Runge-Lenz vector in the Calogero-Coulomb problem

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We construct the Runge-Lenz vector and the symmetry algebra of the rational Calogero-Coulomb problem using the Dunkl operators. We reveal that they are proper deformations of their Coulomb counterpart. Together with similar correspondence between the Calogero oscillator and oscillator models, this observation permits the claim that most of the properties of the Coulomb and oscillator systems can be lifted to their Calogero-extended analogs by the proper replacement of the momenta by the Dunkl momenta operators.

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I. INTRODUCTION

The N-dimensional oscillator and Coulomb problems are the best-known bound systems with maximal number $(2N-1)$ of functionally independent constants of motion. Such systems are called maximally superintegrable. The free particle is the most widely known superintegrable unbound system. It seems that all other superintegrable systems can be obtained somehow from those listed above.

The rational Calogero model [1,2]

$$\mathcal{H}_0=\sum_{i=1}^{N}\frac{p_i^2}{2}+\sum_{i<j}\frac{g(g-1)}{(x_i-x_j)^2} \tag{1}$$

is highlighted among the nontrivial unbound superintegrable systems. Its superintegrability was established in the classical [3] and quantum [4,5] cases.

The generalization of $\mathcal{H}_0$, associated with an arbitrary finite Coxeter group [6], is also superintegrable. Let us mention that the Coxeter group is described as a finite group generated by a set of orthogonal reflections across the hyperplanes $\alpha\cdot x=0$ in the N-dimensional Euclidean space:

$$s_\alpha x = x-\frac{2(\alpha\cdot x)}{\alpha\cdot\alpha}\alpha,\qquad \alpha\in\mathcal{R}_+. \tag{2}$$

Here the vectors α from the set $\mathcal{R}_+$ (called the system of positive roots) uniquely characterize the reflections. In this case the potential in (1) is replaced by

$$V(x_1,\dots,x_N)=\sum_{\alpha\in\mathcal{R}_+}\frac{g_\alpha(g_\alpha-1)(\alpha\cdot\alpha)}{2(\alpha\cdot x)^2}. \tag{3}$$

The coupling constants g_α form a reflection-invariant discrete function. The original Calogero potential in (1) corresponds to the A_{N-1} Coxeter system with the positive roots, defined in terms of the standard basis by $\alpha_{ij}=e_i-e_j$ for $i<j$. The reflections (2) become the coordinate permutations in this particular case [see (6) below].

The rational Calogero model with additional oscillator potential is given by the Hamiltonian

$$\mathcal{H}_\omega=-\frac{1}{2}\partial^2+\frac{\omega^2}{2}x^2+\sum_{i<j}\frac{g(g-1)}{(x_i-x_j)^2}. \tag{4}$$

We refer to it as the *Calogero-oscillator* model,[1] which is also superintegrable [7].

The similarity between the Calogero model and a free particle, as well as between the Calogero-oscillator model and an oscillator, is clearly elucidated from the perspective of the matrix model reduction and the exchange operator formalism (see [6,8] for the review). Let us briefly outline the second approach, elaborated independently by Polychronakos [9] and by Brink, Hansson, and Vasiliev [10], which then has been found to be related with seminal work by Dunkl [11]. Following these authors, we can take into account the Calogero interaction, replacing the momenta $p_i=-\imath\partial_i$ by the Dunkl momenta $-\imath\nabla_i$, defined in terms of the Dunkl operators [11]:

$$\nabla_i=\partial_i-\sum_{j\neq i}\frac{g}{x_i-x_j}s_{ij}. \tag{5}$$

Here s_{ij} is the exchange operator between the ith and jth coordinates:

$$s_{ij}\psi(\dots,x_i,\dots,x_j,\dots)=\psi(\dots,x_j,\dots,x_i,\dots). \tag{6}$$

Amazingly, such operators commute like usual partial derivatives:

$$[\nabla_i,\nabla_j]=0. \tag{7}$$

Meanwhile, their commutations with the coordinates are more involved and are expressed through the permutations:

$$[\nabla_i,x_j]=S_{ij}=\begin{cases}-gs_{ij} & \text{for } i\neq j,\\ 1+g\sum_{k\neq i}s_{ik} & \text{for } i=j.\end{cases} \tag{8}$$

In the absence of the inverse-square potential ($g=0$), the above relations define the usual Heisenberg algebra.

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[1] Actually, in the literature this system is referred to as the Calogero model, while the unbound system (1) is referred as the Calogero-Moser system due to Moser who established the Liouville integrability [2]. Our notations are more proper for reflecting the structure of underlying models.

The Calogero-oscillator Hamiltonian (4) can be obtained by the restriction of the generalized Hamiltonian

$$\mathcal{H}^{\rm gen}_\omega = -\frac{1}{2}\nabla^2 + \frac{\omega^2}{2}x^2 \tag{9}$$

to the symmetric wave functions [10]

$$s_{ij}\psi_s(x) = \psi_s(x). \tag{10}$$

In these terms there is a remarkable similarity between the integrals of motion of the Calogero oscillator and ordinary oscillator systems. First, take the overcompleted set of the symmetry generators of $\mathcal{H}^{\rm gen}_\omega$, which are given by the Dunkl angular momentum operator [4,12][2]

$$M_{ij} = x_i\nabla_j - x_j\nabla_i, \tag{11}$$

satisfying the deformed commutation relations [4,13]

$$[M_{ij},M_{kl}] = S_{kj}M_{il} + S_{li}M_{jk} - S_{ki}M_{jl} - S_{lj}M_{ik}, \tag{12}$$

and the hidden symmetry generators

$$I_{ij} = -\nabla_i\nabla_j + \omega^2 x_i x_j. \tag{13}$$

Note that in the $g = 0$ limit, the generators M_{ij} and I_{ij} are reduced to the unitary algebra $u(N)$, which describes the symmetry of the N-dimensional isotropic oscillator. The constants of motion of the Calogero-oscillator model $\mathcal{H}_\omega$ can be associated with the symmetric polynomials

$$\mathcal{M}_{2k} = \sum_{i<j} M_{ij}^{2k}, \tag{14}$$

$$\mathcal{I}_k = \sum_i^N I_{ii}^k, \qquad \mathcal{I}'_k = \sum_{i<j}^N I_{ij}^k. \tag{15}$$

The symmetrization ensures their valid action on the wave functions (10).

The symmetries of the Calogero model without oscillator $\mathcal{H}_0$ are related to the symmetries of the free-particle system in the same way. Being restricted to the symmetric wave functions, the Hamiltonian (1) coincides with the analog of the free-particle Hamiltonian with the Dunkl derivatives used instead of the standard ones [9]:

$$\mathcal{H}^{\rm gen}_0 = -\tfrac{1}{2}\nabla^2. \tag{16}$$

It commutes both with the Dunk operators (5) and the Dunkl momenta generators (11). The symmetric polynomials

$$I_k|_{\omega=0} = \sum_i \nabla_i^k \tag{17}$$

mutually commute, in contrast to the Dunkl angular momentum polynomials (14). For $k \leqslant N$ they form the set of the Liouville integrals of motion of the Calogero Hamiltonian (1). The operators (14) complete them to the full set of integrals of motion [9].

Hence, the symmetries of the rational Calogero model without and with the oscillator potential, formulated in terms of the Dunkl operators, are in one-to-one correspondence with those of the free particle and the oscillator, respectively. This holds also for a more general rational potential (3), associated with an arbitrary Coxeter group [13].

On the other hand, it has been known for years that the rational Calogero model, extended by any other central potential, remains an integrable system [14,15]. The integrability is more or less obvious. It proceeds from the integrability of the angular part of the generalized rational Calogero model [16]. Meanwhile, only few of them preserve the superintegrability property. In a recent paper with Lechtenfeld, we have shown that the oscillator and Coulomb potentials are unique in this context [17].

Moreover, we have observed that the Calogero-oscillator and *Calogero-Coulomb* models are, in fact, the only isospectral deformations of the original oscillator and Coulomb systems.

Let us remember that the hidden symmetries of the Coulomb problem are given by the Runge-Lenz vector, forming a quadratic algebra together with the angular momentum operators [18]. It is reduced to the orthogonal so$(N+1)$ or pseudo-orthogonal so$(N,1)$ algebra, respectively, on the bound (negative-energy) or unbound (positive-energy) states.

Having in mind the similarity between the symmetries of the isotropic oscillator and Calogero-oscillator models, and the fact that the Calogero-oscillator and Calogero-Coulomb systems are highlighted from an integrability viewpoint [17], we can ask whether the symmetries of the conventional Coulomb problem can be deformed to the symmetries of the Calogero-Coulomb model.

In this article we will show the following:

(1) The symmetry generators of the Calogero-Coulomb system, formulated in terms of the Dunkl operators, are given by the deformed angular momentum tensor (11) and by the deformed Runge-Lenz vector.

(2) The symmetry algebra of the Calogero-Coulomb model is a deformation of the symmetry algebra of the initial Coulomb problem.

(3) The functional relation between the Coulomb Hamiltonian, Runge-Lenz vector, and the angular momentum has an analog in the Calogero-Coulomb problem.

In fact, this means that the Calogero-Coulomb problem is as fundamental as the Calogero-oscillator problem. Due to such profound similarity with the conventional Coulomb problem, we expect that most of the applications of the Coulomb system can be extended somehow to the Calogero-Coulomb system.

II. SYMMETRIES

In this section we demonstrate that all symmetries of the N-dimensional quantum Coulomb model can be deformed to those of the Calogero-Coulomb problem [14,15],[3]

$$\mathcal{H}_\gamma = -\frac{1}{2}\partial^2 - \frac{\gamma}{r} + \sum_{i<j}\frac{g(g-1)}{(x_i-x_j)^2}, \tag{18}$$

using the Dunkl operator formalism.

[2]For simplicity, we omit the imaginary unit $\imath$ in the definition (11) so that M_{ij} becomes anti-Hermitian in our notations.

[3]In our notations, the subscript in $\mathcal{H}_\omega$ and $\mathcal{H}_\gamma$ is not just an argument, but it also defines the type of confining potential.

The generalized Calogero-Coulomb model is described by the following Hamiltonian:

$$\mathcal{H}^{\rm gen}_\gamma = -\frac{\nabla^2}{2} - \frac{\gamma}{r} \qquad \text{with} \qquad r = \sqrt{x^2}. \tag{19}$$

As in the previously discussed Calogero-oscillator case, it preserves the Dunkl angular momentum operators:

$$\left[\mathcal{H}^{\rm gen}_\gamma, M_{ij}\right] = 0. \tag{20}$$

We define the following deformation of the ordinary Runge-Lenz vector:

$$A_i = -\frac{1}{2}\sum_j \{M_{ij},\nabla_j\} + \frac{1}{2}[\nabla_i, S] - \gamma\frac{x_i}{r}, \tag{21}$$

where

$$S = \sum_{i<j} S_{ij}. \tag{22}$$

Here and in the following, the curly brackets mean an anticommutator of two operators:

$$\{a,b\} = ab + ba. \tag{23}$$

The operator S is invariant with respect to the permutations and is a constant on the symmetric wave functions (10):

$$[S,s_{ij}] = 0, \qquad S\psi_s(x) = g\frac{N(1-N)}{2}\psi_s(x). \tag{24}$$

In the absence of inverse-square interaction, $g = 0$, the operators S_{ij} are reduced to δ_{ij}, and the second term in (21) vanishes. Respectively, the conserving quantities A_i are reduced to the usual Runge-Lenz vector of the N-dimensional Coulomb system.

In the Appendix we prove that the generalized Runge-Lenz vector provides the system with the hidden symmetry:

$$\left[\mathcal{H}^{\rm gen}_\gamma, A_i\right] = 0. \tag{25}$$

Therefore the operators M_{ij} and A_i generate an entire symmetry algebra of the generalized Calogero-Coulomb system (19). It appears that they obey the following commutation relations:

$$\begin{aligned} [A_i, M_{kl}] &= A_l S_{ki} - A_k S_{li}, \\ [A_i, A_j] &= -2\mathcal{H}^{\rm gen}_\gamma M_{ij}. \end{aligned} \tag{26}$$

At the pure Coulomb point (i.e., in the $g = 0$ limit) these relations together with (12) are reduced to the symmetry algebra of the N-dimensional Coulomb problem.

The second commutation relation in Eq. (26) is proven in the Appendix.

Consider the first commutator in (26). It can be viewed as a deformation of the infinitesimal rotation of the vector A_i. Note that the coordinates ($u_i = x_i$) and Dunkl operators ($u_i = \nabla_i$) obey the same relation, as it follows from Eqs. (5), (7), (8), and (11):

$$[u_i, M_{kl}] = S_{ik}u_l - S_{il}u_k = u_l S_{ki} - u_k S_{li}. \tag{27}$$

Now we express A_i in terms of the coordinates and Dunkl operators. The following formula is proven in the Appendix:

$$A_i = \left(r\partial_r + \frac{N-1}{2}\right)\nabla_i - x_i\left(\nabla^2 + \frac{\gamma}{r}\right). \tag{28}$$

Evidently, the operator-valued coefficients of x_i and ∇_i in the above expressions commute with M_{kl} and S_{kl}. Hence, the first commutation relation in (26) follows directly from the identities (27).

Like in the oscillator case, we are forced to combine the conserving quantities A_i and M_{ij} into the symmetric polynomials

$$\mathcal{A}_k = \sum_{i=1}^N A_i^k \tag{29}$$

and (14) in order to get the well-defined constants of motion for the original model (18).

The first member of the family set (29) is independent of the S term and is given by the expression [17]

$$\mathcal{A}_1 = \sum_i x_i\left(2\mathcal{H}^{\rm gen}_\gamma + \frac{\gamma}{r}\right) + \left(r\partial_r + \frac{N-1}{2}\right)\sum_i \partial_i. \tag{30}$$

The constant of motion $\mathcal{M}_2$ does not commute with M_{ij} but is related with the Casimir element $\mathcal{M}_2'$ of the algebra (11) in a rather simple way [13]:

$$\begin{aligned} \mathcal{M}_2' &= \mathcal{M}_2 - S(S - N + 2) \\ &= r^2\nabla^2 - r^2\partial_r^2 - (N-1)r\partial_r. \end{aligned} \tag{31}$$

It describes the angular part of the Calogero model, studied from various perspectives in [19,20].

The constant of motion $\mathcal{A}_2$ does not commute with M_{ij} as well. However, the corrected integral

$$\mathcal{A}_2' = \mathcal{A}_2 + 2\mathcal{H}^{\rm gen}_\gamma S \tag{32}$$

becomes commutative with the Dunkl angular momentum, as was proven in the Appendix:

$$[\mathcal{A}_2', M_{ij}] = 0. \tag{33}$$

This suggests that a certain combination of these invariants may commute with A_i too.

In the Appendix we prove the following relation between the symmetry generators, which generalizes a similar relation in the conventional Coulomb problem:

$$\mathcal{A}_2' = \gamma^2 - 2\mathcal{H}^{\rm gen}_\gamma\left(\mathcal{M}_2' - \frac{(N-1)^2}{4}\right). \tag{34}$$

Presumably, it can be used for the pure algebraic derivation of the spectrum of the Calogero-Coulomb problem.

III. CONCLUDING REMARKS

In this article we have proven that all relations between the symmetry generators of the Coulomb problem can be extended to the Calogero-Coulomb model. To obtain them we should replace the momenta operators $-\imath\partial_i$ by the Dunkl momenta $-\imath\nabla_i$ and make proper corrections depending on the permutation operators. It is straightforward to extend our consideration to the Calogero-Coulomb model associated with arbitrary Coxeter group. Note that the two-dimensional Calogero-Coulomb problem associated with the dihedral group D_2 was investigated recently using the Dunkl operators [21]. The same correspondence holds for the Calogero-oscillator model [13].

Both the Calogero-oscillator and Calogero-Coulomb models have superintegrable counterparts on (pseudo)spheres [17], and we have no doubt that the symmetry algebras of (pseudo)spherical oscillator and Coulomb systems can be lifted, in the same way, to those with Calogero term. However, due to technical difficulties we are unable to complete these calculations.

This remarkable similarity between the Calogero-oscillator (Calogero-Coulomb) model and oscillator (Coulomb) permits us to claim that *most of the properties of the oscillator and Coulomb systems can be extended to their counterparts supplemented by the Calogero interaction term.* We are sure that in this way one can construct the superintegrable extensions of three- or five-dimensional Calogero-Coulomb problems, specified by the presence, respectively, of the Dirac and Yang monopoles. Moreover, it seems that acting in the suggested way, we can relate the two-, four-, and eight-dimensional Calogero-oscillator models with the two-, three-, and five-dimensional Calogero-Coulomb models, including those specified by the presence of anyon and Dirac, Yang monopoles in the spirit of Ref. [22]. (For previous treatments see Ref. [15].)

Recently, the superintegrability of the (relativistic) Dirac oscillator [23] and Coulomb [24] systems has been established. It would be interesting to study them in the presence of Calogero interaction from the superintegrability point of view. Moreover, we expect that in this way one can solve the problem of $\mathcal{N}=4,8$ supersymmetrization of the Calogero model, which was treated by many authors (see [25] and references therein).

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APPENDIX: DERIVATIONS

1. Conservation of Runge-Lenz vector (21)

Here we prove that the Calogero-Coulomb Hamiltonian preserves the deformed Runge-Lenz vector (21). First we compute the commutator between the Hamiltonian and A_i. The commutator with the first term in the right-hand side of the equation (21) can be simplified using the following identity:

$$\left[\sum_j\{M_{ij},\nabla_j\}\,,\,\frac{1}{r}\right] = -\frac{1}{r^3}\sum_j\{M_{ij},x_j\} = \left\{\frac{1}{r},\nabla_i\right\} - \sum_j\left\{\frac{x_ix_j}{r^3},\nabla_j\right\}. \tag{A1}$$

In the derivation we have used the vanishing of the two commutators [13]:

$$[\nabla^2,M_{ij}] = [r,M_{ij}] = 0. \tag{A2}$$

Next, we can calculate the commutator of the Hamiltonian with the last term in the deformed Rune-Lenz vector expression (30) using the following identity:

$$\left[\frac{x_i}{r},\nabla^2\right] = \sum_j\left\{\frac{x_ix_j}{r^3} - \frac{S_{ij}}{r},\nabla_j\right\}. \tag{A3}$$

Combining together the relations (A1) and (A3), we obtain

$$\left[-\frac{1}{2}\sum_j\{M_{ij},\nabla_j\} - \frac{\gamma x_i}{r}\,,\,\mathcal{H}_\gamma^{\text{gen}}\right] = \sum_j\left\{\frac{\gamma S_{ij}}{2r},\nabla_i-\nabla_j\right\} = \frac{1}{2}\left[[S,\nabla_i],\mathcal{H}_\gamma^{\text{gen}}\right], \tag{A4}$$

where S is defined in (22). In the last equation we have used the identity

$$\sum_j(\nabla_j-\nabla_i)S_{ij} = [S,\nabla_i].$$

The relation (A4) completes the proof of conservation of the deformed Runge-Lenz operator (25).

2. Second commutation relation in Eq. (26)

Let us derive the commutation relation between the components of the Runge-Lenz vector in (26). For convenience, we present the deformed Runge-Lenz vector (28) in the following form:

$$A_i = a\nabla_i - x_ib,\quad a\equiv r\partial_r + \frac{N-1}{2},\quad b\equiv\nabla^2+\frac{\gamma}{r}. \tag{A5}$$

Then

$$[A_i,A_j] = [a\nabla_i,a\nabla_j] + [x_ib,x_jb] + [a\nabla_j,x_ib] - [a\nabla_i,x_jb]. \tag{A6}$$

Note that the commutator $[a,f]$ counts the total degree in coordinates of the quantity f:

$$[a,\nabla_i] = -\nabla_i,\quad [a,x_i] = x_i,\quad [a,b] = -2\nabla^2 - \frac{\gamma}{r}. \tag{A7}$$

The commutators in (A6), containing the observable b, are simplified to

$$[b,x_i] = 2\nabla_i,\qquad [b,\nabla_i] = \frac{\gamma x_i}{r^3}. \tag{A8}$$

Using the above equations, we obtain

$$[a\nabla_i,a\nabla_j] = 0,\qquad [x_ib,x_jb] = 2M_{ij}b, \tag{A9}$$

$$[a\nabla_j,x_ib] = x_j\nabla_i\nabla^2 + \frac{\gamma x_ix_j}{r^3}(a+1) + aS_{ij}b. \tag{A10}$$

The last two terms on the right-hand side of (A10) are symmetric on the indexes i and j and hence disappear in the commutator (A6). Substituting Eqs. (A9) and (A10) into (A6), we arrive at the relation sought:

$$[A_i,A_j] = -2\mathcal{H}_\gamma^{\text{gen}}M_{ij}. \tag{A11}$$

3. Relation (28)

Here we derive the relation (28), which expresses the Runge-Lenz invariant in terms of the coordinates and Dunkl momenta.

First we calculate the first term of (21):

$$\sum_j\{M_{ij},\nabla_j\} = \{\nabla^2,x_i\} - (x\cdot\nabla)\nabla_i - \nabla_i(\nabla\cdot x)$$
$$= \{\nabla^2,x_i\} - (2r\partial_r + (N+1))\nabla_i + [\nabla_i,S]. \qquad \text{(A12)}$$

We have used the following identities in the derivation:

$$x\cdot\nabla = r\partial_r + S, \qquad \nabla\cdot x = r\partial_r - S + N. \qquad \text{(A13)}$$

Finally, substituting them into (21), we arrive at the equation (28).

4. Relation (32)

Let us calculate the commutator of M_{ij} with $\mathcal{A}_2$:

$$[M_{ij},\mathcal{A}_2] = \sum_k\{A_iS_{jk} - A_jS_{ik},A_k\}. \qquad \text{(A14)}$$

Each term from the right-hand side of this equation can be presented as

$$\sum_k\{A_iS_{jk},A_k\} = 2A_iA_j - 2\mathcal{H}_\gamma^{\text{gen}}\sum_k M_{ki}S_{kj}. \qquad \text{(A15)}$$

Here we take into account the identity

$$\sum_k\{S_{ik},u_k\} = 2u_i + \sum_{k\neq i}\{S_{ik},u_k - u_i\} = 2u_i, \qquad \text{(A16)}$$

which is fulfilled for any local operator u_k. Applying the above relation, one can further simplify the commutator (A14):

$$[M_{ij},\mathcal{A}_2] = -2\mathcal{H}_\gamma^{\text{gen}}\left(\sum_k(M_{ki}S_{jk} - M_{kj}S_{ik}) + 2M_{ij}\right)$$
$$= -2\mathcal{H}_\gamma^{\text{gen}}\left[M_{ij},\sum_{k\neq i,j}(S_{kj}+S_{ki})\right]$$
$$= -2\mathcal{H}_\gamma^{\text{gen}}[M_{ij},S]. \qquad \text{(A17)}$$

This completes the proof of the equation (32).

5. Relation (34)

We use the representation (A5) for $\mathcal{A}_2$:

$$\mathcal{A}_2 = \sum_i(a\nabla_i - x_ib)^2 = (a+1)a\nabla^2$$
$$+ r^2b^2 + 2(x\cdot\nabla)b - \sum_i\{a\nabla_i,x_ib\}$$
$$= r^2b^2 + 2a^2\mathcal{H}_\gamma^{\text{gen}} - 2(x\cdot\nabla)\mathcal{H}_\gamma^{\text{gen}} - 2a\frac{\gamma}{r}. \qquad \text{(A18)}$$

The commutation relations (A7), (A8), and (A13) are used in the derivation.

In the first term in the last expression one can select the Hamiltonian as follows:

$$r^2b^2 = -2r^2\nabla^2\mathcal{H}_\gamma^{\text{gen}} + \frac{\gamma(N-3)}{r} + 2\gamma\partial_r + \gamma^2. \qquad \text{(A19)}$$

Inserting this into Eq. (A18) and simplifying it, we arrive at the desired relation (34).

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