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Mechanical cooling in single-photon optomechanics with quadratic nonlinearity

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In the paper we study the nonlinear mechanical cooling processes in an intrinsic quadratically optomechanical coupling system without linearizing the optomechanical interaction. We apply scattering theory to calculate the transition rates between different mechanical Fock states using the resolvent of the Hamiltonian, which allows for a direct identification of the underlying physical processes, where only even-phonon transitions are permitted and odd-phonon transitions are forbidden. We verify the feasibility of the approach by comparing the steady-state mean phonon number obtained from transition rates with the simulation of the full quantum master equation, and also discuss the analytical results in the weak coupling limit that coincide with two-phonon mechanical cooling processes. Furthermore, to evaluate the statistical properties of steady mechanical state, we respectively apply the Mandel Q parameter to show that the oscillator can be in nonclassical mechanical states, and the phonon number fluctuations F to display that the even-phonon transitions favor suppressing the phonon number fluctuations compared to the linear coupling optomechanical system.

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I. INSTRUCTION

Realization of cooling of mechanical motion within the vicinity of the quantum regime is a key ingredient for a broad range of applications, including quantum information processing (QIP) [1–3], high-precision measurement [4,5], probing quantum behavior of macroscopic systems [6], etc. Recently, theoretical and experimental studies have been mainly focused on the linearized treatment of linear coupling optomechanical system, where the coupling term is proportional to the mechanical displacement x [7–9]. However, some studies have begun to generalize the investigations of nonlinear mechanical cooling processes, such as second-sideband laser cooling of trapped ions [10] and cooling in the single-photon strong optomechanical coupling regime [11], which can lead to nonthermal steady states, and even the nonclassical, sub-Poissonian states. Quantum systems far from thermal equilibrium hold great promise for the investigation of fundamental physics and the implementation of practical devices [12,13].

Now, a new type of optomechanical system, i.e., optomechanics with quadratic nonlinearity (herein "quadratical optomechanics"), has been proposed, where the coupling term is proportional to the square of mechanical displacement x^2, which is realizable in the setups of membrane-in-the-middle optomechanical systems, ultracold atomic ensembles, and superconducting electrical circuit systems [14–16]. Distinct from the displacement of mechanical equilibrium position in linear coupling optomechanics, the quadratic coupling will modify the resonant frequency of the mechanical oscillator [17–19]. Thus the quadratical optomechanical system can be used for quantum nondemolition (QND) measurement of individual quantum jumps [20]. In addition, robust stationary mechanical squeezing and electromagnetically induced transparency (EIT) from two-phonon processes in a quadratical optomechanical system are also achievable [21,22]. For practical applications one should minimize the influence of thermal noise, and the linearized approach to the cooling of a quadratically coupled mechanical oscillator has been studied in Ref. [23]. With the advancement of experimental technology, now it will become attractive to theoretically investigate the nonlinear cooling processes induced by intrinsic quadratical coupling. For example, utilizing avoided crossings effects in a multimode optical cavity containing a flexible dielectric membrane enables a significantly enhanced quadratical coupling strength that can reach >30 MHz/nm^2 [24,25]. Further, via mapping the "membrane-in-the-middle" optomechanics in the optical domain onto thae superconducting electrical circuit system, the quadratical optomechanical coupling strength g compared to the dissipation rate κ could be raised to $g/\kappa > 0.1$ by tuning the bias flux and coupling capacitance [16]; Purdy *et al.* provide the first characterization of quadratical optomechanical effects in cavity optomechanics with ultracold atoms by localizing the gas within a submicron region positioned variably along the cavity axis [15]; in the double-slotted photonic crystal structure the quadratical coupling rate is measured to be as large as 1 THz/nm^2 by integrating electrostatic actuators to allow for tuning of the tunnel coupling strength between the cavity modes around the two slots [26]. These platforms offer the possibility to enter the single-photon strong quadratically optomechanical coupling regime, where a single photon can produce observable effects on a mechanical oscillator and the nature of optomechanical effects remains poorly understood.

Here, we investigate the nonlinear mechanical cooling processes in the intrinsic quadratically optomechanical coupling regime without linearizing the optomechanical interaction. We apply scattering theory to calculate the transition rates between different mechanical Fock states using the Hamiltonian resolvent method [27,28]. This approach allows for a clear identification of the underlying physical processes, where only even-phonon transitions are permitted while the odd-phonon transitions are prohibited. We first derive the mechanical transition rates, verify the consistence of transition rates with the results of a simulation of the full quantum master equation

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with the consideration of thermal mechanical damping, and then discuss the analytical results in the weak coupling limit that coincide with two-phonon mechanical cooling processes [29,30]. We indicate the difference from a linear coupling optomechanical system by evaluating the Mandel Q parameter; it tends to achieve nonclassical mechanical states in quadratical optomechanics. Furthermore, we apply the phonon number fluctuations F to present that the even-phonon transitions favor suppressing the phonon number fluctuations.

The paper is organized as follows. In Sec. II the quadratical optomechanical system is introduced. In Sec. III the scattering theory of the single-photon quadratically optoemchanical cooling process is presented and the cooling limits in the weak and strong coupling regimes are discussed. The nonthermal and nonclassical statistical properties of the mechanical oscillator are analyzed in Sec. IV. Last, the conclusions are given.

II. DESCRIPTION OF PHYSICAL MODEL

We consider a quadratically coupled optomechanical system in which an optical cavity mode is parametrically coupled to the square of the position of a mechanical oscillator, and the Hamiltonian is written as

$$\hat{H}_0 = \omega_R \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b})^2, \tag{1}$$

where ω_R is the frequency of the cavity mode $\hat{a}$, ω_m is the frequency of the mechanical mode $\hat{b}$, and g is the quadratically optomechanical coupling strength between the cavity mode and mechanical motion of the oscillator. The coupling allows for a direct measurement of the square of the oscillator's displacement for the introduction of phase shift of the cavity field. In the rotating-wave approximation the average mechanical motion is $(\hat{b}^\dagger + \hat{b})^2/2 = \hat{n}_b + 1/2$ with $\hat{n}_b$ denoting the number of quanta, and the interaction between the cavity and mechanical oscillator becomes analogous to the optical Kerr effect, which is practicable for QND measurements. Thus in principle the read-out of the oscillator's energy eigenstate is feasible, which is a major goal in the field of quantum measurement. In order to make the present system stable, the Routh-Hurwitz criterion implies that the condition $\omega_m + 4sg > 0$ should be satisfied, in which s is the number of photons inside the cavity. In the frame rotating at the driving laser's frequency ω_L, the optical drive is described by the Hamiltonian $\hat{H}' = \Omega(\hat{a}^\dagger + \hat{a})$ with the driving strength Ω. We are considering the regime $\Omega \ll \kappa$, where κ is the damping rate of the cavity field, and thus the cavity states with more than one photon can be neglected.

The damping of the cavity field is described by the interaction between the mode $\hat{a}$ and the environmental mode $\hat{c}_{k\epsilon}$, where $\hat{H}_{\rm emf} = \sum_{k\epsilon} (\omega_k - \omega_L) \hat{c}^\dagger_{k\epsilon} \hat{c}_{k\epsilon}$ is the free dynamics of the electromagnetic field external to the resonator in the reference frame rotating at the laser frequency ω_L. Here, the sum runs over all modes, identified by the wave vector k and orthogonal polarization ϵ, the operators $\hat{c}_{k\epsilon}$ and $\hat{c}^\dagger_{k\epsilon}$ annihilate and create photons in the corresponding mode, and $\omega_k = ck$ denotes the mode frequency. The cavity-environment interaction

$$\hat{W}_\kappa = \sum_{k\epsilon} g^{(\kappa)}_{k\epsilon} (\hat{a}^\dagger \hat{c}_{k\epsilon} + \hat{a} \hat{c}^\dagger_{k\epsilon}) \tag{2}$$

accounts for the coupling of the cavity mode with the modes of the external radiation field, and the explicit expressions of coupling constant $g^{(\kappa)}_{k\epsilon}$, which is connected to the reflectivity amplitude of the cavity mirror, can be found in the literature [31,32]. In the high-Q cavity limit, it demonstrates that the Markovian approximation is a good approach to describe the cavity damping process, where $g^{(\kappa)}_{k\epsilon} = \sqrt{\kappa/2\pi}$, with cavity damping rate κ defined as [33]

$$\begin{aligned} \kappa &= \int_0^\infty \sum_{k\epsilon} \left| g^{(\kappa)}_{k\epsilon} \right|^2 e^{-i(\omega_k - \omega_R)t} dt \\ &= 2\pi \sum_{k\epsilon} \left| g^{(\kappa)}_{k\epsilon} \right|^2 \delta(\omega_k - \omega_R). \end{aligned} \tag{3}$$

The delta function physically implies that the emission is centered about the cavity frequency ω_R. The mechanical dissipation rate is usually much smaller than the cavity damping rate, and we will take it into consideration in the following.

III. RESOLVENT METHOD ON NONLINEAR COOLING PROCESSES

In this paper we will resort to scattering theory to explicitly evaluate the transition rates which quantitatively determine the cooling and heating processes. This approach will allow for a direct identification of the underlying physical processes. For convenience we separate the total system Hamiltonian according to $\hat{H} = \hat{H}_0 + \hat{V}$, where the redefined main operator $\hat{H}_0$ consists of Eq. (1) andhtbe free electromagnetic field, i.e.,

$$\hat{H}_0 = -\Delta \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b})^2 + \hat{H}_{\rm emf}, \tag{4}$$

and the small interaction part is

$$\hat{V} = \hat{W}_\kappa + \hat{H}', \tag{5}$$

in which $\Delta = \omega_L - \omega_R$ is the detuning between laser frequency ω_L and resonator frequency ω_R. In the case of weak optical drive strength Ω, it is sufficient to work to the second order of Ω in the transition rate to well describe the scattering processes. Moreover, the cooling mechanism relies on the energy dissipation, i.e., cavity damping rate κ that is proportional to the square of $g^{(\kappa)}_{k\epsilon}$. Thus we choose $\hat{V}$ as the perturbation to calculate the transition rates, which are proportional to the cavity damping rate κ and the second order of Ω, by using the scattering theory.

First, the initial system is in the state $|i\rangle = |0; n; 0_{k\epsilon}\rangle$, where the cavity mode is in the vacuum state $|0\rangle$, $|n\rangle$ are phononic excitations and $|0_{k\epsilon}\rangle$ is the environmental vacuum mode. We are interested in the processes that change the motion state of the membrane induced by one photon followed by dissipation to the environment, and we denote the final states $|f\rangle = |0; m; 1_{k\epsilon}\rangle$. For the weak optical driving and cavity-reservoir coupling, i.e., $\{\Omega, g^{(\kappa)}_{k\epsilon}\} \ll g$, it is sufficient to derive the transition amplitude up to the second-order perturbation of interaction $\hat{V}$ to achieve transition rates proportional to Ω^2

and κ, which is given by the form

$$\mathcal{T}_{\rm fi} = \langle f|\hat{V} + \hat{V}\hat{G}_0(E_i)\hat{V}|i\rangle, \tag{6}$$

where $\hat{G}_0(z) = 1/(z - \hat{H}_0^{\rm eff})$ is the resolvent of the effective Hamiltonian $\hat{H}_0^{\rm eff} = \hat{H}_0 - i\frac{\kappa}{2}\hat{a}^\dagger\hat{a}$ with the consideration of cavity damping, and $E_i = n\omega_m$ is the energy of the initial state [33]. By substituting the initial and final states $|i\rangle$ and $|f\rangle$, it can be easily verified that

$$\langle f|\hat{V}|i\rangle = 0, \tag{7}$$

since $\hat{W}_\kappa$ and $\hat{H}'$ contain an annihilation or creation cavity operator while the cavity number states before and after the scattering process are unchanged, and thus only the second-order perturbation is retained.

To obtain the explicit expression of transition amplitude $\mathcal{T}_{\rm fi}$, we have to find the eigensystem of the Hamiltonian $\hat{H}_0$. Since the optical photon is conserved, i.e., $[\hat{a}^\dagger\hat{a},\hat{H}_0] = 0$, the quadratical coupling suggests that the cavity photon squeezes the position of the mechanical oscillator associated with the number of cavity mode $\hat{a}^\dagger\hat{a}$ [18,19]. Then we can find the phonon squeezing operator dependent on photon number to diagonalize the freedom of the mechanical oscillator, which is given by the polaron transformation

$$\hat{U} = \exp\left[-\frac{1}{4}\ln\left(1 + \frac{4\hat{a}^\dagger\hat{a}g}{\omega_m}\right)(\hat{b}^2 - \hat{b}^{\dagger 2})/2\right], \tag{8}$$

with the squeezing factor connected to the photon number.

Then, with the help of the relation

$$\left[\hat{U}\frac{1}{z - \hat{H}_0^{\rm eff}}\hat{U}^\dagger\right]\left[\hat{U}\left(z - \hat{H}_0^{\rm eff}\right)\hat{U}^\dagger\right] = \mathbf{I}, \tag{9}$$

where $\mathbf{I}$ is the identity matrix, the resolvent of the effective Hamiltonian can be reexpressed as

$$\hat{G}_0(z) = \frac{1}{z - \hat{H}_0^{\rm eff}} = \hat{U}^\dagger\hat{U}\frac{1}{z - \hat{H}_0^{\rm eff}}\hat{U}^\dagger\hat{U} = \hat{U}^\dagger\frac{1}{\hat{U}\left(z - \hat{H}_0^{\rm eff}\right)\hat{U}^\dagger}\hat{U}. \tag{10}$$

By applying a form of polaron transformation, the effective Hamiltonian $\hat{H}_0^{\rm eff}$ is diagonalized as

$$\hat{H}_{\rm diag} = \hat{U}\left(z - \hat{H}_0^{\rm eff}\right)\hat{U}^\dagger = z - \left[-\Delta\hat{a}^\dagger\hat{a} + \omega_m\sqrt{1 + 4\hat{a}^\dagger\hat{a}g/\omega_m} \times (\hat{b}^\dagger\hat{b} + 1/2) - \frac{1}{2}\omega_m - i\frac{\kappa}{2}\hat{a}^\dagger\hat{a}\right] - \hat{H}_{\rm emf}, \tag{11}$$

where the eigenfrequency of the oscillator is dependent on the cavity photon number. With the explicit forms of the interaction part $\hat{V}$ in Eq. (5) and the transition amplitude in Eq. (6), for the initial state $|i\rangle$ where the photon number in cavity and reservoir modes is zero, $\hat{W}_\kappa|i\rangle = 0$, and the operation of $\hat{H}'$ on $|i\rangle$ induces one cavity excitation. Following the operation of $\hat{G}_0(E_i)$, the cavity excitation is conserved since $[\hat{a}^\dagger\hat{a},\hat{H}_0^{\rm eff}] = 0$, and there is still no excitation of the reservoir modes. To connect with the final state which contains zero cavity and one reservoir excitation, we should apply the operator $\hat{W}_\kappa$ since the matrix elements of $\hat{H}'$ are zero. Then the transition amplitude in Eq. (5) can be simplified as

$$\mathcal{T}_{\rm fi} = \langle f|\hat{W}_\kappa\hat{G}_0(E_i)\hat{H}'|i\rangle = \Omega g_{k\epsilon}^{(\kappa)}\langle 1;m;0_{k\epsilon}|\hat{U}^\dagger\hat{H}_{\rm diag}^{-1}\hat{U}|1;n;0_{k\epsilon}\rangle. \tag{12}$$

Since under the operations of $\hat{U}$ and $\hat{H}_{\rm diag}$ the states of the cavity and reservoir modes are unchanged, we can insert the completeness of mechanical Fock states $\sum_l |l\rangle\langle l| = \mathbf{I}$. The transition amplitude becomes

$$\mathcal{T}_{\rm fi} = \Omega g_{k\epsilon}^{(\kappa)}\sum_{l,l'}\langle 1;m|\hat{U}^\dagger|1;l\rangle\langle 1;l|\hat{H}_{\rm diag}^{-1}|1;l'\rangle\langle 1;l'|\hat{U}|1;n\rangle, \tag{13}$$

and the matrix elements

$$\langle 1;l|\hat{H}_{\rm diag}^{-1}|1;l'\rangle = \frac{1}{\delta^{(1)}(l,n) + \Delta + i\frac{\kappa}{2}}\delta_{ll'}, \tag{14}$$

where

$$\delta^{(1)}(l,n) = \left(n + \tfrac{1}{2}\right)\omega_m - \left(l + \tfrac{1}{2}\right)\omega_m^{(1)}, \tag{15}$$

and

$$\omega_m^{(1)} = \omega_m\sqrt{1 + 4g/\omega_m} \tag{16}$$

is the membrane's eigenfrequency modified by one cavity photon [19]. The polaron transformation $\hat{U}$ in the phonon Fock representation is the coefficient of the squeezed number state, and we denote $S_{l,n} = \langle 1,l|\hat{U}|1;n\rangle$, which is given by

$$S_{l,n} = \langle l|\exp[-\xi(\hat{b}^2 - \hat{b}^{\dagger 2})/2]|n\rangle \tag{17}$$

with $\xi = \frac{1}{4}\ln(1 + 4g/\omega_m)$. The coefficient can be expressed in the explicit form [17,18,34]

$$S_{l,n} = \frac{\sqrt{l!n!}}{(\cosh\xi)^{n+1/2}}\sum_{k'=0}^{{\rm Floor}[\frac{l}{2}]}\sum_{k=0}^{{\rm Floor}[\frac{n}{2}]}\frac{(-1)^k}{k'!k!} \times \frac{\left(\frac{1}{2}\tanh\xi\right)^{k'+k}}{(n-2k)!}(\cosh\xi)^{2k}\delta_{l-2k',n-2k}, \tag{18}$$

where the function Floor$[x]$ gives the greatest integer less than or equal to x. By substituting the Eqs. (14) and (17) into Eq. (13), the transition amplitude becomes

$$\mathcal{T}_{\rm fi} = \Omega g_{k\epsilon}^{(\kappa)}\sum_l\frac{S_{l,m}S_{l,n}}{\delta^{(1)}(l,n) + \Delta + i\frac{\kappa}{2}}. \tag{19}$$

Then the corresponding transition rate is defined as

$$R_{\rm fi} = 2\pi\sum_{k\epsilon}|\mathcal{T}_{\rm fi}|^2\delta(E_i - E_f), \tag{20}$$

where the δ function guarantees the energy conservation and the sum covers all relevant polarizations and wave vectors [33]. The transition rate becomes

$$R_{\rm fi} = 2\pi\Omega^2\sum_{k\epsilon}\left|g_{k\epsilon}^{(\kappa)}\right|^2 \times \delta(\omega_k - \omega_R)\left|\sum_l\frac{S_{l,m}S_{l,n}}{\delta^{(1)}(l,n) + \Delta + i\frac{\kappa}{2}}\right|^2. \tag{21}$$

With the definition of cavity damping in Eq. (3), the transition rate which denotes the change of the motional state of mechanical oscillator from n to m phonons becomes

$$R_{fi} = \Gamma_{n\to m\neq n} = \kappa\Omega^2 \left| \sum_l \frac{S_{l,m}S_{l,n}}{\delta^{(1)}(l,n) + \Delta + i\frac{\kappa}{2}} \right|^2. \quad (22)$$

Equation (22) presents a clear physical view of the process of an incident photon scattered by the quadratical cavity-optomechanical system with the change of the motion of the mechanical oscillator from n to m phonons. When a laser photon turns into the cavity photon, the phonon state changes from the initial number n to an intermediate number l demanding that $|l-n|$ is even due to the term of two-phonon operators, and the amplitude is inversely proportional to the frequency detuning $\delta^{(1)}(l,n)+\Delta$ which relates to the photon-coupled membrane's resonant frequency $\omega_m^{(1)}$, and is proportional to the matrix element of squeezed number state $S_{l,n}$. As the photon leaves the cavity and dissipates into the continuum of modes in free space, it induces a transition in mechanical motion from l to m phonons which also demands that $|l-m|$ is even, and its amplitude is independent of the driving detuning but dependent on the cavity-reservoir coupling strength $g_{k\epsilon}^{(\kappa)}$, and is also determined by the coefficient of squeezed number state $S_{l,m}$ for the two-phonon term. The output photon possesses the energy $\omega_L + (n-m)\omega_m$ and carries away $(n-m)$ phonons, in which $(n-m)$ should be even since $|l-n|$ and $|l-m|$ are both even.

Now we stress the differences of nonlinear cooling rates between quadratical and linear coupling optomechanical systems. The quadratical optomechanical coupling modifies the mechanical frequency of the oscillator while the linear coupling just displaces the mechanical equilibrium position. Thus the frequency detuning of the transition $\delta^{(1)}(l,n)$ is closely dependent on the coupling strength g while the effective detuning in the linear coupling optomechanics is independent of the coupling strength in the single-photon level. Moreover, the transition amplitude of the linear coupling optomechanics is proportional to the Franck-Condon overlap factor for the single-phonon operators while the transition amplitude is proportional to the matrix elements of the squeezed number state for the two-phonon operators. Therefore, the form of the transition rate in Eq. (22) is different from that of the linear coupling optomechanical system in Ref. [11,28], and may be helpful to generate the more interesting mechanical states.

Taking into account the thermal damping of the mechanical oscillator with rate γ_m and thermal phonon number n_{th}, the set of rate equations for the mechanical oscillator becomes

$$\begin{aligned}\dot{P}_n =& -\gamma_m n_{\text{th}}(n+1)P_n - \gamma_m(n_{\text{th}}+1)nP_n \\ &+ \gamma_m n_{\text{th}} n P_{n-1} + \gamma_m(n_{\text{th}}+1)(n+1)P_{n+1} \\ &- \sum_{m\neq n}\Gamma_{n\to m}P_n + \sum_{m\neq n}\Gamma_{m\to n}P_m \;(|m-n| \text{ is even}),\end{aligned} \quad (23)$$

where P_n is the phonon number distribution in the n phonon state.

A. Cooling process in the resolved-sideband and weak quadratical coupling limits

The occurrence of cooling processes in the strong quadratical coupling regime can absorb multiple two-phonons, and modify the eigenfrequency of the mechanical oscillator simultaneously, which can lead to the generation of uncommon mechanical states. To understand this behavior we first consider the case of the resolved-sideband limit $\omega_m \gg \kappa$ and weak quadratical coupling limit $g \ll \omega_m$, where the scattering processes are mainly dominated by the two-phonon transitions $\Gamma_{n\to n\pm 2}$, and the rates in Eq. (22) are explicitly simplified as $\Gamma_{n\to n-2} = n(n-1)\Gamma_\downarrow$ and $\Gamma_{n\to n+2} = (n+1)(n+2)\Gamma_\uparrow$ in these limits, in which

$$\begin{aligned}\Gamma_\downarrow &= \frac{\kappa\Omega^2 g^2/\omega_m^2}{(\kappa/2)^2 + (\Delta+2\omega_m)^2}, \\ \Gamma_\uparrow &= \frac{\kappa\Omega^2 g^2/\omega_m^2}{(\kappa/2)^2 + (\Delta-2\omega_m)^2}\end{aligned} \quad (24)$$

describe the strengths of two-phonon absorption and emission processes. The results are consistent with effective two-phonon quadratically mechanical cooling theory [29], in which the cavity mode is linearized and then served as the two-phonon reservoir in the weak optomechanical coupling limit. The set of rate equations in Eq. (23) becomes

$$\begin{aligned}\dot{P}_n =& \gamma_m(n_{\text{th}}+1)[(n+1)P_{n+1} - nP_n] \\ &- \gamma_m n_{\text{th}}[(n+1)P_n - nP_{n-1}] \\ &+ \Gamma_\downarrow[(n+1)(n+2)P_{n+2} - n(n-1)P_n] \\ &- \Gamma_\uparrow[(n+1)(n+2)P_n - n(n-1)P_{n-2}].\end{aligned} \quad (25)$$

For the high-Q mechanical oscillator and low initial temperature, i.e., $\gamma_m n_{\text{th}} \ll \Gamma_{\downarrow\uparrow}$, we can ignore the thermal damping effects. The cooling process turns out to be the pure two-phonon transitions, and the phonon number distribution P_n can be analytically solved and expressed in the form

$$P_{2n+j}^{(2)} = (1-r)r^n(\gamma + j - 1)(-1)^{j-1}, \; j = 0,1, \quad (26)$$

where $r = \Gamma_\uparrow/\Gamma_\downarrow$, and γ characterizes the relative weight of the odd phonon states determined by the initial conditions [35]. In the resolved-sideband regime, when the detuning satisfies $\Delta = -2\omega_m$, the two-phonon absorption process dominates the emission process, i.e., $r \ll 1$, and the phonon number distribution is concentrated at zero- and one-phonon states $P_0 = 1-\gamma$, $P_1 = \gamma$, which indicates that two-phonon cooling processes preserve the phonon-number parity, leading to the initial odd phonon states cooled to the one-phonon state and the even phonon states cooled to the zero-phonon state independently.

In the strong two-phonon absorption regime $\Gamma_\downarrow \gg \gamma n_{\text{th}} \gg \Gamma_\uparrow$, which is achievable with $\Delta = -2\omega_m$ in the resolved-sideband limit, finally we can obtain [35]

$$\begin{aligned}P_0 &= \frac{1+2\xi}{1+3\xi} + O\left(\frac{\gamma_m n_{\text{th}}}{\Gamma_\downarrow}\right), \\ P_1 &= \frac{\xi}{1+3\xi} + O\left(\frac{\gamma_m n_{\text{th}}}{\Gamma_\downarrow}\right), \quad \xi = \frac{n_{\text{th}}}{n_{\text{th}}+1}.\end{aligned} \quad (27)$$

Thus the minimal mean phonon number $\langle\hat{n}\rangle = 1/(4+1/n_{\text{th}})$. To investigate the statistical properties of the phonon field, we

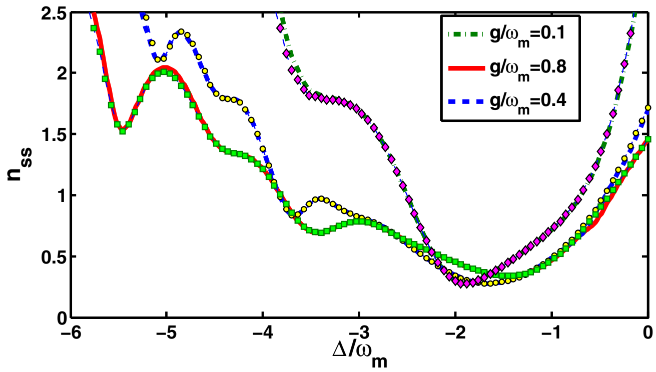

FIG. 1. (Color online) Steady-state phonon number $n_{ss} = \langle \hat{b}^\dagger \hat{b} \rangle$ in a single-photon quadratical optomechanical system as a function of detuning Δ with the coupling strengths $g/\omega_m = 0.8$ (red solid line and green squares), $g/\omega_m = 0.4$ (blue dashed line and yellow circles), and $g/\omega_m = 0.1$ (green dash-dotted line and magenta diamonds). The other parameters are $\gamma_m/\omega_m = 10^{-6}$, $n_{\text{th}} = 10$, $\kappa/\omega_m = 0.25$, and $\Omega/\kappa = 0.4$. [In the figure the lines are obtained from the simulation of the master equation (29) by using the Quantum Optics Toolbox and the markers are obtained by solving the rate equations in Eq. (23)].

resort to the Mandel Q factor that is defined as [36]

$$Q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle} - 1 \tag{28}$$

with $\langle \hat{n} \rangle = \sum_{n=0}^{\infty} n P_n, \langle \hat{n}^2 \rangle = \sum_{n=0}^{\infty} n^2 P_n$. Here, the $Q \approx -\langle \hat{n} \rangle$ becomes negative and the phonon number distribution is sub-Poissonian, which implies that the membrane is in a nonclassical mechanical state.

B. Cooling process in the strong coupling limit

In the strong coupling regime higher-order mechanical sidebands appear and the multiple two-phonon transitions begin to work. In Fig. 1 we plot the steady-state phonon number $n_{ss} = \langle \hat{b}^\dagger \hat{b} \rangle$ as a function of detuning $\Delta = \omega_L - \omega_R$ with different coupling strengths g/ω_m. We observe that in the weak coupling and the resolved-sideband limits, i.e., $g/\omega_m = 0.1$, the optimal cooling occurs at the detuning $\Delta = -2\omega_m$ and the cooling limit reaches a value close to 0.25, which is in accordance with Eq. (27). In the strong coupling limit, the efficient cooling processes occur at the multiple two-phonon resonances $\Delta = -\delta^{(1)}(l,n)$, where the multiple dips of the evolution of n_{ss} indicate the occurrence of the multiple two-phonon transitions, which will further influence the statistical properties of the mechanical oscillator.

To indicate the validity of the set of rate equations obtained by scattering theory in Eq. (23), we compare the steady-state phonon number with numerical simulation of the quantum master equation

$$\frac{d}{dt}\hat{\rho} = -i[\tilde{\tilde{H}},\hat{\rho}] + \frac{\kappa}{2}\mathcal{L}[\hat{a}]\hat{\rho} + \gamma_m(n_{\text{th}}+1)\mathcal{L}[\hat{b}]\hat{\rho} + \gamma_m n_{\text{th}}\mathcal{L}[\hat{b}^\dagger]\hat{\rho}, \tag{29}$$

performed with the Quantum Optics Toolbox [37], where the Hamiltonian is $\tilde{\tilde{H}} = -\Delta\hat{a}^\dagger\hat{a} + \omega_m\hat{b}^\dagger\hat{b} + g\hat{a}^\dagger\hat{a}(\hat{b}+\hat{b}^\dagger)^2 + \Omega(\hat{a}+\hat{a}^\dagger)$, and $\mathcal{L}[\hat{o}]\hat{\rho} = \hat{o}\hat{\rho}\hat{o}^\dagger - \frac{1}{2}\{\hat{o}^\dagger\hat{o},\hat{\rho}\}$ is the Lindblad operator of photon (phonon) dissipation. In Fig. 1, we plot the results of master equation (29) in various styles of lines, which are respectively red solid, blue dashed, and green dash-dotted lines for the different coupling strengths $g/\omega_m = 0.8, 0.4, 0.1$, and specific markers, which are green squares, yellow circles, and magenta diamonds, are obtained by solving the set of rate equations (23). These two results are well matched, and the rate equation approach coincides with the master equation. Thus it demonstrates that scattering theory is feasible in describing nonlinear quadratical cooling processes, and the clear view of the underlying physics in the cooling process presented by the analytical expressions is also acceptable.

IV. STATISTICAL PROPERTIES OF STEADY MECHANICAL STATE

The occurrence of the multiple two-phonon transitions in cooling processes, especially in the strong coupling regime, decreases the membrane's vibrating energy containing several two-phonons simultaneously, and together with the modification of the resonant frequency of the membrane due to the quadratical coupling, the mechanical oscillator can present different statistical properties compared with linear coupling and linearized optomechanics. In this section we mainly resort to the Mandel Q parameter and phonon number fluctuations F to discuss the mechanical statistical properties.

By using the Mandel Q parameter it is convenient to characterize nonclassical states with negative values, which indicate a sub-Poissonian statistics and have a nonclassical analogue. In Fig. 2, we plot the Mandel Q parameter as a function of detuning with different quadratical coupling strengths. In the weak coupling regime, i.e., $g/\omega_m = 0.1$, the minimum value of Q becomes negative, that is, close to the value $-\langle \hat{n} \rangle$, which agrees with the results of Eq. (27). With the increasing coupling strength, Q can present a negative value as well. For example, at $g/\omega_m = 0.4$ it takes a smaller value. Thus the mechanical oscillator is in a nonclassical state. However, in the strong linear coupling optomechanics [11,28], it is relatively difficult to present the nonclassical properties in the weak driving regime.

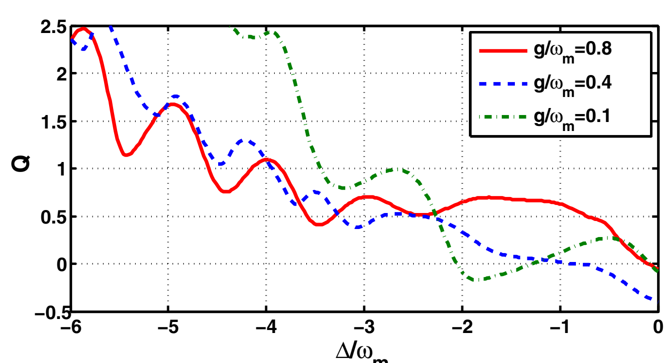

FIG. 2. (Color online) Mandel Q parameter in a single-photon quadratical optomechanical system as a function of detuning Δ with the coupling strengths $g/\omega_m = 0.8$ (red solid line), $g/\omega_m = 0.4$ (blue dashed line), and $g/\omega_m = 0.1$ (green dash-dotted line). The other parameters are $\gamma_m/\omega_m = 10^{-6}$, $n_{\text{th}} = 10$, $\kappa/\omega_m = 0.25$, and $\Omega/\kappa = 0.4$.

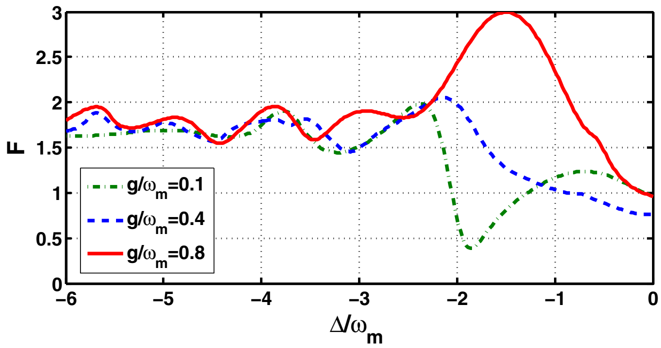

FIG. 3. (Color online) Phonon number fluctuations F in a single-photon quadratical optomechanical system as functions of detuning Δ with the coupling strengths $g/\omega_m = 0.8$ (red solid line), $g/\omega_m = 0.4$ (blue dashed line), and $g/\omega_m = 0.1$ (green dash-dotted line). The other parameters are $\gamma_m/\omega_m = 10^{-6}$, $n_{\rm th} = 10$, $\kappa/\omega_m = 0.25$, and $\Omega/\kappa = 0.4$.

In the linearized treatment of the mechanical cooling regime, the rate equation satisfies the detailed balance condition which implies that the final phonon state is in a thermal state. For the thermal state, $\langle\hat{n}\rangle = n_{\rm th}$, $\langle\hat{n}^2\rangle = 2n_{\rm th}^2 + n_{\rm th}$, yielding the phonon number fluctuations $F = \langle\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}\rangle/\langle\hat{b}^\dagger\hat{b}\rangle^2 = 2$ [38]. Here the rate equations (23) obviously do not obey the detailed balance condition and the steady mechanical state will be in a nonthermal state. It is more clearly exhibited in the single two-phonon cooling process of the weak coupling limit in Eq. (25). In Fig. 3 we plot the phonon number fluctuations F as functions of detuning Δ with different coupling strengths. Since phonon number fluctuations $F = 2$ for the thermal mechanical state, in nonlinear intrinsic quadratical optomechanics we can both decrease and increase the number fluctuations ($F < 2$ and $F > 2$), and the fluctuations can be suppressed for a large range of detuning. Moreover, compared with the optomechanical cooling induced by single-photon linear coupling where the fluctuation is mainly enlarged, here the mechanical cooling occurs with the mainly decreased phonon number fluctuations, which possesses potential applications of QIP and measurement. The suppression is induced by the (multiple) two-phonon cooling processes that decrease the phonon occupation in the zero- and one-phonon number states, and compared to one-phonon cooling processes it is more achievable to suppress the phonon number fluctuations [11,29].

Finally, the phonon number distributions are detectable by using the QND measurement, such as in methods of cold atoms in an optical lattice system [39,40], where another auxiliary probe cavity mode should also quadratically couple to the membrane. For the interaction between the probe field and the membrane, the rotating-wave approximation should be justified and the cross-Kerr coupling between the cavity and membrane modes is achievable. Then it is feasible for the QND measurement to probe the phonon states [41].

V. CONCLUSIONS

In summary, we have studied the nonlinear mechanical cooling processes in the intrinsic quadratically optomechanical coupling regime without linearizing the optomechanical interaction. We apply scattering theory to calculate the transition rates between different mechanical Fock states using the Hamiltonian resolvent method, since the approach can present a direct identification of the underlying physical processes. Due to the preservation of phonon number parity, only even-phonon transitions are permitted and the odd-phonon transitions are forbidden. In the paper we first derive the phononic transition rates, and verify the feasibility of the scattering approach by comparing with the results of simulation of the full quantum master equation. We also discuss the analytical mechanical cooling limits in the weak coupling limit, and find that they coincide with two-phonon mechanical cooling processes. Finally, the statistical properties of the mechanical state are presented. The mechanical state can be in a sub-Poissonian distribution that characterizes the nonclassical state, and the even-phonon transitions favor suppressing the phonon number fluctuations for a large range of detuning as well.

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