ in terms of energy, it suffices to choose $\Pi_{\rm max} = O(\sqrt{\frac{V'E}{\varepsilon}})$. The number of qubits needed for the simulation is then $n = V'b = O[V' \log_2(\frac{V'E}{m_0\epsilon})]$. In the massive case, two point correlators decay exponentially with separation, and V' need only scale logarithmically with ε.

The asymptotic scaling for the number of quantum gates needed to simulate particle scattering is found by summing the gates for the following steps: free field ground-state preparation, excited-state preparation by adiabatic turn-on of particle creation interaction, adiabatic turn-on of interaction terms in the Hamiltonian, and, finally, measurement of scattering probabilities. It is shown that the total number of gates is a small polynomial in $1/\varepsilon$ in the weak-coupling regime, and in the strong-coupling regime there is an additional overhead of a polynomial in the momentum p of the colliding particles, the number of outgoing particles, and the distance from the phase transition such that the overall scaling for a simulation of duration t is $O(p^{d+1+o(1)}(tV')^{1+o(1)})$.

In the wavelet basis, for the one-dimensional case $d = 1$, the number of modes is $V = L2^{l_{\rm max}+1}$ [Eq. (25)] and for arbitrary dimension, $V = (L2^{l_{\rm max}+1})^d$. The longest wavelength physics that can be captured is La and the highest momentum scale, from Eq. (44), is $2^{l_{\rm max}}a$. In order to compare the resource scaling with the case of discrete basis we need to equate the longest wavelength and highest momentum scales in the two descriptions, namely

$$\begin{aligned} Na' &= La, \\ \frac{1}{a'} &= \frac{2^{l_{\rm max}}}{a}, \end{aligned} \tag{50}$$

which implies

$$N = L2^{l_{\rm max}} \Rightarrow V = 2^d V'. \tag{51}$$

In dimensions $d = 1,2$, or 3 the number of modes used in both simulations are very similar. The same arguments that led to the scaling of the maximum field amplitude $\Phi_{\rm max}$ apply, namely we are truncating a field on V modes by cutoffs in the field amplitude at $\Phi_{\rm max}$. Hence the scalings of $\Phi_{\rm max}, \Pi_{\rm max}$ and the total number of qubits b is the same as in the discretized position basis where V' is replaced by V.

The number of quantum gates to perform a quantum simulation incurs only a penalty of replacing V with $V \log_2(V)$ in the scaling formulas relative to the discretized basis encoding. The reason is that in the wavelet basis the terms in the free field and interacting Hamiltonians couple across all scales as opposed to the discretized position basis where only nearest-neighbor modes are coupled. Because the wavelets have compact support, the number of summands in $\hat{H}$ scales like $O[V \log_2(V)]$. The first step of constructing the ground state of the free field Hamiltonian has time cost $O(V^{2.376})$, the same form as in the discretized bases, which is obtained from the worse case scaling assuming a dense correlation matrix. During particle creation and simulated evolution steps, the aforementioned additional terms in the Hamiltonian using the wavelet basis means scaling with respect to V in the discretized basis should be replaced by $V \log_2(V)$. Finally, measurement has the same scaling in either basis. A notable advantage of using the wavelet basis is that particle creation

and measurement can be done at a variety of different length or energy scales without further transformations on the system.

V. CONCLUSIONS

We have shown that scalar bosonic quantum field theories can be simulated efficiently on a quantum computer using a wavelet basis. Without compromising overall efficiency, our algorithm reorganizes the quantum state and its evolution into sectors at different length scales. We anticipate this could be useful to study renormalization flow and is a natural setting for characterizing fields in terms of finite-bandwidth detectors. Furthermore, the wavelet basis is well adapted to study entanglement in the free field theory at different length scales, including at the critical massless point [16]. We note that early work [17] on classical simulations of the $d = 2\,\phi^4$ theory using wavelets by found notable improvements in simulation times. Specifically, those authors showed that local Metropolis simulations of autocorrelation times showed an order of magnitude improvement in computation time cost using Haar wavelets vs discretized bases. The origin of this improvement was the ability to optimize the updates over each scale independently. It could well be that quantum simulations using wavelets could also see improvements using real-time propagation with scale-dependent parameters, e.g., using adaptive Trotter time steps that depend on the scale of the wavelet excitations.

For future work we note that an efficient quantum algorithm is known [18–20] for performing Daubechies$\mathcal{K}$-wavelet transforms on an m qubit register in $O(m^2)$ gates [21]. This algorithm translates between the discretized position space representation of a single particle and the wavelet representation. It would be of interest to adapt this to quantum simulations of multiparticle strongly correlated systems.

Resolving the description of a system according to length scale has also led to a successful numerical approach—the multi-scale renormalization ansatz (MERA) [22]—primarily for classical simulation of both discrete quantum many-body systems and also field theories [23] (in the latter case, the success of the ansatz has been demonstrated for free field theories). We remark that there could be interesting connections between the multiscale representation of quantum many-body states using the MERA and the wavelet basis described here. In the wavelet basis, the wavelet modes capture the short-range entangement at any given length scale, while in the MERA the same role is played by local disentangling and coarse-graining transformations.

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APPENDIX A: PREPARING THE GROUND STATE USING A DISCRETIZED POSITION BASIS

In this appendix we review how the state representing the quantum field is encoded in the discretized position basis. The Hamiltonian $\hat{H}^{(0)}$ can be obtained as the continuum limit of a discrete system with V' bosons with canonical position and momenta variables $\{\hat{q}_j\}$ and $\{\hat{p}_j\}$, respectively, which satisfy $[\hat{q}_j, \hat{p}_k] = i\delta_{j,k}$. Consider the following Hamiltonian for bosons on the sites of a cubic lattice of size $V' = L^d$ with uniform lattice spacing a':

$$\hat{H}^{(0)} = \sum_{m=1}^{V'} \frac{\hat{p}_m^2}{2\mu} + \frac{\lambda}{2}\sum_m \hat{q}_m^2 + \kappa \sum_{\langle m,n\rangle} (\hat{q}_m - \hat{q}_n)^2, \tag{A1}$$

where the sum over $\langle m,n\rangle$ is over nearest-neighbor pairs. The continuum limit is obtained by taking

$V' \to \infty$
Positions $\boldsymbol{x}_m = a'\boldsymbol{m}, \quad \boldsymbol{m} \in \mathbb{Z}^d$
$\hat{q}_m \to \hat{\Phi}(\boldsymbol{x}_m)$
$\sum_m \to \frac{1}{a'^d}\int d^d x$
$(\hat{q}_m - \hat{q}_n) \to a'[\nabla\hat{\Phi}(\boldsymbol{x})]_{\langle m,n\rangle}$
$\kappa = a'^{d-2}\bar{\kappa}, \quad \mu = a'^d\bar{\mu}, \quad \lambda = a'^d\bar{\lambda}$
Rescaling: $\hat{\Phi}(\boldsymbol{x}) \to \bar{\kappa}^{-1/2}\hat{\Phi}(\boldsymbol{x})$,

which leads to the Hamiltonian density

$$\hat{\mathcal{H}}^{(0)} = \frac{1}{2}\{\bar{\mu}\bar{\kappa}^{-1}\hat{\Pi}(\boldsymbol{x},t)^2 + [\nabla\hat{\Phi}(\boldsymbol{x},t)]^2 + \bar{\lambda}\bar{\kappa}^{-1}\hat{\Phi}(\boldsymbol{x},t)^2\}. \tag{A2}$$

Setting $\bar{\mu}\bar{\kappa}^{-1} \to 1$ and $\bar{\lambda}\bar{\kappa}^{-1} = \lambda\mu^{-1} = m_0^2$, we obtain the Hamiltonian density for the free field interaction in Eq. (11).

The descretized version of the Hamiltonian [Eq. (A1)] is compactly written:

$$\hat{H}^{(0)} = \tfrac{1}{2}\hat{\mathbf{s}}^T A\hat{\mathbf{s}}, \tag{A3}$$

where $\hat{\mathbf{s}}$ is the $2V'$ dimensional vector of position and momenta operators, $\hat{\mathbf{s}} = (\hat{q}_1, \ldots \hat{q}_{V'}, \hat{p}_1, \ldots \hat{p}_{V'})^T$, and

$$A = \begin{pmatrix} K & 0 \\ 0 & \mathbf{1}_{V'} \end{pmatrix}, \tag{A4}$$

where

$$K_{i,j} = \left(4d + m_0^2\right)\delta_{i,j} - 2\delta \;\; (i \in \text{neighborhood}\, j). \tag{A5}$$

The covariance matrix associated with a state ρ is defined $\Gamma_{j,k} = \mathrm{Re}[\mathrm{tr}[\rho(\hat{\mathbf{s}}_j - \langle\hat{\mathbf{s}}_j\rangle)(\hat{\mathbf{s}}_k - \langle\hat{\mathbf{s}}_k\rangle)]]$, where $\langle\hat{\mathbf{s}}_j\rangle$ is the expectation value of j-th element of $\hat{\mathbf{s}}$ and where K is defined in Eq. (A5). The ground state (vacuum) of this system then can be expressed as a Gaussian in the position basis:

$$|G\rangle = \mathcal{N}^{-1}\int_{-\infty}^{\infty} dq_1 \ldots \int_{-\infty}^{\infty} dq_{V'}\, e^{-\frac{1}{2}\mathbf{q}^T K^{1/2}\mathbf{q}}|q_1\rangle \ldots |q_{V'}\rangle, \tag{A6}$$

where $\mathcal{N}^{-1} = \det(K^{1/2})^{1/4}/\pi^{V'/4}$ is the normalization and $\boldsymbol{q} = (q_1, \ldots, q_{V'})^T$.

The values of q_j are discretized via a b bit string $x_j = x_{j,0}x_{j,1}\ldots x_{j,b-1}$ according to $q_j(x_j) = \delta_\Phi(-1)^{x_{j,0}}\sum_{r=1}^{b-1} 2^{x_{j,r}}$. The ground state can be represented as a state of $b \times V'$ qubits:

$$|G\rangle \approx \mathcal{N}^{-1} \sum_{\{x_{j,r}\in\{0,1\}\}_{j=1,r=0}^{V',b-1}} e^{-\frac{1}{2}\boldsymbol{q}(\{x_j\})^T K^{1/2}\boldsymbol{q}(\{x_j\})} |x_{0,0}\ldots x_{0,k-1}\rangle \ldots |x_{V'-1,0}\ldots x_{V'-1,k-1}\rangle. \tag{A7}$$

APPENDIX B: WAVELETS FOR HIGHER DIMENSIONS $d > 1$

The wavelet representation for scalar field theories can be straightforwardly generalized to higher dimensions as described in Ref. [10]. For completeness, we include the argument. In $d = 3$, for example, $\boldsymbol{n} = (n_x,n_y,n_z) \in \mathbb{Z}^3$ and the scale functions $s^{l_{\min}}_{\boldsymbol{n}}(\boldsymbol{x}) = s^{l_{\min}}_{n_1}(x_1)s^{l_{\min}}_{n_2}(x_2)s^{l_{\min}}_{n_3}(x_3)$, where $\boldsymbol{x}$ is now a position vector in $\mathbb{R}^3$ and $\boldsymbol{n} = (n_x,n_y,n_z) \in \mathbb{Z}^3$ becomes a displacement vector. The generalized wavelets $w^m_{\boldsymbol{n},\alpha}(\boldsymbol{x})$ are defined by seven different forms (distinguished by the collective index α):

$$\begin{aligned}
w^m_{\boldsymbol{n},(1,k_3)} &= s^{l_{\min}}_{n_1}(x_1)s^{l_{\min}}_{n_2}(x_2)w^{k_3}_{n_2}(x_3) \quad m = \max(k,k_3)\\
w^m_{\boldsymbol{n},(2,k_3)} &= s^{l_{\min}}_{n_1}(x_1)w^{k_2}_{n_2}(x_2)s^{l_{\min}}_{n_2}(x_3) \quad m = \max(k,k_2)\\
w^m_{\boldsymbol{n},(3,k_3)} &= w^{k_1}_{n_1}(x_1)s^{l_{\min}}_{n_2}(x_2)s^{l_{\min}}_{n_2}(x_3) \quad m = \max(k,k_1)\\
w^m_{\boldsymbol{n},(4,k_2,k_3)} &= s^{l_{\min}}_{n_1}(x_1)w^{k_2}_{n_2}(x_2)w^{k_3}_{n_2}(x_3) \quad m = \max(k,k_2,k_3)\\
w^m_{\boldsymbol{n},(5,k_1,k_2)} &= w^{k_1}_{n_1}(x_1)w^{k_2}_{n_2}(x_2)s^{l_{\min}}_{n_2}(x_3) \quad m = \max(k,k_1,k_2)\\
w^m_{\boldsymbol{n},(6,k_1k_3)} &= w^{k_1}_{n_1}(x_1)s^{l_{\min}}_{n_2}(x_2)w^{k_3}_{n_2}(x_3) \quad m = \max(k,k_1,k_3)\\
w^m_{\boldsymbol{n},(7,k_1,k_2,k_3)} &= w^{k_1}_{n_1}(x_1)w^{k_2}_{n_2}(x_2)w^{k_3}_{n_2}(x_3) \quad m = \max(k_1,k_2,k_3)
\end{aligned} \tag{B1}$$

The mode operators $\hat{\Phi}$ and $\hat{\Pi}$ are now indexed as:

$$\begin{aligned}
&\hat{\Phi}^{[\mathbf{s}]l_{\min}}(n,t) \to \hat{\Phi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{n},t), \quad \hat{\Phi}^{[\mathbf{w}]l}(n,t) \to \hat{\Phi}^{[\mathbf{w}]l}(\boldsymbol{n},\alpha,t)\\
&\hat{\Pi}^{[\mathbf{s}]l_{\min}}(n,t) \to \hat{\Pi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{n},t), \quad \hat{\Pi}^{[\mathbf{w}]l}(n,t) \to \hat{\Pi}^{[\mathbf{w}]l}(\boldsymbol{n},\alpha,t),
\end{aligned} \tag{B2}$$

where the discrete field operators satisfy the following equal time commutation relations (assuming here that $k \leqslant r,s$):

$$\begin{aligned}
&[\hat{\Phi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{n}),\hat{\Phi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{m})] = 0, \quad [\hat{\Pi}^{k}(\boldsymbol{n}),\hat{\Pi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{m})] = 0\\
&[\hat{\Phi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{n}),\hat{\Pi}^{[\mathbf{s}]l_{\min}}(\boldsymbol{m})] = i\delta_{\boldsymbol{n},\boldsymbol{m}},\\
&[\hat{\Phi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Phi}^{[\mathbf{w}]s}(\boldsymbol{m},\beta)] = 0\\
&\hat{\Pi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Pi}^{[\mathbf{w}]s}(\boldsymbol{m},\beta)] = 0\\
&[\hat{\Phi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Pi}^{[\mathbf{w}]s}(\boldsymbol{m},\beta)] = i\delta_{\alpha,\beta}\delta_{r,s}\delta_{\boldsymbol{n},\boldsymbol{m}}\\
&[\hat{\Phi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Phi}^{[\mathbf{w}]s}(\boldsymbol{m})] = 0, \quad [\hat{\Pi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Pi}^{[\mathbf{w}]s}(\boldsymbol{m})] = 0\\
&[\hat{\Phi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Pi}^{[\mathbf{w}]s}(\boldsymbol{m})] = 0, \quad [\hat{\Pi}^{[\mathbf{w}]r}(\boldsymbol{n},\alpha),\hat{\Phi}^{[\mathbf{w}]s}(\boldsymbol{m})] = 0.
\end{aligned} \tag{B3}$$

APPENDIX C: ADAPTATION OF THE SIMULATION TO BOSONIC ENCODING

Rather than discretizing the amplitude of the register modes using qubits we could instead opt to directly use V distinguishable bosonic modes with position basis states $\{|q\rangle_{\hat{q}_0},\ldots,|q_{V-1}\rangle_{\hat{q}_{V-1}}\}$. In this case the mode operators in Eq. (26) are just the position operators $\{\hat{q}_j\}$ acting on the modes according to $\hat{q}_j|q\rangle_{\hat{q}_j} = q|q\rangle_{\hat{q}_j}$.

The ground state is a multimode Gaussian state, which we rewrite for clarity:

$$|G\rangle = \mathcal{N}^{-1}\int dq_0 \ldots \int dq_{V-1}\, e^{-\frac{1}{2}\boldsymbol{q}^T K^{1/2}\boldsymbol{q}}|q_0\rangle_{\hat{q}_0}\ldots|q_{V-1}\rangle_{\hat{q}_{V-1}}, \tag{C1}$$

where $\boldsymbol{q} = (q_0,\ldots q_{V-1})^T$ and the coupling matrix K is given in Eq. (28). The ground state $|G\rangle$ is obtained by a unitary transformation on the V mode vacuum state, described by the following symplectic transformation on the initially decoupled position and momentum mode operators:

$$\hat{\boldsymbol{v}} \to Y\hat{\boldsymbol{v}}. \tag{C2}$$

The transformation acts to transform the vacuum correlation function as

$$\Gamma_{\text{vac}} = \tfrac{1}{2}\mathbf{1}_{2V} \to \Gamma = \tfrac{1}{2}YY^T = \tfrac{1}{2}(K^{-1/2}\oplus K^{1/2}), \tag{C3}$$

The symplectic transformation Y is composed of the $2V$ column vectors which are eigenvectors of the matrix $\Omega A\Omega^T A = K\oplus K$, where $A = K\oplus\mathbf{1}$. There is a canonical decomposition for Y written as one round of beam splitters and phase shifters, followed by parallel single mode squeezing, followed by a second round of beam splitters and phase shifters [24]. This decomposition is efficient, costing $O(V^2)$ elementary operations.

Particle excitations above the ground state can also be created using the bosonic encoding. Here the Hamiltonian used to create excitations is a simple quadratic interaction,

$$\hat{H}_\psi = \hat{f}^\dagger\hat{c} + \hat{f}\hat{c}^\dagger, \tag{C4}$$

where $\hat{c}^\dagger,\hat{c}$ are creation and annihilation operators that act on an ancillary bosonic mode. We prepare the ancillary mode in the Fock state $|n=1\rangle$ and evolve by $\hat{H}_\psi$, such that $e^{-i\hat{H}_\psi\pi/2}|G\rangle|n=1\rangle = -i\hat{f}^\dagger|G\rangle|n=0\rangle$ and we have the excited state up to a phase with no entanglement left between the ancilla and the register. Note the Fock state $|n=1\rangle$ is a non-Gaussian state; however, it can be prepared efficiently by a variety of techniques (see Ref. [25] and references therein).

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