Trade-off between information gain and fidelity under weak measurements

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It is of general interest how a quantum measurement may disturb a quantum system while it gives information on the state of the system. We study a trade-off relation between the information gain and the output fidelity for a quantum nondemolition measurement scheme for photon numbers. Toward this aim, we obtain general expressions for the information gain and the output fidelity for an arbitrary initial state. We particularly investigate how these two quantities vary with measurement strength for some specific classes of states, through a single measurement or successive measurements.

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I. INTRODUCTION

When one makes a measurement on a quantum system, the system is generally disturbed while one obtains information on its state. This relation between information gain and state disturbance is of fundamental and practical interest, particularly in quantum information processing, as it may provide, e.g., a basis for secure communication. Among all possible measurements, the best known is the von Neumann measurement [1], which enables one to gain information on a certain observable $\hat{O}$ in a strong form. Under this measurement, the state of the system is projected to one of the eigenstates of $\hat{O}$. There is no way to recover the original state even in a probabilistic way, except when the system is initially in an eigenstate. On the other hand, one may minimize the disturbance of the system by reducing the measurement strength—the so-called *weak* measurement—however, this leads to the gain of less information.

Recently, numerous efforts have been devoted to a rigorous characterization of the information-disturbance relation [2–11]. In Refs. [3,5,7,11], the information gain is measured by the fidelity of the state estimated from the measurement outcome with respect to an initial state. From an information-theoretic point of view, the information gain can also be quantified by the mutual information between the prepared and the measured distributions of an observable [6] or by the decrease in the entropy of the system after measurement [2,9,10]. The reversibility—the probability of reversing the state after measurement back to its initial state—has also been studied [12–20]. In addition, the fidelity of the output state with respect to an initial state can be used as a measure of disturbance. In these works [2,3,5–11], a quantitative balance, or an upper bound, by which the maximization of information gain together with the minimization of state disturbance is limited, has been studied for a given measurement. On the other hand, how the trade-off relation can vary upon changing the strength of the measurement or repeating weak measurements sequentially is of some interest but has not yet been addressed.

In this paper, we study the trade-off between the information gain and the output fidelity in a quantum nondemolition (QND) measurement of photon numbers [21]. In a QND measurement based on a cavity-QED setup, the measurement outcomes are binary, i.e., distinction between two atomic states, and an initial cavity state may collapse to a Fock state only after many successive measurements [22]. In this respect, a single QND measurement in our study is a weak measurement and the measurement strength can be adjusted by changing the experimental parameters.

We consider that the cavity is initially prepared in an unknown pure state $|\psi\rangle$ with a given probability density $p(\psi)$. The information gain may be quantified as the decrease in the Shannon entropy of the cavity state under the QND measurement, i.e., mutual information. We also introduce another measure of information gain, which is closely related to the concept of classical fidelity. We then investigate the trade-off relation between the information gain for the unknown state and the disturbance of the state by varying the coupling strength or the number of successive measurements. Our results show that more information gain does not always lead to worse fidelity.

This paper is organized as follows. In Sec. II, we present our scheme for a QND measurement of photon numbers and give general expressions for information gain and output fidelity after N sequential measurements. We illustrate the trade-off relations with two specific classes of cavity states in Sec. III. The main results of this work are summarized in Sec. IV.

II. THE SCHEME AND THEORY ANALYSIS

A. QND measurements of photon numbers

We first introduce the scheme used for a QND measurement of photon numbers based on a cavity-QED setup [21]. This scheme makes it possible to gain information on the distribution of photon numbers of a cavity field without absorption of the photons (Fig. 1). Suppose that the state of the system prepared in a high-Q cavity [C in Fig. 1(a)] is a pure state $|\psi\rangle$ with its probability distribution $p(\psi)$. A three-level atom (measuring device), with the level diagram in Fig. 1(b), is initially prepared in the $|e\rangle$ state. The initial state of the system and the device is given by

$$\rho_{sd} = \rho_s \otimes \rho_d = \sum_{\psi} p(\psi)|\psi\rangle\langle\psi| \otimes |e\rangle\langle e|, \quad (1)$$

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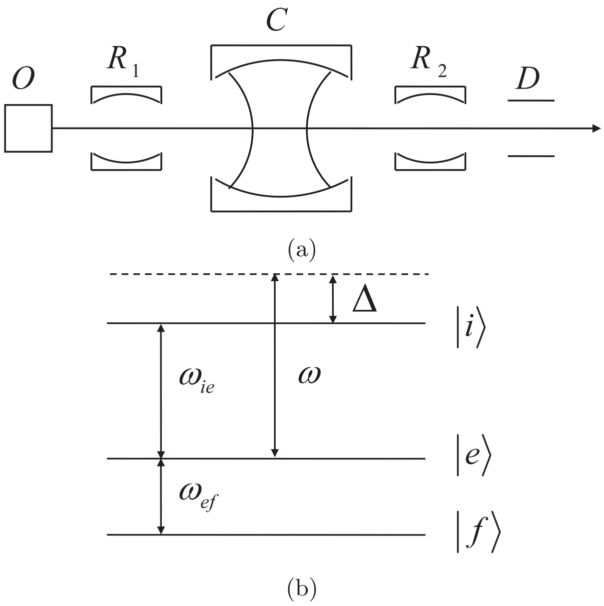

FIG. 1. (a) QND measurement scheme. The field is initially prepared in a high-Q cavity C interacting with traveling atoms. The atoms are prepared and velocity-selected in the box O, then pass through the three cavities. In each of cavities R_1 and R_2, the atom undergoes a $\pi/2$ transformation. Finally, the ionized counter D detects whether each atom is in state $|e\rangle$ or state $|f\rangle$. (b) Atomic level diagram used for a QND measurement of cavity photon numbers. ω is the cavity resonant frequency, which is largely detuned by Δ from the atomic transition frequency $|e\rangle$ to $|i\rangle$. This results in a dispersive coupling which adds a phase to level $|e\rangle$, while atomic state $|f\rangle$ is not involved in the interaction.

where $p(\psi)$ is the probability of each state $|\psi\rangle$ being prepared inside the cavity. In the remainder of this article, we use the term "information" to mean how much we know about which state $|\psi\rangle$ is prepared in the cavity out of the ensemble $\sum_\psi p(\psi)|\psi\rangle\langle\psi|$. That is, we measure a single system to guess which $|\psi\rangle$ is the most likely input state given a measurement output.

Inside the cavity, the atom interacts with the field, which is described by

$$H = \tfrac{1}{2}\hbar\omega_{ie}\sigma_z + \hbar\omega a^\dagger a + \hbar g(a\sigma_+ + a^\dagger\sigma_-), \tag{2}$$

where a ($a^\dagger$) is the cavity photon annihilation (creation) operator, $\sigma_- = |e\rangle\langle i|$, $\sigma_+ = |i\rangle\langle e|$, $\sigma_z = \sigma_+\sigma_- - \sigma_-\sigma_+$, and g the atom-cavity coupling strength. The atomic level $|f\rangle$ is not involved in the interaction. When the cavity-field frequency ω is detuned by an amount Δ from the atomic transition frequency ω_{ie}, the effective interaction becomes a dispersive coupling described by [23]

$$V = \frac{\hbar g^2}{\Delta} a^\dagger a|e\rangle\langle e|. \tag{3}$$

After an interaction time τ, the evolution operator is given by

$$U_I = \exp(-iV\tau/\hbar) = \exp(-i\varphi a^\dagger a|e\rangle\langle e|), \tag{4}$$

where $\varphi \equiv g^2\tau/\Delta$ is the phase shift caused by one photon, which characterizes the coupling strength between the atom and the field. This interaction leads to a phase shift of the $|e\rangle$ state, which is proportional to the photon numbers.

The phase shift can be detected by the Ramsey interferometric method [24], in which the atom undergoes $U_{\pi/2}$ transformations before and after the cavity C, where

$$U_{\pi/2} = \frac{1}{\sqrt{2}}(|e\rangle\langle e| + |f\rangle\langle f| + i|e\rangle\langle f| + i|f\rangle\langle e|). \tag{5}$$

Finally, we use an ionized detector to make a projective measurement of the atom represented by the operators:

$$P_m = |m\rangle\langle m|, \quad m = e, f. \tag{6}$$

Given the outcome m, the operation made is described by

$$U_m = P_m U_{\pi/2} U_I U_{\pi/2}. \tag{7}$$

After such an operation, the density operator evolves to

$$\rho_{sd,m} = U_m \rho_{sd} U_m^\dagger. \tag{8}$$

We are interested in the state of the system only. Tracing over the device and normalizing, we obtain the density operator of the system,

$$\rho_{s,m} = \frac{\mathrm{Tr}_d(U_m\rho_{sd}U_m^\dagger)}{\mathrm{Tr}(U_m\rho_{sd}U_m^\dagger)} = \frac{M_m\rho_s M_m^\dagger}{\mathrm{Tr}_s(M_m\rho_s M_m^\dagger)}, \tag{9}$$

where Tr_s (Tr_d) denotes tracing over the system (device) and Tr represents tracing over both the system and the device. The measurement operators M_m are derived according to Kraus representation theory as

$$\begin{aligned} M_e &= \langle e|U_e|e\rangle = [\exp(-i\varphi a^\dagger a) - 1]/2, \\ M_f &= \langle f|U_f|e\rangle = [\exp(-i\varphi a^\dagger a) + 1]/2. \end{aligned} \tag{10}$$

In the Fock-state basis, they can be expressed as

$$\begin{aligned} M_e &= \sum_{n=0}^{n_{\max}} \frac{\exp(-in\varphi) - 1}{2}|n\rangle\langle n|, \\ M_f &= \sum_{n=0}^{n_{\max}} \frac{\exp(-in\varphi) + 1}{2}|n\rangle\langle n|. \end{aligned} \tag{11}$$

It can be readily checked that the relation $\sum_{m=e,f} M_m^\dagger M_m = I$ is satisfied, implying that our QND measurement is a general quantum measurement with two outcomes.

The probability of obtaining outcome m is given by

$$p(m) = \mathrm{Tr}_s(M_m\rho_s M_m^\dagger) = \sum_\psi p(\psi)p(m|\psi), \tag{12}$$

where the conditional probability of output m given input $|\psi\rangle$ is

$$p(m|\psi) \equiv \mathrm{Tr}_s(M_m|\psi\rangle\langle\psi|M_m^\dagger). \tag{13}$$

B. Measurements with N successive atoms

Suppose now that we make N successive QND measurements. Given that N_e atoms are found in state $|e\rangle$ and $N_f = N - N_e$ atoms in state $|f\rangle$, the state of the cavity field becomes (we omit the index s on the density operator of the system henceforth)

$$\rho_{N_e} = \frac{M_{N_e}\rho M_{N_e}^\dagger}{\mathrm{Tr}_s\left(M_{N_e}\rho M_{N_e}^\dagger\right)}, \tag{14}$$

where the measurement operator reads

$$M_{N_e} = M_e^{N_e} M_f^{N-N_e}. \tag{15}$$

We here assume that the coupling strength φ is the same for each measurement, for simplicity. We note that M_e and M_f commute with each other, so the order of the operators does not affect the results. From an information-theoretic perspective, this means that no extra information can be obtained by keeping a record of the sequence of measurement outputs; therefore one only needs to count N_e, the number of atoms in $|e\rangle$. There are $C_N^{N_e} \equiv \frac{N!}{N_e!(N-N_e)!}$ cases to obtain N_e counts and it follows that

$$\sum_{N_e=0}^{N} C_N^{N_e} M_{N_e}^\dagger M_{N_e} = I, \tag{16}$$

meaning that counting the number N_e also represents a general quantum measurement with $N+1$ different outcomes.

The probability of getting output N_e is

$$p(N_e) \equiv C_N^{N_e}\mathrm{Tr}_s\left(M_{N_e}\rho M_{N_e}^\dagger\right) = \sum_\psi p(\psi)p(N_e|\psi), \tag{17}$$

where the conditional probability of N_e given ψ is

$$p(N_e|\psi) \equiv C_N^{N_e}\mathrm{Tr}_s\left(M_{N_e}|\psi\rangle\langle\psi|M_{N_e}^\dagger\right). \tag{18}$$

In the Fock-state basis, a random state can be expressed as $|\psi^k\rangle = \sum_{n=0}^{n_{\max}} b_n^k|n\rangle$. The above probability is then expressed as

$$P(N_e|\psi^k) = C_N^{N_e}\sum_n \left|b_n^k\right|^2 c_n^{2N_e} d_n^{2(N-N_e)}, \tag{19}$$

where $c_n = \sin\left(\frac{n\varphi}{2}\right)$ and $d_n = \cos\left(\frac{n\varphi}{2}\right)$.

C. Information gain via mutual information

Next we introduce a measure of information gain: mutual information. Suppose that we have a black box (an operation) with an input port and an output port. The input random variable x is chosen from the set $\{x \in X\}$ with prior probability $p(x)$. The output is chosen from the set $\{y \in Y\}$. The black box is modeled by the transfer probability $p(y|x)$. The Shannon entropy [25] for the set $\{x \in X\}$ is defined as

$$H(X) = -\sum_x p(x)\log_2 p(x), \tag{20}$$

which quantifies the lack of information on events X. Given an output y, we obtain a conditional probability $p(x|y) = p(y|x)p(x)/p(y)$, which leads to the conditional entropy

$$H(X|Y) = -\sum_y p(y)\sum_x p(x|y)\log_2 p(x|y). \tag{21}$$

The conditional probability measures how much information on X is still missing after we have known Y. Thus the difference between the original entropy $H(X)$ and the conditional entropy $H(X|Y)$ may be regarded as the information gain. This so-called mutual information [25–27] is defined as

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= \sum_x\sum_y p(x,y)\log_2\frac{p(x,y)}{p(x)p(y)}. \end{aligned} \tag{22}$$

In our case, the input random variable is chosen from all possible cavity states $\{|\psi^k\rangle\}$ with prior probability $p(\psi^k)$. The output random variable is the count N_e under N measurements. Equation (22) is then written by replacing $x \to \psi^k$ and $y \to N_e$, which gives the mutual information as

$$I_M = \sum_{N_e=0}^{N}\sum_k p(N_e|\psi^k)p(\psi^k)\log_2\frac{p(N_e|\psi^k)}{p(N_e)}. \tag{23}$$

Plugging Eqs. (17) and (19) into Eq. (23), we can calculate the information gain.

D. Information gain via the probability overlap

Instead of the mutual information I_M in Eq. (23) adopting the entropies of relevant probability distributions, we may characterize the information gain in another form, which can have a conceptual connection to fidelity. Given two probability distributions, $p_1(x)$ and $p_2(x)$, their distinction can be measured by $1 - F_c^2$, where F_c is the classical fidelity quantifying their overlap as

$$F_c(p_1,p_2) = \sum_x\sqrt{p_1(x)p_2(x)}. \tag{24}$$

If the probability distribution $p_1 = p(\psi^k|N_e)$ conditioned on the measurement outcome N_e becomes more distinguishable from a completely random distribution $p_2(x) = 1/N$ than the initial distribution $p_1 = p(\psi^k)$, it may represent information gain through our QND measurement. Thus, using

$$I_F[p(x)] \equiv 1 - \left(\sum_x\sqrt{p(x)/N}\right)^2, \tag{25}$$

we may define another measure of information gain as

$$I_F = \sum_k p(\psi^k)\sum_{N_e} p(N_e)I_F[p(\psi^k|N_e)] - I_F[p(\psi^k)], \tag{26}$$

where the subscript F refers to the conceptual connection to fidelity. In this paper, we use two quantifiers, I_M in Eq. (23) and I_F in Eq. (26), to measure information gain under QND measurements.

E. Fidelity

On the other hand, we use the output fidelity as a measure of disturbance due to quantum measurement. The fidelity [3] of the output state is defined to be the average overlap between the input state $|\psi^k\rangle$ and the output state $M_{N_e}|\psi^k\rangle$, given by

$$F = \sum_k p(\psi^k)\sum_{N_e=0}^{N} C_N^{N_e}\left|\langle\psi^k|M_{N_e}|\psi^k\rangle\right|^2. \tag{27}$$

By substituting Eq. (15), we obtain the expression of fidelity in the Fock-state basis as

$$F = \sum_k p(\psi^k) \sum_{N_e=0}^{N} C_N^{N_e} \left| \sum_n^{n_{\max}} |b_n^k|^2 e^{-\frac{iNn\varphi}{2}} c_n^{N_e} d_n^{N-N_e} \right|^2. \quad (28)$$

III. TRADE-OFF RELATIONS

Having obtained the general expressions for the information gain and the output fidelity, we now consider some classes of states to investigate the trade-off relation under varying experimental conditions.

A. Photonic qubit states

We here consider an initial state of the cavity field as an unknown superposition of the vacuum and the single-photon states chosen from the following mixture with probability density $p(\theta,\phi)$,

$$\rho = \sum_{\theta,\phi} p(\theta,\phi) |\psi_{\theta,\phi}\rangle\langle\psi_{\theta,\phi}|, \quad (29)$$

where the photonic qubit state

$$|\psi_{\theta,\phi}\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle. \quad (30)$$

We assume that this unknown state is uniformly distributed in the Bloch sphere so that the probability density is

$$p(\theta,\phi) = 1/4\pi. \quad (31)$$

Since it is a continuous distribution, we convert the summation to an integral as

$$\sum_{\theta,\phi} p(\theta,\phi) \to \frac{1}{4\pi} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \quad (32)$$

and study the trade-off under different coupling strengthes and the number of measurements. It is straightforward to obtain

$$F(N,\varphi) = \frac{2}{3} + \frac{1}{3}\cos(N\varphi/2)\cos^N(\varphi/2),$$
$$I_M(N,\varphi) = 1 - \frac{1}{2\ln 2} - \frac{1}{2}\cos^{4N}(\varphi/2)\log_2\cos^{2N}(\varphi/2) \times [1-\cos^{2N}(\varphi/2)]^{-1} - \frac{1}{2}[1+\cos^{2N}(\varphi/2)] \times \log_2[1+\cos^{2N}(\varphi/2)],$$
$$I_F(N,\varphi) = 1 - \frac{\pi^2}{128} - \frac{4}{9}\left[1 + \frac{\cos^{2N}(\varphi/2)}{1+\cos^N(\varphi/2)}\right]^2 + \frac{\pi^2}{128}\cos^{2N}(\varphi/2), \quad (33)$$

where the information gain and the output fidelity are given as functions of the measurement strength $\varphi \equiv g^2\tau/\Delta$ after N successive measurements.

First, we consider a single measurement $N = 1$ with one atom passing through the cavity. Both quantifiers of information gain, I_M and I_F, increase monotonically with the measurement strength $\varphi \in [0,\pi]$, while the output fidelity F decreases monotonically as shown in Fig. 2(a). When $\varphi = \pi$, the QND scheme becomes a photon-number-parity measurement, as seen from Eq. (11) with $\frac{e^{-in\varphi}\pm 1}{2} = \frac{(-1)^n \pm 1}{2}$. Detecting the atom to be in $|e\rangle$ or $|f\rangle$ designates the cavity photon number to be odd or even, respectively. In the case of a qubit state, this parity measurement effectively distinguishes $|0\rangle$ and $|1\rangle$. We obtain the highest information gain and the lowest fidelity when applying this parity measurement.

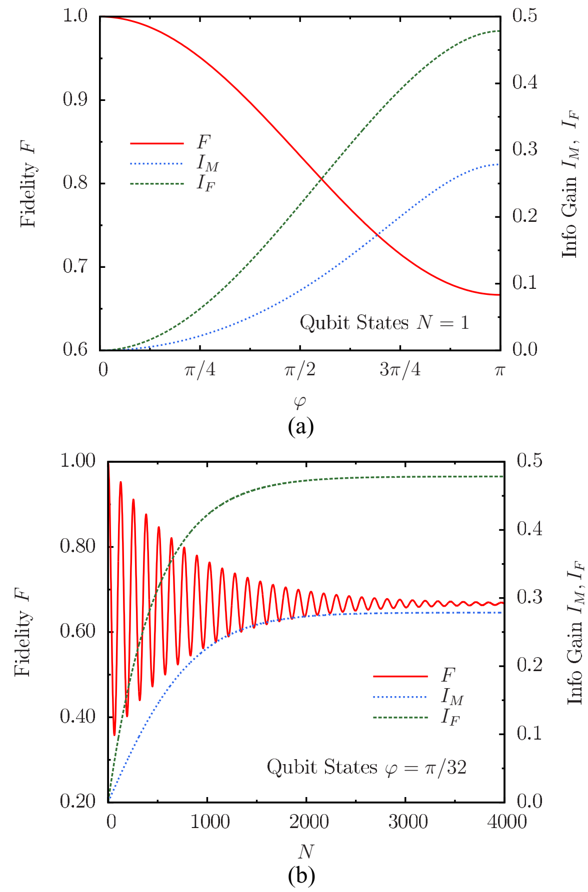

FIG. 2. (Color online) Trade-off relation under measurements on a qubit. (a) F, I_M, and I_F with respect to measurement strength φ under a single measurement on a qubit. As φ increases, the fidelity monotonically decreases while the information gain increases. When $\varphi = \pi$, the measurement discriminates states $|0\rangle$ and $|1\rangle$ perfectly, so the greatest information gain is obtained. (b) F, I_M, and I_F with respect to measurement times N under a successive measurement on a qubit with measurement strength $\varphi = \pi/32$. A larger N value results in a greater information gain and less fidelity, while the fidelity also shows oscillating behavior.

Second, we consider the case of successively performing the QND measurements many times for a given measurement strength $\varphi = g^2\tau/\Delta$. We see from Eq. (33) that $I_M(N,\varphi)$ and $I_F(N,\varphi)$ monotonically increase with the number N of measurements, whereas $F(N,\varphi)$ exhibits oscillating behavior, as illustrated in Fig. 2(b). As the number N of successive measurements increases, we also see that a greater information gain does not always entail a greater disturbance of the state, as evidenced by the oscillating behavior with respect to N.

From Eq. (33), for $\varphi \neq \pi$, the limiting values of the information gain ($I_M = 1 - \frac{1}{2\ln 2} \approx 0.28$ and $I_F = 1 - \frac{4}{9} - \frac{\pi^2}{128} \approx 0.48$) and the output fidelity ($F = \frac{2}{3}$) are achieved as the number of measurements becomes increasingly large as shown in Fig. 2(b). For $\varphi = \pi$ (parity measurement), these values are achieved with only a single measurement $N = 1$. It indicates that the parity measurement extracts the most information from an unknown qubit state and leaves the state the most deeply disturbed.

B. Coherent states

Next we consider that the cavity field is prepared in an unknown coherent state $|\alpha\rangle \equiv ||\alpha|e^{i\beta}\rangle$ with a probability density $p(|\alpha|,\beta)$, i.e.,

$$\rho = \sum_{|\alpha|,\beta} p(|\alpha|,\beta)||\alpha|e^{i\beta}\rangle\langle|\alpha|e^{i\beta}|, \tag{34}$$

where a coherent state is given by

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{35}$$

We assume that the coherent state of complex amplitude $\alpha = |\alpha|e^{i\beta}$ is prepared with a Gaussian distribution,

$$p(|\alpha|,\beta) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}\sigma|\alpha|}\exp\left[-\frac{(|\alpha|-|\alpha_0|)^2}{2\sigma^2}\right]; \tag{36}$$

i.e., its magnitude $|\alpha|$ has a distribution centered at $|\alpha_0|$ with a width σ, whereas its phase β is randomly distributed over $[0,2\pi]$. Since it is a continuous distribution, we convert the summation to an integral as

$$\sum_{|\alpha|,\beta} p(|\alpha|,\beta) \to A\int_0^{\infty}|\alpha|d|\alpha|\int_0^{2\pi} d\beta p(|\alpha|,\beta), \tag{37}$$

where A is a normalized factor.

In this case, the analytical expressions of the information gain and the fidelity are tedious to obtain, thus we study the numerical results of those quantities. As an illustration, we consider the case where the initial coherent state distribution is centered at $|\alpha_0| = 0$ with $\sigma^2 = 2$.

First, for a single measurement $N = 1$, we plot the information gain and the fidelity versus the measurement strength φ in Fig. 3(a). We observe that the fidelity decreases when the measurement strength increases, which is similar to the qubit case. At $\varphi = \pi$, the fidelity is the minimum since most elements of M_e and M_f shown in Eq. (11) are 0. However, the information gain at $\varphi = \pi$ is not maximum, because such a measurement can only discriminate odd or even number states; e.g., it cannot discriminate $n = 1$ or $n = 3$. From the plot we see that the maximum points are around $\pi/8$ for both kinds of information gain. [See the paragraphs below discussing Eq. (38).]

Second, we consider successive weak measurements with $\varphi = \pi/32$ for the same Gaussian distribution with $|\alpha_0| = 0$ and $\sigma^2 = 2$. As shown in Fig. 3(b), the information gain and the output fidelity behave similarly to those in the qubit case. I_M and I_F increase monotonically with the number of measurements, while the fidelity curve exhibits a decreasing trend but with oscillating behavior. The information gain in both forms, I_M and I_F, increases monotonically with respect to N for all values of φ as illustrated in Fig. 4.

The optimal $\varphi_{\rm opt}$ to obtain the greatest information gain depends on the range of possible photon numbers. It may

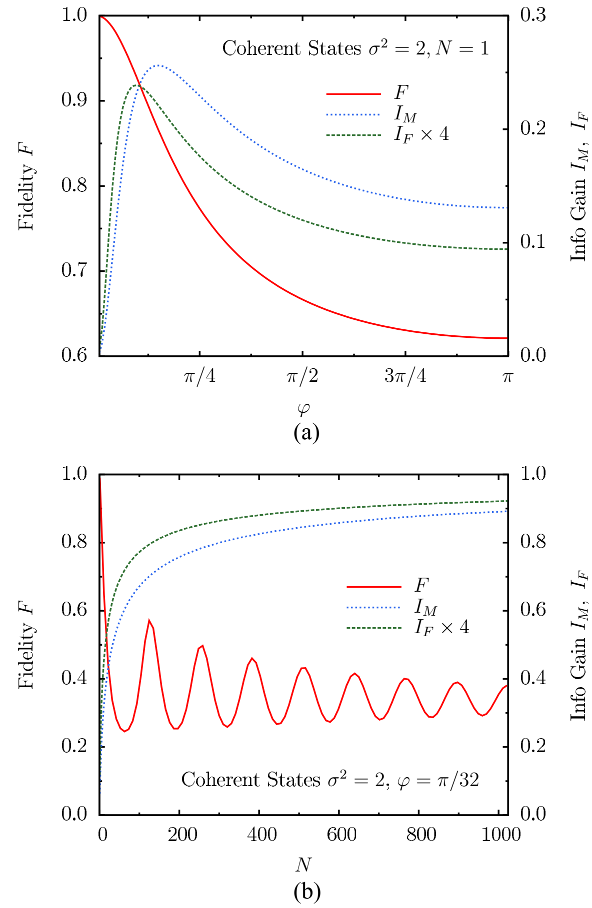

FIG. 3. (Color online) Trade-off relation under measurements on coherent states ($|\alpha_0| = 0$, $\sigma^2 = 2$). I_F is multiplied by 4 for convenience of comparison. (a) F, I_M, and I_F with respect to measurement strength φ under a single measurement. The fidelity decreases when φ increases. The peaks of I_M and I_F slightly depart from each other, however, both of them are around $\pi/8$. (b) F, I_M, and I_F with respect to measurement times N under successive measurements when $\varphi = \pi/32$. The results are similar to those for qubit states shown in Fig. 2(b).

be ascribed to the mechanism of the QND measurements on photon numbers. It discriminates photon numbers n by a mapping from N_e to n. To accomplish this task best, it is required that n and N_e be bijective; i.e., a one-to-one correspondence between n and N_e must be established. The analytical relation is given by the conditional probability in Eq. (19). If the factors $c_n^2 = \sin^2(n\varphi/2)$ and $d_n^2 = \cos^2(n\varphi/2)$ are monotonous in each range of π for $n\varphi$, then n and N_e are bijective. If $\varphi > \pi/(n_{\max} - n_{\min})$, we could not discriminate n perfectly, because two different n's could be mapped to the same N_e. In the opposite limit, a φ value that is too small means a weaker measurement, resulting in less information gain. Therefore, the optimal φ is estimated to be

$$\varphi_{\rm opt} \approx \frac{\pi}{n_{\max} - n_{\min}}. \tag{38}$$

This again explains why the optimal φ is π for qubit states ($n_{\max} - n_{\min} = 1$). For the prior Gaussian distribution of coherent states with zero mean and σ^2 variance, the range of $|\alpha|$ can be cut off at about $\sqrt{2}\sigma$. The photon number of

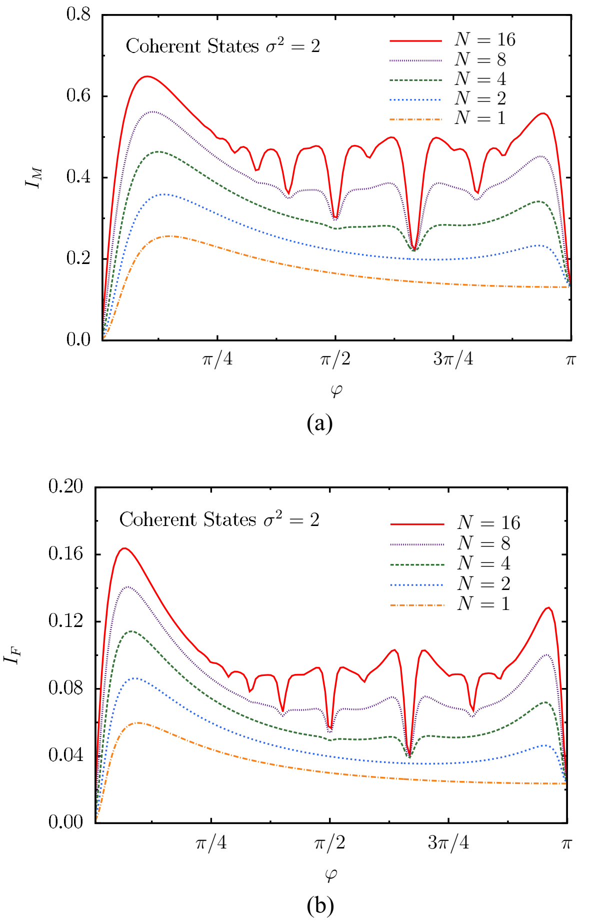

FIG. 4. (Color online) Information gain I_M and I_F with respect to measurement strength φ under successive measurements for different times ($N = 1, 2, 4, 8, 16$) on coherent states ($|\alpha_0| = 0$, $\sigma^2 = 2$). The optimal values agree with results in Eq. (39), although they deviate slightly when N increases.

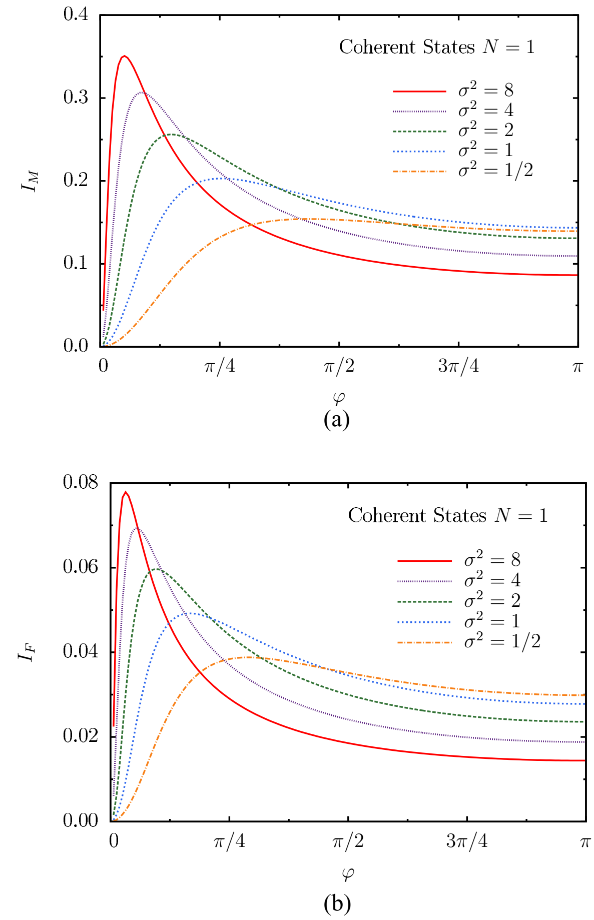

FIG. 5. (Color online) Information gain I_M and I_F with respect to measurement strength φ under a single measurement on coherent states ($|\alpha_0| = 0$, different σ). As the variance σ^2 increases, the positions of peaks shift leftward. The peaks of both I_M and I_F for the same σ are around $\pi/(2\sigma^2 + 2\sigma)$, although they slightly depart from each other.

the coherent state is then effectively distributed between 0 and $2\sigma^2 + 2\sigma$. Therefore the optimal measurement strength for obtaining the highest information gain can be estimated as

$$\varphi_{\text{opt}} \approx \frac{\pi}{2\sigma^2 + 2\sigma}. \tag{39}$$

For successive measurements, φ_{opt} shifts a little leftward with the increase in measurement times but is still around $\pi/(2\sigma^2 + 2\sigma)$ as shown in Fig. 4. To see the dependence on σ, we plot the information gains I_M and I_F after a single measurement versus φ for different σ values in Fig. 5. We observe that with a larger σ the optimal φ becomes smaller. The analytical ansatz values are $\varphi_{\text{opt}} \approx 1.30, 0.79, 0.46, 0.26$, and 0.15. The peak values for I_M are $\varphi_{\text{opt}}^M \approx 1.37, 0.80, 0.47, 0.28$, and 0.16, while those for I_F are $\varphi_{\text{opt}}^F \approx 0.91, 0.52, 0.30, 0.17$, and 0.10. Therefore φ_{opt} in Eq. (39) is a very good estimate of φ_{opt}^M. A cutoff larger than $\sqrt{2}\sigma$ can be chosen to better fit φ_{opt}^F, and the discrepancy between the peak of I_M and that of I_F may be due to their different contexts as information measures.

Through our QND measurements, we gain information on which states are more likely to be the initial state. In other words, given different measurement outputs, the conditional

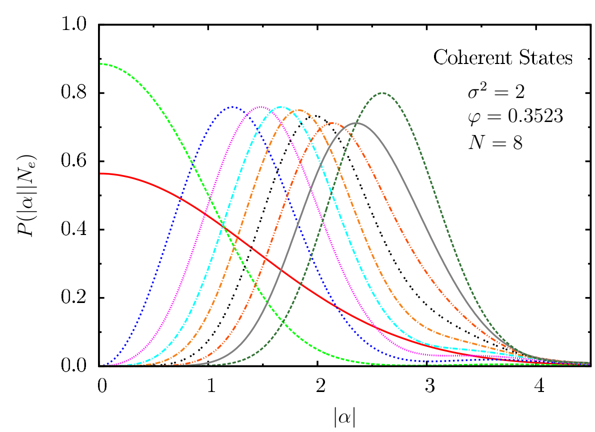

FIG. 6. (Color online) Probability density distributions when $|\alpha_0| = 0$, $\sigma^2 = 2$, $\varphi = 0.3523$, and $N = 8$. The bold solid (red) curve represents the prior distribution. The other nine curves are conditional ones given the outcomes $N_e = 0, 1, \ldots, 8$, from left to right. Each peak indicates a most probable $|\alpha|$.

probability density distributions become different from, and sharper than (with less Shannon entropy), the prior probability distribution. To see this, we plot in Fig. 6 the conditional probability density distributions after eight successive weak measurements with the numerically obtained optimal $\varphi^M_{\text{opt}} = 0.3523$. As a comparison, we also plot the prior probability distribution in the same figure by the bold solid (red) line. We observe that these conditional curves are well separated and sharper than the prior distribution. Each peak indicates a most probable $|\alpha|$.

IV. CONCLUSION

We have studied the trade-off relation between information gain and output fidelity for the case of a QND measurement of photon numbers based on a cavity-QED setup. The information gain has been quantified either by I_M, based on the mutual information, or by I_F, based on the concept of classical fidelity. In particular, we have investigated from an information theoretic viewpoint how the information gain and the output fidelity behave as we vary the measurement strength or the number N of successive measurements. We have shown that both the information quantifiers, I_M and I_F, exhibit very similar behaviors for all cases considered here (qubit states and coherent states), thus the trade-off relation between the information gain and the output fidelity remains almost the same regardless of the measure of information gain.

To illustrate our analysis, we have considered two specific classes of initial states: qubit states and coherent states. In either case, the cavity is initially prepared in an unknown pure state with a certain probability. For a single weak measurement, the optimal measurement strength depends on the range of all possible photon numbers and a stronger measurement does not necessarily lead to more information gain. For the case of successive weak measurements, the information gain increases monotonically with respect to the number N of measurements, while the fidelity shows oscillatory decreasing behavior. This results from the interference terms with different photon numbers. Thus, a greater information gain does not always lead to a worse fidelity. This may deserve further study to gain more insight into the trade-off relation occurring in quantum measurements, which may also have some practical implications.

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