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Detecting multipartite spatial entanglement with modular variables

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Interference phenomena of quantum systems have been studied in the context of fundamental aspects of quantum physics and are considered a necessary resource for quantum information. Here we investigate the interference of multiparticle wave packets in terms of modular variables, which is a natural and convenient way to describe two or more interfering wave functions. Through the modular-variable description, interesting phenomena appear such as the complementarity between the number of wave packets and the width of the peaks of the momentum distribution. In the multipartite case, this effect produces quantum entanglement. We derive entanglement criteria that test for bipartite entanglement in generic bipartitions of a multipartite quantum state and use these criteria to test for genuine D-partite entanglement.

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I. INTRODUCTION

The Young double-slit experiment with individual quanta has been at the heart of fundamental discussions since the beginning of quantum mechanics. The early discussions between Bohr and Einstein concerning the validity of the complementarity principle and wave-particle duality centered around the appearance (or not) of double-slit interference when one attempts to mark the passage of the particle through the slits [1]. Later, it was shown that the entanglement between the particles and a quantum "which-path" marker allows one to register the passage of the particle through the double slit and subsequently erase that information, recovering an interference pattern [2–6].

Since the appearance of the seminal Einstein-Podolsky-Rosen (EPR) paper in 1935 [7], quantum entanglement took the stage as perhaps the most curious and controversial aspect of quantum physics [8,9]. The development of Bell's inequality in 1965 showed that quantum entanglement between remote systems could produce experimental results that are not possible with only classically correlated systems [10]. Quantum entanglement between two systems leads to novel interference effects. Entangled particles passing through double-slit apertures were shown to exhibit two-particle conditional interference fringes, where "conditional" means that the interference pattern depends upon the detection position of both particles [11–15]. Moreover, a number of complementarity relations between the one-particle and two-particle fringes have been demonstrated [16–18].

The difference between classical and quantum physics is even more striking when one considers systems of three or more parties. The Greenberger-Horne-Zeilinger (GHZ) paradox was one of the first examples of an "all-versus-nothing" proof of the incompatibility of quantum physics with classical reasoning [19]. Many of the gedanken experiments considered in the discussion of entanglement also consider interference between wave packets of two or more particles, which is a multiparticle analog to double-slit interference [16].

These concepts have been experimentally tested using pairs of entangled photons that pass through $\mathcal{N}$-slit apertures [12,14,15,20–24], which exhibit nonlocal interference effects. For example, it has been shown that D entangled photons can be produced so as to produce interference fringes with period proportional to the de Broglie wavelength of the D-photon wave packet [12,23,25,26], which has been shown to be potentially useful for quantum lithography [27]. Additional experiments include spatial antibunching [20,21], fractional topological phase of entangled photons [28], tests of quantum complementarity [4–6,15], quantum contextuality [29], and quantum information protocols [30]. It is also possible to produce high-dimensional entangled photons using temporal wave packets [31,32].

At first, the entanglement in two-particle wave packets were demonstrated using arguments based on conditionality of the interference fringes. That is, it was shown that the phase of the interference fringes obtained by measuring one particle depended on the detection position of the other [12,14,33,34]. However, entanglement of these particles was never explicitly demonstrated. More recently, entanglement criteria were developed that are especially well suited to nonlocal Young-like interference effects [24,35]. These employ modular variables (MVs), which are an alternative representation of the usual position and momentum variables and have been shown to be particularly relevant in interference phenomena [36]. For example, it is well known that increasing the number of interfering wave packets decreases the width of the interference peaks. This is well described by the uncertainty relations between integer and modular components of the position and momentum variables [24,35,37]. However, these effects are not quantified by the usual uncertainty relations based on variances or entropies based on the position and momentum.

MVs have also been used to develop tests of the GHZ paradox [38] and quantum contextuality [39,40] for continuous variables. They provide a natural way to discretize the continuum, and provide a possible way to apply results from

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quantum information of discrete systems to continuous ones [41]. It is important to note that this discretization does not discard information, since complete knowledge of the continuous variable can be retrieved from the integer and modular components.

In this paper, we develop entanglement criteria for genuine entanglement of multipartite wave packets. In Sec. II, we describe interfering wave packets in terms of MVs and provide a MV description of a general quantum state. We extend this description to the multipartite case in Sec. III. In Sec. IV, we develop an entanglement criteria for genuine multipartite entanglement and provide numerical results showing its utility. In Sec. VI, we provide concluding remarks.

II. MODULAR VARIABLE DESCRIPTION OF WAVE PACKETS

Consider a wave function of the form

$$\Psi(x) = \sum_{n=0}^{\mathcal{N}-1} c_n \psi_n(x), \tag{1}$$

where

$$\psi_n(x) = \psi_0(x - n\ell) \tag{2}$$

is a normalized wave function centered at $n\ell$ and $\sum_n |c_n|^2 = 1$. This wave function is an $\mathcal{N}$-toothed comb of ψ functions, and might describe a particle passing through an $\mathcal{N}$-slit aperture [12,14,15,20–24], or multiple temporal wave packets [31,32]. An example of wave function (1) is shown in Fig. 1. Here we consider generic position and momentum variables, which we assume to be dimensionless for convenience. The operators associated with these variables obey the commutation relation $[\mathsf{x},\mathsf{p}] = i$ (we set $\hbar = 1$ throughout this paper). The Fourier

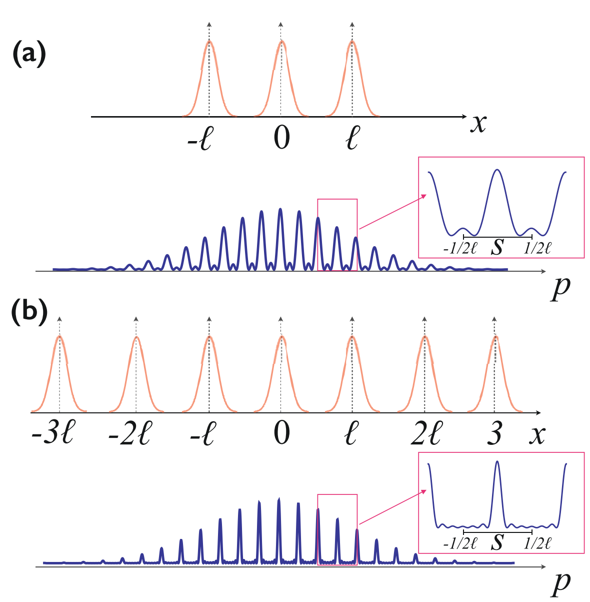

FIG. 1. (Color online) Probability distributions for (a) $\mathcal{N} = 3$ and (b) $\mathcal{N} = 7$ interfering wave packets. Shown are probability distributions $|\Psi(x)|^2$ (top) and $|\Phi(p)|^2$ (bottom) for dimensionless position x and momentum p variables. Insets: The distributions $|g(s)|^2$. The variable ℓ is a dimensionless length parameter.

transform of Eq. (1) gives the wave function in momentum space,

$$\Phi(p) = \sum_{n=0}^{\mathcal{N}-1} c_n \phi_n(p), \tag{3}$$

where

$$\phi_n(p) = \exp(2i\pi nlp)\phi_0(p) \tag{4}$$

is the Fourier transform of $\psi_n(x)$. Then,

$$\Phi(p) = \phi_0(p) \sum_{n=0}^{\mathcal{N}-1} c_n \exp(2i\pi n\ell p). \tag{5}$$

An example of wave function (5) is shown in Fig. 1. Much can be learned about these wave functions if one recasts them in terms of modular variables. Choosing a scale factor ℓ with dimension of length, one can define modular variables for the continuous x and p variables [35]:

$$x = n\ell + r, \tag{6a}$$

$$p = \frac{m}{\ell} + s, \tag{6b}$$

where n is the integer component of x/ℓ and the modular part is defined as $r = (x + \ell/2) \mod (\ell) - \ell/2$ so that $-\ell/2 < r < \ell/2$. Similarly, m is the integer component of $p\ell$ and $s = (p + 1/2\ell) \mod (1/\ell) - 1/2\ell$. Though the variables r and s are the "modular" components, we will refer to definitions (6) as the "modular variable description" of the continuous variables x and p. Changing to modular variables (6), we can rewrite wave function (5) as

$$\Phi(m,s) = \phi_0\left(\frac{m}{\ell} + s\right) \sum_{n=0}^{\mathcal{N}-1} c_n \exp(2i\pi n\ell[m/\ell + s]). \tag{7}$$

If the $\psi_0(x)$ function is well localized around $x = 0$, with a support that is small compared to ℓ, then its Fourier transform $\phi_0(p) \sim$ const over the range $-1/2\ell$ to $1/2\ell$, and we can write $\phi_0(m/\ell + s) \approx \phi_0(m/\ell)$. In this case, the momentum-space wave function can be rewritten as

$$\Phi(m,s) \approx \phi_0\left(\frac{m}{\ell}\right) \sum_{n=0}^{\mathcal{N}-1} c_n \exp(2i\pi n\ell s). \tag{8}$$

In other words, $\Phi(m,s) \approx \phi_0(m/\ell)g(s)$, where

$$g(s) = \sum_{n=0}^{\mathcal{N}-1} c_n \exp(2i\pi n\ell s). \tag{9}$$

The sum in (9) is a Fourier series of the function $g(s)$, with discrete coefficients c_n. Note that the assumption that $\phi_0(p) \sim$ const allows us to write the total wave function as a product of the integer and modular wave functions. In this respect, we can consider a pure state $|g\rangle$, such that $g(s) = \langle s|g\rangle$.

The modular variables (6) and wave function (9) present a number of interesting properties. For instance, the modular operators r and s associated with the modular variables r and s commute [36]. On the other hand, the integer part of the momentum (position) does not commute with the modular part of the position (momentum). In fact, these noncommuting pairs satisfy a number of uncertainty relations [24,35–37]. In Ref. [35], it was shown that wave function (9) presents a type of "squeezing" relation between the integer component n of the position x, and the modular component s of the momentum. In this case, the width of the peaks $\Delta s \sim 1/\mathcal{N}$. This can be seen in Fig. 1, where we see that the momentum peaks for $\mathcal{N}=7$ wave packets are narrower than for only $\mathcal{N}=3$ interfering wave packets.

In Ref. [24], it was shown that the continuous variable s and discrete variable n obey the uncertainty relation

$$H[n] + h[s] \geqslant \ln\frac{1}{\ell}, \tag{10}$$

where

$$H[n] = -\sum_n P_n \ln P_n \tag{11}$$

is the discrete Shannon entropy and

$$h[s] = -\int ds\, P(s)\ln P(s) \tag{12}$$

is the differential Shannon entropy of continuous variable s [42], where here $P_n = |c_n|^2$, $P(s) = |g(s)|^2$. This uncertainty relation is saturated for a state with one $c_n = 1$ and all the rest zero. Then, $H(n) = 0$ and $h(s) = -\ln\ell$.

The above discussion assumed that the wave function is a superposition of $\mathcal{N}$ identical wave packets (shifted in position space), which allowed for a convenient description in terms of MVs and the separation of the wave function $\Phi(m,s) = \phi_0(m)g(s)$. In Appendix A, we provide the MV description of a general quantum state ρ and show that it satisfies the uncertainty relation (10).

Measuring modular variables

We note briefly that probability distributions describing the integer and modular variables can be determined from their continuous counterparts in a straightforward way. For example, if one measures the distribution $P(x)$, then the distributions $P(n)$ and $P(r)$ can be calculated directly using definitions (6). The same is true for the momentum variables.

Below we will consider the integer and modular parts of global variables—say, X—given by the linear combination of local variables x of a multipartite system. In general, the global distribution $P(X)$ is a marginal of the joint distribution $P(x_1,x_2,\dots)$, which can be obtained from measurements on the particles. Then, calculation of $P(X)$ and the probability distributions corresponding to the integer and modular parts of X is straightforward.

III. MULTIPARTITE WAVE PACKETS

Consider now a D-partite state of the form

$$|\Psi_D\rangle = \sum_{n_1,\dots,n_D=0}^{\mathcal{N}-1} C_{n_1,\dots,n_D}|\psi_{1n_1}\rangle|\psi_{2n_2}\rangle\otimes\cdots\otimes|\psi_{Dn_D}\rangle, \tag{13}$$

where $|\psi_{kn_k}\rangle$ describes the state of particle k localized around some position $n_k\ell$. This could be the passage of D particles through $\mathcal{N}$-slit apertures. This type of state has been produced for $D=2$ photons [33] and, in Ref. [24], it was shown that MVs were useful (if not necessary) for demonstrating quantum correlations such as entanglement and EPR steering from interference patterns.

The wave function $\langle x_1,\dots,x_D|\Psi_D\rangle$ of the D-partite state (13) is

$$\Psi_D(x_1,\dots,x_D) = \sum_{n_1,\dots,n_D=0}^{\mathcal{N}-1} C_{n_1,\dots,n_D}\prod_{k=1}^{D}\psi_{n_k}(x_k). \tag{14}$$

Each wave packet ψ_{n_k} is given by Eq. (2) and its Fourier transform ϕ_{n_k} by Eq. (4). The momentum-space wave function of the D-partite state is

$$\Phi_D(p_1,\dots,p_D) = \sum_{n_1,\dots,n_D=0}^{\mathcal{N}-1} C_{n_1,\dots,n_D}\prod_{k=1}^{D}\phi_0(p_k)e^{2i\pi\ell n_k p_k}. \tag{15}$$

Rewriting in terms of modular variables, $p_j = \frac{m_j}{l} + s_j$, and assuming $\phi_0(p_j) \sim$ const over the range $-1/2\ell$ to $1/2\ell$, we have

$$\Phi_D(p_1,\dots,p_D) = \left\{\prod_{j=1}^{D}\phi_0\left(\frac{m_j}{\ell}\right)\right\} \times g(\vec{s}), \tag{16}$$

with

$$g(\vec{s}) = \sum_{\vec{n}} C_{n_1,\dots,n_D}e^{2i\pi\ell\vec{n}\cdot\vec{s}}, \tag{17}$$

where $\vec{n}\cdot\vec{s} = n_1s_1 + \cdots + n_Ds_D$ and the sum is over all values of the vector $\vec{n}$. The entanglement present in the state depends upon the coefficients $C_{n_1,\dots,n_D}$. For example, if $C_{n_1,\dots,n_D} \propto \delta_{n_1,n}\times\cdots\times\delta_{n_D,n}$, then all particles go through the nth slit, corresponding to an entangled state, and we have

$$g_n(\vec{s})\frac{1}{\sqrt{\mathcal{N}}}\sum_n e^{2i\pi\ell\vec{n}\cdot\vec{s}}. \tag{18}$$

This is similar to a GHZ state [16], where all particles are in either the zero state or the one state, corresponding to $\mathcal{N}=2$. In the case of more than two slits, one might expect novel multislit multiparticle interference effects to appear.

A number of methods exist to identify entanglement in discrete variable systems [43]. However, states of the form (13) describe continuous variable systems. The wave function corresponding to these entangled states is similar to the one-particle wave functions illustrated in Fig. 1. In the multipartite case, we have the same type of squeezing relation between the number of slits $\mathcal{N}$ and the width of the function $|g(\vec{s})|^2$. However, now the squeezing is in terms of the global variable $\vec{s}$ and appears as quantum entanglement between the particles.

It is well known that squeezing in global variables can produce entanglement. The most well-known example is the bipartite EPR state [7], which is simultaneously perfectly correlated in the variables $x_1 - x_2$ and $p_1 + p_2$. The entanglement can be identified using criteria that are based on the variances or entropies of these global variables [44–47]. For the EPR state, both the entropy and the variance of the global variables above are zero. The EPR state is the limiting case of the more general two-mode squeezed state.

In the case of state (16), the squeezing is revealed only when we isolate the integer and modular parts of the position and momentum variables. For example, we note in Fig. 1 that the width of the entire momentum function $\Phi(p)|^2$ does not change when $\mathcal{N}$ increases. As $\mathcal{N}$ increases, so does the squeezing in $\vec{s}$ and the correlations between the individual modular variables s_j. However, this does not appear if we consider criteria that are based solely on the complete position x and momentum p. It is necessary to consider modular variables. As an example, consider a state of the form (16) with $D = 2$. Moreover, let us consider that the wave functions ψ can be approximated by Dirac δ functions, and that $\mathcal{N} = \infty$. In this case, it is straightforward to calculate that the x variables are perfectly correlated, $x_1 - x_2 = \text{const}$, suggesting entanglement similar to the EPR state. However, the probability distribution of the momentum variable $P(p_1 + p_2)$ is an infinite comb of Dirac δ functions centered at integer multiples of $1/\ell$, which has infinite entropy and infinite variance. Thus, any of the criteria [44–47] will not detect entanglement. However, isolating the modular part of the momentum, one can easily check that they are perfectly anticorrelated, $s_1 + s_2 = \text{const}$. Thus, entanglement criteria for modular variables such as [24,35] can be successfully employed.

The same is true for multipartite entanglement. Entanglement criteria available in the literature for continuous variables [48–51] involve variances or correlation functions of x and p. However, as described above, the correlations present in states of the form (13) are only revealed when modular variables are considered. In the next section, we provide a criterion to test for genuine multipartite entanglement that is especially suited to states of the form (15).

IV. GENUINE ENTANGLEMENT

A state that is biseparable in bipartition $A|B$ can be written as

$$\sigma_{A|B} = \sum_k \lambda_k \rho_{Ak} \otimes \rho_{Bk}, \tag{19}$$

where $\lambda_k \geqslant 0$. Here, partition A consists of d subsystems and B consists of $D - d$ subsystems, such that all D systems are contained in either A or B. Identifying genuine multipartite entanglement then can be achieved by showing that the quantum state cannot be written as a convex sum of biseparable states (19):

$$\sigma_{bs} = \sum_{\Pi} \eta_{\Pi} \sigma_{\Pi}, \tag{20}$$

where the index Π runs over all possible bipartitions of the form $A|B$, $\eta_\Pi \geqslant 0$, and $\sum_\Pi \eta_\Pi = 1$. For D systems, there are $2^{(D-1)} - 1$ possible bipartitions. Figure 2 shows the possible bipartitions for $D = 3$ and $D = 4$.

We will start by considering an entanglement criteria for the biseparable pure state $|\phi_A\rangle|\phi_B\rangle$ (subscripts have been dropped for notational convenience) and then extend the result to biseparable mixed states.

A. General bipartitions

Let us divide the D-partite system into two bipartitions A and B, so that A contains d subsystems and B contains $D - d$ subsystems. Let us define an arbitrary position variable X_j and momentum variable P_j for bipartitions $j = A, B$. In general, these variables can be linear combinations of the individual position and momentum variables x and p of each subsystem contained in the partition j. Explicit definition of the variables is given in Appendix B. We consider the modular-variable decomposition of each of these partition variables,

$$X_j = N_j \ell + R_j \tag{21a}$$

and

$$P_j = \frac{1}{\ell} M_j + S_j. \tag{21b}$$

We now define global modular variables of the whole system:

$$N_\pm = N_A \pm N_B \tag{22a}$$

and

$$S_\pm = S_A \pm S_B. \tag{22b}$$

Let us consider first the global variables $S_\pm$. For the separable state $|\sigma\rangle = |\phi_A\rangle|\phi_B\rangle$, we have the probability distribution $P(S_A, S_B) = |\langle S_A, S_B|\sigma\rangle|^2 = P_A(S_A)P_B(S_B)$. From this, one can calculate the probability density $P(S_\pm)$ for global variables, which is

$$P_\pm(S_\pm) = [P_A * P_B^\pm](S_\pm), \tag{23}$$

where $P^\pm = P(\pm S)$ and $*$ represents a convolution operation. A continuous variable whose probability density is the convolution of the probability densities satisfies the entropy

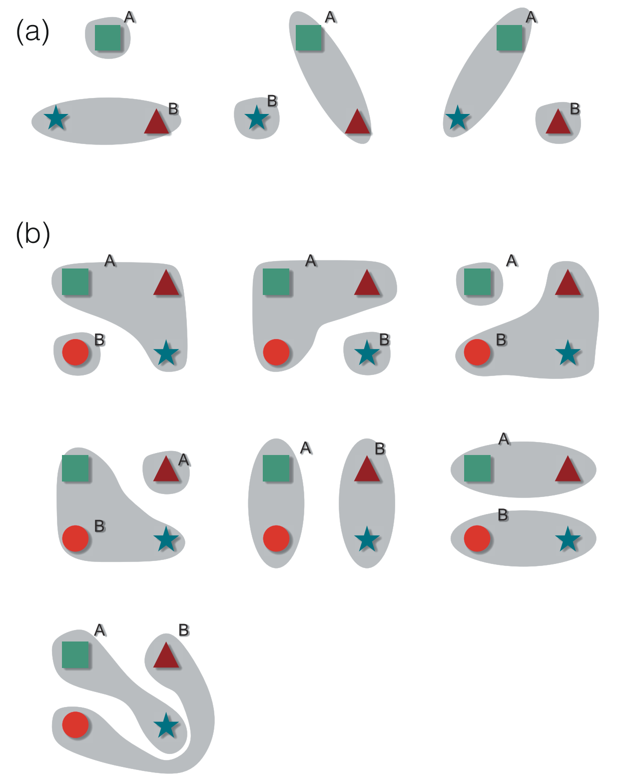

FIG. 2. (Color online) Possible bipartitions for (a) $D = 3$ and (b) $D = 4$ subsystems.

power inequality [42]. Thus, we can write, for the continuous Shannon entropy [47],

$$h(S_{\mp}) \geqslant \tfrac{1}{2}\ln[e^{2h(S_A)} + e^{2h(S_B)}]. \qquad (24)$$

For the discrete variables $N_{\pm}$, we exploit the fact that the entropy of the sum or difference of two independent random discrete variables is always greater than the entropy of each variable [42]. Therefore, we have $H(N_{\pm}) \geqslant H(N_A)$ and $H(N_{\pm}) \geqslant H(N_B)$, and we can write

$$H(N_{\pm}) \geqslant \tfrac{1}{2}\ln\left[\tfrac{1}{2}e^{2H(N_A)} + \tfrac{1}{2}e^{2H(N_B)}\right]. \qquad (25)$$

Then, we can write

$$H(N_{\pm}) + h(S_{\mp}) \geqslant \tfrac{1}{2}\ln[e^{2h(S_A)} + e^{2h(S_B)}] + \tfrac{1}{2}\ln\left[\tfrac{1}{2}e^{2H(N_A)} + \tfrac{1}{2}e^{2H(N_B)}\right]. \qquad (26)$$

In Appendix B, we show that variables X_j and P_j defined in Eq. (21) can be chosen so that the following uncertainty relation is satisfied:

$$H(N_j) + h(S_j) \geqslant -\ln \ell, \qquad (27)$$

where $j = A,B$. Then, using inequality (27) in inequality (26) gives

$$H(N\pm) + h(S\mp) \geqslant \ln\frac{1}{\ell} + \frac{1}{2}\ln\{1 + \cosh[2h(S_A) - 2h(S_B)]\}. \qquad (28)$$

Since $\cosh(x) \geqslant 1$, we have

$$H(N\pm) + h(S\mp) \geqslant \ln\frac{\sqrt{2}}{\ell}, \qquad (29)$$

which is satisfied by a generic biseparable pure state. Since any mixed state can be written as a convex sum of pure states, it follows that from the concavity of the Shannon entropy, for any state

$$\sigma_{A|B} = \sum \eta_k |\phi_{Ak}\rangle\langle\phi_{Ak}| \otimes |\phi_{Bk}\rangle\langle\phi_{Bk}| \qquad (30)$$

separable in partition $A|B$, we have

$$H(N\pm)_{\sigma_{A|B}} + h(S\mp)_{\sigma_{A|B}} \geqslant \sum \lambda_k[H(N\pm)_k + h(S\mp)_k]. \qquad (31)$$

It follows from $\eta \geqslant 0$, $\sum_k \eta_k = 1$, and Eq. (29) that

$$H(N\pm)_{\sigma_{A|B}} + h(S\mp)_{\sigma_{A|B}} \geqslant \ln\frac{\sqrt{2}}{\ell}. \qquad (32)$$

Thus the criteria also holds for biseparable mixed states. We note that we made no restriction on the size of the subsystems A and B. Thus, general inequality (29) can be used for different definitions of N_A, N_B, S_A, and S_B to produce a number of different inequalities for any bipartition $A|B$ of the system.

The one restriction in our criteria is the choice of X_j and P_j $(j = A,B)$ and thus N_j and S_j so that inequality (27) is satisfied. In Appendix B, we show that this amounts to defining these variables as orthogonal transforms of the original position and momentum variables of the D subsystems. Thus, our criteria can be tested for a general class of variables.

Below we will use these inequalities to construct entanglement criteria for the most general biseparable states of form (20). To do so, it will be convenient to exponentiate both sides of inequality (32), giving

$$\mathcal{W}_{A|B}(\sigma_{A|B}) \geqslant \frac{\sqrt{2}}{\ell}, \qquad (33)$$

where

$$\mathcal{W}_{A|B}(\sigma_{A|B}) = \exp\{H(N\pm)_{\sigma_{A|B}} + h(S\mp)_{\sigma_{A|B}}\}. \qquad (34)$$

Here we include $\sigma_{A|B}$ in the notation to explicitly denote that inequality (33) is valid for states separable in partition $A|B$. Since the exponential of any real number is real and nonnegative, in general, any state ρ satisfies

$$\mathcal{W}_{A|B}(\rho) \geqslant 0, \qquad (35)$$

which is a property that will be helpful below.

B. Multipartite entanglement tests

We will now use the inequalities developed in the last section, in particular inequality (33), to show how one can test for genuine D-partite entanglement using the MV description. Our results are inspired by the continuous variable entanglement criteria developed in Ref. [51].

For a system composed of D subsystems, there are $2^{(D-1)} - 1$ possible bipartitions. Thus, using the method described in Sec. IV A, we can define variables $N_{\pm}$ and $S_{\pm}$ that test each of these bipartitions separately, resulting in an inequality of form (33) for each set of variables. Consider an entanglement criteria designed to identify entanglement in the bipartition Π' applied to the general biseparable state (20). Using the concavity of the Shannon entropy [42], we have

$$\mathcal{W}_{\Pi'}(\sigma_{bs}) \geqslant \sum_{\Pi} \eta_{\Pi}\mathcal{W}_{\Pi'}(\sigma_{\Pi}), \qquad (36)$$

where $\mathcal{W}_{\Pi'}(\rho_{\Pi})$ is the entanglement criteria, designed for bipartition Π' applied to the state σ_{Π} that is separable in bipartition Π. The sum in $\mathcal{W}_{\Pi'}(\sigma_{bs})$ thus contains the bipartition test $\mathcal{W}_{\Pi'}$ applied to all possible bipartitions of the general biseparable state (20). By definition, at least one of the terms in sum (20) will satisfy inequality (33). That is, if for a certain term in the sum we have $\Pi = \Pi'$, then $\mathcal{W}_{\Pi,\Pi} \geqslant \sqrt{2}/\ell$ by inequality (33). If $\Pi \neq \Pi'$, we have $\mathcal{W}_{\Pi',\Pi} \geqslant 0$ by inequality (35), which is satisfied by any state. Thus, for any Π', we have

$$\mathcal{W}_{\Pi'}(\sigma_{bs}) \geqslant \eta_{\Pi'}\mathcal{W}_{\Pi'}(\sigma_{\Pi'}) \geqslant \eta_{\Pi'}\frac{\sqrt{2}}{\ell}. \qquad (37)$$

Testing for all possible bipartitions Π' we can sum over the results, giving

$$\mathcal{W}_{\text{sum}}(\sigma_{bs}) = \sum_{\Pi'}\mathcal{W}_{\Pi'}(\sigma_{bs}) \geqslant \sum_{\Pi}\eta_{\Pi}\mathcal{W}_{\Pi}(\sigma_{\Pi}) \geqslant \frac{\sqrt{2}}{\ell}, \qquad (38)$$

where we used $\sum_{\Pi}\eta_{\Pi} = 1$ in the last step. Inequality (38) is satisfied by any biseparable state of form (20). Violation of the inequality $\mathcal{W}_{\text{sum}}(\rho) \geqslant \sqrt{2}/\ell$ is thus a criteria for genuine D-partite entanglement of state ρ. The challenge is to construct proper variables that identify this entanglement in interesting quantum states. In Sec. V, we illustrate this for GHZ-like states.

V. EXAMPLE

Let us consider state (13), described by wave functions (14) and (15) for $D = 3$ and $C_{n_1,n_2,n_3} = \delta_{n_1,n}\delta_{n_2,n}\delta_{n_3,n}\mathcal{N}^{-1/2}\ell^{3/2}$. This corresponds to a state where the three particles are all found in the same wave packet ψ_n, similar to a GHZ state. The three possible bipartitions can be written $(aa')|b$, for $a,a',b = 1,2,3$. We define $X_1 = (x_a + x_{a'})/2$, $X_2 = (x_a - x_{a'})/2$, $P_1 = p_a + p_{a'}$, and $P_1 = p_a - p_{a'}$, so that $X_1P_1 + X_2P_2 = x_a p_a + x_{a'}p_{a'}$. Then, from Eq. (15), the wave function $\Phi_3(P_1,P_2,p_b)$ can be written

$$\Phi_3(P_1,P_2,p_b) = \sqrt{\frac{\ell^3}{\mathcal{N}}}\sum_{n=0}^{\mathcal{N}-1} e^{2\pi n i\ell(P_1+p_b)}\phi_0\left(\frac{P_1+P_2}{2}\right) \times \phi_0\left(\frac{P_1-P_2}{2}\right)\phi_0(p_b). \tag{39}$$

Changing to MVs as defined in Eqs. (21), we have

$$\Phi_3(P_1,P_2,p_b) \longrightarrow F(M_1,M_2,m_b)g(S_1,S_2,s_b), \tag{40}$$

where

$$g(S_1,S_2,s_b) = \sqrt{\frac{\ell^3}{\mathcal{N}}}\sum_{n=0}^{\mathcal{N}-1} e^{2\pi in\ell(S_1+s_b)}. \tag{41}$$

The momentum-space wave function ϕ_0 is broad, so we can approximate $\phi_0(p) \approx \phi_0(m/\ell)$ and we write

$$F(M_1,M_2,m_b) = \phi_0\left(\frac{M_1+M_2}{2\ell}\right)\phi_0\left(\frac{M_1-M_2}{2\ell}\right)\phi_0\left(\frac{m_b}{\ell}\right). \tag{42}$$

We can change now to the global variables $N_\pm$ and $S_\pm$ defined in Eq. (22), where $N_A = N_1$, $N_B = n_b$, $S_A = S_1$, and $S_B = s_b$. Since the function $g(S_1,S_2,s_b)$ does not depend upon S_2, we can write

$$g(S_+,S_-) = \frac{\ell}{2\sqrt{\mathcal{N}}}\sum_{n=0}^{\mathcal{N}-1} e^{2\pi in\ell S_+}. \tag{43}$$

The function $g(S_+,S_-)$ factors into a product of the functions $f(S_-) = \sqrt{\ell/2}$ and

$$g(S_+) = \sqrt{\frac{\ell}{2\mathcal{N}}}\sum_{n=0}^{\mathcal{N}-1} e^{2\pi in\ell S_+}. \tag{44}$$

$f(S_-) = \sqrt{\ell/2}$ is analogous to passage through a single slit aperture of width 2ℓ centered at the origin, so that $P(N_-) = \delta_{N_-,0}$ and $H(N_-) = 0$. The entropy $h(S_+)$ defined in Eq. (12) can be calculated from the probability distribution $P(S_+) = |g(S_+)|^2$, giving

$$P(S_+) = \frac{\ell}{2\mathcal{N}}\sum_{n,n'=0}^{\mathcal{N}-1} e^{2\pi i(n-n')\ell S_+}. \tag{45}$$

The entropy $h(S_+)$ can be calculated using Eq. (45). Values of $\mathcal{W}_{A|B} - \sqrt{2}/\ell$ are given in Fig. 3. Violation can be achieved for $\mathcal{N} \geqslant 2$. From symmetry, all bipartitions violate by the same amount, giving $\mathcal{W}_{\rm sum} = \mathcal{W}_{A|B} \leqslant \sqrt{2}/\ell$, which identifies genuine tripartite entanglement.

For GHZ states with D subsystems, similar calculations can be performed. For any bipartition, variables can be defined so that $N_- = N_A - N_B = \text{const}$ and $P(S_+)$ is given by Eq. (45). Thus, for any bipartition, we obtain the same values of $\mathcal{W}_{A|B}$ reported in Fig. 3, and we can identify genuine D-partite entanglement since $W_{\rm sum} = \mathcal{W}_{A|B} \leqslant \sqrt{2}/\ell$.

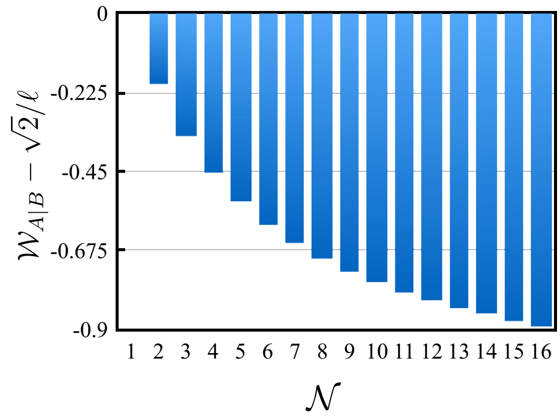

FIG. 3. (Color online) The dimensionless quantity $\mathcal{W}_{A|B} - \sqrt{2}/\ell$ as a function of the number of interfering wave packets $\mathcal{N}$ for GHZ-like states. Negative values indicate entanglement. The variable ℓ is a dimensionless length parameter. Entanglement is identified for all $\mathcal{N} \geqslant 2$. For $\mathcal{N} = 1$, the state is separable.

VI. CONCLUSION

We derived entanglement criteria for bipartite entanglement in generic bipartitions of a multipartite quantum state using modular variables. These inequalities were used to construct entanglement tests for genuine multipartite entanglement. Our results should be useful for exploring novel quantum interference phenomena involving multiple particles. For example, we showed that these criteria detect genuine entanglement in a generalized GHZ-like state of D particles all passing through the same slit of an $\mathcal{N}$-slit aperture. Since these states exhibit correlations in both discrete and modular (continuous) variables, our entanglement criteria might find use in novel quantum information protocols. For example, the discrete correlations might be used to construct a shared cryptographic key between several users, while the correlations in the modular components are used only for security checks.

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APPENDIX A: MODULAR-VARIABLE DESCRIPTION OF ARBITRARY WAVE FUNCTIONS

In Sec. II, we showed that MVs appear naturally for periodic wave functions. Here we show that the MV description can be safely used to analyze any wave function. In other words, the uncertainty relation (10) is valid in the most general sense.

Consider the generic wave function $\psi(x)$, and its Fourier transform

$$\phi(p) = \int_{-\infty}^{\infty} dx e^{2\pi i x p}\psi(x). \tag{A1}$$

Using the MVs defined in Eqs. (6), we have

$$\phi(m,s) = \sum_n e^{2\pi i \ell n s}\int_{-\ell/2}^{\ell/2} dr e^{2\pi i r(m/\ell+s)}\psi_n(r), \tag{A2}$$

where $\psi_n(r)$ is the original wave function in the region $n\ell/2 \leqslant x < (n+1)\ell/2$. In the example given in Sec. II, the function $\phi(m,s)$ factored into functions $\phi_0(m/\ell)$ and $g(s)$. In the general case, this is no longer true, and the variable s cannot be written in terms of a "modular wave function" $g(s)$. Still, the probability distribution $P(s)$ can be calculated by summing over the m variable as

$$P(s) = \sum_m |\phi(m,s)|^2, \tag{A3}$$

giving

$$P(s) = \frac{1}{\ell}\sum_{n,n'} I_{n,n'} e^{2\pi i \ell s(n-n')}. \tag{A4}$$

Here we have used the fact that

$$\sum_{m=-\infty}^{\infty} e^{2\pi i m(r-r')/\ell} = \frac{1}{\ell}\delta(r-r'), \tag{A5}$$

and defined

$$I_{n,n'} = \int_{-\ell/2}^{\ell/2} dr \psi_n(r)\psi^*_{n'}(r). \tag{A6}$$

Using the normalization of $\psi(x)$,

$$\int |\psi(x)|^2 dx = \sum_n I_n dr = 1, \tag{A7}$$

we can rewrite $P(s)$ as

$$P(s) = \frac{1}{\ell} + \frac{2}{\ell}\sum_{n,n',n<n'} \mathrm{Re}(I_{n,n'})\cos 2\pi\ell s(n-n'). \tag{A8}$$

One can see that $0 \leqslant P(s) \leqslant 1$ and that $P(s)$ is normalized in the region $-1/2\ell \leqslant s \leqslant 1/2\ell$, showing that $P(s)$ is a bona fide probability distribution. In fact, $P(s)$ is analogous to the probability distribution of a general quantum state written in the plane-wave basis $\exp(i2\pi n\ell s)$. The probability distribution $p(n)$ for this state is $p(n) = I_{n,n}$. Since probability distributions $p(n)$ and $P(s)$ correspond to a quantum state, they satisfy the uncertainty relation (10). Moreover, a general mixed state ϱ can be written as a convex sum of pure states $|\psi_j\rangle$, which leads to probability distributions

$$P_\varrho(s) = \sum_j \lambda_j P_j(s) \tag{A9}$$

and

$$p_\varrho(n) = \sum_j \lambda_j p_j(n), \tag{A10}$$

with $\lambda_j \geqslant 0$ and $\sum \lambda_j = 1$. Since each $P_j(s)$ and $p_j(n)$ satisfy (10), $P_\varrho(s)$ and $p_\varrho(n)$ also satisfy the uncertainty relation (10).

APPENDIX B: ENTROPIC UNCERTAINTY RELATION FOR COMPOSITE SYSTEM

Here we prove the general uncertainty relation (27) with $j = A,B$ for the variables defined in Eqs. (22). Consider the most general pure state for partition A, given by wave function $\Psi(\vec{x})$. Here, $\vec{x} = (x_1,\ldots,x_d)$ is a d-dimensional vector describing the position of the d subsystems contained in partition A. The Fourier transform is

$$\Phi(\vec{p}) = \int d\vec{x} e^{2\pi i \vec{x}\cdot\vec{p}}\Psi(\vec{x}), \tag{B1}$$

where $\vec{p} = (p_1,\ldots,p_d)$. Let us change to a new, generic set of variables

$$X_k = \sum_{j=1}^{d} a_{kj}x_j \tag{B2a}$$

and

$$P_k = \sum_{j=1}^{d} \alpha_{kj}p_j, \tag{B2b}$$

so that $\vec{x}\cdot\vec{p} = \vec{X}\cdot\vec{P}$. We can rewrite the wave function as

$$\Phi(\vec{P}) = \int d\vec{X} e^{2\pi i \vec{X}\cdot\vec{P}}\Psi(\vec{X}). \tag{B3}$$

Let us now define modular variables for these new variables as

$$X_k = N_k\ell + R_k \tag{B4}$$

and

$$P_k = \frac{1}{\ell}M_k + S_k. \tag{B5}$$

Following the calculation in Appendix A, we can calculate the probability distribution $P(\vec{S})$, giving

$$P(\vec{S}) = \frac{1}{\ell^d}\sum_{\vec{N},\vec{N}'} I_{\vec{N},\vec{N}'} e^{2\pi i \ell(\vec{N}-\vec{N}')\cdot\vec{S}}, \tag{B6}$$

where

$$I_{\vec{N},\vec{N}'} = \int_{-\ell}^{\ell} d\vec{R}\,\Psi_{\vec{N}}(\vec{R})\Psi^*_{\vec{N}'}(\vec{R}), \tag{B7}$$

and $\Psi_{\vec{N}(\vec{R})}$ is the wave function $\Psi(\vec{X})$ in the region $\ell\vec{N} \leqslant \vec{X} \leqslant \ell\vec{N} + \vec{1}$. Suppose now that X_k and P_k were defined so that $X_1 = X_A$ and $P_1 = P_A$, giving $N_1 = N_A$ and $S_1 = S_A$. The probability distribution $P(S_A)$ can be obtained by integrating over all of the S_k variables with $k \neq 1$. We then have $d-1$ integrals of the form

$$\int_{-1/2\ell}^{1/2\ell} e^{2\pi i \ell N S} dS = \frac{\sin N\pi}{\ell N\pi} = \frac{1}{\ell}\delta_{N,0}, \tag{B8}$$

where the right-hand side is true for integer values of N, as is the case. We have

$$P(S_A) = \frac{1}{\ell}\sum_{N_2,\ldots,N_d}\sum_{N_A,N'_A} I^{N_2,\ldots,N_d}_{N_A,N'_A} e^{2\pi i \ell(N_A-N'_A)S_A}, \tag{B9}$$

where we defined

$$I^{N_2,\ldots,N_d}_{N_A,N'_A} = \delta_{N_2,N'_2}\cdots\delta_{N_d,N'_d} I_{\vec{N},\vec{N}'}. \tag{B10}$$

Probability distribution $P(S_A)$ is completely analogous to that of a mixed state, given by $P_\varrho(S)$ of Eq. (A10). Thus, $P(S_A)$ satisfies the uncertainty relation

$$H(N_A) + h(S_A) \geqslant -\ln \ell. \tag{B11}$$

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