Coherent quantum dynamics: What fluctuations can tell

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(Received 13 March 2015; published 10 August 2015)

Coherent states provide a natural connection of quantum systems to their classical limit and are employed in various fields of physics. Here we derive general systematic expansions, with respect to quantum parameters, of expectation values of products of arbitrary operators within both oscillator coherent states and SU(2) coherent states. In particular, we generally prove that the energy fluctuations of an arbitrary Hamiltonian are in leading order entirely due to the time dependence of the classical variables. These results add to the list of well-known properties of coherent states and are applied here to the Lipkin-Meshkov-Glick model, the Dicke model, and to coherent intertwiners in spin networks as considered in loop quantum gravity.

DOI: 10.1103/PhysRevA.92.022108 PACS number(s): 03.65.Sq, 04.60.Pp

I. INTRODUCTION

Coherent states are at the heart of semiclassical descriptions of generic quantum systems and have proven to be a versatile tool in a multitude of physical problems. In the general literature [1–5], mainly two types of coherent states are typically distinguished: The first type, the coherent states of the harmonic oscillator, was already investigated by Schrödinger [6] shortly after the birth of quantum mechanics, while SU(2) coherent states in the Hilbert space of a spin of general length S were added in the early 1970s [7,8].

Both types of coherent states share a list of well-known properties which constitute the basis for their prominent role in semiclassics: (i) The coherent states can be generated by a unitary transformation from an appropriate reference state. In the oscillatory case this state is the ground state of a harmonic, while for spins one uses the highest-weight state in some arbitrary basis. As a result, the coherent states are (ii) (over)complete, (iii) eigenstates of simple operators generic to the system, and (iv) they have minimum uncertainty products with respect to an obvious choice of variables. Moreover, (v) coherent states show a coherent time evolution perfectly mimicking the classical limit under appropriate Hamiltonians. For oscillator coherent states such a Hamiltonian is the one of the harmonic oscillator itself, and for the spin case the Zeeman Hamiltonian (coupling the spin to an external magnetic field) plays an analogous role.

In the present work we argue that one can extend the above list by general statements about correlations and fluctuations within coherent states. Specifically we consider the coherent expectation value of a product of two arbitrary operators. For such expectation values we derive systematic expansions in the quantum parameters $\hbar$ or $1/S$ which involve only coherent expectation values of single operators and their commutators with the system variables. These expansions are a versatile tools for the study of the semiclassical regime of generic quantum systems. As an important finding, the energy fluctuations of an arbitrary Hamiltonian are generally proven to be in leading order entirely due to the time dependence of the classical variables. These results add to the above list of properties of coherent states. Reflecting the widespread use of the latter objects, we apply our findings to the Lipkin-Meshkov-Glick model originating from nuclear physics [9], to the Dicke model describing superradiance in quantum optics [10], and to coherent intertwiners of spin networks occurring in the loop approach to quantum gravity [11,12].

This paper is organized as follows: In Sec. II we review and summarize important properties of oscillator coherent states and SU(2) coherent states. The announced results on the coherent expectation values of arbitrary operator products are derived in Sec. III and discussed there on a general footing. Some technical details of the calculations are deferred to Appendix A. Section IV contains the application of our general findings to the Lipkin-Meshkov-Glick model, and the Dicke model is treated in Sec. V. In Sec. VI we turn to the study of coherent intertwiners of spin networks investigated in the covariant approach advocated by loop quantum gravity. Here we derive semiclassical corrections to expectation values in terms of universal expansion coefficients depending only on the network geometry. We close with a summary and an outlook in Sec. VII.

II. COHERENT STATES

We now briefly review, using standard notation, distinctive properties of oscillator coherent states and SU(2) coherent states.

A. Coherent oscillator states

The harmonic oscillator is described by

$$\mathcal{H}_h = \tfrac{1}{2}(p^2 + \omega^2 q^2) = \hbar\omega\left(a^+ a + \tfrac{1}{2}\right) \tag{1}$$

with

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega}{\hbar}}q + \frac{i}{\sqrt{\hbar\omega}}p\right), \quad a^+ = (a)^+ \tag{2}$$

fulfilling

$$[p,q] = \frac{\hbar}{i} \Leftrightarrow [a,a^+] = 1. \tag{3}$$

The system has an equidistant spectrum labeled by $n \in \{0,1,2,\ldots\}$,

$$\mathcal{H}_h|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)|n\rangle. \tag{4}$$

Coherent states of the harmonic oscillator are eigenstates of the lowering operator a with complex eigenvalues α,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{5}$$

They are generated from the ground state via

$$|\alpha\rangle = \exp(\alpha a^{+} - \alpha^{*} a)|0\rangle \tag{6}$$

$$= \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{7}$$

The parameter α is naturally decomposed into its real and imaginary parts as

$$\alpha = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega}{\hbar}}\xi + \frac{i}{\sqrt{\hbar\omega}}\pi\right). \tag{8}$$

Denoting an expectation value within a coherent state (6) by $\langle\cdot\rangle$ it holds that

$$\langle q\rangle = \xi, \quad \langle p\rangle = \pi. \tag{9}$$

Coherent states maintain their shape in the time evolution of the harmonic oscillator,

$$e^{-(i/\hbar)\mathcal{H}_h t}|\alpha\rangle = e^{-(i/2)\omega t}|\alpha e^{-i\omega t}\rangle, \tag{10}$$

and the time dependence of the expectation values (9) follows exactly the classical motion of the harmonic oscillator. This fact justifies the term "coherent states" and relies on the equidistance of the spectrum. The latter property is shared by a quantum spin of arbitrary length in a magnetic field and leads there to a coherent Larmor precession, as we will discuss in Sec. II B.

Moreover, coherent states minimize uncertainty products,

$$\Delta p \Delta q = \hbar/2, \tag{11}$$

and fulfill an (over)completeness relation,

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \mathbf{1}. \tag{12}$$

B. SU(2) coherent states

In the Hilbert space of a spin of length S an SU(2) (or spin) coherent state $|\vartheta,\varphi\rangle$ is defined by the equation

$$\vec{s}\cdot\vec{S}|\vartheta,\varphi\rangle = \hbar S|\vartheta,\varphi\rangle \tag{13}$$

for the direction $\vec{s} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$. For generic systems expectation values within these states provide a natural approach to the classical limit given by $\hbar \to 0$, $S \to \infty$ while $\hbar S$ is kept constant.

Introducing the usual basis of eigenstates of S^z ($S^z|m\rangle = \hbar m|m\rangle$) coherent states can be generated from $|S\rangle$ by a unitary rotation,

$$|\vartheta,\varphi\rangle = U(\vartheta,\varphi)\,|S\rangle \tag{14}$$

$$= \frac{1}{(1+|z|^2)^S}e^{zS^-}|S\rangle \tag{15}$$

with [2,4]

$$U(\vartheta,\varphi) = \exp\left(\frac{i}{\hbar}\vartheta(\sin\varphi S^x - \cos\varphi S^y)\right) \tag{16}$$

$$= e^{zS^-/\hbar}e^{\eta S^z/\hbar}e^{-\bar{z}S^+/\hbar} \tag{17}$$

$$= e^{-\bar{z}S^+/\hbar}e^{-\eta S^z/\hbar}e^{zS^-/\hbar} \tag{18}$$

and

$$z(\vartheta,\varphi) = \tan\frac{\vartheta}{2}e^{i\varphi}, \quad \eta(\vartheta) = 2\ln\cos\frac{\vartheta}{2}. \tag{19}$$

Expanded in the above basis SU(2) coherent states read

$$|\vartheta,\varphi\rangle = \frac{1}{(1+|z|^2)^S}\sum_{m=-S}^{S}\binom{2S}{S+m}^{1/2}z^m|m\rangle \tag{20}$$

$$= \sum_{m=-S}^{S}\left\{\binom{2S}{S+m}^{1/2}\left[\cos\left(\frac{\vartheta}{2}\right)\right]^{S+m}\right.$$
$$\left.\times\left[\sin\left(\frac{\vartheta}{2}\right)\right]^{S-m}e^{\imath\varphi(s+m)}\,|m\rangle\right\}. \tag{21}$$

The analog of the harmonic oscillator for SU(2) coherent states is the Zeeman Hamiltonian

$$\mathcal{H}_z = -\vec{S}\cdot\vec{h} \tag{22}$$

coupling the spin to a magnetic field $\vec{h}$. The spectrum consists of $2S+1$ equidistant energy levels, and the corresponding time evolution of SU(2) coherent states is a coherent Larmor precession, which is most easily seen when putting, without loss of generality, the field direction along the z axis,

$$e^{-(i/\hbar)\mathcal{H}_z t}|\vartheta,\varphi\rangle = e^{-i\varphi Sht}|\vartheta,\varphi+ht\rangle. \tag{23}$$

The latter finding is completely analogous to the harmonic oscillator having a semi-infinite equidistant spectrum.

As further standard properties shared with coherent oscillator states, SU(2) coherent states have a minimum uncertainty product

$$\Delta(\vec{e}_1\cdot\vec{S}\,)\,\Delta(\vec{e}_2\cdot\vec{S}\,) = \frac{\hbar^2 S}{2} \tag{24}$$

with $\vec{e}_1,\vec{e}_2,\vec{s}$ being an orthonormal system, and their (over)completeness can be expressed as

$$\mathbf{1} = \frac{2S+1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\vartheta\,\sin\vartheta\,|\vartheta,\varphi\rangle\langle\vartheta,\varphi| \tag{25}$$

$$= \frac{2S+1}{\pi}\int\frac{d^2z}{(1+|z|^2)^2}|z\rangle\langle z| \tag{26}$$

$$= \frac{2S+1}{\pi}\int d^2z\frac{e^{zS^-/\hbar}|S\rangle\langle S|e^{\bar{z}S^+/\hbar}}{(1+|z|^2)^{2(S+1)}}, \tag{27}$$

where $|z\rangle = |\vartheta,\varphi\rangle$. Further below it will be useful to change reference state $|S\rangle$ in Eq. (27) to an arbitrary SU(2) coherent state by applying the unitary transformation given in Eqs. (16)–(18):

$$\mathbf{1} = \frac{2S+1}{\pi}U\int d^2w\frac{e^{wS^-/\hbar}|S\rangle\langle S|e^{\bar{w}S^+/\hbar}}{(1+|w|^2)^{2(S+1)}}U^+$$

$$= \frac{2S+1}{\pi}\int d^2w\frac{e^{w\tilde{S}^-/\hbar}|z\rangle\langle z|e^{\bar{w}\tilde{S}^+/\hbar}}{(1+|w|^2)^{2(S+1)}} \tag{28}$$

with $\vec{\tilde{S}} = U\vec{S}U^+$.

III. CORRELATIONS

We now derive general theorems for the expectation values of operator products within coherent states.

A. Oscillatory systems

1. General correlation functions

Let A, B be two operators being functions of the two canonical operators p, q (or, equivalently, a, a^+). Using the completeness relation (12) the expectation value of AB within a coherent oscillator state can be formulated as

$$\langle\alpha|AB|\alpha\rangle = \frac{1}{\pi}\int d^2\beta e^{-|\beta|^2}\langle 0|U_\alpha^+ A U_\alpha U_\alpha^+ e^{\beta a^+}|0\rangle \times\langle 0|e^{\bar\beta a}U_\alpha U_\alpha^+ B U_\alpha|0\rangle, \tag{29}$$

where U_α is the unitary operator on the right-hand side of Eq. (6), and

$$U_\alpha^+ e^{\beta a^+}|0\rangle = e^{-(1/2)|\alpha|^2+\bar\alpha\beta}e^{(\beta-\alpha)a^+}|0\rangle \tag{30}$$

such that

$$\begin{aligned}\langle\alpha|AB|\alpha\rangle &= \frac{1}{\pi}\int d^2\beta e^{-|\beta-\alpha|^2}\langle 0|U_\alpha^+ A U_\alpha e^{(\beta-\alpha)a^+}|0\rangle \\ &\quad\times\langle 0|e^{(\bar\beta-\bar\alpha)a}U_\alpha^+ B U_\alpha|0\rangle \\ &= \frac{1}{\pi}\int d^2\beta e^{-|\beta|^2}\langle 0|e^{-\beta a^+}U_\alpha^+ A U_\alpha e^{\beta a^+}|0\rangle \\ &\quad\times\langle 0|e^{\bar\beta a}U_\alpha^+ B U_\alpha e^{-\bar\beta a}|0\rangle,\end{aligned} \tag{31}$$

where we have shifted the integration variable and used $e^{-\bar\beta a}|0\rangle = |0\rangle$. The remaining operator products can be expanded into series of iterated commutators according to

$$e^X Y e^{-X} = \sum_{n=0}^{\infty}\frac{1}{n!}[X,Y]_n \tag{32}$$

with $[X,Y]_0 = Y$ and $[X,Y]_n = [X,[X,Y]_{n-1}]$. Upon performing the integration the two infinite series shrink to a single one yielding

$$\langle\alpha|AB|\alpha\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\langle 0|[-a^+,U_\alpha^+ A U_\alpha]_n|0\rangle \times\langle 0|[a,U_\alpha^+ B U_\alpha]_n|0\rangle \tag{33}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\langle\alpha|[iU_\alpha a^+ U_\alpha^+,A]_n|\alpha\rangle\langle\alpha|[iU_\alpha a U_\alpha^+,B]_n|\alpha\rangle$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\langle\alpha|[ia^+,A]_n|\alpha\rangle\langle\alpha|[ia,B]_n|\alpha\rangle. \tag{34}$$

In the last step we took into account that $U_\alpha a^+ U_\alpha^+$ and a^+ differ just by a constant which commutes with any operator. Thus, we have arrived at an expression for the expectation value of product of two operators within coherent states in terms of a sum over products of such coherent-state expectation values which involve only one of the operators. An alternative form of the above expansions can be given via Eq. (33) as

$$\langle\alpha|AB|\alpha\rangle = \sum_{n=0}^{\infty}\langle 0|U_\alpha^+ A U_\alpha|n\rangle\,\langle n|U_\alpha^+ B U_\alpha|0\rangle, \tag{35}$$

which of course just expresses the completeness of the states $|n\rangle$ and provides an alternative way to derive Eq. (34).

Moreover, using the definition (2) Eq. (34) can be rewritten as

$$\langle\alpha|AB|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\hbar^n}{n!2^n}\langle\alpha|\left[\frac{i}{\hbar}\left(\sqrt{\omega}q - i\frac{p}{\sqrt{\omega}}\right),A\right]_n|\alpha\rangle \times\langle\alpha|\left[\frac{i}{\hbar}\left(\sqrt{\omega}q + i\frac{p}{\sqrt{\omega}}\right),B\right]_n|\alpha\rangle. \tag{36}$$

Since each commutation of p, q with A or B yields a factor of $\hbar$ all expectation value on the above right-hand side are of the same order in $\hbar$. Thus, Eq. (36) is indeed a systematic expansion in $\hbar$ of the coherent-state expectation value of an arbitrary product of two operators. The zeroth order equals the classical result, and for a general correlation function one has the semiclassical expansion

$$\begin{aligned}C_{AB} &:= \langle\alpha|AB|\alpha\rangle - \langle\alpha|A|\alpha\rangle\langle\alpha|B|\alpha\rangle \\ &= \sum_{n=1}^{\infty}\frac{\hbar^n}{n!2^n}\langle\alpha|\left[\frac{i}{\hbar}\left(\sqrt{\omega}q - i\frac{p}{\sqrt{\omega}}\right),A\right]_n|\alpha\rangle \\ &\quad\times\langle\alpha|\left[\frac{i}{\hbar}\left(\sqrt{\omega}q + i\frac{p}{\sqrt{\omega}}\right),B\right]_n|\alpha\rangle.\end{aligned} \tag{37}$$

Choosing $A = B$ we obtain a general expression for the variance of a Hermitian operator A,

$$(\Delta A)^2 = \sum_{n=1}^{\infty}\frac{\hbar^n}{n!2^n}\left|\langle\alpha|\left[\frac{i}{\hbar}\left(\sqrt{\omega}q - i\frac{p}{\sqrt{\omega}}\right),A\right]_n|\alpha\rangle\right|^2, \tag{38}$$

where each term in the semiclassical expansion is nonnegative.

2. Energy fluctuations

Considering $A = \mathcal{H}$ as a Hamiltonian, the corresponding energy fluctuation reads in leading order in $\hbar$

$$(\Delta\mathcal{H})^2 = \frac{\hbar}{2}\left(\left\langle\frac{i}{\hbar}[\sqrt{\omega}q,\mathcal{H}]\right\rangle^2 + \left\langle\frac{i}{\hbar}\left[\frac{p}{\sqrt{\omega}},\mathcal{H}\right]\right\rangle^2\right) + O(\hbar^2) \tag{39}$$

$$= \frac{\hbar}{2}\left(\omega\langle\partial_t q\rangle^2 + \frac{\langle\partial_t p\rangle^2}{\omega}\right), \tag{40}$$

where we have replaced, according to the Heisenberg equations of motion, the commutators with time derivatives. Indeed, if the system is prepared at some initial time $t = t_i$ in a coherent state we have [cf. Eq. (8)]

$$\langle\partial_t q\rangle = \partial_t\xi, \quad \langle\partial_t p\rangle = \partial_t\pi \tag{41}$$

and

$$(\Delta\mathcal{H})^2 = \frac{\hbar}{2}\left(\omega(\partial_t\xi)^2 + \frac{(\partial_t\pi)^2}{\omega}\right) + \mathcal{O}(\hbar^2) \tag{42}$$

at $t = t_i$. In the subsequent time evolution governed by the Hamiltonian $\mathcal{H}$ the state of the system will, for not too large times, approximately be coherent with time-dependent parameters $\xi(t)$, $\pi(t)$ playing the approximate role of classical Hamiltonian variables. Thus, in this semiclassical regime the fact that a coherent state has a finite energy variance, i.e., it is not an eigenstate of the Hamiltonian, is in leading order in $\hbar$ just expressed by the fact that the classical Hamiltonian variables have a nontrivial time dependence, i.e., the system is moving. This result complements the historical Ehrenfest theorem stating that expectation values of observables follow the classical equations.

Relations of the type (40) and (42) were already found in Ref. [13] on the example of specific Hamiltonians. The results here are derived for arbitrary systems and are based on the very general expansions (37) and (38) for correlation functions and fluctuations.

The fact that the system will in its time evolution in general not strictly remain in a coherent state, i.e., decoherence occurs, is reflected by the higher contributions to the energy variance. Indeed, for a harmonic oscillator (1) the time evolution is strictly coherent and we have as an identity

$$(\Delta\mathcal{H}_h)^2 \equiv \frac{\hbar}{2}\left(\omega(\partial_t\xi)^2 + \frac{(\partial_t\pi)^2}{\omega}\right) \tag{43}$$

for all times $t \geqslant t_i$ and without any higher correction.

Finally, it is straightforward to extend the above results for general operator products to the case of $N > 1$ degrees of freedom; details are sketched in Appendix A. For the energy variance one finds in leading order in $\hbar$

$$(\Delta\mathcal{H})^2 = \frac{\hbar}{2}\sum_{a=1}^{N}\left[\left\langle\frac{i}{\hbar}[\sqrt{\omega_a}q_a,\mathcal{H}]\right\rangle^2 + \left\langle\frac{i}{\hbar}\left[\frac{p_a}{\sqrt{\omega_a}},\mathcal{H}\right]\right\rangle^2\right] + O(\hbar^2) \tag{44}$$

$$= \frac{\hbar}{2}\sum_{a=1}^{N}\left[\omega_a\langle\partial_t q_a\rangle^2 + \frac{\langle\partial_t p_a\rangle^2}{\omega_a}\right] \tag{45}$$

with operator pairs p_a, q_a and frequencies ω_a, and the analog of Eq. (42) reads

$$(\Delta\mathcal{H})^2 = \frac{\hbar}{2}\sum_{a=1}^{N}\left[\omega_a(\partial_t\xi_a)^2 + \frac{(\partial_t\pi_a)^2}{\omega_a}\right] + O(\hbar^2). \tag{46}$$

B. Spin systems

1. General correlation functions

We consider again two arbitrary operators A, B which are now functions of a spin operator $\vec{S}$. The expectation value of the product AB within an SU(2) coherent state $|z\rangle$ for spin length S can be formulated as

$$\langle z|AB|z\rangle = \frac{2S+1}{\pi}\int\frac{d^2w}{(1+|w|^2)^{2(S+1)}} \times\langle z|e^{-w\tilde{S}^-/\hbar}Ae^{w\tilde{S}^-/\hbar}|z\rangle \times\langle z|e^{\bar{w}\tilde{S}^+/\hbar}Be^{-\bar{w}\tilde{S}^+/\hbar}|z\rangle, \tag{47}$$

where we have used the completeness relation in the form (28) and the observation

$$e^{-\bar{w}\tilde{S}^+/\hbar}|z\rangle = Ue^{-\bar{w}S^+/\hbar}|0\rangle = |z\rangle \tag{48}$$

with U given in Eqs. (16)–(18). Employing now again the expansion (32) and performing the integration leads to

$$\langle z|AB|z\rangle = \sum_{n=0}^{2S}\frac{(2S-n)!}{n!(2S)!}\langle z|\left[\frac{i}{\hbar}\tilde{S}^-,A\right]_n|z\rangle\langle z|\left[\frac{i}{\hbar}\tilde{S}^+,B\right]_n|z\rangle. \tag{49}$$

The above equation is the spin analog of the result (36). Again all iterated commutators are of the same order in $\hbar$ and S whereas the prefactor of the nth term carries a product $2S(2S-1)\dots(2S-n+1)$ in its denominator. Thus, Eq. (49) is essentially an expansion in the quantum parameter $1/S$. Note that the spin components $\tilde{S}^x$, $\tilde{S}^y$ represent the direction perpendicular to the spin polarization of the coherent state $|z\rangle$. Alternatively, the result (49) can be written as

$$\langle z|AB|z\rangle = \sum_{n=0}^{2S}\frac{(2S-n)!}{n!(2S)!}\langle S|\left[\frac{i}{\hbar}S^-,U^+AU\right]_n|S\rangle \times\langle S|\left[\frac{i}{\hbar}S^+,U^+BU\right]_n|S\rangle \tag{50}$$

$$= \sum_{n=0}^{2S}\langle S|U^+AU|S-n\rangle\langle S-n|U^+BU|S\rangle. \tag{51}$$

Analogously to Eq. (35) for oscillatory systems, the last formulation is just the completeness relation for the states $|m\rangle$ and allows for an alternative derivation of the central result (49). Using the latter, arbitrary correlation functions within SU(2) coherent states can be expressed in full analogy to Eq. (37).

2. Fluctuations

For the variance of a Hermitian operator A we have

$$(\Delta A)^2 = \sum_{n=1}^{2S}\frac{(2S-n)!}{n!(2S)!}\left|\langle z|\left[\frac{i}{\hbar}\tilde{S}^-,A\right]_n|z\rangle\right|^2. \tag{52}$$

The expectation values occurring in leading order can be rewritten as

$$|\langle z|[i\tilde{S}^-,A]|z\rangle|^2 = \sum_{i=1}^{3}|\langle z|[i\tilde{S}^i,A]|z\rangle|^2 \tag{53}$$

$$= \sum_{i=1}^{3}|\langle z|[iS^i,A]|z\rangle|^2, \tag{54}$$

where we have observed that $|z\rangle$ is an eigenstate of $\tilde{S}^z$, and that $\vec{\tilde{S}}$ and $\vec{S}$ are related by an orthogonal matrix,

$$\tilde{S}^i = \sum_{j=1}^{3}O_{ji}S^j. \tag{55}$$

Thus, we have

$$(\Delta A)^2 = \frac{1}{2S}\sum_{i=1}^{3}\left|\langle z|\left[\frac{i}{\hbar}S^i,A\right]|z\rangle\right|^2 + O\left(\frac{1}{S^2}\right), \tag{56}$$

and by a slight generalization of the above arguments one finds for the expectation value of a product of commuting operators

A, B,

$$\langle z|AB|z\rangle = \frac{1}{2}\langle z|AB+BA|z\rangle = \langle z|A|z\rangle\langle z|B|z\rangle + \frac{1}{2S}\sum_{i=1}^{3}\langle z|\left[\frac{i}{\hbar}S^i,A\right]|z\rangle\langle z|\left[\frac{i}{\hbar}S^i,B\right]|z\rangle + O\left(\frac{1}{S^2}\right). \quad (57)$$

The requirement here for a symmetrized operator product stems from the fact that for an identity analogous to Eq. (53) to hold products of expectation values involving both $\tilde{S}^x$ and $\tilde{S}^y$ should drop out.

Choosing now in Eq. (56) A to be the Hamiltonian $\mathcal{H}$ of the underlying system we can write by the same arguments as for Eq. (40),

$$(\Delta\mathcal{H})^2 = \frac{1}{2S}\langle\partial_t\vec{S}\rangle^2 + O\left(\frac{1}{S^2}\right), \quad (58)$$

with $\langle\cdot\rangle = \langle z|\cdot|z\rangle$. To this result the same comments apply as to its oscillatory counterpart Eq. (40): If the system is initially in an SU(2) coherent state it holds [cf. Eq. (13)] that

$$\langle\partial_t\vec{S}\rangle = \hbar S\partial_t\vec{s} \quad (59)$$

and

$$(\Delta\mathcal{H})^2 = (\hbar S)^2\left[\frac{1}{2S}(\partial_t\vec{s})^2 + O\left(\frac{1}{S^2}\right)\right] \quad (60)$$

at initial time $t = t_i$, and for not too large times $t > t_i$ the system will approximately remain coherent in its time evolution under $\mathcal{H}$ with $\vec{s}(t)$ being a classical vector. Our finding (60) is again a manifestation of our previous result (42): In leading order in the quantum parameter ($\hbar$ or $1/S$) the variance of the energy is due to the classical motion of the system. Findings of the type (60) were also obtained previously in Ref. [14] on the example of specific Hamiltonians. Here we provide a generalization to arbitrary systems based on the very general expansions (49) and (52) for correlation functions and fluctuations.

Decoherence effects, i.e., deviations from the coherent state with time-dependent parameters $\vec{s}(t)$, are again indicated by the higher-order terms in the energy variance, as we shall investigate on a specific example in Sec. IV. Conversely, the Zeeman Hamiltonian (22) generates a strictly coherent time evolution with

$$(\Delta\mathcal{H}_z)^2 \equiv \frac{(\hbar S)^2}{2S}(\partial_t\vec{s})^2 \quad (61)$$

as an identity for arbitrary times $t \geqslant t_i$.

Similarly as for oscillator systems, the above results for general operator products are easily generalized to the situation of $N > 1$ spins of various lengths; details can be found in Appendix A. The leading order of the energy variance is given by

$$(\Delta\mathcal{H})^2 = \sum_{a=1}^{N}\left[\frac{1}{2S_a}\langle\partial_t\vec{S}_a\rangle^2 + O\left(\frac{1}{S_a^2}\right)\right], \quad (62)$$

and Eq. (60) is generalized to

$$(\Delta\mathcal{H})^2 = \sum_{a=1}^{N}\left\{(\hbar S_a)^2\left[\frac{1}{2S_a}(\partial_t\vec{s}_a)^2 + O\left(\frac{1}{S_a^2}\right)\right]\right\}. \quad (63)$$

IV. LIPKIN-MESHKOV-GLICK MODEL

The Lipkin-Meshkov-Glick (LMG) model is an approximate description of N interacting spin-1/2 systems and was originally inspired by nuclear physics [9,15,16]. More recently this model has been argued to describe two-mode Bose-Einstein condensates [17–20], phase transitions in optical cavity QED [21–23], and molecular magnets [24]. Moreover it has been employed to model a spin bath [25,26] and in studies of quenched dynamics [27]. In the last decade a flurry of publications investigating various aspects of the LMG model has appeared; as an entry point to the recent literature we refer to Refs. [28–38].

Concentrating on the sector of maximal spin $S = N/2$, the LMG Hamiltonian reads

$$\mathcal{H} = -hS^z - \frac{1}{2\hbar S}(\gamma_x S^x S^x + \gamma_y S^y S^y), \quad (64)$$

where h can be interpreted as a magnetic field coupling to the z component of the spin while γ_x, γ_y parametrize an anisotropic interaction among the perpendicular components. The factor $\hbar S$ in the denominator is a convention common to the literature and leads to a linear scaling of energies as a function of $S \gg 1$. The expectation value within an SU(2) coherent state is given by (neglecting a constant contribution)

$$\langle\mathcal{H}\rangle = \hbar S\left(-h\cos\vartheta + \frac{\tilde{\gamma}_x}{2}\sin^2\vartheta\cos^2\varphi + \frac{\tilde{\gamma}_y}{2}\sin^2\vartheta\sin^2\varphi\right) \quad (65)$$

and equals the classical energy expression up to the renormalized parameters $\tilde{\gamma}_i = \gamma_i[1 - 1/(2S)]$. Taking coherent expectation values of both sides of the Heisenberg equations of motion one obtains the (semi)classical equations

$$\frac{ds^x}{dt} = h\sin\vartheta\sin\varphi - \tilde{\gamma}_y\cos\vartheta\sin\vartheta\sin\varphi, \quad (66)$$

$$\frac{ds^y}{dt} = -h\sin\vartheta\cos\varphi + \tilde{\gamma}_x\cos\vartheta\sin\vartheta\cos\varphi, \quad (67)$$

$$\frac{ds^z}{dt} = -(\tilde{\gamma}_x - \tilde{\gamma}_y)\sin^2\vartheta\cos\varphi\sin\varphi. \quad (68)$$

For the energy variance one finds by a direct (and somewhat tedious) calculation

$$(\Delta\mathcal{H})^2 = \Omega_1 + \Omega_2 \quad (69)$$

with

$$\Omega_1 = (\hbar S)^2\frac{1}{2S}[h^2\sin^2\vartheta - 2h\tilde{\gamma}_x\cos\vartheta\sin^2\vartheta\cos^2\varphi - 2h\tilde{\gamma}_y\cos\vartheta\sin^2\vartheta\sin^2\varphi + \tilde{\gamma}_x^2(\sin^4\vartheta\cos^2\varphi\sin^2\varphi + \cos^2\vartheta\sin^2\vartheta\sin^2\varphi) + \tilde{\gamma}_y^2(\sin^4\vartheta\cos^2\varphi\sin^2\varphi + \cos^2\vartheta\sin^2\vartheta\cos^2\varphi) - 2\tilde{\gamma}_x\tilde{\gamma}_y\sin^4\vartheta\cos^2\varphi\sin^2\varphi] \quad (70)$$

being of leading order $1/S$ while the contributions summarized in

$$\Omega_2 = (\hbar S)^2 \frac{1}{8S^2}\left(1-\frac{1}{2S}\right)(-4\gamma_x\gamma_y\cos^2\vartheta + [\gamma_x(1-\sin^2\vartheta\cos^2\varphi)+\gamma_y(1-\sin^2\vartheta\sin^2\varphi)]^2) \quad (71)$$

are of order $1/S^2$ and higher. Using now Eqs. (66)–(68) we can identify the leading contribution to the energy uncertainty as

$$\Omega_1 = (\hbar S)^2\frac{1}{2S}\left(\frac{d\vec{s}}{dt}\right)^2, \quad (72)$$

in accordance with the general result (60). The subleading contributions Ω_2 indicate decoherence effects, i.e., departures from the submanifold of the coherent states in the Hamiltonian time evolution, as we now discuss explicitly the example of the isotropic LMG model.

Isotropic case

Putting $\gamma_x = \gamma_y =: \gamma$ the Hamiltonian becomes diagonal in the states $|m\rangle$ with eigenvalues

$$\varepsilon_m/\hbar = -hm + \frac{\gamma}{2S}m^2 - \frac{\gamma}{2}(S+1). \quad (73)$$

This eigensystem is simple enough to analytically compute the exact time evolution of coherent expectation values $\langle\vec{S}(t)\rangle$: Due to symmetry, the z component is constant,

$$\langle S^z(t)\rangle \equiv \hbar S\cos\vartheta, \quad (74)$$

while for the perpendicular components one finds

$$\langle S^+(t)\rangle = \hbar S\sin\vartheta\, e^{i(\varphi-\{h-\gamma[1-1/(2S)]\cos\vartheta\}t)} \times\left\{e^{-i(\gamma\cos\vartheta/2S)t}\left[\cos\left(\frac{\gamma}{2S}t\right) + i\cos\vartheta\sin\left(\frac{\gamma}{2S}t\right)\right]\right\}^{2S-1}. \quad (75)$$

The above closed result relies on the fact that S^+ couples only eigenstates with neighboring indices such that all occurring energy differences are, apart from a constant term, linear in m. The first line in Eq. (75) describes a classical rotation of the spin according to Eqs. (66)–(68) whereas the second line contains quantum effects: The "spin length"

$$|\langle S^+(t)\rangle| = \hbar S\sin\vartheta\left[1-\sin^2\vartheta\sin^2\left(\frac{\gamma}{2S}t\right)\right]^{S-1/2} \quad (76)$$

composed from the perpendicular components breathes sinusoidally in time. Quantum (quasi)revivals occur at times at $t = 2\pi kS/\gamma$ for any integer k where the state returns precisely to the submanifold of the coherent states. These times are large in the semiclassical regime as they are proportional to S.

Regarding small times, we define $t =: \sqrt{S}\tau$ and consider the regime $\gamma\tau \ll \sqrt{S}$, such that for large $S \gg 1$ it follows that

$$\frac{|\langle S^+(t)\rangle|}{\hbar S\sin\vartheta} \approx \left(1-\frac{(\gamma\tau\sin\vartheta)^2/4}{S}\right)^{S-1/2} \approx e^{-(\gamma\tau\sin\vartheta)^2/4}, \quad (77)$$

i.e., the spin expectation value $\langle\vec{S}(t)\rangle$ shows a Gaussian decay with time scale $\Delta t = \sqrt{2S}/(\gamma\sin\vartheta)$. On this time scale, sometimes known as Ehrenfest time [39], departures between classical and quantum dynamics become sizable. The above finding for Δt is consistent with a heuristic uncertainty argument in the following sense: Replacing in $\Delta H\Delta t \geqslant \hbar$ the energy uncertainty with $\sqrt{\Omega_2}$ one obtains a lower bound for Δt being proportional to $\hbar S$ which is a constant independent of S in the semiclassical regime. Thus, this lower bound is consistent with the above result which grows with the square root of S.

V. DICKE MODEL

The Dicke model describes the superradiant interaction of a single cavity mode of a radiation field with N two-level systems (atoms) [10]. Although introduced already in the 1950s, this model continues to be investigated under various aspects; as a guide to the recent literature see, e.g., Refs. [40–43].

Focusing again on the sector of maximal spin $S = N/2$, the Dicke Hamiltonian can be formulated as

$$\mathcal{H} = \hbar\omega a^+a + \Omega S^z + \frac{\lambda}{\sqrt{2S}}S^x(a^+ + a) \quad (78)$$

$$= \frac{1}{2}(p^2+\omega^2q^2) + \Omega S^z + \lambda\sqrt{\frac{\omega}{\hbar S}}S^x q, \quad (79)$$

where the parameters ω, Ω, and λ have all dimension of inverse time. In the classical limit, the superradiant phase, characterized by a finite bosonic occupation in the ground state, occurs for $\lambda^2 > \Omega\omega$. The expectation value of the Hamiltonian within a tensor product of an oscillator and an SU(2) coherent state reads

$$\langle\mathcal{H}\rangle = \hbar\omega|\alpha|^2 + \Omega\hbar S\cos\vartheta + \frac{\lambda}{\sqrt{2S}}\hbar S\sin\vartheta\cos\varphi(\bar{\alpha}+\alpha), \quad (80)$$

which perfectly matches the classical expression. The (semi)classical equations of motion can be obtained analogously as Eqs. (66)–(68),

$$\frac{d\bar{\alpha}}{dt} = i\omega\bar{\alpha} + \frac{i}{\hbar}\frac{\lambda}{\sqrt{2S}}\hbar S\sin\vartheta\cos\varphi, \quad (81)$$

$$\frac{ds^x}{dt} = -\Omega\sin\vartheta\sin\varphi, \quad (82)$$

$$\frac{ds^y}{dt} = \Omega\sin\vartheta\cos\varphi - \frac{\lambda}{\sqrt{2S}}\cos\vartheta(\bar{\alpha}+\alpha), \quad (83)$$

$$\frac{ds^z}{dt} = \frac{\lambda}{\sqrt{2S}}\sin\vartheta\sin\varphi(\bar{\alpha}+\alpha), \quad (84)$$

and a direct (but again quite lengthy) calculation of the energy variance yields

$$(\Delta\mathcal{H})^2 = \Omega_1 + \Omega_2 \quad (85)$$

with the leading-order term

$$\Omega_1 = (\hbar\omega)^2|\alpha|^2 + \frac{(\hbar S)^2}{2S}\Omega^2 \sin^2\vartheta + (\hbar S)^2\frac{\lambda^2}{2S}\left[\frac{1}{2S}(\sin^2\vartheta\sin^2\varphi + \cos^2\vartheta)(\bar\alpha+\alpha)^2 + \sin^2\vartheta\cos^2\varphi\right] + \hbar\omega\frac{\lambda}{\sqrt{2S}}\hbar S\sin\vartheta\sin\varphi(\bar\alpha+\alpha) - \Omega\frac{\lambda}{\sqrt{2S}}\frac{(\hbar S)^2}{S}\cos\vartheta\sin\vartheta\cos\varphi(\bar\alpha+\alpha) \tag{86}$$

and the subleading contributions

$$\Omega_2 = (\hbar S)^2\frac{\lambda^2}{(2S)^2}(\sin^2\vartheta\sin^2\varphi + \cos^2\vartheta) \tag{87}$$

$$= \frac{\lambda^2}{(2S)^2}(\langle S^y\rangle^2 + \langle S^z\rangle^2). \tag{88}$$

Finally, comparison with Eqs. (81)–(84) shows

$$\Omega_1 = \frac{\hbar}{2}\left(\omega(\partial_t\xi)^2 + \frac{(\partial_t\pi)^2}{\omega}\right) + \frac{(\hbar S)^2}{2S}(\partial_t\vec s)^2, \tag{89}$$

in accordance with the general results (46), (63), and (A7).

Note that the higher-order contributions (88) describing decoherence effects depend only on the coupling parameter λ but not on the frequencies ω, Ω, in accordance with the fact that spin and oscillator show perfectly coherent time evolutions in the absence of coupling. Another distinctive feature of the result (88) [compared to, e.g., Eq. (71)] is its simplicity which calls for further applications. In fact, an extensive numerical study of the dynamics of the Dicke model in the semiclassical regime was performed recently in Ref. [43]. Here the initial condition was, as in the present work, a tensor product of an oscillator and a spin coherent state, and it is straightforward to evaluate the above Ω_2 in terms of such dynamical data. In particular, it is an interesting speculation whether or not Ω_2 behaves differently in the regular versus (quantum) chaotic regime as studied in Ref. [43]. Another aspect is to compare the Ehrenfest times Δt found numerically with estimates according to $\sqrt{\Omega_2}\Delta t \geqslant \hbar$.

VI. COHERENT INTERTWINERS IN SPIN NETWORKS

We now apply our general findings on coherent expectation values of operator products to spin network states as studied in loop quantum gravity (LQG) [12,44,45]. In brief, a spin network is a collection of points (called *vertices* or *nodes*) in (typically) three-dimensional space connected by one-dimensional curves (*edges*). Each edge is assigned a spin of individual length, and a spin network state in the tensor product of all those SU(2) representations is defined by the additional requirement that all spins joining in a given node are coupled to a total singlet. The latter property implements the Gauss constraint on the holonomy and flux variables used in LQG [46,47].

A convenient parametrization of spin network states are coherent intertwiners as introduced by Livine and Speziale [11]. Fixing an N-valent node (connecting N edges), one considers a tensor product

$$|\Phi\rangle := \bigotimes_{a=1}^{N}|\vartheta_a,\varphi_a\rangle \tag{90}$$

of SU(2) coherent states describing the spin on each edge. A coherent intertwiner is then defined by the projection of this object onto the singlet subspace [11]

$$|\Phi\rangle_s = \frac{P|\Phi\rangle}{\sqrt{\langle\Phi|P|\Phi\rangle}}, \tag{91}$$

where the denominator takes care of the normalization. The projection operator can be formalized by a Haar integration over all uniform rotations of the N spins (group averaging),

$$P = \int_{SU(2)} d\mu\,\exp\left(i\psi\vec n\sum_a\vec S_a\right) \tag{92}$$

$$= \frac{1}{4\pi^2}\int_0^\pi d\vartheta\sin\vartheta\int_0^{2\pi}d\varphi\int_0^{2\pi}d\psi\sin^2\frac{\psi}{2} \times\exp\left(i\psi\vec n\sum_a\vec S_a\right) \tag{93}$$

with $\vec n = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$. Here and in what follows we take all spin operators to be dimensionless (as a factor of $\hbar$ will occur below in the Planck length squared). In particular, a coherent intertwiner is by construction invariant under arbitrary rotations of all spins meaning

$$\left(\sum_{a=1}^N\vec S_a\right)|\Phi\rangle_s = 0. \tag{94}$$

Moreover, nodes in a spin network allow for a geometric interpretation in terms of convex polyhedra [48]. From a classical point of view, this relies on a theorem due to Minkowski [49]. It states that given N unit vectors $\vec s_a$ and N positive numbers A_a fulfilling $\sum_a A_a\vec s_a = 0$, there is a unique convex polyhedron with N faces such that $\vec s_a$ is the normal to the ath face and A_a is its area. Thus choosing as areas the quantum numbers S_a, the classical closure relation

$$\sum_{a=1}^N S_a\vec s_a = 0 \tag{95}$$

ensures that the geometric information contained in the state (90) encodes a convex polyhedron. The quantum counterpart of the relation (95) is Eq. (94) giving rise to the notion of a *quantum polyhedron* [48]. In the framework of LQG, the spin operators representing the faces of the polyhedron are, up to a prefactor, considered to be flux operators [46,47]

$$\vec E_a = 8\pi\gamma\ell_P^2\vec S_a \tag{96}$$

with γ being the Immirzi parameter and the squared Planck length $\ell_P^2 = \hbar G/c^3$.

Let us now explore expectation values within coherent intertwiners. Here one can concentrate without loss of generality on operators unchanged by uniform rotations since for any operator being the sum of a rotationally invariant part and terms without this property, only the former will contribute. Any rotationally invariant operator Q commutes with the projector onto the singlet space, $[Q,P]=0$, such that $PQP=QP=PQ$. Therefore we can use the result (57) to obtain a semiclassical approximation to the expectation value within a coherent intertwiner,

$$ {}_s\langle\Phi|Q|\Phi\rangle_s = \frac{\langle\Phi|PQ+QP|\Phi\rangle}{2\langle\Phi|P|\Phi\rangle} \tag{97}$$

$$ = \langle\Phi|Q|\Phi\rangle + \sum_{a=1}^{N}\frac{1}{2S_a} \times \sum_{i=1}^{3}\langle\Phi|\big[iS_a^i,Q\big]|\Phi\rangle C_a^i(\Phi)+\cdots \tag{98}$$

with

$$ C_a^i(\Phi) = \frac{\langle\Phi|\big[iS_a^i,P\big]|\Phi\rangle}{\langle\Phi|P|\Phi\rangle}. \tag{99}$$

Thus, the expectation value of Q is in leading order just given by the expectation value of the unprojected state (90), and the normalization factor in the definition (91) drops out. For the subleading corrections one needs to determine the coefficients (99). Here both numerator and denominator are conveniently formulated in terms of Haar integrations as shown explicitly in Eq. (93). In the semiclassical regime studied here where all spins are long, $\forall a S_a \gg 1$, these integrals become amenable to a saddle-point approximation as worked out in Ref. [11]. For the denominator one finds for a general N-valent node

$$ \langle\Phi|P|\Phi\rangle = \frac{1}{\sqrt{\pi}\,\det H} + \frac{\mathrm{tr}(H^{-1})}{4\sqrt{\pi}\,\det H}+\cdots, \tag{100}$$

where

$$ H^{ij} = \sum_{a=1}^{N} S_a\big(\delta^{ij}-s_a^i s_a^j\big) \tag{101}$$

is twice the negative Hessian of the saddle-point expression, and the details of the calculation can be found in Appendix B. Since H is a linear combination of geometric projection operators $(\delta^{ij}-s_a^i s_a^j)$ with positive coefficients, its eigenvalues are non-negative, and zero eigenvalues only occur in the degenerate case where all vectors $\vec{s}_a$ are collinear, which we shall not consider here. Thus, the eigenvalues of H can be taken to be positive, and the determinant can be formulated more explicitly as [11]

$$ \det H = \frac{T}{2}\sum_{ab} S_a S_b(\vec{s}_a\times\vec{s}_b)^2 - \frac{1}{6}\sum_{abc} S_a S_b S_c |(\vec{s}_a\times\vec{s}_b)\cdot\vec{s}_c|^2 \tag{102}$$

with $T=\sum_a S_a$. The semiclassical limit of a quantum polyhedron is obtained by rescaling all quantum numbers as $S_a \mapsto \lambda S_a$ with some integer $\lambda \gg 1$. Thus the leading term in Eq. (100) (already obtained in Ref. [11]) is of order $\lambda^{-3/2}$ while the subleading correction scales like $\lambda^{-5/2}$. The numerator in Eq. (99) can be evaluated via saddle-point approximation in a similar fashion (see Appendix B) giving, again for a general N-valent node fulfilling the classical closure relation (95),

$$ \langle\Phi|[iS_a,P]|\Phi\rangle = S_a\frac{\vec{s}_a\times(H^{-1}\vec{s}_a)}{\sqrt{\pi}\,\det H} \tag{103}$$

such that for the coefficients themselves we have the amazingly simple result

$$ \vec{C}_a(\Phi) = S_a\vec{s}_a\times(H^{-1}\vec{s}_a). \tag{104}$$

The expression (103) is of order $\lambda^{-3/2}$ while the coefficients (104) are independent of λ and vanish if the matrix H is proportional to the unit matrix. Thus, polyhedra where all eigenvalues of H are degenerate enjoy an enhanced classical character in the sense that the leading order of semiclassical corrections to general expectation values (98) vanishes. In the general case, Eq. (98) tells us that the coherent-intertwiner expectation value of any (rotationally invariant) operator is in leading order given by the expectation value of the unprojected tensor product of SU(2) coherent states, and the leading correction scales with the inverse of the spin lengths.

The coefficients (104) are universal in the sense that they are the same for any operator Q. Making use of the symmetry of H they can also be formulated as

$$ C_a^i(\Phi) = S_a\sum_{jkl}\epsilon^{ijk}(H^{-1})^{kl}\big(s_a^l s_a^j-\delta^{lj}\big) \tag{105}$$

implying the sum rule

$$ \sum_{a=1}^{N} C_a^i(\Phi) = -\sum_{jkl}\epsilon^{ijk}(H^{-1})^{kl}H^{lj}=0, \tag{106}$$

which also follows from the definition (99) and the quantum closure relation (94). Thus, the sum rule (106) holds independently of the fulfillment of the classical closure relation (95) which underlies the explicit result (104).

Moreover, it is interesting to note that the matrix H can be interpreted as the inertia tensor of a distribution of masses S_a whose positions are given by the unit vectors $\vec{s}_a$. By the same token, the classical closure relation (95) states that the center of mass of this distribution lies in the origin of the chosen coordinate system. In particular, H is proportional to the unit matrix [such that the expansion coefficients (99) vanish] if the node has the shape of an Archimedian body such as a regular tetrahedron. We leave it to further studies to explore further possible consequences of the above analogy.

Very typical examples of rotationally invariant operators are volume operators of polyhedra [48,50,51]. The simplest nontrivial case of a quantum polyhedron is given by a tetrahedron, i.e., a four-valent node [52]. The volume operator can be formulated as

$$ V = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1\cdot(\vec{E}_2\times\vec{E}_3)|} \tag{107}$$

using any three of the four flux operators. Squaring this expression and strippng all prefactors one is led to consider the expression

$$ Q = \vec{S}_1\cdot(\vec{S}_2\times\vec{S}_3), \tag{108}$$

acting on the Hilbert space defined by the constraint (94). The study of this operator in the semiclassical limit has attracted quite a deal of interest recently [48,53–56]. For the expectation value within coherent intertwiners one finds from Eqs. (98) and (104)

$$
{}_s\langle\Phi|\vec{S}_1(\vec{S}_2\times\vec{S}_3)|\Phi\rangle_s
= S_1S_2S_3\big[\vec{s}_1(\vec{s}_2\times\vec{s}_3)+\tfrac{1}{2}(\vec{s}_1\times(\vec{s}_2\times\vec{s}_3)
\cdot[\vec{s}_1\times(H^{-1}\vec{s}_1)]+\text{c.p.})\big]+\cdots \tag{109}
$$

As before, the form of the subleading corrections here holds if the classical closure relation (95) is fulfilled.

VII. SUMMARY AND OUTLOOK

We have derived general systematic expansions with respect to quantum parameters of expectation values of products of arbitrary operators within both oscillator coherent states and SU(2) coherent states. These results are versatile tools for the study of the semiclassical regime of generic quantum systems. In particular, we prove that the energy fluctuations of an arbitrary Hamiltonian are in leading order entirely due to the time dependence of the classical variables, a result very general and very intuitive at the same time.

Our findings offer many possibilities for application in various fields of physics. Here we have specifically studied the Dicke model stemming from quantum optics, and the LMG model originating from nuclear physics. For the latter system we have investigated decoherence effects (i.e., deviations from the submanifold of coherent states) via an exact solution of the dynamics. Finally we have applied our general results to coherent intertwiners in spin networks as investigated in LQG. For expectation values of rotationally invariant operators (and these are the only ones contributing) one finds here a subleading correction to the classical limit given in terms of universal (i.e., operator-independent) expansion coefficients which contain only geometric information about the network node.

APPENDIX A: $N > 1$ DEGREES OF FREEDOM

Let us now extend our results on the coherent-state expectation values of operator products to systems with $N > 1$ degrees of freedom. We start by two oscillatory degrees of freedom q_a, p_a with frequencies ω_a, $a \in \{1,2\}$. Iterating the arguments leading to Eq. (36) one finds

$$
\langle\alpha|AB|\alpha\rangle = \sum_{m,n=0}^{\infty}\frac{\hbar^{m+n}}{m!n!}\langle\alpha|[Q_2,[Q_1,A]_n]_m|\alpha\rangle
\times\langle\alpha|[Q_2^+,[Q_1^+,B]_n]_m|\alpha\rangle. \tag{A1}
$$

with $|\alpha\rangle = |\alpha_1\rangle\otimes|\alpha_2\rangle$ and

$$
Q_a = \frac{i}{\sqrt{2\hbar}}\left(\sqrt{\omega_a}q_a - i\frac{p_a}{\sqrt{\omega_a}}\right). \tag{A2}
$$

Since Q_1, Q_2 commute, the corresponding left arguments in the above nested commutators can be freely interchanged such that

$$
\frac{(m+n)!}{m!n!}\langle\alpha|[Q_2,[Q_1,A]_n]_m|\alpha\rangle\langle\alpha|[Q_2^+,[Q_1^+,B]_n]_m|\alpha\rangle
= \sum_{P_{mn}}\langle\alpha|\big[Q_{P_{mn}(1)},\big[Q_{P_{mn}(2)},\ldots\big[Q_{P_{mn}(m+n)},A\big]\ldots\big]\big]|\alpha\rangle
\times\langle\alpha|\big[Q^+_{P_{mn}(1)},\big[Q^+_{P_{mn}(2)},\ldots\big[Q^+_{P_{mn}(m+n)},B\big]\ldots\big]\big]|\alpha\rangle, \tag{A3}
$$

where the sum goes over all functions $P_{mn} : \{1,\ldots,m+n\}\to\{1,2\}$ taking m times the value 2 and n times the value 1. Thus we arrive at

$$
\langle\alpha|AB|\alpha\rangle
= \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\sum_{P_n}\langle\alpha|\big[Q_{P_n(1)},\big[Q_{P_n(2)},\ldots\big[Q_{P_n(n)},A\big]\ldots\big]\big]|\alpha\rangle
\times\langle\alpha|\big[Q^+_{P_n(1)},\big[Q^+_{P_n(2)},\ldots\big[Q^+_{P_n(n)},B\big]\ldots\big]\big]|\alpha\rangle, \tag{A4}
$$

where the second sum extends now over all functions $P_n : \{1,\ldots,n\}\to\{1,2\}$. Moreover, it is straightforward to see that the above expression also holds for an arbitrary number N of oscillatory degrees of freedom with $|\alpha\rangle = |\alpha_1\rangle\otimes\cdots\otimes|\alpha_N\rangle$ and functions $P_n : \{1,\ldots,n\}\to\{1,\ldots,N\}$. In particular, the variance of a Hermitian operator A can be expressed as

$$
(\Delta A)^2 = \sum_{n=1}^{\infty}\frac{\hbar^n}{n!}\sum_{P_n}|\langle\alpha|\big[Q_{P_n(1)},
\big[Q_{P_n(2)},\ldots\big[Q_{P_n(n)},A\big]\ldots\big]\big]|\alpha\rangle|^2, \tag{A5}
$$

and the leading-order results for energy fluctuations are given in Eqs. (44)–(46).

The counterpart of Eq. (A1) for two spins $\vec{S}_1$, $\vec{S}_2$ reads

$$
\langle z|AB|z\rangle = \sum_{m=0}^{2S_2}\sum_{n=0}^{2S_1}\frac{(2S_2-m)!}{m!(2S_2)!}\frac{(2S_1-n)!}{n!(2S_1)!}\langle z|\left[\frac{i}{\hbar}\tilde{S}_2^-,\left[\frac{i}{\hbar}\tilde{S}_1^-,A\right]_n\right]_m|z\rangle\langle z|\left[\frac{i}{\hbar}\tilde{S}_2^+,\left[\frac{i}{\hbar}\tilde{S}_1^+,B\right]_n\right]_m|z\rangle. \tag{A6}
$$

with $|z\rangle = |z_1\rangle\otimes|z_2\rangle$. Due to the more complicated prefactors, a similarly compact form as in Eq. (A4) for the full expansion seems to be unachievable for spin systems. The leading terms of energy fluctuations given in Eqs. (62) and (63), however, are again rather simple and allow for an intuitive interpretation.

Finally, combining both types of systems, the leading-order contribution to the fluctuation of a Hamiltonian depending on N oscillatory degrees of freedom and M spins reads

$$
(\Delta\mathcal{H})^2 = \frac{\hbar}{2}\sum_{a=1}^{N}\left[\omega_a\langle\partial_t q_a\rangle^2 + \frac{\langle\partial_t p_a\rangle^2}{\omega_a}\right] + O(\hbar^2)
+\sum_{b=1}^{M}\left\{(\hbar S_b)^2\left[\frac{1}{2S_b}(\partial_t\vec{s}_b)^2 + O\left(\frac{1}{S_b^2}\right)\right]\right\}. \tag{A7}
$$

APPENDIX B: NORMALIZATION OF COHERENT INTERTWINERS AND RELATED INTEGRALS

In order to evaluate the normalization integral of coherent intertwiners in the semiclassical regime, we shall use a slightly different version of SU(2) coherent states generated by

$$V(\vartheta,\varphi) = e^{-\varphi S^z} e^{-\vartheta S^y} \quad \text{(B1)}$$

such that compared to Eq. (14) one has (dropping again factors of $\hbar$)

$$V(\vartheta,\varphi)|S\rangle = e^{i\varphi S} U(\vartheta,\varphi)|S\rangle, \quad \text{(B2)}$$

i.e., the coherent states generated by the operators (16) and (B1) just differ by a phase factor which drops out from all expectation values. The operator (B1) fulfills

$$V^+ \vec{S} V = \vec{u} S^x + \vec{v} S^y + \vec{s} S^z \quad \text{(B3)}$$

with

$$\vec{u} = \frac{(\vec{e}_z \times \vec{s}) \times \vec{s}}{|\vec{e}_z \times \vec{s}|}, \quad \vec{v} = \frac{\vec{e}_z \times \vec{s}}{|\vec{e}_z \times \vec{s}|} \quad \text{(B4)}$$

such that $\vec{u}$, $\vec{v}$, $\vec{s}$, form an orthonormal system. If one had used the original operator (16) for generating coherent states the form of the vectors $\vec{u}$, $\vec{v}$ would be less transparent. Now the normalization integral can be written as

$$\langle\Phi|P|\Phi\rangle = \int d\mu \prod_{a=1}^{N} \langle S_a|e^{i\psi\vec{n}_a\vec{S}_a}|S_a\rangle \quad \text{(B5)}$$

with

$$n_a^x = \vec{n}\vec{u}_a, \quad n_a^y = \vec{n}\vec{v}_a, \quad n_a^z = \vec{n}\vec{s}_a, \quad \text{(B6)}$$

where $\vec{n}$ is the rotation axis occurring in Eqs. (92) and (93). Taking into account the explicit form of the rotation matrix element [57]

$$\langle S|e^{i\psi\vec{n}\vec{S}}|S\rangle = \left(\cos\frac{\psi}{2} + in^z \sin\frac{\psi}{2}\right)^{2S}, \quad \text{(B7)}$$

elementary manipulations lead to

$$\langle\Phi|P|\Phi\rangle = \frac{1}{2\pi^2}\int_0^\pi d\vartheta\,\sin\vartheta \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\psi\,\sin^2\psi \times \sum_{\eta=\pm}\prod_{a=1}^{N}(\eta\cos\psi + i\vec{n}\vec{s}_a\sin\psi)^{2S_a}, \quad \text{(B8)}$$

where the cosine of ψ is non-negative in the entire integration interval. Following Ref. [11] we introduce $\vec{p} := \vec{n}\sin\psi$ fulfilling

$$d^3p = \sin\vartheta\,\sin^2\psi\,\cos\psi\, d\vartheta\, d\varphi\, d\psi \quad \text{(B9)}$$

such that

$$\langle\Phi|P|\Phi\rangle = \frac{1}{2\pi^2}\sum_{\eta=\pm}\int_{p\leqslant 1}\frac{d^3p}{\sqrt{1-p^2}} e^{S_\eta(\vec{p})}, \quad \text{(B10)}$$

where

$$S_\eta(\vec{p}) = \sum_{a=1}^{N} 2S_a \ln(\eta\sqrt{1-p^2} + i\vec{p}\vec{s}_a). \quad \text{(B11)}$$

In this form the integral can be evaluated via saddle-point approximation to $S_\eta(\vec{p})$. As discussed in detail in Ref. [11], provided that the classical closure relation (95) holds, the maximum of $S_\eta(\vec{p})$ occurs at $\vec{p} = 0$ with

$$S_+(0) = 0, \quad S_-(0) = 2\pi i \sum_{a=1}^{N} S_a \quad \text{(B12)}$$

and since the latter sum must be an integer for a nontrivial singlet space we have $\exp(S_\pm(0)) = 1$. The Hessian is given by [cf. Eq. (101)]

$$\left(\frac{\partial^2 S_\pm(\vec{p})}{\partial p^i \partial p^j}\right)_{\vec{p}=0} =: -2H^{ij} = -2\sum_{a=1}^{N} S_a(\delta^{ij} - s_a^i s_a^j). \quad \text{(B13)}$$

Extending now the integration domain in Eq. (B10) to the infinite space (as the integrand falls off rapidly), we are left with simple Gaussian integrals leading to the result (100) where the leading first term was already obtained in Ref. [11] while the subleading correction stems from expanding the square root in Eq. (B10).

To compute the numerator of the coefficients (99) we consider

$$\langle\Phi|[iV_a\vec{S}_aV_a^+,P]|\Phi\rangle = \int d\mu \langle S_a|[i\vec{S}_a, e^{i\psi\vec{n}_a\vec{S}_a}]|S_a\rangle \times \prod_{b\neq a}\langle S_b|e^{i\psi\vec{n}_b\vec{S}_b}|S_b\rangle \quad \text{(B14)}$$

with $V_a = V(\vartheta_a,\varphi_a)$. With the help of the rotation matrix element [57]

$$\langle S-1|e^{i\psi\vec{n}\vec{S}}|S\rangle = \sqrt{2S}\left(\cos\frac{\psi}{2} + in^z\sin\frac{\psi}{2}\right)^{2S-1} \times (n^x + in^y)\sin\frac{\psi}{2} \quad \text{(B15)}$$

one derives

$$\langle S_a|\big[iS_a^x, e^{i\psi\vec{n}_a\vec{S}_a}\big]|S_a\rangle = -i\left(\cos\frac{\psi}{2} + in_a^z\sin\frac{\psi}{2}\right)^{2S-1} \times 2S_a n_a^y \sin\frac{\psi}{2}, \quad \text{(B16)}$$

$$\langle S_a|\big[iS_a^y, e^{i\psi\vec{n}_a\vec{S}_a}\big]|S_a\rangle = i\left(\cos\frac{\psi}{2} + in_a^z\sin\frac{\psi}{2}\right)^{2S-1} \times 2S_a n_a^x \sin\frac{\psi}{2}, \quad \text{(B17)}$$

$$\langle S_a|\big[iS_a^z, e^{i\psi\vec{n}_a\vec{S}_a}\big]|S_a\rangle = 0. \quad \text{(B18)}$$

Now proceeding as before, the two nontrivial expectation values can be formulated as

$$\langle\Phi|\big[iV_aS_a^xV_a^+,P\big]|\Phi\rangle = \frac{-iS_a}{\pi^2}\sum_{\eta=\pm}\int\frac{d^3p}{\sqrt{1-p^2}}\frac{\vec{p}\vec{v}_a}{\eta\sqrt{1-p^2}+i\vec{p}\vec{s}_a}e^{S_\eta(\vec{p})}, \quad \text{(B19)}$$

$$\langle\Phi|\big[iV_aS_a^yV_a^+,P\big]|\Phi\rangle = \frac{iS_a}{\pi^2}\sum_{\eta=\pm}\int\frac{d^3p}{\sqrt{1-p^2}}\frac{\vec{p}\vec{u}_a}{\eta\sqrt{1-p^2}+i\vec{p}\vec{s}_a}e^{S_\eta(\vec{p})}. \quad \text{(B20)}$$

Performing again a saddle-point approximation to the exponential and expanding the remaining integrand in quadratic order around $\vec{p} = 0$ leads to

$$\langle\Phi|\big[iV_aS_a^xV_a^+,P\big]|\Phi\rangle = -S_a\frac{\big(\vec{v}_a^TH^{-1}\vec{s}_a\big)}{\sqrt{\pi}\det H}, \tag{B21}$$

$$\langle\Phi|\big[iV_aS_a^yV_a^+,P\big]|\Phi\rangle = S_a\frac{\big(\vec{u}_a^TH^{-1}\vec{s}_a\big)}{\sqrt{\pi}\det H}, \tag{B22}$$

and using Eq. (B3) along with elementary geometric relations it follows for the coefficients (99) that

$$\begin{aligned}\vec{C}_a(\Phi) &= -S_a\big[\vec{u}_a\big(\vec{v}_a^TH^{-1}\vec{s}_a\big) - \vec{v}_a\big(\vec{u}_a^TH^{-1}\vec{s}_a\big)\big]\\ &= S_a\vec{s}_a \times \big[\vec{v}_a\big(\vec{v}_a^TH^{-1}\vec{s}_a\big) + \vec{u}_a\big(\vec{u}_a^TH^{-1}\vec{s}_a\big)\big].\end{aligned} \tag{B23}$$

Finally, observing that $\vec{u}_a$, $\vec{v}_a$ span the plane perpendicular to $\vec{s}_a$ we obtain the result (104), and the numerator of Eq. (99) is given by Eq. (103).

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