Polarization singularities in a sum-frequency light beam generated by a bichromatic singular beam in the bulk of an isotropic nonlinear chiral medium

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Expressions for the electric field at a sum frequency generated by a collinear elliptically polarized Gaussian beam and circularly polarized Laguerre-Gaussian beam in an isotropic chiral nonlinear medium are obtained in quadratures. The amount and locations of C points in the cross section of a signal beam at a sum frequency are shown to be dependent on frequency and diffraction lengths ratios of fundamental beams and on the ellipticity degree of the Gaussian beam's polarization ellipse. Possible values of total topological charges of the emergent C points are determined by the topological charge of the Laguerre-Gaussian beam and remain constant while the radiation propagates in nonlinear media. In case of nonzero total topological charge C lines form helical structures, the parameters of which depend on the wave-vector mismatch. Otherwise, C lines form a loop. As the wave-vector mismatch grows the loop undergoes deformation and breaks up, creating new C lines.

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I. INTRODUCTION

The space where an inhomogeneously polarized monochromatic light beam propagates can possess certain lines, along which the polarization of the beam's electric field is purely circular. These lines are called C lines [1]. Their intersections with the cross section of the propagating beam are points of circular polarization singularity, or C points. Polarization singularity is characterized by its topological charge, which is equal to the total winding number of the polarization ellipse during one full counter-clockwise loop around the point of singularity. If the rotation of the ellipse is clockwise, the charge is considered to be negative. C points of the least possible absolute value of topological charge $(1/2)$ are stable under slight perturbations of the field, while singularities with greater charges $(n/2)$ split under the perturbations of the field into $|n|$ C points, each having the topological charge equal to $(1/2)\,\mathrm{sgn}(n)$. While analyzing different cross sections of the light beam, which correspond to increasing values of the propagation coordinate, one can observe continuous motion of C points. In places where the C line is tangential to the cross-section plane the pairwise creation or annihilation of C points with opposite topological charges takes place. Many other interesting properties of polarization singularities are well known at the present time [2], and C points' emergence and annihilation behavior have been studied in a great diversity of problems in linear optics [3–6].

The features of emergence and evolution of polarization singularities in nonlinear media remained unexplored for a long time because the account of a polarization state's changes in a propagating wave leads to rather cumbersome calculations. As a rule, authors consider interaction of linearly polarized waves, thereby proceeding to the more simple case of scalar field singularities—optical vortices. In [7], second-harmonic generation by superposition of optical vortices was studied, and the authors pointed out characteristic doubling of the topological charge of the singularities. A similar effect was experimentally observed in [8]. Interesting features of the birth and evolution of optical vortices in photorefractive crystal were theoretically described and experimentally confirmed in research [9]. Also the scenarios of parametric interaction of noncoaxial optical vortices and the impact of incident radiation parameters on the singular pattern of propagating waves were thoroughly investigated in [10]. Meanwhile, taking into account the vectorial nature of interacting fields allows one to observe a rich palette of analogous polarization effects even in media belonging to higher classes of symmetry. In particular, it was shown [11–14] that polarization singularities may emerge in the bulk and on the surface of nonlinear media with nonlocal optical response in processes of sum-frequency and second-harmonic generation even if the fundamental beams are homogeneously polarized. Moreover, the results of numerical modeling of the propagation of light beams initially containing polarization singularities demonstrate interesting peculiarities of their birth and annihilation processes in isotropic media with nonlocality of its cubic nonlinear response [15].

The aim of the present paper is to analyze the creation, evolution, and interaction of polarization singularities in a sum-frequency beam generated in an isotropic chiral medium by a regular elliptically polarized beam and circularly polarized beam with phase singularity. We call their superposition a bichromatic singular beam. Generally speaking, canonical definition of polarization singularity becomes less applicable in our case because the tip of the total oscillating electric-field vector of the collinear signal and fundamental beams in a given point of space draws out a Lissajous figure but not an ellipse. Fundamental properties of polychromatic polarization states and singularities were studied in original research [16,17]. However, we limit ourselves by considering only the canonical polarization singularities of a signal beam, which is indeed monochromatic. The tensor of local quadratic susceptibility of isotropic chiral media $\hat{\chi}^2(\omega_1+\omega_2;\omega_1,\omega_2)$ is proportional to a Levi-Civita tensor, and the Fourier component of the field of

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nonlinear polarization vector $\mathbf{P}^{\mathrm{NL}}$ at a sum frequency has the following form:

$$\mathbf{P}^{\mathrm{NL}}(\omega_1+\omega_2;\omega_1,\omega_2)=\chi^{(2)}[\mathbf{E}_1(\omega_1)\times\mathbf{E}_2(\omega_2)]. \quad (1)$$

Here $\mathbf{E}_1(\omega_1)$ and $\mathbf{E}_2(\omega_2)$ are Fourier components of electric fields of the first and the second fundamental beams, and $\chi^{(2)}$ is a real parameter in the absence of absorption [18]. We emphasize that if both of the fundamental waves are plane waves then the medium's nonlinear polarization vector $\mathbf{P}^{\mathrm{NL}}$ is parallel to their propagation direction. This forbids the generation of a transversal wave at a sum frequency in the bulk of a nonlinear medium. However, a real laser beam is a superposition of transversal plane Fourier harmonics having slightly noncollinear wave vectors, which form small angles with the beam's propagation direction. In order to satisfy the Maxwell equation ($\operatorname{div}\mathbf{E}=0$), each fundamental beam must possess a small longitudinal component of the electric-field vector. These components actually provide the generation of a transversal signal beam at a sum frequency when fundamental beams propagate collinearly. The problems related to the efficiency of such processes and analysis of a polarization state of a signal beam were discussed in detail in [12,13], where, in particular, the sum-frequency field was obtained in quadratures in the case of a nondepleted pump of two coaxial Gaussian beams.

II. MAIN EQUATIONS

Let the first part of a bichromatic singular fundamental beam be a regular elliptically polarized Gaussian beam,

$$\mathbf{E}_1(x,y,z,t)=E_1^{(0)}\big[\mathbf{e}_1+ik_1^{-1}\mathbf{e}_z(\mathbf{e}_1\cdot\nabla)\big]G_1(x,y,z)\times\exp[-i\omega_1 t+ik_1(z-l_0)], \quad (2)$$

with maximum intensity on its axis Oz, having the major axes of all polarization ellipses parallel to the Ox axis. The second part of the bichromatic beam is a right-hand circularly polarized Laguerre-Gaussian beam,

$$\mathbf{E}_2(x,y,z,t)=E_2^{(0)}\big[\mathbf{e}_+ +ik_2^{-1}\mathbf{e}_z(\mathbf{e}_+\cdot\nabla)\big]\frac{x+imy}{w_2\beta_2(z)}\times G_2(x,y,z)\exp[-i\omega_2 t+ik_2(z-l_0)], \quad (3)$$

which has a zero amplitude of its transversal electric field on the Oz axis. Here $E_{1,2}^{(0)}$ are constants, and

$$G_{1,2}(x,y,z)=\frac{1}{\beta_{1,2}(z)}\exp\left[-\frac{(x^2+y^2)}{w_{1,2}^2\beta_{1,2}(z)}\right] \quad (4)$$

are solutions of the parabolic equation describing linear diffraction of scalar Gaussian beams with waist sizes $w_{1,2}$ and waist z coordinates $z=l_0$ (Fig. 1).

Polarization unit vectors $\mathbf{e}_\pm=(\mathbf{e}_x\mp i\mathbf{e}_y)/\sqrt{2}$ describe the right- and left-hand circularly polarized waves, respectively, as defined from the point of view of the receiver. Here $\mathbf{e}_{x,y,z}$ are Cartesian unit vectors, $\beta_{1,2}(z)=1+i(z-l_0)/l_{1,2}$, $l_{1,2}=k_{1,2}w_{1,2}^2/2$ are diffraction lengths, $m=\pm1$ is known as the charge of the Laguerre-Gaussian mode, and vector $\nabla=\{\partial/\partial x,\partial/\partial y\}$. The polarization unit vector in Eq. (2) is

$$\mathbf{e}_1=(\sqrt{1+M_0}\mathbf{e}_+ +\sqrt{1-M_0}\mathbf{e}_-)/\sqrt{2}, \quad (5)$$

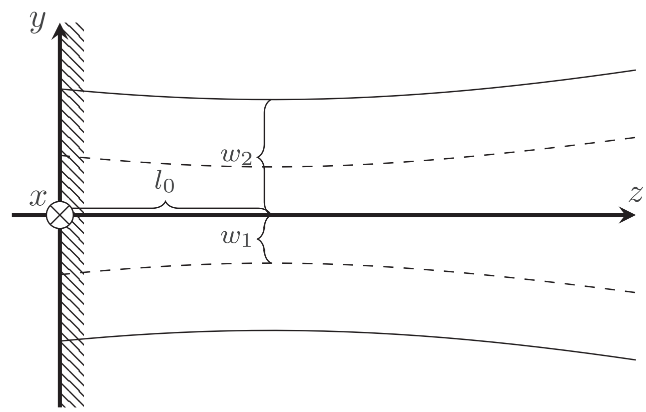

FIG. 1. The geometry of the fundamental beams interaction.

where parameter M_0 is an ellipticity degree of Gaussian beam polarization ellipses ($M_0=\pm1$ correspond to right- and left-hand circularly polarized radiation, respectively, and $M_0=0$ corresponds to the linearly polarized wave). We notice that the fundamental beams (2) and (3) have small longitudinal components of the electric field. Their form is chosen in order to satisfy the equations $\operatorname{div}\mathbf{E}_{1,2}=0$ within the first-order approximation on the parameters $\lambda_{1,2}/w_{1,2}\ll1$, where $\lambda_{1,2}$ are the wavelengths of the fundamental beam parts.

One finds the nonlinear polarization vector $\mathbf{P}^{\mathrm{NL}}$ substituting Eqs. (2) and (3) in Eq. (1). It is necessary to separate its solenoidal part $\mathbf{P}^{(\mathrm{v})}$, so $\operatorname{div}\mathbf{P}^{(\mathrm{v})}=0$. The procedure is carried out within the first-order approximation on the parameter $1/k_3w_3$ in the following way: $\mathbf{P}^{(\mathrm{v})}=\mathbf{P}^{\mathrm{NL}}+k_3^{-2}\nabla(\nabla\cdot\mathbf{P}^{\mathrm{NL}})$. Here $w_3=(w_1^{-2}+w_2^{-2})^{-1/2}$, and $k_3=k_1+k_2$. It is convenient for our research to decompose this beam into circularly polarized components $P_\pm^{(\mathrm{v})}=(P_x^{(\mathrm{v})}\pm iP_y^{(\mathrm{v})})/\sqrt{2}$:

$$P_+^{(\mathrm{v})}=\frac{P_0}{2}\frac{G_3(x,y,z)}{w_2\beta_2(z)}\big\{(x+imy)\times[d_{12}\mu_+(x-iy)+d_{13}\mu_-(x+iy)]+k_2^{-1}\mu_+(1+m)+k_3^{-1}\mu_-(1-m)\big\}, \quad (6a)$$

$$P_-^{(\mathrm{v})}=\frac{P_0}{2}\frac{G_3(x,y,z)}{w_2\beta_2(z)}\mu_-\big\{(x+imy)(x-iy)d_{23}-\big(k_2^{-1}-k_3^{-1}\big)(1+m)\big\}. \quad (6b)$$

Here $\mu_\pm=\sqrt{1\pm M_0}$, $P_0=\chi^{(2)}E_1^{(0)}E_2^{(0)}$, $d_{ij}=\beta_i^{-1}(z)/l_i-\beta_j^{-1}(z)/l_j$, $\beta_3^{-1}(z)=w_3^2[\beta_1^{-1}(z)w_1^{-2}+\beta_2^{-1}(z)w_2^{-2}]$, $l_3=k_3w_3^2/2$, and $G_3(x,y,z)=G_1(x,y,z)G_2(x,y,z)$. The value of l_3 is always between two values of l_1 and l_2.

It was shown [12,13] that the following system of differential equations for slowly varying envelopes $E_\pm^{(\mathrm{v})}=(E_x^{(\mathrm{v})}\pm iE_y^{(\mathrm{v})})/\sqrt{2}$ of right- and left-hand circularly polarized components can be written for the solenoidal part of the sum-frequency beam:

$$\left(\frac{\partial}{\partial z}-\frac{i}{2k_{SF}}\Delta_\perp\right)E_\pm^{(\mathrm{v})}=\frac{2\pi ik_{SF}}{\varepsilon(\omega_3)}P_\pm^{(\mathrm{v})}(\omega_3)\exp[i\Delta k(z-l_0)]. \quad (7)$$

Here the Oz axis is orthogonal to the plane surface of the medium and is directed into its bulk, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transversal Laplace operator, $\omega_3 = \omega_1 + \omega_2$, ε is the linear dielectric permittivity of the medium, $k_{SF} = \omega_3 \varepsilon(\omega_3)/c$, k_1 and k_2 are wave vectors of signal and fundamental waves, and $\Delta k = k_1 + k_2 - k_{SF}$ is the wave-vector mismatch. If the medium has normal frequency dispersion, then $\varepsilon(\omega_{1,2}) < \varepsilon(\omega_3)$ and $\Delta k < 0$. In case of anomalous dispersion the reversed inequalities are valid. We neglect the linear gyration effects, as it was done in [12,13], and assume wave-vector mismatch to be the same for both components $E_\pm^{(v)}$. The initial conditions are trivial $E_\pm^{(v)}(x,y,0) = 0$, and in approximation of small pump depletion Eq. (7) become linear heterogeneous ones. This allows us to write their solutions using the Green's function.

III. DISCUSSION OF THE RESULTS

The expressions for circularly polarized components of a sum-frequency beam can only be obtained in quadratures:

$$E_+^{(v)}(x,y,z) = i\alpha\{(x + imy) \times [J_{12}^{(1)}\mu_+(x - iy) + J_{13}^{(1)}\mu_-(x + iy)] + (J^{(2)}k_2^{-1} + iJ_{12}^{(3)}k_{SF}^{-1})\mu_+(1 + m) + (J^{(2)}k_3^{-1} + iJ_{13}^{(3)}k_{SF}^{-1})\mu_-(1 - m)\}, \quad (8a)$$

$$E_-^{(v)}(x,y,z) = i\alpha\mu_-\{(x + imy)(x - iy)J_{23}^{(1)} - [J^{(2)}(k_2^{-1} - k_3^{-1}) - iJ_{23}^{(3)}k_{SF}^{-1}](1+m)\}. \quad (8b)$$

It is readily seen that the distribution of polarization ellipses in each transversal cross section of the signal beam does not depend on the nonlinearity parameter $\alpha = \pi k_{SF} l_3 P_0 / w_2 \varepsilon(\omega_3)$. The expressions in braces in Eq. (8) contain the following integrals:

$$J_{12,13,23}^{(1)}(x,y,z) = \int_{-l_0/l_3}^{(z-l_0)/l_3} \frac{\tilde\beta_1^2(\zeta')\tilde\beta_2(\zeta')}{[B(\zeta,\zeta')]^3} K(\zeta,\zeta')\tilde d_{12,13,23} d\zeta', \quad (9)$$

$$J_{12,13,23}^{(3)}(x,y,z) = \int_{-l_0/l_3}^{(z-l_0)/l_3} \frac{(\zeta - \zeta')\tilde\beta_1(\zeta')}{[B(\zeta,\zeta')]^2} l_3 K(\zeta,\zeta')\tilde d_{12,13,23} d\zeta', \quad (10)$$

$$J^{(2)}(x,y,z) = \int_{-l_0/l_3}^{(z-l_0)/l_3} \frac{K(\zeta,\zeta')}{\tilde\beta_2(\zeta')B(\zeta,\zeta')} d\zeta'. \quad (11)$$

In Eqs. (9)–(11) function $B(\zeta,\zeta') = \tilde\beta_1(\zeta')\tilde\beta_2(\zeta') + ik_3(\zeta - \zeta')\tilde\beta_A(\zeta')/k_{SF}$, $\zeta = (z - l_0)/l_3$, $\tilde\beta_{1,2}(\zeta) = 1 + i\zeta l_3/l_{1,2}$, $\tilde\beta_A(\zeta) = 1 + i\zeta l_3^2/l_1 l_2$, the kernel

$$K(\zeta,\zeta') = \exp\left[i\nu\zeta' - \frac{x^2 + y^2}{w_3^2}\frac{\tilde\beta_A(\zeta')}{B(\zeta,\zeta')}\right], \quad (12)$$

$\tilde d_{ij} = \tilde\beta_i^{-1}(\zeta)/l_i - \tilde\beta_j^{-1}(\zeta)/l_j$, and $\nu = l_3\Delta k$ is proportional to the wave-vector mismatch. The solution (8) has an axial symmetry $E_\pm^{(v)}(x,y,z) = E_\pm^{(v)}(-x,-y,z)$, and, because of the exponent inside the integrals, it is Gaussian asymptotic on the transversal coordinates x and y. Each of the circularly polarized components of the sum-frequency beam consists of a "central core" with a Gaussian-like intensity profile and a "frame," similar to Laguerre-Gaussian modes of second order, which depend quadratically on the transversal coordinates. If the diffraction lengths of two fundamental beams are equal, then the following equalities are valid: $d_{12} = d_{13} = d_{23} = 0$. In this case the signal beam does not contain any polarization singularities.

The search for C points in the signal beam's cross section and the reconstruction of C lines were done by the following numerical algorithm. At first, the grid values of integrals in Eqs. (9)–(11) were found numerically for the consequence of the propagation coordinate values z_k. After that the singularity patterns were reconstructed using piecewise interpolation methods. Near the medium's border the problem is remarkably simpler because the electric field (8) of the signal beam has almost the same form as the vector field $\mathbf{P}^{(v)}$. This similarity takes place because for the small values of z the solution of Eq. (7) can be written in the following form:

$$E_\pm^{(v)}(x,y,z) = \frac{2\pi i k_{SF}}{\varepsilon(\omega_3)}\int_0^z dz' \exp[i\Delta k(z' - l_0)] \times \iint dx'dy' \, \mathrm{Gr}(x - x', y - y', z - z')P_\pm^{(v)}(x',y',z') \approx \frac{2\pi i k_{SF} z}{\varepsilon(\omega_3)} P_\pm^{(v)}(x,y,0)e^{-i\Delta k l_0}, \quad (13)$$

where $\mathrm{Gr}(x,y,z) = -ik_{SF}/(2\pi z)\exp[ik_{SF}(x^2 + y^2)/(2z)]$ is the Green's function of Eq. (7). Polarization singularities emerge in the signal beam if the vector field $\mathbf{P}^{(v)}$ itself has polarization singularities at the medium's border. From now on we will call the latter singularities *the generators* or G points (their trajectories being G lines). Polarization singularities in the signal beam have the same topological charges as the corresponding generators and are located close to them in the cross sections which are close enough to the medium's border. The positions of all G points at any z coordinate are determined by the following systems:

$$\begin{cases} A_{1_\pm}x^2 + 2C_{1_\pm}xy + B_{1_\pm}y^2 + D_\pm = 0 \\ A_{2_\pm}x^2 + 2C_{2_\pm}xy + B_{2_\pm}y^2 = 0 \end{cases}. \quad (14)$$

The first system [the top sign "+" in Eq. (14)] determines the position of left-hand generators (which are zeros of $P_+^{(v)}$) and the second one [the bottom sign "−" in Eq. (14)] determines the position of right-hand generators (which are zeros of $P_-^{(v)}$). Real coefficients A–D are the following:

$$A_{1_+} = (l_1 l_2 - z'^2)(\mu_+ + \mu_- k_2/k_3),$$

$$B_{1_+} = m(l_1 l_2 - z'^2)(\mu_+ - \mu_- k_2/k_3),$$

$$C_{1_+} = 0.5z'(l_1 + l_2)[(m-1)\mu_+ + (m+1)\mu_- k_2/k_3],$$

$$D_+ = [k_2^{-1}\mu_+(1+m) + k_3^{-1}\mu_-(1-m)]\frac{(z'^2 + l_1^2)(z'^2 + l_2^2)}{l_2 - l_1},$$

$$A_{2_+} = z'(l_1 + l_2)(\mu_+ + \mu_- k_2/k_3),$$

$$B_{2_+} = mz'(l_1 + l_2)(\mu_+ - \mu_- k_2/k_3),$$

$$C_{2_+} = 0.5(z'^2 - l_1 l_2)[(m-1)\mu_+ + (m+1)\mu_- k_2/k_3],$$
$$A_{1_-} = l_1 l_2 - z'^2,$$
$$B_{1_-} = m(l_1 l_2 - z'^2),$$
$$C_{1_-} = 0.5z'(l_1 + l_2)(m-1),$$
$$D_- = (1+m)\left(k_2^{-1}\right)\frac{\left(z'^2 + l_1^2\right)\left(z'^2 + l_2^2\right)}{l_2 - l_1},$$
$$A_{2_-} = z'(l_1 + l_2),$$
$$B_{2_-} = mz'(l_1 + l_2),$$
$$C_{2_-} = 0.5(z'^2 - l_1 l_2)(m-1), \qquad (15)$$

where $z' = z - l_0$. Generally, the second equations in Eq. (14) determine two straight lines crossing in the origin at $z = \text{const}$, and the first equations determine a second-order curve with its center in the origin. Thus, the distribution of G points in the transversal plane $z = \text{const}$ has the center of symmetry, and their amount cannot exceed 4 for each handedness of rotation of the $\mathbf{P}^{(v)}$ vector. Symmetrically positioned G points have identical topological charges. The analysis of Eq. (6) shows that the amount and characteristics of generators depend on wave vectors and waist sizes ratios of two fundamental beams. Varying the ellipticity degree M_0 of the Gaussian fundamental beam, one can control the position and amount of only left-hand G points (zeros of $P_+^{(v)}$). The positions and amount of right-hand generators (zeros of $P_-^{(v)}$) do not depend on M_0, as this parameter affects only the absolute value of $|P_-^{(v)}|$.

Consider the structure of the right-hand circularly polarized component of the field $\mathbf{P}^{(v)}$. The first of two systems (14) becomes significantly simpler when $z = l_0$: the coefficients A_{2_+} and B_{2_+} become zero. The left-hand generators (zeros of $P_+^{(v)}$) determined by this system lie on Ox and Oy axes and their coordinates are determined by the following equalities:

$$x^2 = -D_+/A_{1_+}, \qquad (16a)$$
$$y^2 = -D_+/B_{1_+}, \qquad (16b)$$

in which A_{1_+}, B_{1_+}, and D_+ are calculated at $z = l_0$. Each equality determines two symmetrically positioned generators, if its right part is positive. If the inequality $m(M_0 - M^*) < 0$ is valid, where $M^* = (k_2^2 - k_3^2)/(k_2^2 + k_3^2)$, then only Eq. (16a) (if $l_1 > l_2$) or Eq. (16b) (if $l_1 < l_2$) has physical meaning. In both of these cases two corresponding generators have identical topological charges equal to $m/2$. In contrast, if $m(M_0 - M^*) > 0$, then both of the equalities (16) have physical meaning (if $l_1 > l_2$) or do not have it (if $l_1 < l_2$) simultaneously. In the first case two of four generators have topological charges of $1/2$, and the other two have the charge of $-1/2$, so the total topological charge of all generators is zero.

The second of the systems, Eq. (14), becomes so simple when $z = l_0$ that the distribution of right-hand generators (zeros of $P_-^{(v)}$) has a degenerate form. When charge $m = 1$ the circularly polarized component $P_-^{(v)}$ is zero in the points (x,y) of the plane $z = l_0$ satisfying the condition $x^2 + y^2 = -D_-/(l_1 l_2)$, where D_- is calculated at $z = l_0$. In other parallel planes there are no right-hand G points. This means that if $l_1 > l_2$ then vector field $\mathbf{P}^{(v)}$ is characterized by one G line having the shape of the circumference, the plane of which is orthogonal to the propagation direction of the beam. Numerical analysis of the solutions (8) allows us to conclude that such a G line generates no right-hand polarization singularities in the beam at the sum frequency. If the charge $m = -1$, then $P_-^{(v)} \propto \mu_-(x - iy)^2 d_{23} G_3$ and $E_-^{(v)} \propto \mu_-(x - iy)^2 J_{23}^{(1)}$, and at any z there is a right-hand singularity with topological charge equal to 1 on the signal beam's axis. This case corresponds to doubling of the topological charge, which is typical for nonlinear transformation of singularly polarized radiation. Both of the described cases are not structurally stable and emerge due to the assumption of the Laguerre-Gaussian fundamental beam to be purely circularly polarized. A small variation of the polarization unit vector in Eq. (3) will lead to the destruction of the described structures. In the case of positive $m = 1$ instead of the ring-shaped G line in the $z = l_0$ plane there will be retained only four generators with total topological charge equal to zero lying symmetrically on Ox and Oy axes. If $m = -1$, then instead of a sole G point with the charge of 1 on the beam's axis there will be two G points each having the charge of $1/2$. These two points will lie symmetrically close to the origin on the Ox (if $l_1 > l_2$) or Oy (if $l_1 < l_2$) axis. We notice that both cases can be realized for the left-hand component of the signal beam as well. For this reason we will further discuss only left-hand polarization singularities.

Assuming the wave-vector ratio of the fundamental beams to be constant, one can create one of the three possible types of configurations of the medium's nonlinear polarization field, varying the waists of the beams $w_{1,2}$ and ellipticity degree M_0. The first type is realized with $m(M_0 - M^*) < 0$. The beam of nonlinear polarization contains one pair of left-hand G points in the $z = l_0$ plane with total topological charge equal to m. These generators also exist at any other z [Figs. 2(a) and 2(b)]. This configuration of vector field $\mathbf{P}^{(v)}$ generates an analogous polarization structure of the electric field at the sum frequency. However, unlike G lines each C line has a helical structure if the dimensionless wave-vector mismatch $\nu \neq 0$ and the mean "step" of the spiral decreases as the absolute value $|\nu|$ grows. There also can be a more complicated transformation resulting in such a curving of C lines that they start to intersect some of the transversal planes more than once. This transformation leads to the appearance of additional C points' birth and annihilation events. It is possible to create the first type of configuration of the vector field $\mathbf{P}^{(v)}$ for both values of the charge m. In these two cases C points have opposite topological charges and the helicity of each spiral is also opposite. Finally, if we assume the Laguerre-Gaussian fundamental beam to be left-hand circularly polarized (instead of right-hand), it will be also possible to create vector beams $\mathbf{P}^{(v)}$ differing from previously discussed ones only in the extent of the electric-field vector rotation. In this case the change of the C points' polarization state is followed by the corresponding change of C lines' helicity. Figure 3 illustrates C lines in the sum-frequency beam and G lines of vector field $\mathbf{P}^{(v)}$ (the latter being drawn thinner than C lines) for different values of the wave-vector mismatch. White and black stars designate the points of space where pairwise creation (white stars) or annihilation (black stars) of C points takes place. Red (solid) lines in (a, c) and markers in (b,d) designate the singularities with positive (1/2) topological charge and blue

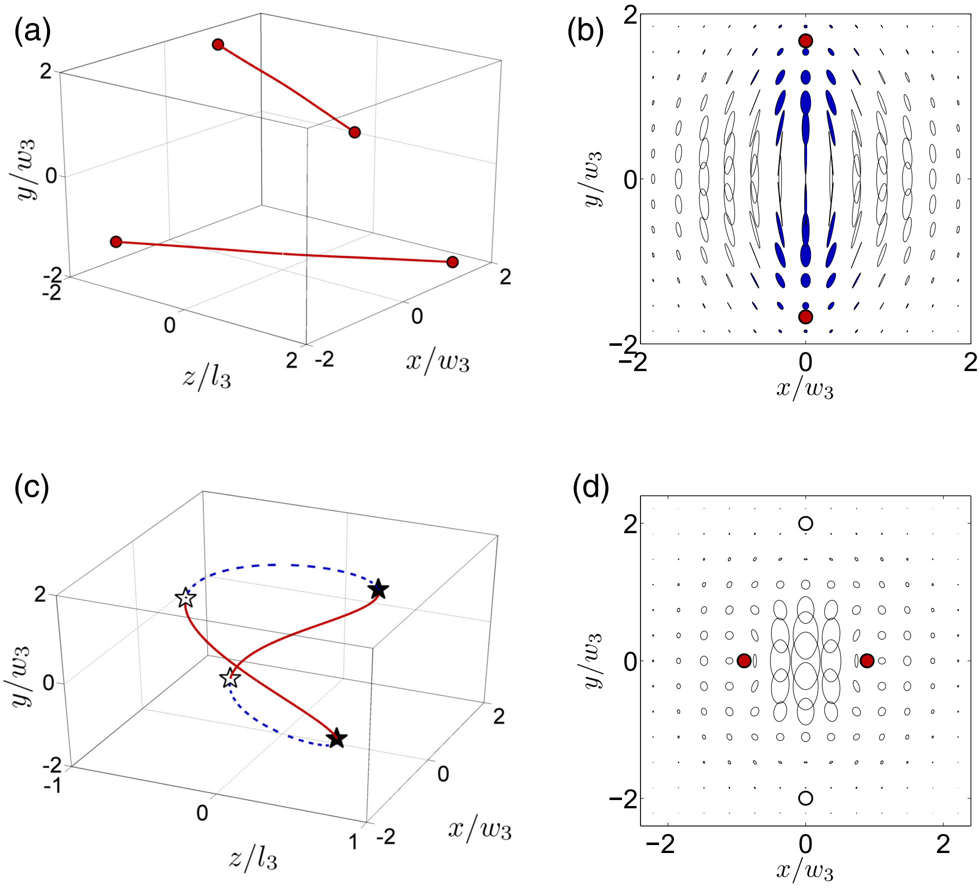

FIG. 2. (Color online) Left-hand G lines (a, c) and transversal distributions of the nonlinear medium polarization field at the position of the waist of the fundamental beams (b, d) for the first (a, b) and the second (c, d) types of vector field $\mathbf{P}^{(\nu)}$ configurations. The charge $m = 1$, and beams' parameters are $w_2/w_1 = 2$, $k_2/k_1 = 1.4$, $M_0 = -0.7$ (a, b) and $w_2/w_1 = 0.3$, $k_2/k_1 = 2$, $M_0 = 0$ (c, d). The plane $z = 0$ in the figure corresponds to the position of the fundamental beams' waists. Red (full) and blue (dashed) lines correspond to C lines with positive and negative topological charges, respectively. C points with positive and negative charges are marked by red (filled) and white circles. Blue (filled) ellipses are left-hand polarized and empty ellipses are right-hand polarized.

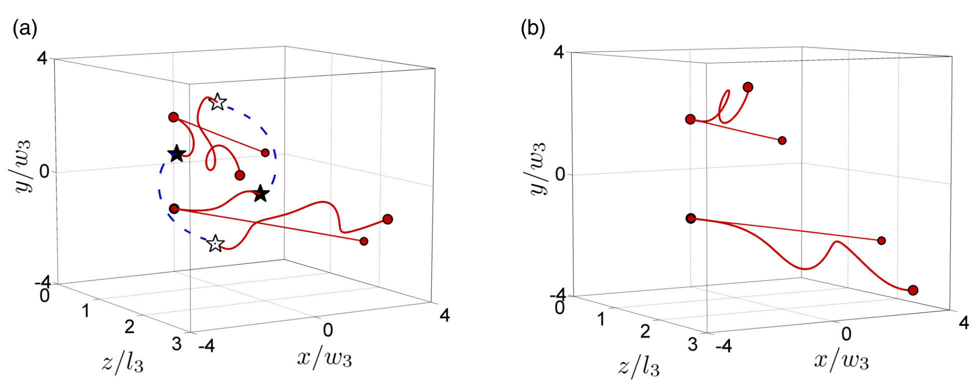

FIG. 3. (Color online) Left-hand C lines (bold) and G lines (thin) for $\nu = -5$ (a), $\nu = 5$ (b), and $w_2/w_1 = 2$, $k_2/k_1 = 1.4$, $m = 1$, $M_0 = -0.7$, $l_0 = 0$. In figure (a) each C line forms a left-hand screw, and in figure (b) it forms a right-hand one. Red (full) and blue (dashed) lines correspond to C lines with positive and negative topological charges, respectively. C points with positive and negative charges are marked by red (filled) and white circles.

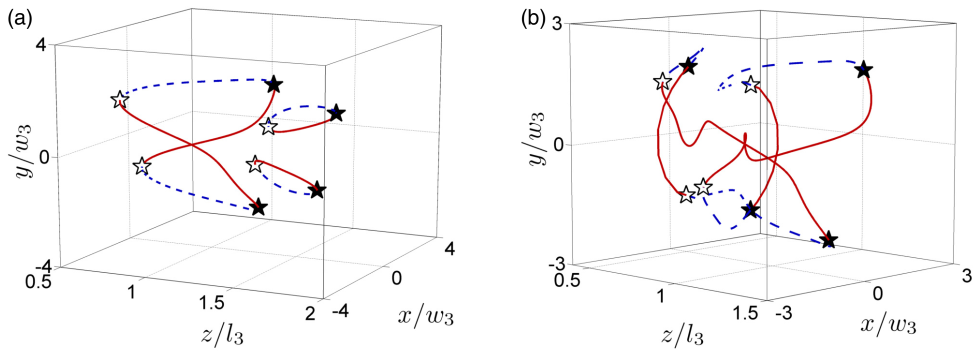

FIG. 4. (Color online) Left-hand C lines in the sum-frequency beam for $\nu = -4$ (a), $\nu = -8$ (b), and $w_2/w_1 = 0.3$, $k_2/k_1 = 2$, $m = 1$, $M_0 = 0$, $l_0 = l_3$. C lines form three separate loops in figure (a) and two loops in figure (b). Red (full) and blue (dashed) lines correspond to C lines with positive and negative topological charges, respectively.

(dashed) lines and white markers designate ones with negative $(-1/2)$ charge. We notice that each star connects two cuts of C lines with opposite topological charges.

The second type of the nonlinear medium's polarization field configuration is realized when both $m(M_0 - M^*) > 0$ and $l_1 > l_2$ are valid. In this case there are two pairs of G points in the $z = l_0$ plane with zero total topological charge. The generators exist only in the planes $z = z_1$, if z_1 is in the interval $[-z_{\rm II}, z_{\rm II}]$. In the plane $z = l_0 - z_{\rm II}$ the birth of two pairs of G points takes place, and in $z = l_0 + z_{\rm II}$ their pairwise annihilation occurs [Figs. 2(c) and 2(d)]. The value $z_{\rm II}$ is readily obtained from the systems (14): $z_{\rm II} = \sqrt{l_1 l_2}(1 + \lambda - \sqrt{\lambda^2 + 2\lambda})^{1/2}$. In this formula

$$\lambda = \begin{cases} (M_0 - M^*)(l_1 + l_2)^2[l_1 l_2 \mu_-^2 (1 + M^*)]^{-1}, & \text{if } m = 1 \\ (M^* - M_0)(l_1 + l_2)^2[l_1 l_2 \mu_+^2 (1 - M^*)]^{-1}, & \text{if } m = -1 \end{cases}. \tag{17}$$

Four corresponding cuts of G lines start and end in places of birth and annihilation of G points ($z = l_0 \mp z_{\rm II}$), forming a closed loop. If $l_0 > z_{\rm II}$, then the fundamental beams are focused in such a way that the whole loop is located inside the medium, where $z > 0$. As in the configuration of the first type, when $k_3 = k_{SF}$, C lines in the sum-frequency beam have almost the same structure as the G lines do. The presence of wave-vector mismatch in this case leads to the deformation of the C lines loop, and the growth of ν causes the formation of additional pairs of C points and appearance of new loops of C lines which are not connected with the primal loop. Further increasing of the absolute value of ν leads to the splitting of the primal loop into two new loops, and the cuts of C lines become helical. The loops of left-hand C lines for different values of ν and fixed parameters of the fundamental beams are shown in Fig. 4.

Finally, the third type of configuration of nonlinear polarization field $\mathbf{P}^{(\nu)}$ is realized when both $m(M_0 - M^*) > 0$ and $l_1 < l_2$ are valid. The beam of this type does not contain singularities in the waist plane $z = l_0$, although two pairs of G points exist in the planes $z = z_1$, where z_1 satisfies the following relations: $z_1 < l_0 - z_{\rm III}$ or $z_1 > l_0 + z_{\rm III}$. In this case total topological charge of G points is zero as in the second type of $\mathbf{P}^{(\nu)}$ field configuration. In the $z = l_0 + z_{\rm III}$ plane birth of G points takes place, and in $z = l_0 - z_{\rm III}$ their pairwise annihilation does. The characteristic coordinate $z_{\rm III} = \{l_1 l_2[1 + \lambda + (\lambda^2 + 2\lambda)^{1/2}]\}^{1/2}$ is found in a similar way as $z_{\rm II}$. Here λ is still determined by Eq. (17). The value of $z_{\rm III}$ cannot be less than the minimum of two diffraction lengths $l_{1,2}$ of the fundamental beams. Despite the fact that C points' formation induced by G points is still possible in this type of configuration, it is not of practical interest, because $\mathbf{P}^{(\nu)}$ tends to zero as $|z - l_0|$ increases and the efficiency of the sum-frequency generation is relatively small.

IV. CONCLUSION

Analytical expressions of the electric field at a sum frequency generated in an isotropic chiral medium by a collinear elliptically polarized Gaussian beam and circularly polarized Laguerre-Gaussian beam were obtained. It was shown that the amount and positions of C points in the cross section of a signal beam are governed by the ratio of fundamental frequencies and the ellipticity degree of the Gaussian beam. The topological charge of a Laguerre-Gaussian beam determines possible values of topological charges of generated C points. The configurations of the signal beam polarization structure were classified based on the value of total topological charge of all singularities, which remains constant for each configuration as the beam propagates in the nonlinear medium. Numerical investigations of the solution demonstrated strong impact of the wave-vector mismatch on the form of C lines. In one of the configurations they have a helical structure; the greater is the absolute value of the wave-vector mismatch the shorter is the mean step of each spiral. If the wave-vector mismatch tends to zero, then the mean step is sufficiently greater than the diffraction lengths of the two fundamental beams. The helicity of each spiral changes to opposite when changing the sign of the wave-vector mismatch. In the other configuration C lines of the signal beam form a loop, which undergoes deformation, if the wave-vector mismatch increases in absolute value. As it reaches sufficiently large values the topological features of the signal beam change: the primal loop breaks up and new loops are formed.

In the present work only the circular polarization singularities are considered. There are also lines of linear polarization in the transversal section of the beam, which correspond to surfaces in three-dimensional space (so-called L surfaces). Such singularities separate the regions with opposite handedness of polarization rotation. As it can be seen in Fig. 2(b), these regions are present in the medium's nonlinear polarization field $\mathbf{P}^{(v)}$. Following the introduced "generator-singularity" concept, one can show that L surfaces appear in the signal beam as well. However, detailed analysis of these singularities is more complicated compared to C lines, because L surfaces are objects of higher dimension and they are harder to be captured by numerical methods. Thus, we do not include the discussion on L surfaces in the present paper, and we aim to consider them in the future.

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