Gamma-ray laser based on the collective decay of positronium atoms in a Bose-Einstein condensate

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We consider, in general, the collective two-photon annihilation decay of positronium atoms arising from the second quantized formalism. It is shown that two-photon annihilation of positronium atoms in a Bose-Einstein condensate (BEC) is unstable. Due to the BEC coherence, an absolute instability in such system takes place, i.e., the number of photons created as a result of positronium decay grows in every point within a BEC. The latter leads to an exponential buildup of a macroscopic population into the certain modes. Cooperative effects start for densities much smaller than the Dicke limit of spontaneous super radiation. For laserlike action, i.e., for directional radiation, we consider the BEC with elongated shape when the spontaneously emitted entangled and oppositely directed photon pairs are amplified, leading to an exponential buildup of a macroscopic population into the end-fire modes. We also consider the roles of confinement and interaction among positronium atoms in the amplification process.

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I. INTRODUCTION

Positronium (Ps), a bound state of an electron and a positron [1], being a unique physical system, has attracted enormous interest since the origination of quantum electrodynamics (QED). Being a pure leptonic atom, Ps is of interest for revealing QED effects with great precision [2]. Besides, Ps is a matter-antimatter compound and may play a central role for achieving a fundamental understanding of diverse phenomena in many branches of contemporary physics, ranging from the elementary particle physics [3] to astrophysics [4–6], as well as condensed-matter physics [7–11]. In the dilute limit, the Ps atoms behave as weakly interacting bosons and may form a Bose-Einstein condensate (BEC) [7]. The successful realization of BEC in the trapped alkali atoms [12,13], which are several thousand times heavier than a Ps, provides an enormous stimulus for the research in this direction [8–11] because the critical BEC temperature of a Ps gas is much higher than that of an alkali atom gas with the same number density, which is a crucial advantage from an experimental point of view. For instance, for a Ps density 10^{19} cm^{-3}, the critical temperature would be $T_c \simeq 70$ K. Nevertheless, to realize a BEC of Ps atoms, two problems should be overcome. First, one should find effective sources of slow positrons. For practical use in BEC studies, positrons need to be cooled down to a few electron volts by means of a moderator and then trapped. The traditional sources for positrons, such as the β^+ decay of radioactive isotopes [14] and the Surko trap method [15], seem to be a promising route. Second, one should form bound states of the electron positron and cool the system in a time scale much smaller than the lifetime of a Ps. The singlet (1^1S_0) state of a Ps, i.e., the so-called parapositronium (p-Ps), mainly decays into the two photons with the lifetime of $\tau_p \simeq 125$ ps, while the triplet (1^3S_1) state of a Ps, i.e., the so-called orthopositronium (o-Ps), mainly decays into the three photons with a relatively long lifetime of $\tau_o \simeq 142$ ns [16,17]. The recent advances in this direction show that the positrons implanted into a porous silica film may efficiently form Ps atoms [8–11] or molecules [18]. Regarding the fast cooling of Ps atoms, the current technique allows a laser cooling in the time scales smaller than the o-Ps lifetime [19]. In Ref. [20], the possibility to produce a BEC made of Ps atoms in a porous silica material containing nanometric cavities was investigated. The numerical simulations showed that the condensation process is compatible with the o-Ps lifetime. Besides, the lifetime of Ps strongly depends on the external conditions. In particular, one can achieve considerable enhancement of the lifetime of the Ps in the laser fields [21–24] or in the magnetic fields [25].

Another issue is the interaction of Ps atoms with the mutual spin-conversion reactions. In Ref. [26], the obtained s-wave scattering lengths were $a_s \simeq 8.44 a_B$ for the singlet-singlet and $a_t \simeq 3a_B$ for the triplet-triplet states (a_B is the Bohr radius). The positive scattering length means that a stable BEC of the o-Ps atoms is physically possible. The large value of the o-Ps,o-Ps to the p-Ps,p-Ps cross section, $\sim 7.4\pi a_B^2$, suggests for BEC the use of spin-polarized positrons, as proposed in Ref. [7]. The two-component BEC of the o-Ps and p-Ps atoms has been considered in Ref. [27] and it was shown that for an initially unpolarized condensate in an isotropic cavity, there is a threshold density of BEC at which the spin mixing between the o-Ps and p-Ps atoms occurs. For a stimulated o-Ps to p-Ps transition, they suggest a specific choice of an external field.

Regarding the other sources of positrons, it is worthwhile to note the experiments [28] and further theoretical investigations [29] showing that one can achieve the production of positrons with the density of 10^{20} cm^{-3} at the laser-solid interactions. In the astrophysical conditions, there are many sources of positrons and even Ps atoms. Electron-positron plasmas are an inherent feature of the winds from the pulsars and black holes [30,31]. Besides, Ps atoms are connected with the cosmic electron-positron annihilation radiation first detected from the Galactic Center direction during the 1970s. Since these times, the International Gamma-Ray Astrophysics Laboratory has greatly refined these measurements [4–6] and has shown that the line center is $\sim$511 keV with the annihilation rate $\sim 3 \times 10^{42}$ electron-positron pairs per second. The data analysis suggests that annihilations through Ps formation dominate (on average 90%), resulting in a narrow 511 keV line [5,6]. However, it follows that the origin of these positrons and the formation of Ps atoms in astrophysical conditions remain a mystery.

Because of the ongoing progress in physics for the production and manipulation of positronium atoms, one can expect the realization of BEC of Ps atoms in the near future. The latter is a very challenging project and could lead to various fundamental applications. In particular, it is of great interest to investigate the process of collective annihilation decay of Ps atoms from the BEC state, as a coherent ensemble of lasing atoms, towards the generation of intense coherent γ rays in the MeV domain of energies. The creation of a γ-ray laser has been the subject of extreme interest since the realization of the first lasers. The annihilation of electron-positron pairs has been considered as one of the basic processes for the intense γ-ray sources. The induced annihilation of a pair was already considered by Dirac [32]. For the observed γ-ray lines from the astrophysical objects, the radiation through the spontaneous [33] and stimulated annihilation [34,35] of an electron-positron plasma was considered. Then the ideas of Ps BEC and subsequent annihilation in the context of a γ-ray laser were considered in Refs. [19,36–38]. In these papers, the lasing gain coefficient has been obtained from the rate equations. The latter is applicable to lasing systems with drivers (initial seed) and resonators and cannot be extrapolated to the exponential gain regime [39]. Meanwhile, because there are no drivers or mirrors operable at γ-ray frequencies, one should realize single pass lasers operating in the so-called self-amplified spontaneous-emission regime. A mechanism of a γ-ray laser at the collective annihilation of Ps atoms in a BEC state in the self-amplified spontaneous-emission regime has been proposed in Ref. [40]. It has been shown that at the coupling of two macroscopic coherent ensembles of bosons—the BEC of Ps atoms and photons—there is an instability at which, starting from the vacuum state of the photonic field, the expectation value of the photon's mode occupation grows exponentially for a narrow interval of frequencies around the 511 keV line. In the present paper, a more detailed and thorough study of the γ-ray generation at the collective annihilation of Ps atoms in a BEC state in the self-amplified spontaneous-emission regime is presented. Here we utilize the more general Hamiltonian including the stimulated o-Ps to p-Ps transition. For the elongated shape of the BEC, it shows a laserlike action, i.e., directional radiation when the spontaneously emitted entangled and the oppositely directed photon pairs are amplified, leading to an exponential buildup of a macroscopic population into the end-fire modes. We also investigate the influence of an external potential and interaction between the Ps atoms on the γ-ray self-amplification process.

The paper is organized as follows. In Sec. II, the main Hamiltonian is constructed. In Sec. III, two-photon decay of a Ps atom is analyzed. In Sec. IV, we consider the intrinsic instability of recoilless collective two-photon decay and present the setup for a γ-ray laser. In Sec. V, we consider the influence of confinement and interaction between the positronium atoms on the considered process. Finally, conclusions are given in Sec. VI.

II. BASIC HAMILTONIAN

We begin our study with construction of the Hamiltonian which governs the quantum dynamics of the considered process. Here and below, except where it is stated otherwise, we employ natural units ($c = \hbar = 1$). Since o-Ps has a relatively long lifetime, in a laboratory-based experiment it will be more suitable to obtain a Bose-Einstein condensate for o-Ps. As was proposed in Ref. [7], the use of spin-polarized positrons will eventually lead to a gas of spin-polarized Ps, which does not undergo the mutual spin-conversion reaction. Thus, in the ensemble of Ps atoms, rapid annihilation of the singlet states and collisions among the various triplet substates will cause the Ps atoms to become completely polarized into a pure $m = 1$ triplet state. Then, to trigger two-photon annihilation, one should induce the triplet-to-singlet transition. The latter can be realized via the ground-state hyperfine transition either by the resonant sub-THz radiation (0.2 THz) or strong off-resonant electromagnetic field. Thus, in Fig. 1, the energy levels of interest are schematically shown. The upper two levels represent hyperfine splitting of the ground state of a Ps atom. The applied electromagnetic wave field drives the o-Ps $\Longrightarrow$ p-Ps transition. Then annihilation decay of the p-Ps into the two entangled photons of the same helicity are shown.

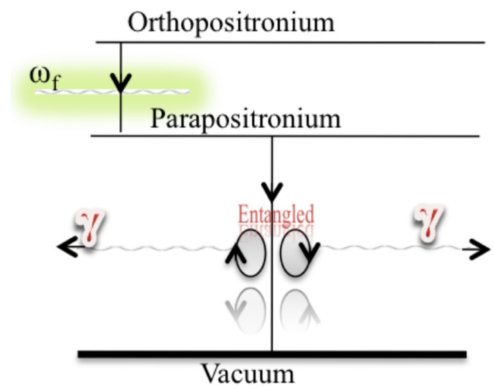

FIG. 1. (Color online) The energy levels of interest. The upper two levels represent hyperfine splitting of the ground state of Ps. The applied electromagnetic field drives the o-Ps $\Longrightarrow$ p-Ps transition. The annihilation decay of the p-Ps into two entangled photons of the same helicity is shown.

To obtain dynamic equations, we will utilize the second quantized formalism. For this purpose, let us introduce the creation and annihilation operators for p-Ps and o-Ps. The operator describing the creation of p-Ps in the internal ground state with the total center-of-mass momentum $\mathbf{p}$ can be written as

$$\widehat{\Pi}_{\mathbf{p}}^{+} = \frac{1}{\sqrt{2\mathcal{V}}} \int d\Phi_{\mathbf{p}'} \varphi\left(\mathbf{p}' - \frac{\mathbf{p}}{2}\right) [\widehat{a}_{\mathbf{p}',s_+}^{+} \widehat{b}_{\mathbf{p}-\mathbf{p}',s_-}^{+} - \widehat{a}_{\mathbf{p}',s_-}^{+} \widehat{b}_{\mathbf{p}-\mathbf{p}',s_+}^{+}], \quad (1)$$

where $\varphi(\mathbf{p})$ is the Fourier transform of the ground-state wave function,

$$\varphi(\mathbf{p}) = \frac{8\sqrt{\pi a_0^3}}{\left(1 + \mathbf{p}^2 a_0^2\right)^2}, \quad (2)$$

$a_0 = 2/(m\alpha_0)$ is the Bohr radius for Ps, m is the electron mass, and α_0 is the fine-structure constant. For the phase-space integration, we have introduced the notation $d\Phi_{\mathbf{q}} = \mathcal{V} d^3\mathbf{q}/(2\pi)^3$ ($\mathcal{V}$ is the quantization volume). In Eq. (1), $\widehat{a}_{\mathbf{p},s}^{+}$ and $\widehat{b}_{\mathbf{p},s}^{+}$ are the creation operators for electrons and positrons, respectively. The quantum number s describes the spin state of the particles. The operators $\widehat{a}^{+}_{\mathbf{p},s}$ and $\widehat{b}^{+}_{\mathbf{p},s}$ satisfy the fermionic anticommutation rules

$$\{\widehat{a}_{\mathbf{p},s},\widehat{a}^{+}_{\mathbf{p}',s'}\} = \frac{(2\pi)^3}{\mathcal{V}}\delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}. \tag{3}$$

The commutator for the p-Ps operator is

$$\widehat{\Pi}_{\mathbf{p}}\widehat{\Pi}^{+}_{\mathbf{p}'} - \widehat{\Pi}^{+}_{\mathbf{p}'}\widehat{\Pi}_{\mathbf{p}} \simeq \frac{(2\pi)^3}{\mathcal{V}}\delta(\mathbf{p}-\mathbf{p}') - O\left(a_0^3\frac{N_0}{\mathcal{V}}\right). \tag{4}$$

This is a bosonic commutation relation for a relatively small number of p-Ps atoms N, i.e., at $N/\mathcal{V} << a_0^{-3} \sim 10^{24}$ cm^{-3}. However, at high densities, one should take into account the deviations from the bosonic nature. The operator describing the creation of o-Ps in the pure $m = 1$ triplet state can be written as

$$\widehat{\Xi}^{+}_{\mathbf{p}} = \frac{1}{\sqrt{2\mathcal{V}}}\int d\Phi_{\mathbf{p}'}\varphi\left(\mathbf{p}'-\frac{\mathbf{p}}{2}\right)\widehat{a}^{+}_{\mathbf{p}',s_+}\widehat{b}^{+}_{\mathbf{p}-\mathbf{p}',s_+}. \tag{5}$$

The total Hamiltonian consists of four parts,

$$\widehat{H} = \widehat{H}_{\mathrm{Ps}} + \widehat{H}_{\mathrm{ph}} + \widehat{H}_{o\to p} + \widehat{H}_{2\gamma}. \tag{6}$$

Here the first part is the Hamiltonian of free Ps atoms of two species,

$$\widehat{H}_{\mathrm{Ps}} = \int d\Phi_{\mathbf{p}}\mathcal{E}_{\Pi}(\mathbf{p})\widehat{\Pi}^{+}_{\mathbf{p}}\widehat{\Pi}_{\mathbf{p}} + \int d\Phi_{\mathbf{p}}\mathcal{E}_{\Xi}(\mathbf{p})\widehat{\Xi}^{+}_{\mathbf{p}}\widehat{\Xi}_{\mathbf{p}}, \tag{7}$$

where

$$\begin{aligned}\mathcal{E}_{\Pi}(\mathbf{p}) &= \sqrt{(2m+\mathcal{E}_{S_0})^2+\mathbf{p}^2},\\ \mathcal{E}_{\Xi}(\mathbf{p}) &= \sqrt{(2m+\mathcal{E}_{S_1})^2+\mathbf{p}^2},\end{aligned} \tag{8}$$

are the total energies of the p-Ps and o-Ps with the momentum $\mathbf{p}$ of the center-of-mass motion, and $\mathcal{E}_{S_0}$, $\mathcal{E}_{S_1}$ are the binding energies, respectively. The origin of the energy difference between the ground states of the o-Ps and p-Ps (hyperfine splitting) is the spin-spin interaction [16]. In the lowest order of α_0, the latter is

$$\mathcal{E}_{S_1} - \mathcal{E}_{S_0} \equiv \varepsilon_{\mathrm{hfs}} = \tfrac{7}{12}m\alpha_0^4 \simeq 0.85\ \mathrm{meV}. \tag{9}$$

The second term in Eq. (6) is the Hamiltonian of the free photons,

$$\widehat{H}_{\mathrm{ph}} = \sum_{\zeta}\int d\Phi_{\mathbf{k}}\omega(\mathbf{k})\widehat{c}^{+}_{\mathbf{k},\zeta}\widehat{c}_{\mathbf{k},\zeta}, \tag{10}$$

where $\widehat{c}_{\mathbf{k},\zeta}$ ($\widehat{c}^{+}_{\mathbf{k},\zeta}$) is the annihilation (creation) operator of the photon with the momentum

$$\mathbf{k} = \omega(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta). \tag{11}$$

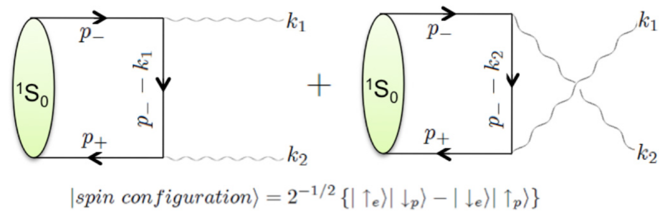

FIG. 2. (Color online) Feynman diagrams for the two-photon decay of a p-Ps.

As the two independent basis vectors of the polarization, we have chosen

$$\begin{aligned}\epsilon_{(\zeta)} = \frac{1}{\sqrt{2}}\{&\zeta\cos\vartheta\cos\varphi + i\sin\varphi, \zeta\cos\vartheta\sin\varphi\\ &-i\cos\varphi, -\zeta\sin\vartheta\},\end{aligned} \tag{12}$$

which corresponds to the certain helicity ($\zeta = \pm 1$) of photons,

$$\epsilon^{(\zeta)}\epsilon^{*(\zeta')} = \delta_{\zeta\zeta'}; \quad \mathbf{k}\epsilon^{(\zeta)} = 0.$$

The third part in Eq. (6) is the Hamiltonian that is responsible for the triplet-to-singlet transition,

$$\widehat{H}_{o\to p} = \int d\Phi_{\mathbf{p}}[\Lambda(t)\widehat{\Pi}^{+}_{\mathbf{p}}\widehat{\Xi}_{\mathbf{p}} + \Lambda^{*}(t)\widehat{\Xi}^{+}_{\mathbf{p}}\widehat{\Pi}_{\mathbf{p}}]. \tag{13}$$

Here it is assumed that the o-Ps $\Longrightarrow$ p-Ps transition is recoilless (the generalization of obtained results for the transition with momentum transfer is straightforward) and

$$\Lambda(t) = \Lambda_0 e^{i\omega_f t}; \quad \Lambda_0 = \tfrac{1}{2}\mu_B B_0, \tag{14}$$

where Λ_0 is the amplitude of the spin-magnetic field interaction, $\mu_B = e/2m = 5.8\times 10^{-5}$ eV $\times$ T^{-1} is the Bohr magneton, B_0 is the amplitude of the applied magnetic field, and ω_f is the frequency of the applied wave field. The last term in Eq. (6),

$$\begin{aligned}\widehat{H}_{2\gamma} = \sum_{\zeta,\zeta'}\int d\Phi_{\mathbf{k}}\int d\Phi_{\mathbf{p}}[&\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p})\widehat{c}^{+}_{\mathbf{k},\zeta}\widehat{c}^{+}_{\mathbf{p}-\mathbf{k},\zeta'}\widehat{\Pi}_{\mathbf{p}}\\ &+\mathcal{M}^{*}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p})\widehat{\Pi}^{+}_{\mathbf{p}}\widehat{c}_{\mathbf{p}-\mathbf{k},\zeta'}\widehat{c}_{\mathbf{k},\zeta}],\end{aligned} \tag{15}$$

is the Hamiltonian of the two-photon decay of a p-Ps. The amplitude $\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p})$ for the annihilation of a p-Ps into the two photons is given by the Feynman diagrams in Fig. 2. The latter can be derived from the amplitude for annihilation of a free electron-positron pair with the momenta $\mathbf{p}_-$ and $\mathbf{p}-\mathbf{p}_-$ into the two photons with the polarizations $\epsilon_{(\zeta)}$, $\epsilon_{(\zeta')}$ and momenta $\mathbf{k}$, $\mathbf{k}' = \mathbf{p}-\mathbf{k}$ [17]. Taking into account definition (1), we obtain

$$\begin{aligned}\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p}) = \frac{\pi\alpha_0}{\mathcal{V}^{3/2}}\int &\frac{d\Phi_{\mathbf{p}_-}\varphi\left(\mathbf{p}_- - \frac{\mathbf{p}}{2}\right)}{\sqrt{2\omega\omega'\varepsilon(\mathbf{p}-\mathbf{p}_-)\varepsilon(\mathbf{p}_-)}}\\ &\times\left\{\bar{v}^{(s_+)}(\mathbf{p}-\mathbf{p}_-)\left[\not{\epsilon}^{*}_{(\zeta')}\frac{1}{\not{p}_- - \not{k} - m}\not{\epsilon}^{*}_{(\zeta)} + \not{\epsilon}^{*}_{(\zeta)}\frac{1}{\not{p}_- - \not{k}' - m}\not{\epsilon}^{*}_{(\zeta')}\right]u^{(s_-)}(\mathbf{p}_-) - (s_+ \Leftrightarrow s_-)\right\},\end{aligned} \tag{16}$$

where $\not{a} \equiv a_\mu\gamma^\mu$, $\gamma^\mu \equiv \{\gamma^0,\gamma^1,\gamma^2,\gamma^3\}$ are the Dirac matrices, $\varepsilon(\mathbf{p})$ is given by the free-electron dispersion relation, and $u^{(\alpha)}(\mathbf{p})$ and $v^{(\alpha)}(\mathbf{p})$ are the bispinor amplitudes of a free Dirac particle corresponding to the electron and positron, respectively.

We consider a dilute system of Ps atoms when $na_t^3 \ll 1$, and interaction between the Ps atoms is neglected. For the considered process of γ-ray annihilation decay, this is justified for a uniform system of Ps atoms and for the condensate confined by a box with sufficiently (infinitely) hard walls (see Sec. VI).

III. SPONTANEOUS TWO-PHOTON DECAY OF A PARAPOSITRONIUM

Before considering collective annihilation decay of the p-Ps, it will be useful to consider the spontaneous decay of single p-Ps from the quantum dynamic point of view. For this purpose, we need Hamiltonian (6), without the $\widehat{H}_{o\to p}$ and o-Ps parts in Eq. (7):

$$\widehat{H} = \int d\Phi_{\mathbf{p}}\mathcal{E}_\Pi(\mathbf{p})\widehat{\Pi}_{\mathbf{p}}^{+}\widehat{\Pi}_{\mathbf{p}} + \widehat{H}_{\mathrm{ph}} + \widehat{H}_{2\gamma}. \tag{17}$$

For the spontaneous decay, we consider the initial condition in which the photonic field begins in the vacuum state, while the p-Ps field is prepared in a Fock state with one p-Ps in the rest ($\mathbf{p}=\mathbf{0}$). From Eq. (1), it follows that such state can be represented as $|\Psi(0)\rangle = |0_{\mathrm{ph}}\rangle \otimes \widehat{\Pi}_{\mathbf{0}}^{+}|0_{\mathrm{Ps}}\rangle$. Then the state vector for times $t>0$ is just given by the expansion

$$|\Psi\rangle = C_0(t)e^{-i\mathcal{E}_\Pi(\mathbf{0})t}|0_{\mathrm{ph}}\rangle \otimes \widehat{\Pi}_{\mathbf{0}}^{+}|0_{\mathrm{Ps}}\rangle + \sum_{\alpha,\alpha'}\int d\Phi_{\mathbf{k}}d\Phi_{\mathbf{k}'} \times C_{\mathbf{k},\alpha;\mathbf{k}',\alpha'}(t)e^{-i(\omega+\omega')t}\widehat{c}_{\mathbf{k},\alpha}^{+}\widehat{c}_{\mathbf{k}',\alpha'}^{+}|0_{\mathrm{ph}}\rangle \otimes |0_{\mathrm{Ps}}\rangle, \tag{18}$$

where $C_{\mathbf{k}_1,\alpha_1;\mathbf{k}_2,\alpha_2}(t)$ is the probability amplitude for the photonic field to be in the two-photon state, while the p-Ps field is in the vacuum state. From the Schrödinger equation, one can obtain evolution equations,

$$i\frac{\partial C_{\mathbf{k},\alpha;\mathbf{k}',\alpha'}}{\partial t} = \frac{\mathcal{M}_{\alpha,\alpha'}(\mathbf{k},\mathbf{0}) + \mathcal{M}_{\alpha',\alpha}(\mathbf{k}',\mathbf{0})}{2} \times C_0 e^{i(2\omega-\mathcal{E}_\Pi(\mathbf{0}))t}\frac{(2\pi)^3}{\mathcal{V}}\delta(\mathbf{k}+\mathbf{k}'), \tag{19}$$

$$i\frac{\partial C_0}{\partial t} = 2\sum_{\zeta,\zeta'}\int d\Phi_{\mathbf{k}}\mathcal{M}^{*}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0})e^{i[\mathcal{E}_\Pi(\mathbf{0})-2\omega]t}C_{\mathbf{k},\zeta;-\mathbf{k},\zeta'}. \tag{20}$$

Here we have taken into account the bosonic nature of photons: $C_{\mathbf{k}_1,\alpha_1;\mathbf{k}_2,\alpha_2} = C_{\mathbf{k}_2,\alpha_2;\mathbf{k}_1,\alpha_1}$.

The calculation of the amplitude $\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0})$ is substantially simplified if one takes into account the nonrelativistic nature of the Ps internal degrees of freedom. As follows from Eq. (2), the wave function $\varphi(\mathbf{p})$ takes sizable values for momenta $p \lesssim 1/a_0 \sim m\alpha_0 \ll m$. Meanwhile, the momentum scale for positronium annihilation is of the order of m. This corresponds to the well-known fact that positronium decay is only sensitive to the value of the wave function at zero separation of the electron and positron,

$$\phi(0) = \frac{1}{\mathcal{V}}\int d\Phi_{\mathbf{p}_-}\varphi(\mathbf{p}_-) = \sqrt{\frac{m^3\alpha_0^3}{8\pi}}. \tag{21}$$

Hence, one can make an approximation for the amplitude $\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0})$ as follows:

$$\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0}) = \frac{\pi\alpha_0}{\sqrt{2\mathcal{V}}m^2}\left[\frac{1}{\mathcal{V}}\int d\Phi_{\mathbf{p}_-}\varphi(\mathbf{p}_-)\right]\left\{\bar{v}^{(s_+)}(\mathbf{0})\left[\not{\epsilon}^{*}_{(\zeta')}\frac{1}{\not{p}_- - \not{k} - m}\not{\epsilon}^{*}_{(\zeta)} + \not{\epsilon}^{*}_{(\zeta)}\frac{1}{\not{p}_- - \not{k}' - m}\not{\epsilon}^{*}_{(\zeta')}\right]u^{(s_-)}(\mathbf{0}) - (s_+ \Leftrightarrow s_-)\right\}. \tag{22}$$

In Eq. (22), $p_- = \{m,0,0,0\}$, $k = \{m, m\widehat{\mathbf{k}}\}$, and $k' = \{m, -m\widehat{\mathbf{k}}\}$. After long but straightforward calculations, we arrive at

$$\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0}) = i\frac{4\pi\alpha_0\phi(0)}{m^2\sqrt{2\mathcal{V}}}\widehat{\mathbf{k}}\cdot[\epsilon^{*}_{(\zeta)} \times \epsilon^{*}_{(\zeta')}]. \tag{23}$$

Then, taking into account Eqs. (11), (12), and (21), we have

$$\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{0}) = -\sqrt{\frac{\pi\alpha_0^5}{m\mathcal{V}}}\zeta\delta_{\zeta,\zeta'}. \tag{24}$$

Then, according to perturbation theory, we take $C_0(t) \simeq 1$, and for the amplitude $C_{\mathbf{k},\alpha;\mathbf{k}',\alpha'}(t\to\infty)$ from Eq. (19), we obtain

$$C_{\mathbf{k},\alpha;\mathbf{k}',\alpha'} = i\sqrt{\frac{\pi\alpha_0^5}{m\mathcal{V}}}\frac{(2\pi)^4}{\mathcal{V}}\alpha\delta_{\alpha,\alpha'} \times \delta(\mathbf{k}+\mathbf{k}')\delta[2\omega(\mathbf{k}) - \mathcal{E}_\Pi(\mathbf{0})]. \tag{25}$$

Then, returning to expansion (18), one can write

$$|\Psi\rangle \simeq C_0 e^{-i\mathcal{E}_\Pi(\mathbf{0})t}|0_{\mathrm{ph}}\rangle \otimes \widehat{\Pi}_{\mathbf{0}}^{+}|0_{\mathrm{Ps}}\rangle + i\frac{\sqrt{\mathcal{V}m^3\alpha_0^5}}{8\pi^{3/2}}|0_{\mathrm{Ps}}\rangle \otimes \int d\widehat{\mathbf{k}}e^{-2imt}[\widehat{c}_{\mathbf{k},+}^{+}\widehat{c}_{-\mathbf{k},+}^{+}|0_{\mathrm{ph}}\rangle - \widehat{c}_{\mathbf{k},-}^{+}\widehat{c}_{-\mathbf{k},-}^{+}|0_{\mathrm{ph}}\rangle]. \tag{26}$$

As is seen from Eq. (24), the two-photon annihilation amplitude does not depend on $\mathbf{k}$; as a result, the two-photon state (26) resulting from the p-Ps decay is a maximally entangled (over the helicity) state of the two oppositely propagating photons.

For the decay rate of the process, $p\text{-Ps} \to 2\gamma$, one can write

$$\Gamma = \frac{1}{2}\sum_{\alpha_1,\alpha_2}\int d\Phi_{\mathbf{k}_1}d\Phi_{\mathbf{k}_2}\frac{|C_{\mathbf{k}_1,\alpha_1;\mathbf{k}_2,\alpha_2}|^2}{T},$$

where T is the interaction time and the symmetry factor $1/2!$ takes into account that in the final state there are two identical photons. With the help of Eq. (25), we obtain the well-known result [16]

$$\Gamma = \frac{m\alpha_0^5}{2}. \tag{27}$$

IV. COLLECTIVE TWO-PHOTON DECAY

For analysis of the collective two-photon decay, we will use the Heisenberg representation, where the evolution operators are given by the following equation:

$$i\frac{\partial \widehat{L}}{\partial t} = [\widehat{L},\widehat{H}], \tag{28}$$

and the expectation values are determined by the initial wave function Ψ_0,

$$\langle \widehat{L}\rangle = \langle \Psi_0|\widehat{L}|\Psi_0\rangle.$$

We will assume that the photonic field starts up in the vacuum state, while the Ps field is in the Bose-Einstein condensate state. Taking into account Hamiltonian (6) from Eq. (28), we obtain a set of equations,

$$i\frac{\partial \widehat{c}_{\mathbf{k},\zeta}}{\partial t} = \omega(\mathbf{k})\widehat{c}_{\mathbf{k},\zeta} + \sum_{\zeta_1}\int d\Phi_{\mathbf{p}}\{\mathcal{M}_{\zeta,\zeta_1}(\mathbf{k},\mathbf{p}) + \mathcal{M}_{\zeta_1,\zeta}(\mathbf{p}-\mathbf{k},\mathbf{p})\}\widehat{c}^{+}_{\mathbf{p}-\mathbf{k},\zeta_1}\widehat{\Pi}y_{\mathbf{p}}, \tag{29}$$

$$i\frac{\partial \widehat{\Pi}_{\mathbf{p}}}{\partial t} = \mathcal{E}_{\Pi}(\mathbf{p})\widehat{\Pi}_{\mathbf{p}} + \Lambda(t)\widehat{\Xi}_{\mathbf{p}} + \sum_{\zeta_1,\zeta_2}\int d\Phi_{\mathbf{k}}\mathcal{M}^{*}_{\zeta_1,\zeta_2}(\mathbf{k},\mathbf{p})\widehat{c}_{\mathbf{p}-\mathbf{k},\zeta_2}\widehat{c}_{\mathbf{k},\zeta_1}, \tag{30}$$

$$i\frac{\partial \widehat{\Xi}_{\mathbf{p}}}{\partial t} = \mathcal{E}_{\Xi}(\mathbf{p})\widehat{\Xi}_{\mathbf{p}} + \Lambda^{*}(t)\widehat{\Pi}_{\mathbf{p}}. \tag{31}$$

These equations are a nonlinear set of equations with the photonic and Ps fields' operators defined self-consistently. As we are interested in the quantum dynamics of the considered system in the presence of instabilities, we can decouple the photonic and Ps fields treating the dynamics of a photonic field. Passing to the interaction picture,

$$\widehat{\Xi}_{\mathbf{p}} = \widehat{\Theta}_{\mathbf{p}}e^{-i\mathcal{E}_{\Xi}(\mathbf{p})t}, \quad \widehat{\Pi}_{\mathbf{p}} = \widehat{F}_{\mathbf{p}}e^{-i[\mathcal{E}_{\Xi}(\mathbf{p})-\omega_f]t}, \quad \widehat{c}_{\mathbf{k},\zeta} = \widehat{a}_{\mathbf{k},\zeta}e^{-i\omega(\mathbf{k})t}, \tag{32}$$

for the new operators $\widehat{a}_{\mathbf{k},\zeta}$, $\widehat{F}_{\mathbf{p}}$, and $\widehat{\Theta}_{\mathbf{p}}$, we obtain

$$i\frac{\partial \widehat{a}_{\mathbf{k},\zeta}}{\partial t} = \sum_{\zeta_1}\int d\Phi_{\mathbf{p}}\{\mathcal{M}_{\zeta,\zeta_1}(\mathbf{k},\mathbf{p}) + \mathcal{M}_{\zeta_1,\zeta}(\mathbf{p}-\mathbf{k},\mathbf{p})\} \times \widehat{a}^{+}_{\mathbf{p}-\mathbf{k},\zeta_1}\widehat{F}_{\mathbf{p}}e^{i[\omega(\mathbf{k})+\omega(\mathbf{p}-\mathbf{k})-\mathcal{E}_{\Xi}(\mathbf{p})+\omega_f]t}, \tag{33}$$

$$i\frac{\partial \widehat{F}_{\mathbf{p}}}{\partial t} + \Delta_{\mathbf{p}}\widehat{F}_{\mathbf{p}} = \Lambda_0\widehat{\Theta}_{\mathbf{p}} + \sum_{\zeta_1,\zeta_2}\int d\Phi_{\mathbf{k}}\mathcal{M}^{*}_{\zeta_1,\zeta_2}(\mathbf{k},\mathbf{p}) \times \widehat{a}_{\mathbf{p}-\mathbf{k},\zeta_2}\widehat{a}_{\mathbf{k},\zeta_1}e^{-i[\omega(\mathbf{k})+\omega(\mathbf{p}-\mathbf{k})-\mathcal{E}_{\Xi}(\mathbf{p})+\omega_f]t}, \tag{34}$$

$$i\frac{\partial \widehat{\Theta}_{\mathbf{p}}}{\partial t} = \Lambda_0\widehat{F}_{\mathbf{p}}, \tag{35}$$

where

$$\Delta_{\mathbf{p}} = \mathcal{E}_{\Xi}(\mathbf{p}) - \omega_f - \mathcal{E}_{\Pi}(\mathbf{p})$$

is the resonance detuning for the triplet-to-singlet transition. We assume that the Ps atoms are initially in the triplet state ($m = 1$). For driving the triplet-to-singlet transition, we will consider both resonant and nonresonant interactions. At the resonant case $|\Delta_{\mathbf{p}}|^2 \ll \Lambda_0^2$ and in the ultrafast excitation regime (smaller than the lifetime of the o-Ps) when relaxation processes are not relevant, the Rabi oscillation provides a direct control of the states' populations. Thus, with the π pulse $\int \Lambda_0 dt = \pi$, the population can be transferred from the o-Ps to p-Ps state and, instead of Eqs. (34) and (35), one can consider equation

$$i\frac{\partial \widehat{F}_{\mathbf{p}}}{\partial t} = \sum_{\zeta_1,\zeta_2}\int d\Phi_{\mathbf{k}}\mathcal{M}^{*}_{\zeta_1,\zeta_2}(\mathbf{k},\mathbf{p}) \times \widehat{a}_{\mathbf{p}-\mathbf{k},\zeta_2}\widehat{a}_{\mathbf{k},\zeta_1}e^{i[\mathcal{E}_{\Pi}(\mathbf{p})-\omega(\mathbf{k})-\omega(\mathbf{p}-\mathbf{k})]t}. \tag{36}$$

At the nonresonant case $|\Delta_{\mathbf{p}}|^2 \gg \Lambda_0^2$, the pump electromagnetic field is sufficiently far detuned from the resonance for the p-Ps state population to remain small at all times. The intermediate level can then be eliminated in the standard way,

$$\widehat{F}_{\mathbf{p}} \simeq \frac{\Lambda_0}{\Delta_{\mathbf{p}}}\widehat{\Theta}_{\mathbf{p}},$$

and, from Eqs. (33)–(35), we get

$$i\frac{\partial \widehat{a}_{\mathbf{k},\zeta}}{\partial t} = \sum_{\zeta'}\int d\Phi_{\mathbf{p}}[\mathcal{M}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p}) + \mathcal{M}_{\zeta',\zeta}(\mathbf{p}-\mathbf{k},\mathbf{p})] \times \widehat{a}^{+}_{\mathbf{p}-\mathbf{k},\zeta'}\widehat{\Theta}_{\mathbf{p}}\frac{\Lambda_0}{\Delta_{\mathbf{p}}}e^{i[\omega(\mathbf{k})+\omega(\mathbf{p}-\mathbf{k})-\mathcal{E}_{\Xi}(\mathbf{p})+\omega_f]t}, \tag{37}$$

$$i\frac{\partial \widehat{\Theta}_{\mathbf{p}}}{\partial t} = \frac{\Lambda_0}{\Delta_{\mathbf{p}}}\sum_{\zeta,\zeta'}\int d\Phi_{\mathbf{k}}\mathcal{M}^{*}_{\zeta,\zeta'}(\mathbf{k},\mathbf{p}) \times \widehat{a}_{\mathbf{p}-\mathbf{k},\zeta'}\widehat{a}_{\mathbf{k},\zeta}e^{i[\mathcal{E}_{\Xi}(\mathbf{p})-\omega_f-\omega(\mathbf{k})-\omega(\mathbf{p}-\mathbf{k})]t}. \tag{38}$$

To decouple the photonic and Ps fields, we just use the Bogoliubov approximation. If a lowest-energy single-particle state has a macroscopic occupation, we can separate the field operators ($\widehat{F}_{\mathbf{p}},\widehat{\Theta}_{\mathbf{p}}$) into the condensate term and the noncondensate components, i.e., the operator $\widehat{F}_{\mathbf{p}}$ in Eq. (33) or $\widehat{\Theta}_{\mathbf{p}}$ in Eq. (37) is replaced by the c number as follows:

$$\widehat{F}_{\mathbf{p}} = \sqrt{n_0}\frac{(2\pi)^3}{\mathcal{V}^{1/2}}\delta(\mathbf{p}), \tag{39}$$

where n_0 is the number density of atoms in the condensate. Making the Bogoliubov approximation, we arrive at a finite set of the Heisenberg equations,

$$i\frac{\partial \widehat{a}_{\mathbf{k},\zeta}}{\partial t} = \chi_\zeta(\mathbf{k})\widehat{a}^{+}_{-\mathbf{k},\zeta}e^{i\delta(\mathbf{k})t}, \tag{40}$$

$$i\frac{\partial \widehat{a}^{+}_{-\mathbf{k},\zeta}}{\partial t} = -\chi_\zeta(\mathbf{k})\widehat{a}_{\mathbf{k},\zeta}e^{-i\delta(\mathbf{k})t}, \tag{41}$$

which couples the modes $\widehat{a}_{\mathbf{k},\zeta}$ to the modes $\widehat{a}_{-\mathbf{k},\zeta}$ with the coupling constant

$$\chi_\zeta(\mathbf{k}) = 2\sqrt{n_{\mathrm{eff}}}\mathcal{V}^{1/2}\mathcal{M}_{\zeta,\zeta}(\mathbf{k},\mathbf{0}). \tag{42}$$

Here,

$$\delta(\mathbf{k}) = 2\omega - \mathcal{E}_\Pi(\mathbf{0}) \simeq 2(\omega - m_*) \tag{43}$$

is the resonance detuning for the two-photon annihilation, m_* is the half of the Ps mass, which is the electron (positron) mass diminished by the Coulomb attraction: $m_* = m + \mathcal{E}_{S_0}/2$ ($\mathcal{E}_{S_0} = -6.8$ eV). For the joint consideration of resonant and nonresonant cases, we have introduced the effective BEC density $n_{\text{eff}} = \varrho n_0$, where the factor $\varrho = 1$ for the resonant triggering and $\varrho = \Lambda_0^2/\Delta_{\mathbf{p}}^2$ for the off-resonant one.

Equations (40) and (41) are a set of linearly coupled operator equations that can be solved by the method of characteristics whose eigenfrequencies define the temporal dynamics of the photonic field. The existence of an eigenfrequency with an imaginary part would indicate the onset of instability at which the initial spontaneously emitted entangled photon pairs are amplified, leading to an exponential buildup of a macroscopic mode population. Solving Eqs. (40) and (41), we obtain

$$\widehat{a}_{\mathbf{k},\zeta}(t) = e^{i\frac{\delta(\mathbf{k})}{2}t}\left[\widehat{a}_{\mathbf{k},\zeta}(0)\cos\lambda t + \frac{1}{i\lambda}\left\{\chi_\zeta(\mathbf{k})\widehat{a}^{+}_{-\mathbf{k},\zeta}(0) + \frac{\delta(\mathbf{k})}{2}\widehat{a}_{\mathbf{k},\zeta}(0)\right\}\sin\lambda t\right], \tag{44}$$

where

$$\lambda = \sqrt{\frac{\delta^2(\mathbf{k})}{4} - \chi_\zeta^2(\mathbf{k})}. \tag{45}$$

The condition for the exponential gain is therefore

$$|\chi_\zeta(\mathbf{k})| > |\omega - m_*|,$$

leading to the exponential growth of the modes in the narrow interval of frequencies

$$m_* - |\chi_\zeta(\mathbf{k})| < \omega < m_* + |\chi_\zeta(\mathbf{k})|. \tag{46}$$

For the interval (46), we find that the expectation value of the mode occupation grows exponentially,

$$N_{\mathbf{k},\zeta}(t) = \langle 0_{\text{ph}}|\widehat{a}^{+}_{\mathbf{k},\zeta}(t)\widehat{a}_{\mathbf{k},\zeta}(t)|0_{\text{ph}}\rangle = \frac{\chi_\zeta^2(\mathbf{k})}{4\chi_\zeta^2(\mathbf{k}) - \delta^2(\mathbf{k})} \times \left(e^{2\sqrt{\chi_\zeta^2(\mathbf{k}) - \frac{\delta^2(\mathbf{k})}{4}}\,t} + e^{-2\sqrt{\chi_\zeta^2(\mathbf{k}) - \frac{\delta^2(\mathbf{k})}{4}}\,t} - 2\right). \tag{47}$$

For the central frequency [$\delta(\mathbf{k}) = 0$], the exponential growth rate is

$$G = 2|\chi_\zeta(\mathbf{k})|. \tag{48}$$

Taking into account Eq. (42) and derived expression (24) for the decay amplitude, we obtain the compact expression for the exponential growth rate,

$$G = \sqrt{\frac{16\pi n_{\text{eff}}\alpha_0^5}{m}}. \tag{49}$$

We have solved the issue considering uniform BEC without boundary conditions and, as a consequence, according to Eqs. (49) and (47), we have isotropic exponential gain. Due to the BEC coherence, here we have an absolute instability, i.e., the number of photons grows in every point within a BEC. In the meantime, in earlier works [19,36], it was considered γ-ray amplification due to the propagation through BEC. As seen from Eq. (49), the gain is scaled as $\sqrt{n_{\text{eff}}}$, which means that one might observe the startup of an annihilation γ-ray laser at lower densities than would be the case for a gain proportional to the density. Indeed, the Dirac rate from the earlier works [19,36–38] can be written as

$$G_0 = \frac{2\pi}{m^2}n_{\text{eff}}. \tag{50}$$

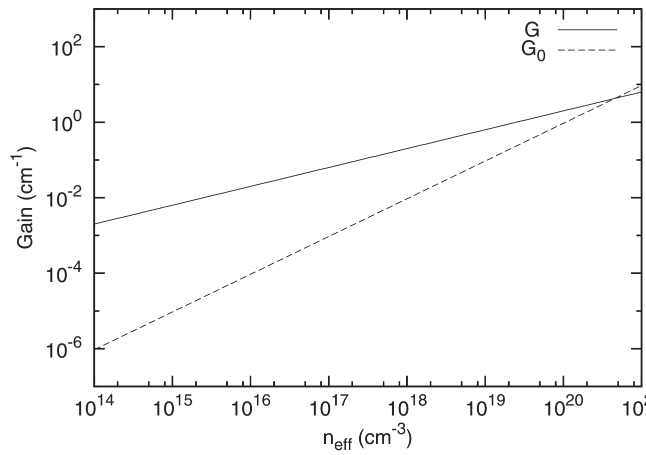

FIG. 3. Comparison of gains. In the logarithmic scale, we plot G and G_0 vs the effective density of p-Ps atoms in BEC.

For the visualization in Fig. 3, we plot G and G_0 as a function of n_{eff}. As is seen from this figure, the gain G is larger than Dirac rate G_0 up to densities 4.53×10^{20} cm^{-3}. Besides, as is seen from (47), the generation process starts without the initial seed.

For laserlike action, i.e., for directional radiation, we should take an elongated shape of the BEC. In this case, boundary conditions can be incorporated into the derived equations (40) and (41) by introducing mode damping. The latter is simply due to the propagation of the photonic field, which escapes from the active medium and is inversely proportional to the transit time of a photon in the active medium. This transit time strictly depends on the direction. The latter is equivalent to the finite interaction time strictly depending on the shape of the BEC. For concreteness, we consider a cigar-shaped BEC of width L_w and length L ($L \gg L_w$). A feasible experimental setup for the γ-ray laser is shown in Fig. 4. It is assumed that initially we have a BEC of spin-polarized o-Ps atoms. Then the applied electromagnetic field triggers collective annihilation of the BEC. Due to the intrinsic instability of recoilless two-photon decay and shape of the condensate, the initial spontaneously emitted entangled photon pairs are amplified, leading to an exponential buildup of a macroscopic population into the end-fire modes. In this case, due to an azimuthal symmetry for the effective interaction time, one can write

$$t_{\text{int}}(\vartheta,\chi,L) = \frac{L}{\sqrt{\cos^2\vartheta + \chi^2\sin^2\vartheta}}, \tag{51}$$

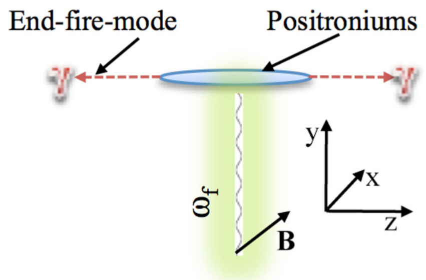

FIG. 4. (Color online) Feasible experimental setup for a γ-ray laser with a cigar-shaped BEC of initially spin-polarized o-Ps atoms. The applied electromagnetic field initiates the transition from the o-Ps BEC to the p-Ps one, triggering collective annihilation of the condensate.

where $\chi = L/L_w \gg 1$. In this case, for the photon number density in the frequency interval (46),

$$n_\gamma \simeq \sum_\zeta \int \frac{d^3\mathbf{k}}{(2\pi)^3} N_{\mathbf{k},\zeta}[t_{\mathrm{int}}(\vartheta,\chi,L)], \tag{52}$$

we have

$$n_\gamma \simeq \frac{G}{2\pi^2\lambda_c^2} \int_0^1 dx \int_0^\pi d\vartheta \frac{\sin\vartheta}{1-x^2} \times \sinh^2\left(\frac{\sqrt{1-x^2}\Lambda}{2\sqrt{\cos^2\vartheta + \chi^2\sin^2\vartheta}}\right), \tag{53}$$

where $\lambda_c = \hbar/mc$ is the electron Compton wavelength and the dimensionless interaction parameter $\Lambda = GL$ defines the amplification regime. In Fig. 5, we show the density of generated γ-ray photons n_γ versus the effective density of p-Ps atoms in a BEC for the given length $L = 1.5$ cm and various widths. The ratio $\chi = L/L_w$ defines the angular width of the end-fire modes. For the densities, when $\Lambda > 1$, we have high gain regime and the radiation is concentrated in the end-fire modes. In Fig. 6, the angular distribution of the density of generated γ-ray photons $dn_\gamma/d\vartheta$ for the given interaction length $L = 1.5$ cm and density 10^{21} cm^{-3} is shown. As is seen, due to the intrinsic instability of the two-photon collective decay of BEC and its shape, the initial spontaneously emitted entangled photon pairs are amplified, leading to an exponential buildup of a macroscopic population into the end-fire modes. Since we have not considered BEC depletion, the obtained solution (53) is applicable for the time scales when the number of photons N_γ is much smaller than the total number of Ps atoms (N): $N_\gamma \ll N$.

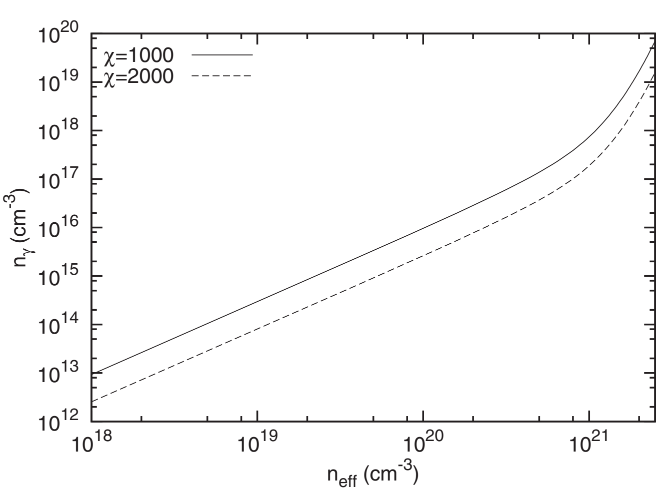

FIG. 5. The density of generated γ-ray photons vs effective density of p-Ps atoms in BEC for the given interaction length $L = 1.5$ cm and various widths, $L_w = L/\chi$, in the logarithmic scale.

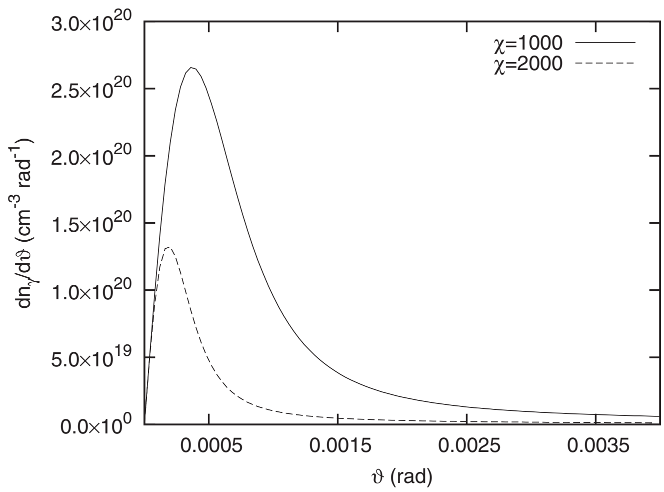

FIG. 6. Angular distribution of the density of generated γ-ray photons for the given interaction length $L = 1.5$ cm, density 10^{21} cm^{-3}, and various widths, $L_w = L/\chi$. There is a similar peak close to $\vartheta \simeq \pi$.

Let us consider the parameters required for an efficient γ-ray laser. The BEC occurs below a critical temperature, which for a uniform gas of Ps atoms with the density n_0 is given by the formula

$$T_c \simeq 1.66 \frac{\hbar^2}{mk_B} n_0^{2/3}, \tag{54}$$

where k_B is the Boltzmann constant. The maximal amplification length is taken to be $L_m \simeq c\tau_p \simeq 3.75$ cm. For an exponential amplification, we need $GL_m > 1$, which defines minimal densities $\sim 2 \times 10^{18}$ cm^{-3} for the realization of the γ-ray laser. Figure 7 shows the temperature-density diagram for the realization of BEC of Ps atoms. This diagram also shows the range of parameters where such γ-ray laser may be implemented. As a limiting density, we take $n_0 \simeq 1/(4a_s^3) \simeq 2.8 \times 10^{21}$ cm^{-3}. With the further increase of the density, the deviation from the bosonic nature of Ps atoms becomes considerable [see Eq. (4)]. At high densities, the bound states of electron-positron pairs do not survive making electron-positron plasma. It should be noted that for the BEC realized in the trap with the potential that varies relatively smoothly in the space, the critical temperature and the number of Ps atoms in the condensate are strongly defined by the parameters of the trap (see next section).

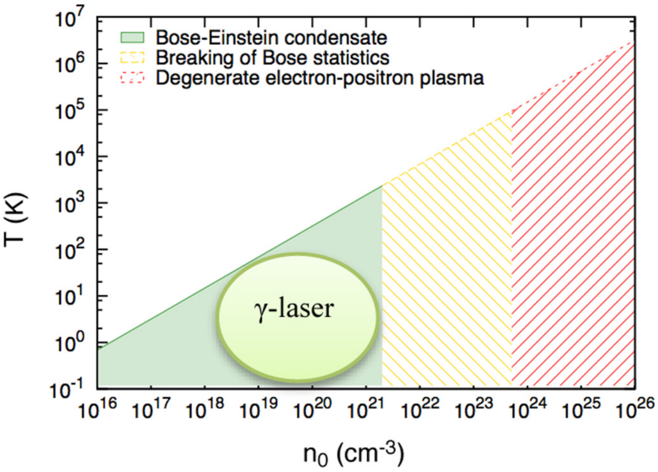

FIG. 7. (Color online) Temperature-density diagram for the formation of BEC of Ps atoms, in the logarithmic scale. With the increase of the density, the deviation from the bosonic nature of Ps atoms becomes considerable. At high densities, the bound states of electron-positron pairs do not survive making electron-positron plasma. The range of parameters where the γ-ray laser may be implemented is also shown.

V. THE INFLUENCE OF THE CONFINEMENT AND INTERACTION BETWEEN THE POSITRONIUM ATOMS ON THE GAMMA-RAY GENERATION PROCESS

Although we consider dilute system of Ps atoms when $na_t^3 \ll 1$, for the trapped atoms the interaction can have a deep influence on the ground state of the BEC [13] and on the critical temperature [41] of condensation. In this case, the starting point is the Gross-Pitaevskii equation [13] for the order parameter of an inhomogeneous BEC well below the critical temperature. The Gross-Pitaevskii equation for the order parameter $\Psi(\mathbf{r})$ of a BEC has the well-known form [13]

$$\left[-\frac{\hbar^2}{2m_a}\Delta + V_{\mathrm{tr}}(\mathbf{r}) + \frac{4\pi\hbar^2 a_t}{m_a}|\Psi(\mathbf{r})|^2\right]\Psi(\mathbf{r}) = \mu\Psi(\mathbf{r}), \quad (55)$$

where $m_a = 2m_*$ is the Ps mass and $V_{\mathrm{tr}}(\mathbf{r})$ is the trap confining potential. The nonlinear term takes into account the interaction between the Ps atoms parametrized by the s-wave scattering length a_t. The chemical potential μ is fixed by the normalization condition,

$$\int n(\mathbf{r})d\mathbf{r} = N; \quad n(\mathbf{r}) = |\Psi(\mathbf{r})|^2, \quad (56)$$

where $n(\mathbf{r})$ is the density of the atoms with the total number N. When the number of atoms is large and the interaction is repulsive ($a_t > 0$), an accurate expression for the ground state $\Psi(\mathbf{r})$ may be obtained within the Thomas-Fermi approximation. The latter is valid when the dimensionless parameter $Na_t/\overline{a}$ is very large. Here, $\overline{a}$ is the characteristic length of the confining potential. In this case, the kinetic-energy term $\sim\Delta$ can be neglected in the Gross-Pitaevskii equation (55), and we have

$$n(\mathbf{r}) = \frac{m_a}{4\pi\hbar^2 a_t}[\mu - V_{\mathrm{tr}}(\mathbf{r})] \quad (57)$$

in the region where the right-hand side is positive and $n(\mathbf{r}) = 0$ otherwise. The boundary of the BEC cloud is given by the relation $\mu = V_{\mathrm{tr}}(\mathbf{r})$. The Thomas-Fermi approach fails near the edge of the cloud when the kinetic-energy term should be taken into account. In this case, the characteristic length is the healing length $l_h = 1/\sqrt{8\pi n a_t}$, which describes the distance over which the density tends to its bulk value from the boundary. For the considered densities $n = 10^{18}$–10^{21} cm^{-3}, the healing length $l_h \simeq 10^{-6} - 5\times 10^{-8}$ cm $\ll L, L_w$. As a consequence, the boundary effects can be neglected. Thus, for the condensate confined by a box with sufficiently (infinitely) hard walls, the above consideration is valid and one can consider a homogeneous condensate with the density

$$n(\mathbf{r}) = \frac{m_a}{4\pi\hbar^2 a_t}\mu = n_0. \quad (58)$$

For a confining potential that varies relatively smoothly in the space, the inhomogeneous nature of BEC should be taken into account. For an anisotropic three-dimensional harmonic-oscillator potential $V_{\mathrm{tr}}(\mathbf{r})$ given by

$$V_{\mathrm{tr}}(\mathbf{r}) = \frac{1}{2}m_a\omega_0^2\left(x^2 + y^2 + \frac{z^2}{\chi^2}\right), \quad (59)$$

the solution (57) becomes

$$n(x,y,z) = n_{\max}\left[1 - \frac{1}{R_0^2}\left(x^2 + y^2 + \frac{z^2}{\chi^2}\right)\right], \quad (60)$$

with

$$n_{\max} = \frac{15^{2/5}}{8\pi\overline{a}^2 a_t}\left(\frac{Na_t}{\overline{a}}\right)^{2/5}; \quad R_0 = \frac{15^{1/5}}{\chi^{1/3}}\left(\frac{Na_t}{\overline{a}}\right)^{1/5}\overline{a}, \quad (61)$$

where $\overline{a} = \sqrt{\hbar/m_a\overline{\omega}}$ and $\overline{\omega} = \omega_0\chi^{-1/3}$ is the geometrical mean frequency of an anisotropic oscillator. As far as the γ-ray wavelength $\sim\lambda_c \ll L, L_w$, we can use the expression (49) for the exponential growth rate with the density defined through Eq. (60),

$$G(x,y,z) = \sqrt{\frac{16\pi\varrho n(x,y,z)\alpha_0^5}{m}}. \quad (62)$$

Then, the dimensionless interaction parameter in the exponent of Eq. (53) for the end-fire modes can be written as

$$\Lambda = \int_{-\chi R_0}^{\chi R_0} G(0,0,z)dz = \frac{\chi\pi R_0}{2}\sqrt{\frac{16\pi\varrho n_{\max}\alpha_0^5}{m}}. \quad (63)$$

As is seen from Eq. (63), the effective interaction length is $\chi\pi R_0/2$. Taking into account the interaction of Ps atoms, the critical temperature for BEC in the trap (59) is defined as [41]

$$T_c \simeq 0.94\frac{\hbar\overline{\omega}}{k_B}N^{1/3}\left(1 - 1.33\frac{a_t}{\overline{a}}N^{1/6}\right). \quad (64)$$

Thus, for a system of 10^{12} Ps atoms, interacting with a scattering length $a_t \simeq 1.6\times 10^{-8}$ cm, which is trapped in an anisotropic harmonic potential fixed by $\overline{a} \simeq 10^{-5}$ cm with the anisotropy parameter $\chi = 2000$, the dimensionless interaction parameter at $\varrho = 1$ will be $\Lambda \simeq 1.12$. There the critical temperature (64) will be $T_c \simeq 330$ K.

Note that considering two-wave instability has been proven for an infinite and nondamping medium. Meanwhile, due to the finite interaction length with sufficient loss, the system may become stable [42,43]. In our scheme, damping effects are negligible since the annihilation proceeds in vacuum. The γ-ray line broadening due to the uncertainty in the momentum of Ps atoms confined in a trap $\delta\omega \sim \hbar/(2m_a R_0^2)$ is considerably smaller than the rate G. Hence, for instability, G should be larger than the γ-ray line broadening due to finite interaction length: $G > 2\delta k \sim 1/L$. The latter is just the condition for the high gain regime.

VI. CONCLUSION

We have presented a theoretical treatment of the collective annihilation decay of Ps atoms from a BEC state. We have considered coherent γ-ray generation at the collective annihilation of Ps atoms in a BEC state in the self-amplified spontaneous-emission regime arising from the second quantized formalism. We have considered the stimulated o-Ps to p-Ps transition, which modifies the exponential growth rate through the effective BEC density n_{eff} that depends on the pumping scheme. The exponential growth rate has nonlinear dependence on the BEC density, in contrast to the rate equations where the corresponding gain is proportional to the number density. It has been shown that one might observe the startup of an annihilation γ-ray laser at lower densities than would be the case for a gain proportional to the density. For an elongated shape of the BEC, we show the laserlike action, i.e., directional radiation when the spontaneously emitted entangled and oppositely directed photon pairs are amplified, leading to an exponential buildup of a macroscopic population into the end-fire modes. In addition, these photon beams are entangled, i.e., we have a so-called Schrödinger cat state with a macroscopic number of γ-ray photons. The parameters required for an efficient γ-ray laser have been specified. The influence of the external potential and interaction between the Ps atoms on the γ-ray self-amplification process has also been investigated. While the parameters required for an efficient γ-ray laser are certainly very challenging, the ongoing and future progress in the creation, trapping, and cooling of Ps atoms promise clear prospects to reach them.

ACKNOWLEDGMENTS

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