Controlling disorder with periodically modulated interactions

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We investigate the celebrated problem of the one-dimensional tight-binding model in the presence of disorder leading to Anderson localization from a novel perspective. A binary disorder is assumed to be created by immobile, heavy particles that affect the motion of the lighter, mobile species in the limit of no interaction between mobile particles. Fast, periodic modulations of interspecies interactions allow us to produce an effective model with small diagonal and large off-diagonal disorder previously unexplored in cold-atom experiments. We present an expression for an approximate Anderson localization length and verify the existence of the well-known, extended resonant mode. We also analyze the influence of nonzero next-nearest-neighbor hopping terms. We point out that periodic modulation of interaction allows disorder to work as a tunable bandpass filter for momenta.

I. INTRODUCTION

Anderson localization (AL) in disordered systems has fascinated and stimulated physicists for more than 50 yr [1,2]. For one-dimensional (1D) systems, as discussed below, a particle propagating with momentum k, in a disordered medium, undergoes multiple scatterings and eventually localizes with an exponentially decaying density profile [3–5]. AL is a single-particle interference effect which cannot be observed directly in solid-state systems due to the presence of electron-electron and electron-phonon interactions. AL has been widely studied for various systems including tight-binding models with diagonal disorder and nearest-neighbor and/or next-nearest-neighbor hopping [6–9].

Ultracold atomic gases have become a playground where complex systems can be simulated and investigated [10–12]. In particular, optical lattice engineering allows a high degree of controllability. Various techniques, i.e., periodic modulation of lattice positions and amplitudes [13–15] or Feshbach resonance [16], are used to effectively tune parameters of a given model [17–21]. The level of experimental control and detection allows one to build quantum simulators, i.e., experimentally controlled systems that are able to mimic other systems difficult to investigate directly [22,23]. Ultracold atomic gases are ideal systems for theoretical and experimental investigation of the Anderson localization of matter waves. An experimental observation of AL was first realized in one dimension 7 yr ago [24] (a closely related Aubry-André [25] localization was also realized [26] independently). For reviews see Refs. [27,28]. Recently AL was observed also in three dimensions [29,30] in speckle potentials. In all these experiments, in order to get rid of particle interactions, either Feshbach resonances [16] were employed or a low atomic density limit was reached.

Typically in cold-atom disorder experiments, as well as theoretical propositions, the disorder appears in a diagonal form, either on the chemical potential level (lattice site energies) like for quasiperiodic lattices [26] or for interactions [31]. Similarly, a diagonal disorder appears for a binary disorder that is introduced by interactions with other species [32–36]. On the other hand, a more detailed analysis shows that for cold

atoms the off-diagonal disorder, i.e., a disorder in tunnelings, appears in quite a natural way both for the incommensurate superlattice potential [37] and for the speckle random potential perturbing the optical lattice [38]. However in both these cases, the disorder in tunneling is strongly correlated with the diagonal disorder. The aim of this paper is to show that periodic modulations of interactions allow a transfer of the disorder to the kinetic energy (tunneling terms) creating a tunable off-diagonal disorder, opening up the possibility of its study in controllable cold-atom settings.

The interest in off-diagonal disorder stems from the fact that it can profoundly affect the properties of the system. Consider, for example, the case when the disorder is purely off-diagonal. There has been a long debate about the nature of states in the center of the band. The first works [39–41] showed that the localization length diverges in the center of the energy band (i.e., $E = 0$); therefore it was argued that a transparency window appears and even the 1D system exhibits the mobility edge. However, in the early 1980s it was established that even for a purely off-diagonal disorder all states are localized [42–45]. The puzzle of the $E = 0$ state was solved showing that the wave-function envelope scales as $\exp(-\gamma\sqrt{N})$, with N being the system size [42]. In the presence of both diagonal and off-diagonal disorder all states remain localized [46] except very special cases of correlated diagonal and off-diagonal disorder [47]. Additional details, especially in the context of conductance anomalies, may be found in Ref. [48].

Possible correlations in the system and/or in the disorder may also profoundly affect localization properties [49–51]. A famous example is the dimer model [50] in which sites may have energies ϵ_a and ϵ_b with the constraint that ϵ_b sites come only in pairs. This short-range correlation leads to delocalized states. A similar situation occurs for the dual random dimer model (DRDM) in which consecutive sites may not have ϵ_b energy. The model has several applications in different areas from DNA studies [52,53] to photonic systems [54]. A cold-atom version of the DRDM has been proposed by Schaff, Akdeniz, and Vignolo [55], who have shown that by tuning the interaction between atomic species one may observe the localization-delocalization transition and study the

extended resonant mode. In our proposition we modify the approach proposed in Ref. [55] by periodically modulating the interactions which allows us a great freedom in changing the relative importance of diagonal and off-diagonal disorder in the system. For such a model we calculate the approximate AL length and compare it with numerical results. We show that the delocalization window may serve as a narrow energy filter for the particles. In addition we point out that the extended mode is vulnerable to effects due to next-nearest-neighbor tunneling, limiting its existence to relatively deep lattices. Details of numerical methods needed to calculate the AL length with next-nearest-neighbor random tunneling are presented in Appendix B.

II. THE MODEL

We consider a simple, standard 1D noninteracting tight-binding Hamiltonian (with $\hbar = 1$):

$$H = \sum_i [\epsilon_i n_i - J(a_i^\dagger a_{i+1} + \text{H.c.})], \qquad (1)$$

where a_i ($a_i^\dagger$) denotes an annihilation (creation) operator of bosons at site i, $n_i = a_i^\dagger a_i$ is a particle counting operator, ϵ_i are on-site energies, and J is the tunneling amplitude which will serve as our energy scale. Let us apply a periodic modulation of the on-site energies,

$$\epsilon_i \rightarrow \epsilon_i[1 + \delta \sin(\omega t)], \qquad (2)$$

where δ is the modulation amplitude and ω its frequency.

Hamiltonian (1) has periodic time dependence $H(t) = H(t + 2\pi/\omega)$. In such a case we can use the well-established formalism of Floquet theory [56] (see also Ref. [57]) extensively used, in the last century, to study the influence of optical or microwave fields on atoms. Recently it has been applied with great success for controlling properties of ultracold atomic systems [13–15]. Solutions of the equation $i\partial_t|\psi_n(t)\rangle = H(t)|\psi_n(t)\rangle$ have the form of Floquet states: $|\psi_n(t)\rangle = e^{-i\varepsilon_n t}|u_n(t)\rangle$, where ε_n is called the quasienergy and $|u_n(t)\rangle$ have periodicity of the Hamiltonian. The Floquet theorem is a time analog of the Bloch theorem for spatially periodic potentials. Although we cannot treat $|\psi(t)\rangle$ as eigenstates of $H(t)$, $|u(t)\rangle$ are eigenstates of the Floquet Hamiltonian $\mathcal{H}(t) = H(t) - i\partial_t$ existing in the extended space of T-periodic functions. In that space we can number states using a new quantum number $m \in \mathbb{Z}$: $|u_n^m\rangle = |u_n\rangle e^{i\omega m t}$, where $|u_n^m\rangle$ is the eigenstate to the eigenenergy $\varepsilon_n^m = \varepsilon_n + \omega m$. This whole class corresponds to one physical state, $|\psi_n(t)\rangle$, as adding ω to ε_n in physical space only takes us to the next "Brillouin zone" for quasienergies. Therefore, it is sufficient to find a block-diagonal form of a Hamiltonian (in an m-ordered basis) and consider only one block corresponding to a single "Brillouin zone." Unfortunately, couplings between different blocks make the block diagonalization a formidable task. One may, however, make a unitary transformation U:

$$\mathcal{H}' = U\mathcal{H}U^\dagger, \quad U = \exp\left[-i\delta\frac{\cos(\omega t)}{\omega}\sum_i \epsilon_i n_i\right], \qquad (3)$$

$$\mathcal{H}'(t) = \sum_i [\epsilon_i n_i - (e^{i(\epsilon_{i+1}-\epsilon_i)\frac{\delta\cos(\omega t)}{\omega}} a_i^\dagger a_{i+1} + \text{H.c.})]. \qquad (4)$$

In effect, for frequencies $\omega \gg 1$ (in units of J) one we can neglect the couplings between different Floquet blocks and consider only one diagonal block governing the long term ($t \gg 1/\omega$) dynamics:

$$H_{\text{eff}} = \frac{1}{T}\int_0^{\frac{2\pi}{\omega}} dt\,\mathcal{H}'(t) = \sum_i [\epsilon_i n_i - (t_i a_i^\dagger a_{i+1} + \text{H.c.})], \qquad (5)$$

where

$$t_i = \mathcal{J}_0\left(\frac{\delta}{\omega}(\epsilon_{i+1} - \epsilon_i)\right) \qquad (6)$$

is the effective position-dependent hopping and $\mathcal{J}_0$ is the zeroth order Bessel function. It has been verified experimentally [14,15] that the effective time-averaged Hamiltonian governs the dynamics of periodically driven systems for long times. In particular, localization properties of eigenstates of Eq. (5) will be shared by the Floquet eigenstates of the original Hamiltonian (1) [with the fast modulation (2)] after averaging them over the period. This may be understood by inspecting the unitary operator U in Eq. (3) and observing that it consists of a product of local operators acting on single sites. Thus U adds only local phases to single-particle states, which cannot affect the resulting density distribution. Further detailed discussion of the construction of the effective Hamiltonians for high-frequency periodically driven systems may be found in Refs. [58–60].

Note that for a uniform system with all ϵ_i being the same, the tunneling (6) is unaffected, while t_i changes if on-site energies vary from site to site. Such is the case for superlattices [61,62] or external potentials such as a harmonic trap or a linear tilt [13]. In this work we will consider the on-site energy variations due to disorder.

It is worth noting that it is usually possible to adiabatically pass from eigenstates of one H_{eff} to another (e.g., for different δ) if the change of parameters is slow enough [63]. Thus it is possible to prepare a time-independent system and turn on periodic modulation or change modulation parameters during experiments. For completeness let us note that, for the tight-binding description in terms of a single lowest band to be valid in Eqs. (1)–(5), ω, while much larger than the tunneling amplitude, must be smaller than the energy separation to the excited band [13,64].

To create disorder we consider two species of atoms repulsively interacting with each other but allow only one of them to move freely on the lattice. The *frozen* atoms (denoted with a superscript f) give rise to a binary disorder [33–35,65]. The mobile particles are assumed not to interact among themselves. They can be spin-polarized fermions or bosons with interactions turned off by microwave or optical Feshbach resonance [16]. The dynamics of the system can be described by a single-particle Hamiltonian with the on-site energy $\epsilon_i = Vn_i^f$, where V denotes an interspecies contact interaction. When *frozen* particles are fermions or strongly repelling (hard-core) bosons their occupation per lattice site n_i^f takes a value of either 0 or 1 and the on-site energy takes only two values, $\epsilon_i \in \{0, V\}$. Further, we consider the particular case of the DRDM when two adjacent sites cannot be occupied by *frozen* atoms simultaneously [50]. For cold atoms

the DRDM can be created by the method described in Ref. [55]. Referring the reader to that paper for details, let us mention only that to assure no heavy particles reside in the neighboring sites one may first trap the heavy species in an auxiliary long-wavelength lattice with the lattice constant being, e.g., three times bigger than the final lattice to be considered. If these heavy particles are strongly repelling and for sufficiently low densities one may assure no double occupancy. Only then one can switch on the desired shorter wavelength lattice which holds the mobile component.

In Ref. [66] the authors considered fast periodic modulation of interactions induced by an appropriate periodic modulation of the magnetic field $B(t) = B(t + T)$ with period $T = \frac{2\pi}{\omega}$ for interacting bosons in the optical lattice. We assume a similar mechanism applied to light-heavy particle interactions resulting in periodic (for simplicity assumed to be harmonic) time dependence of site energies in the form $\epsilon_i(t) = n_i^f[V_0 + V_1 \sin(\omega t)]$, where V_1 is the amplitude modulation strength and ω is the modulation frequency. For magnetic field values close to the Feshbach resonance [16] large variations of interactions may be expected. In particular, choosing the value of the mean magnetic field around which the oscillations occur one may vary at will the relative importance of the V_0 and V_1 coefficients. Note that we can adjust the magnetic field for this purpose since we assume that the mobile particles are either fermions or bosons with interactions turned off by either microwave or optical Feshbach resonance [16].

Compared to the general model discussed above we have $\epsilon_i = n_i^f V_0$ and $\delta = V_1/V_0$. As $n_i^f \in \{0,1\}$, the effective tunneling, after time averaging, can take only two values:

$$\epsilon_i = \begin{cases} 0, & \text{if } n_i^f = 0, \\ V_0, & \text{if } n_i^f = 1, \end{cases} \quad t_i = \begin{cases} 1, & \text{if } n_i^f = n_{i+1}^f, \\ 1 - \gamma, & \text{if } n_i^f \neq n_{i+1}^f, \end{cases}$$
(7)

where $\gamma = 1 - \mathcal{J}_0(V_1/\omega)$ measures the strength of off-diagonal disorder and varies in range from 0 to about 1.4 [to the minimum of the Bessel function, $1 + \min_x \mathcal{J}_0(x)$].

III. ANDERSON LOCALIZATION LENGTH

To calculate the AL length let us start with the time-independent Schrödinger equation for the disordered tight-binding Hamiltonian (5):

$$-t_i \psi_{i+1} - t_{i-1} \psi_{i-1} + \epsilon_i \psi_i = \tilde{E} \psi_i,$$
(8)

where $\tilde{E}$ is the eigenenergy. In the regime of small diagonal and off-diagonal disorder ($V_0 \ll 1$, $\gamma \ll 1$) we can assume that $\tilde{E}$ is approximately given by the dispersion relation $\tilde{E} \approx -2\cos(k)$, where k is the quasimomentum. In order to define the AL length in a system with off-diagonal disorder we utilize the unitary transformation $\psi_i = \phi_i \eta_i$, where $\eta_i = 1/[t_i \eta_{i-1}]$. We transform Eq. (8) to the diagonal form:

$$\phi_{i+1} + \phi_{i-1} + \tilde{V}_i \phi_i = 0,$$
(9)

where $\tilde{V}_i = |\eta_i|^2[\epsilon_i + 2\cos(k)]$ is a new effective diagonal disorder. In the considered DRDM model, $\eta_i = 1/t_i$. Equation (9)

can be expressed as a 2D Hamiltonian map with position and momentum of the form $x_i = \phi_i$ and $p_i = (\phi_i \cos(k) + \phi_{i-1})/\sin(k)$:

$$\begin{cases} x_{i+1} = -(p_i + A_i x_i)\sin(k) + x_i \cos(k), \\ p_{i+1} = (p_i + A_i x_i)\cos(k) + x_i \sin(k), \end{cases}$$
(10)

where

$$A_i = -\frac{\epsilon_i - 2(1 - |t_i|^2)\cos(k)}{\sin(k)}.$$
(11)

The map (10) describes the behavior of a harmonic oscillator under periodic δ kicks with amplitude A_i. Expressing the map in action-angle variables and iterating it, one may estimate the localization length in the limit of small diagonal and off-diagonal disorder. The details are given in Appendix A. We obtain the approximate inverse localization length by

$$\lambda^{-1} = \frac{\rho}{(1 + \rho)^2} \frac{[V_0 + 2\gamma(2 - \gamma)\cos(k)]^2}{8\sin^2(k)}$$
$$\times \left(1 - 2\frac{\rho[\rho + \cos(2k)]}{1 + \rho^2 + 2\rho\cos(2k)}\right),$$
(12)

where $\rho = \tilde{\rho}/(1 - \tilde{\rho})$ and $\tilde{\rho}$ is the mean occupation number of *frozen* particles. A simple analysis of Eq. (12) indicates an anomalous behavior of the AL length due to the existence of a transparent mode with momentum k_t given by the condition $\cos(k_t) = -\frac{V_0}{2\gamma(2-\gamma)}$ for which the localization length diverges. *Frozen* particles in a lattice increase the on-site energy and the motion of mobile particles can be seen as a sequence of scattering processes on potential barriers. In the DRDM two *frozen* particles are separated by at least one lattice site and the resonance is caused by a lossless transmission through a single barrier. The existence of the transparent mode is given by the condition $V_0 \leqslant 2\gamma(2 - \gamma)$ and the wave with momentum k_t travels through the sample without reflection. A similar anomalous mode was observed in Ref. [55]. The on-site energy modulation allows us to change the off-diagonal disorder significantly while keeping the amplitude of the diagonal disorder small (contrary to models deriving the changes of the tunneling from changes of an effective lattice shape only [55,67]). In Fig. 1 we present the numerically calculated AL lengths (using the standard transfer matrix method, see, e.g., Ref. [68]) and compare them with the analytical expression (12). The left panel corresponds to the weak diagonal ($V_0 \in \{0.05, 0.1\}$) and weak off-diagonal ($\gamma \in \{0.05, 0.1\}$) regimes. The occupation of *frozen* particles is equal to $\tilde{\rho} = 1/3$. The theoretical predictions are in good agreement with the numerical calculations. The right panel corresponds to the weak diagonal ($V_0 = 0.1$) and strong off-diagonal ($\gamma = 0.9$) regimes. The width of the divergent AL length window is significantly narrower than that for the weak disorder case. The position of the transparent mode k_t is properly described by Eq. (12) while the AL length shape is not. Due to the existence of the transparent mode in the system, one can expect that the evolution of an initial wave packet with a given momentum distribution results in AL of all momenta except those very close to the transparent mode k_t. Disorder effectively works as a bandpass filter for momenta. To verify this behavior we integrated the Schrödinger equation in time for the Hamiltonian (5) with
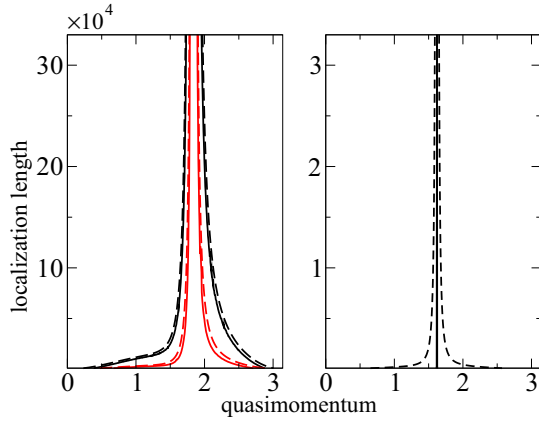
FIG. 1. (Color online) AL length obtained from transfer matrix calculations (solid lines) and from Eq. (12) (dashed lines). Left panel: AL length as a function of quasimomentum for two sets of diagonal and off-diagonal disorder amplitudes: {$V_0 = 0.05$, $\gamma = 0.05$} (black) and {$V_0 = 0.1$, $\gamma = 0.1$} (red [gray]). Filling $\tilde{\rho} = 1/3$. Right panel: AL length as a function of quasimomentum from transfer matrix calculations (black solid line) for diagonal disorder amplitude $V_0 = 0.1$, off-diagonal disorder amplitude $\gamma = 0.9$, and occupation $\tilde{\rho} = 1/3$. For comparison with Eq. (12) we present the analytical expression (black dashed line).

the initial state consisting of one particle in the center of the lattice with uniform momentum distribution. Figure 2 presents momentum distributions of Anderson localized atoms and those that escaped from the disorder after time evolution
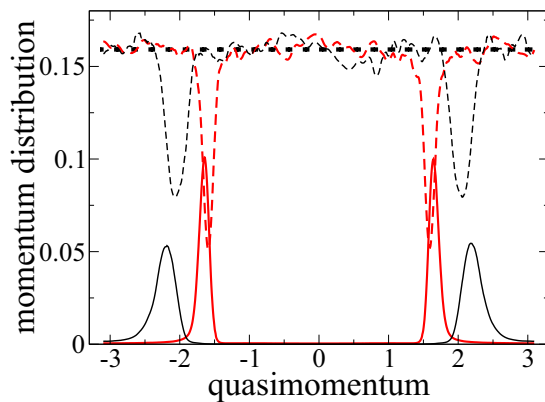

FIG. 2. (Color online) The horizontal dotted thick line represents the initial wave-packet momentum distribution. Solid lines present momentum distribution of wave function outside area of the disorder while dashed line the distribution of momenta that remain in the system. $N = 1000$ lattice sites are considered with filling $\tilde{\rho} = 1/3$. Two cases are plotted. The data for diagonal and off-diagonal amplitudes $V_0 = 0.1$, $\gamma = 0.5$ are represented by red (thick) curves while those for $V_0 = 0.5, \gamma = 0.3$ by black (thin) lines. The evolution time is $t = 1000$. The position of dips in momentum space agree with the position of transparent modes $k_t \approx 1.64$ and $k_t \approx 2.08$, respectively. One can observe two peaks in the momentum distributions of escaped atoms. Due to a marginal difference in the dispersion relation between in and outside of the disorder, the positions of peaks and dips are slightly shifted. These results are obtained as the average over 1000 disorder realizations.

$t = 1000$. The momenta of escaped atoms reveal narrow peaks at the position of the transparent mode while the momentum distribution of Anderson localized atoms reveal dips, positions of which agree with the former peaks. The lattice has $N = 1024$ sites with filling $\tilde{\rho} = 1/3$. We choose two sets of parameters for the amplitude of diagonal and off-diagonal disorder ($V_0 = 0.1$, $\gamma = 0.5$) and ($V_0 = 0.5$, $\gamma = 0.3$). For such parameters the transparent mode appears for $k_t \approx 1.64$ and $k_t \approx 2.08$, respectively. Our system presents, therefore, a promising candidate for obtaining a highly controllable monochromatic gun for matter waves. A similar mechanism was observed in Ref. [69] where coherent multiple scattering processes determined the emitted matter-wave mode. In the following we inspect residual effects that may affect the existence of the transparent mode for a realistic system. In particular, in the next section we analyze the influence of the next-nearest-neighbor tunneling on AL length in the regime of strong off-diagonal disorder.

IV. NEXT-NEAREST-NEIGHBOR TUNNELINGS

For lattice depths $U \approx 10$–$20\,E_R$, where E_R is the lattice recoil energy and the typical optical laser wavelengths, the next-nearest-neighbor hopping amplitude t' ranges between 0.01 and 0.001, as can easily be estimated with appropriate Wannier functions. This is why in typical situations long-range hopping is often negligible. Still let us add the next-nearest-neighbor tunneling term to the Hamiltonian (1):

$$H \rightarrow H + t' \sum_i (a_i^\dagger a_{i+2} + \text{H.c.}). \tag{13}$$

Under the same unitary transformation (3) and after time averaging we obtain the effective Hamiltonian in the form of Eq. (5) plus the additional term reading

$$\sum_i t_i'(a_i^\dagger a_{i+2} + \text{H.c.}), \tag{14}$$

with $t_i' = t' \mathcal{J}_0[\frac{V_0}{\omega}(\epsilon_{i+2} - \epsilon_i)]$. We numerically calculate the AL length for this new effective Hamiltonian. The method we use is applicable to finite but arbitrarily long-range hopping. The transfer matrix approach is not directly applicable so we transform the problem from a 1D to a 2D stripe. This could be achieved with a simple winding of a string (see Fig. 3). In a stripe all particle hoppings are between the neighboring slices, which simplifies numerical calculations.
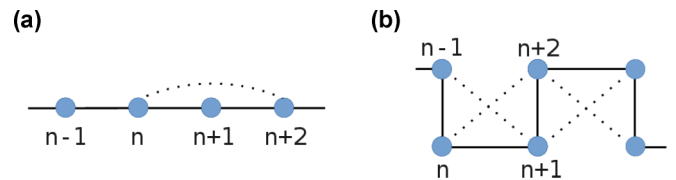

FIG. 3. (Color online) (a) One-dimensional gas on a lattice with nearest-neighbor hopping (solid line) and next-nearest-neighbor hopping (dotted line). (b) The same system can be viewed and described differently, namely, as a stripe with nonuniform tunneling. The advantage is that the hopping of particles takes place between neighboring slices only.
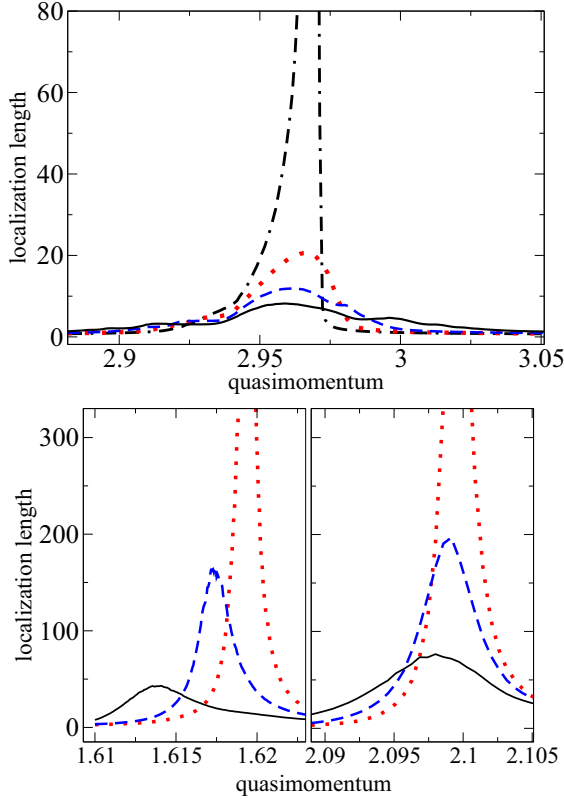
FIG. 4. (Color online) AL length as a function of quasimomentum for diagonal disorder $V_0 = 1.95$ (top panel), $V_0 = 1$ (right bottom panel), and $V_0 = 0.1$ (left bottom panel), $\gamma = 0.9$, occupation $\tilde{\rho} = 1/3$. The values of the next-nearest-neighbor hopping are $t' = 0.01$ (black solid line), $t' = 0.005$ (blue dashed line), $t' = 0.0025$ (red dotted line), and $t' = 0.0005$ (black dashed-dotted line). The resonance diminishes when t' grows. The effect is the strongest for $V_0 = 1.95$ because we are close to a regime without a delocalized mode.

In Appendix B we present the derivation of the equation for a Green's function between the first and the last slice of the stripe. Knowing the Green's function we can easily calculate the localization length [70,71].

As exemplary parameters for numerical analysis we choose $V_0 = \{1.95, 1, 0.1\}$ and $\gamma = 0.9$ in order to obtain a regime where a next-nearest-neighbor hopping significantly affects the dynamics of the system. The results are presented in Fig. 4. We observe that even extremely small values of t' significantly affect the resonance. The presence of a small but nonzero next-nearest-neighbor tunneling t' significantly reduces the localization length for the transparent mode k_t due to a nonzero probability that a mobile particle scatters back on a *frozen* particle through a next-nearest-neighbor hopping. This process can appear when two *frozen* particles are separated by a single site. Indeed, when we exclude such configurations, the resonance reappears. In deep lattices next-nearest-neighbor tunneling also reduces the localization length for the transparent mode, but the effect is weaker and may not affect the selective emission due to a finite optical lattice size (when this size is smaller than the localization length).

V. CONCLUSIONS

In this paper we proposed a method to realize a controllable off-diagonal disorder with binary random potential resulting from time-periodic modulation of interactions between mobile and frozen particles in an optical lattice. Since no interactions between mobile particles are taken into account they may be assumed to be spinless (spin-polarized) fermions. One could also imagine a scheme with bosons with interactions turned off by some microwave or optical Feshbach resonance [16] (note that we already assume using a magnetic field for a standard Feshbach tuning of different species interactions so this method cannot be used simultaneously to control light-light particle collisions). The presented method allows us to obtain models with strong off-diagonal and weak diagonal disorder in a broad regime of relative (off-)diagonal disorder amplitudes. In a regime of weak diagonal and off-diagonal disorder, an analytical expression for the AL length is in very good agreement with numerical simulations. Moreover, we indicate how the DRDM with random hopping can work as a tunable bandpass filter for matter waves. The momentum of escaped atoms reveals narrow peaks in the position of the transparent mode in momentum space while momentum distribution of the Anderson localized atoms reveals dips whose positions agree with former peaks. We indicate the importance of the next-nearest-neighbor hopping for the localization length of the resonant extended mode appearing within the DRDM for strong diagonal disorder.

APPENDIX A: HAMILTONIAN APPROACH TO ANDERSON LOCALIZATION LENGTH

To find the localization length starting from the map (10) [47,72,73] it is more convenient to express it in the action-angle variables (r, θ) using the transformation $x = r \sin(\theta)$, $p = r \cos(\theta)$:

$$\begin{cases} \sin(\theta_{i+1}) = D_i^{-1}[\sin(\theta_i - k) - A_i \sin(\theta_i) \sin(k)], \\ \cos(\theta_{i+1}) = D_i^{-1}[\cos(\theta_i - k) + A_i \sin(\theta_i) \cos(k)], \end{cases} \quad \text{(A1)}$$

where

$$D_i \equiv \frac{r_{i+1}}{r_i} = \sqrt{1 + 2A_i \sin(2\theta_i) + A_i^2 \sin^2(\theta_i)}. \quad \text{(A2)}$$

We define the AL length as

$$\lambda^{-1} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln D_i, \quad \text{(A3)}$$

and after expanding the logarithm to the second order in A_i we get

$$\lambda^{-1} = \tfrac{1}{8}\langle A_i^2 \rangle + \tfrac{1}{2}\langle A_i \sin(2\theta_i) \rangle, \quad \text{(A4)}$$

where $\langle \cdots \rangle$ stands for averaging over i. In order to calculate the "kick-angle" correlator $\langle A_i \sin(2\theta_i) \rangle$ we expand the map (A1) to second order in θ_i,

$$\theta_i = \theta_{i-1} - k + A_{i-1} \frac{\sin^2(\theta_{i-1})}{\sin(k)}, \tag{A5}$$

and express $\langle A_i \sin(2\theta_i) \rangle$ in terms of the preceding $\langle A_i \sin(2\theta_{i-1}) \rangle$ kick-angle correlator. Let us introduce the correlation of the kick's strength A_i with angle θ_i:

$$a_n = -\frac{2i}{\langle A_i^2 \rangle} e^{2ik} \langle A_i e^{2i\theta_{i-n}} \rangle, \tag{A6}$$

where $\langle A_i^2 \rangle$ is variance of the A_i. Multiple applications of Eq. (A5) to a_n give us the recursive relation

$$a_{n-1} = e^{-2ik} a_n + q_n, \tag{A7}$$

where $q_n = \langle A_i A_{i-n} \rangle / \langle A_i^2 \rangle$ is the autocorrelation of A_i. From the definition of a_n we can notice that

$$\langle A_i \sin(2\theta_i) \rangle = \mathrm{Re}\left(\frac{\langle A_i^2 \rangle}{2} e^{-2ik} a_0 \right), \tag{A8}$$

where from Eq. (A7) we obtain $a_0 = \sum_{n=1}^{\infty} q_n e^{-2ik(n-1)}$.

The inverse AL length λ^{-1} takes the form [72]

$$\lambda^{-1} = \frac{\langle A_i^2 \rangle}{8} \left(1 + 2 \sum_{n=1}^{\infty} q_n(k) \cos(2kn) \right). \tag{A9}$$

In the specific case of the DRDM, the variance of A_i and the autocorrelation of A_i read, respectively,

$$\langle A_i^2 \rangle = \frac{\rho}{(1+\rho)^2} \frac{[V_0 + 2\gamma(2-\gamma)\cos(k)]^2}{\sin^2(k)},$$
$$q_n = (-1)^n \rho^n, \tag{A10}$$

and the approximate expression for the inverse localization length takes the form of Eq. (12).

APPENDIX B: ANDERSON LOCALIZATION LENGTH FOR HAMILTONIAN WITH LONG-RANGE RANDOM HOPPING

In this section we describe a method of calculating localization length in a 1D system with long-range tunneling. The idea is to transform the problem from a 1D to a 2D stripe. This could be achieved with a simple winding of a string (see Fig. 3).

1. Mapping to a 2D stripe

To start let us consider a 1D gas on a string of length $\tilde{N}$. We allow long-range tunneling up to the Mth neighboring site. This geometry is equivalent to a 2D stripe of length $N = \tilde{N}/M$ and width M. The evolution of particles in such a stripe is characterized by a Hamiltonian:

$$H(\tilde{N}) = \sum_{n=1}^{N} H_n + \sum_{n=1}^{N-1} (V_n + \mathrm{H.c.}), \tag{B1}$$

where

$$H_n = \sum_{m=1}^{M} \epsilon_m^{(n)} |n,m\rangle\langle n,m|$$
$$- \sum_{m=1}^{M-1} \left(J_{m,m+1}^{(n,n)} |n,m\rangle\langle n,m+1| + \mathrm{H.c.} \right) \tag{B2}$$

is a standard tight-binding Hamiltonian for the nth slice and

$$V_n = -\sum_{m=1}^{M} \sum_{m'=1}^{M'} J_{m,m'}^{(n,n+1)} |n,m\rangle\langle n+1,m'| \tag{B3}$$

describes particles hopping onto the nth slice. $J_{m,m'}^{(n,n')}$ denotes tunneling between $|n,m\rangle$ and $|n',m'\rangle$. Suppose now that we add one extra slice to the system. A new Hamiltonian is straightforward:

$$H(N+1) = H(N) + H_{N+1} + V_N + V_N^\dagger. \tag{B4}$$

For later convenience, let us define the operators

$$H^0(N+1) \equiv H(N) + H_{N+1}, \tag{B5}$$

$$V(N) \equiv V_N + V_N^\dagger, \tag{B6}$$

and finally the Hamiltonian reads

$$H(N+1) = H^0(N+1) + V(N). \tag{B7}$$

2. A recursive equation for a Green's function

Our goal is to find the Green's function $G_E^+(N+1)$ for the Hamiltonian $H(N+1)$. By definition, a Green's function can be obtained from a resolvent of a Hamiltonian,

$$G_E^+(N+1) = \lim_{\epsilon \to 0+} G_{E+i\epsilon}(N+1), \tag{B8}$$

where a resolvent satisfies

$$[z - H(N+1)]G_z(N+1) = 1. \tag{B9}$$

Let's start with an equation for a resolvent of $H(N+1)$:

$$G_z(N+1) = G_z^0(N+1) + G_z^0(N+1)V(N)G_z(N+1), \tag{B10}$$

with G_z^0 being a resolvent of H^0. The resolvent equation (B10) can be derived from a simple operator identity,

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(A-B)\frac{1}{A}, \tag{B11}$$

with a substitution,

$$A = z - H(N+1), \quad B = z - H^0(N+1). \tag{B12}$$

Since we are interested in transport properties of the system, we need to know the Green's function matrix elements between the first and the last slice only. Therefore, we want to obtain $\langle 1|G_z(N+1)|N+1\rangle$. To simplify the equations let us introduce a notation:

$$G_{n,m} \equiv \langle n|G_z(m)|m\rangle. \tag{B13}$$

From equation (B10) we get

$$G_{1,N+1} = G_{1,N} V_N G_{N+1,N+1}, \tag{B14}$$

where we used the observation that

$$G_z^0(N+1) = G_z(N) + \frac{1}{z - H_{N+1}}, \qquad (B15)$$

which stems from the fact that $H(N)$ and H_{N+1} act in orthogonal subspaces of the Hilbert space [see Eq. (B5)].

In order to get a recursive equation for $G_{1,N}$, we need to calculate the $G_{N+1,N+1}$ matrix. It can be obtained from Eq. (B9) by multiplying it from the right side by the projection $P = |N+1\rangle\langle N+1|$ and by P (or $Q = 1 - P$) from the left side:

$$P[z - H(N+1)](P+Q)G_z(N+1)P = P,$$
$$Q[z - H(N+1)](P+Q)G_z(N+1)P = 0. \quad (B16)$$

Solving the set of Eqs. (B16) one gets

$$G_{N+1,N+1} = \frac{1}{z - H_{N+1} - V_N^\dagger G_{N,N} V_N}. \qquad (B17)$$

Combining Eqs. (B14) and (B17),

$$G_{1,N+1}(z - H_N + 1 - V_N^\dagger G_{N,N} V_N) = G_{1,N} V_N \qquad (B18)$$

and, again from Eq. (B14), extracting $G_{N,N}$, we finally obtain a recursive equation:

$$A_{N+2} = (E - H_{N+1})V_N^{-1} A_{N+1} - V_N^\dagger V_{N-1}^{-1} A_N, \qquad (B19)$$

with $A_N = G_{1,N-2}^{-1}$. The initial values of A_N are not relevant, but it is convenient to choose them as

$$A_0 = 0, \quad A_1 = V_0. \qquad (B20)$$

In Eq. (B19) there is no singularity; hence we can replace z with E. From this equation we can calculate $G_{1,N}$, which is connected with the localization length of the system.

3. Calculation of the Anderson localization length

The AL length λ_M in a stripe of width M is defined as

$$\frac{2}{\lambda_M} = -\lim_{n\to\infty} \frac{1}{n} \ln \mathrm{Tr}|G_{1,n}|^2. \qquad (B21)$$

Since our system is in fact 1D the localization length should not depend on the length of the stripe. When a particle reached

the nth slice it covered $\tilde{n} = Mn$ lattice sites. Hence, the localization length λ in one dimension equals $\lambda = M\lambda_M$.

The localization length can be obtained from the recursive equation (B19). We solve the equation iteratively. The problem with this equation is that the elements of A_n grow exponentially for large n, so it requires some regularization. Therefore, in each step we multiply both sides of the equation by some matrix R_n. Starting from $n = 1$ and defining $A_k^{(1)} \equiv A_k R_1$,

$$A_3 = (E - H_2)V_1^{-1} A_2 - V_1^\dagger V_0^{-1} A_1 \mid \times R_1,$$
$$A_3^{(1)} = (E - H_2)V_1^{-1} A_2^{(1)} - V_1^\dagger V_0^{-1} A_1^{(1)}. \qquad (B22)$$

To avoid the exponential growth we put $R_1 = A_2^{-1}$ and

$$A_3^{(1)} = A_3 A_2^{-1}, \quad A_2^{(1)} = 1, \quad A_1^{(1)} = A_1 A_2^{-1}. \qquad (B23)$$

We repeat this procedure in every iteration so that $A_k^{(n)} = A_k^{(n-1)} R_n$, with $R_n = [A_{n+1}^{(n-1)}]^{-1}$, satisfies

$$A_{n+2}^{(n)} = (E - H_{n+1})V_n^{-1} - V_n^\dagger V_{n-1}^{-1} A_n^{(n)}. \qquad (B24)$$

Let us also define the matrix

$$B^{(n)} = B^{(n-1)} R_n / b_n, \quad b_n = \|B^{(n-1)} R_n\|, \qquad (B25)$$

where $\|\cdot\| = \sqrt{\mathrm{Tr}|\cdot|^2}$ is a matrix norm. The $B^{(n)}$ matrix turns out to be very useful, because

$$b_n = \|B^{(n-1)} R_n\| = \frac{1}{b_{n-1}} \|B^{(n-2)} R_{n-1} R_n\|$$
$$= \frac{1}{b_{n-1}} \|B^{(n-2)} [A_{n+1}^{(n-2)}]^{-1}\|$$
$$= \frac{1}{b_{n-1} b_{n-2}} \|B^{(n-3)} [A_{n+1}^{(n-3)}]^{-1}\| = \cdots. \qquad (B26)$$

Continuing this, we notice that

$$\|A_{n+1}^{-1}\| = b_1 b_2 \ldots b_n, \qquad (B27)$$

$$\ln \mathrm{Tr}|G_{1,n}|^2 = 2(\ln b_{n+1} + \cdots + \ln b_1). \qquad (B28)$$

The AL length can be expressed as

$$\lambda = -M \lim_{n\to\infty} \frac{n}{c_{n+1}}, \quad c_{n+1} = c_n + \ln b_{n+1}. \qquad (B29)$$

Iterations should be continued until λ converges within a desired precision.

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[1] P. Anderson, Phys. Rev. **109**, 1492 (1958).

[2] A. Lagendijk, B. van Tiggelen, and D. Wiersma, Phys. Today **62**, 24 (2009).

[3] N. F. Mott and W. D. Twose, Adv. Phys. **10**, 107 (1961).

[4] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[5] K. Ishii, Suppl. Prog. Theor. Phys. **53**, 77 (1973).

[6] J. Biddle, D. J. Priour, Jr., B. Wang, and S. Das Sarma, Phys. Rev. B **83**, 075105 (2011).

[7] R. Sepehrinia, Phys. Rev. B **82**, 045118 (2010).

[8] A. Eilmes, R. A. Romera, and M. Schreiber, Eur. Phys. J. B **1**, 29 (1998).

[9] T. Yamada and H. Fukui, Nucl. Phys. B **679**, 632 (2004).

[10] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).

[11] D. Jaksch and P. Zoller, Ann. Phys. **315**, 52 (2005).

[12] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Many-Body Quantum Systems* (Oxford University Press, London, 2012).

[13] A. Eckardt, C. Weiss, and M. Holthaus, Phys. Rev. Lett. **95**, 260404 (2005).

[14] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. **99**, 220403 (2007).

[15] J. Struck, C. Ölschläger, R. Le Targat, P. Soltan-Panahi, A. Eckardt, M. Lewenstein, P. Windpassinger, and K. Sengstock, Science **333**, 996 (2011).

[16] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).

[17] J. Struck, C. Ölschläger, M. Weinberg, P. Hauke, J. Simonet, A. Eckardt, M. Lewenstein, K. Sengstock, and P. Windpassinger, Phys. Rev. Lett. **108**, 225304 (2012).

[18] P. Windpassinger and K. Sengstock, Rep. Prog. Phys. **76**, 086401 (2013).

[19] K. Sacha, K. Targońska, and J. Zakrzewski, Phys. Rev. A **85**, 053613 (2012).

[20] A. Eckardt, P. Hauke, P. Soltan-Panahi, C. Becker, K. Sengstock, and M. Lewenstein, Europhys. Lett. **89**, 10010 (2010).

[21] A. Przysiężna, O. Dutta, and J. Zakrzewski, New J. Phys. **17**, 013018 (2015).

[22] I. Buluta and F. Nori, Science **326**, 108 (2009).

[23] P. Hauke, M. Cucchietti, F., L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Rep. Prog. Phys. **75**, 082401 (2012).

[24] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature (London) **453**, 891 (2008).

[25] S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).

[26] G. Roati, C. D'Errico, L. Fallani, M. Fattori, C. For, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).

[27] A. Aspect and M. Inguscio, Phys. Today **62**, 30 (2009).

[28] G. Modugno, Rep. Prog. Phys. **73**, 102401 (2010).

[29] F. Jendrzejewski, A. Bernard, K. Müller, P. Cheinet, V. Josse, M. Piraud, L. Pezzé, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, Nat. Phys. **8**, 398 (2012).

[30] S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, Science **334**, 66 (2011).

[31] H. Gimperlein, S. Wessel, J. Schmiedmayer, and L. Santos, Phys. Rev. Lett. **95**, 170401 (2005).

[32] P. Massignan and Y. Castin, Phys. Rev. A **74**, 013616 (2006).

[33] U. Gavish and Y. Castin, Phys. Rev. Lett. **95**, 020401 (2005).

[34] S. Ospelkaus, C. Ospelkaus, O. Wille, M. Succo, P. Ernst, K. Sengstock, and K. Bongs, Phys. Rev. Lett. **96**, 180403 (2006).

[35] B. Horstmann, J. I. Cirac, and T. Roscilde, Phys. Rev. A **76**, 043625 (2007).

[36] K. V. Krutitsky, M. Thorwart, R. Egger, and R. Graham, Phys. Rev. A **77**, 053609 (2008).

[37] T. Roscilde, Phys. Rev. A **77**, 063605 (2008).

[38] D. Semmler, K. Byczuk, and W. Hofstetter, Phys. Rev. B **81**, 115111 (2010).

[39] P. D. Antoniou and E. N. Economou, Phys. Rev. B **16**, 3768 (1977).

[40] R. L. Bush, J. Phys. C: Solid State Phys. **8**, L547 (1975).

[41] G. Theodorou and M. H. Cohen, Phys. Rev. B **13**, 4597 (1976).

[42] C. M. Soukoulis and E. N. Economou, Phys. Rev. B **24**, 5698 (1981).

[43] U. Krey, Z. Phys. B **54**, 1 (1983).

[44] J. B. Pendry, J. Phys. C: Solid State Phys. **15**, 5773 (1982).

[45] F. Delyon, H. Kunz, and B. Souillard, J. Phys. A: Math. Gen. **16**, 25 (1983).

[46] A. Bovier, J. Stat. Phys. **56**, 645 (1989).

[47] J. Flores, J. Phys.: Condens. Matter **1**, 8471 (1989).

[48] L. Alloatti, J. Phys.: Condens. Matter **21**, 045503 (2009).

[49] M. Piraud, and L. Sanchez-Palencia, Eur. Phys. J.: Spec. Top. **217**, 91 (2013).

[50] D. H. Dunlap, H.-L. Wu, and P. W. Phillips, Phys. Rev. Lett. **65**, 88 (1990).

[51] F. A. B. F. de Moura and M. L. Lyra, Phys. Rev. Lett. **81**, 3735 (1998).

[52] W. Zhang and S. E. Ulloa, Phys. Rev. B **69**, 153203 (2004).

[53] R. A. Caetano and P. A. Schulz, Phys. Rev. Lett. **95**, 126601 (2005).

[54] Z. Zhao, F. Gao, R. W. Peng, L. S. Cao, D. Li, Z. Wang, X. P. Hao, M. Wang, and C. Ferrari, Phys. Rev. B **75**, 165117 (2007).

[55] J.-F. Schaff, Z. Akdeniz, and P. Vignolo, Phys. Rev. A **81**, 041604 (2010).

[56] G. Floquet, Gaston Ann. Sci. Ec. Norm. Super. **12**, 47 (1883).

[57] J. H. Shirley, Phys. Rev. **138**, B979 (1965).

[58] S. Rahav, I. Galary, and S. Fishman, Phys. Rev. A **68**, 013820 (2003).

[59] A. Eckardt and E. Anisimovac, arXiv:1502.06477 [New J. Phys. (to be published)].

[60] M. Bukov, L. D'Alessio, and A. Polkovnikov, Adv. Phys. **64**, 139 (2015).

[61] M. Esmann, N. Teichmann, and C. Weiss, Phys. Rev. A **83**, 063634 (2011).

[62] M. Di Liberto, D. Malpetti, G. I. Japaridze, and C. Morais Smith, Phys. Rev. A **90**, 023634 (2014).

[63] D. Poletti and C. Kollath, Phys. Rev. A **84**, 013615 (2011).

[64] M. Łącki and J. Zakrzewski, Phys. Rev. Lett. **110**, 065301 (2013).

[65] B. Horstmann, S. Dürr, and T. Roscilde, Phys. Rev. Lett. **105**, 160402 (2010).

[66] A. Rapp, X. Deng, and L. Santos, Phys. Rev. Lett. **109**, 203005 (2012).

[67] J. Stasińska, M. Łącki, O. Dutta, J. Zakrzewski, and M. Lewenstein, Phys. Rev. A **90**, 063634 (2014).

[68] C. A. Muller and D. Delande, in *Les Houches 2009—Session XCI: Ultracold Gases and Quantum Information* (Oxford University Press, London, 2011), Chap. 9.

[69] M. Płodzień and K. Sacha, Phys. Rev. A **84**, 023624 (2011).

[70] V. I. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1968).

[71] A. MacKinnon and B. Kramer, Z. Phys. B **53**, 1 (1983).

[72] L. Tessieri and F. Izrailev, Phys. E **9**, 405 (2001).

[73] F. M. Izrailev, T. Kottos, and G. P. Tsironis, Phys. Rev. B **52**, 3274 (1995).