

Closing the detection loophole in tripartite Bell tests using the W state

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We study the problem of closing the detection loophole in three-qubit Bell tests, the experimentally most relevant case beyond the usual bipartite scenario, and show that the minimal detection efficiencies required can be considerably lowered compared to the two-qubit case. The lowest reported detection efficiency thresholds for two and three qubits so far are ~ 66.7 and 60% , respectively. Using the three-qubit W state and a three-setting Bell inequality, we beat these thresholds and with an eight-setting Bell inequality we reach 50.13% . We also investigate generic three-qubit states which allow us to attain a detection efficiency of 50% in a four-setting Bell test. We conjecture that the limit of 50% is unbeatable using three-qubit states and any number of measurements.

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I. INTRODUCTION

One of the most surprising features of quantum mechanics is the prediction that distant parties performing measurements on a shared entangled state are able to generate correlations which rule out any local hidden variables explanation. These nonlocal correlations can be witnessed by the violation of Bell inequalities [1,2]. By now, many Bell experiments using various matter systems have been performed (e.g., recently in [3–8]) providing strong indication for the existence of nonlocal correlations in nature [9]. However, imperfections in the technical implementations of these experiments make it possible to reproduce the experimental data by a local hidden variables model. In order to avoid such a classical explanation, all possible loopholes have to be closed simultaneously in a Bell experiment. There are two main technical loopholes, the locality loophole and the detection loophole. The former one can be closed if there is spacelike separation between the observers such that no signal can propagate from one observer to the other. This condition could only be met so far in photonic experiments [10–12].

In the present paper, we would like to address the latter one, the so-called detection loophole. This loophole is most relevant in Bell tests which use photons, in which case measurements frequently give undetected events. These no-click events have to be included in the observed data, and nonlocal correlations are witnessed detection loophole free only if there is no local hidden variables model of the full statistics taking into account the no-click events as well [13]. The detection loophole has been closed in different physical systems such as ions [3], superconductors [4], atoms [5], and more recently in photonic systems as well [6,7].

As we have seen, the only system where both primary loopholes have been closed is that of photons, albeit these were not closed in the same experiment. Though important steps have been made both experimentally (see references above) and theoretically [14], such a loophole-free violation of a Bell inequality has not been performed yet. A comprehensive review on this subject can be found in [15].

Let us mention that closing the detection loophole is also relevant from a practical point of view. The more recent development of device-independent quantum information protocols crucially relies on a detection loophole-free violation of Bell

inequalities. In these protocols, there is no need to assume any knowledge regarding the internal workings of the experimental devices used (see [16] for a recent review of the field). For instance, it would allow two distant parties to establish a certified secret key [17], generate genuinely random numbers [18], or perform black-box state tomography [19].

In order to close the detection loophole, we construct Bell inequalities which are suited to reveal nonlocality using detectors with low efficiencies. We will consider the relatively unexplored case of three-party Bell inequalities involving finite detection efficiencies. In particular, we will focus on the case when each party detects particles with the same η detection efficiency. The critical detection efficiency η_{crit} below which nonlocality cannot be guaranteed depends both on the Bell inequality considered and the quantum state used in the Bell test.

In the two-party case, $\eta_{\text{crit}} \sim 66.7\%$ [20] is required to violate the Clauser-Horne-Shimony-Holt inequality [21] with a partially entangled two-qubit state. For two qubits, to the best of our knowledge, there is no known Bell inequality (with possibly more than two settings and more than two outputs), which would give a lower threshold. Using four-dimensional quantum states and a four-setting Bell inequality, this threshold can be slightly lowered (down to $\sim 61.8\%$ [22]); however, it is still too high when compared to efficiencies achievable with current technology.

One possible approach to go below these threshold values is to consider multipartite Bell tests, i.e., more than two observers. Buhrman *et al.* [23] and more recently [24] have showed that an arbitrarily small efficiency η can be tolerated as the number of parties n and the number of settings m become large. However, these results are interesting mainly from a theoretical point of view. Indeed, in the experimentally more relevant case of a small number of settings, the known results are less promising. For instance, if the number of settings per party is fixed to 2 ($m = 2$), the lowest threshold efficiencies using the Mermin inequality [25] and its generalized version [26] were shown to approach $\eta_{\text{crit}} = 50\%$ for large n [27]. The same limit can be approached if we use the many-site generalization of the Clauser-Horne inequality [28]. Also, a multipartite two-setting Bell test based on single-photon entanglement (i.e., a W state shared between multiple parties) was shown to approach $\eta_{\text{crit}} \simeq 66.7\%$ for large n [29]. These above examples

considered a large number of parties and two settings. There exist other constructions for a class of multipartite two-setting inequalities (e.g., [30,31]). Note, however, that due to [32] the critical efficiency for two-setting inequalities cannot be lower than $n/(2n-1)$. Hence, none of these inequalities may allow us to go below $\eta_{\text{crit}} = 60\%$ for three parties and below $\eta_{\text{crit}} = 50\%$ for an infinite number of parties.

In contrast to two settings and a large number of parties, the case of more than two settings per party and a moderate number of parties is much less explored. Indeed, in the case of three parties ($n=3$) and a few number of settings m , which is the experimentally most interesting setup beyond the usual two-party scenario, only a few results are known. To the best of our knowledge, for three parties the lowest detection efficiency is attained in [28] giving $\eta_{\text{crit}} = 60\%$ using $m=2$ settings. The aim of this paper is to go beyond two measurement settings per party, which opens the door to more efficient multisetting Bell inequalities. In particular, we explore numerically the best detection efficiencies for the emblematic three-qubit W state [34], and also perform a detailed numerical search when the underlying state is a more general symmetric three-qubit pure state. Note that the search for critical detection efficiencies using the famous Greenberger-Horne-Zeilinger (GHZ) [37] state was carried out recently in [24], attaining the lowest efficiency $\eta_{\text{crit}} = 12/17 \simeq 70.59\%$ so far using $m=17$ settings per party (for an explicit construction of the Bell inequality, please see the website [38]). Before this work the best bound of $\eta_{\text{crit}} = 75\%$ for a GHZ state was provided by Larsson [39] using the Mermin inequality.

Here we report a considerable improvement over the above values by showing that detection efficiencies as low as 50% can be tolerated in tripartite Bell tests featuring a reasonable number of measurements. However, our setups turn out to be very fragile to noise; hence, we believe that the experimental implementation remains a challenging issue.

II. SETUP

We consider a Bell scenario with three observers ($n=3$)—Alice, Bob, and Cecil—who carry out experiments in distant laboratories. Each observer can choose among m possible inputs and receive two possible outcomes. Let us identify the inputs of the three parties with $i, j, k = 1, \dots, m$ which correspond to a set of m possible measurements $\{A_i\}$, $\{B_j\}$, $\{C_k\}$ for each party. Without loss of generality, we can label with $+1$ and -1 the two different outcomes α, β , and γ for the respective parties. The experiment is fully characterized by the conditional probabilities $P(\alpha\beta\gamma|A_i B_j C_k)$. We use the shorthand notation $P(A_i B_j C_k) \equiv P(111|A_i B_j C_k)$ and similarly for a subset of the parties, such as $P(A_i B_j) \equiv P(11|A_i B_j)$ and $P(A_i) \equiv P(1|A_i)$, etc. It can be seen that these probabilities fully determine the joint distribution $P(\alpha\beta\gamma|A_i B_j C_k)$; hence, it is enough to consider them.

Throughout this work we stick to symmetric Bell inequalities, that is, inequalities which are symmetric for all permutations of the parties. In addition, our Bell inequalities will not contain single party marginal terms; they are built up only by two-particle and three-particle correlation terms. We will also assume without loss of generality that the classical bound of the Bell inequalities is zero. The Bell inequalities

considered in [24,28] are similarly restricted. As we will see, this simplification allows us to treat the problem with the tools of linear programming. We can write such a Bell inequality as

$$\sum_{i,j=1}^m M_{ij}^{(2)} [P(A_i B_j) + P(A_i C_j) + P(B_j C_k)] + \sum_{i,j,k=1}^m M_{ijk}^{(3)} P(A_i B_j C_k) \leq 0, \quad (1)$$

where

$$M_{ijk}^{(3)} = M_{ikj}^{(3)} = M_{jik}^{(3)} = M_{jki}^{(3)} = M_{kij}^{(3)} = M_{kji}^{(3)}, \\ M_{ij}^{(2)} = M_{ji}^{(2)}, \quad (2)$$

and the Bell coefficients $M_{ij}^{(3)}$ and $M_{ij}^{(2)}$ are chosen such that the classical bound is zero.

A. Local bound

Let us first compute the local limit of the above Bell inequality (1) allowing any classical mechanism. In order to do that, it is enough to consider deterministic strategies: Each of the parameters a_i , b_i , and c_i , where i runs from one to m , may take the value of either zero or one, and a deterministic strategy is defined by a particular choice. This corresponds to a definite outcome for each measurement value for each party. For example, $a_i = 1$ means that the probability for Alice to get the value $+1$ for her i th measurement is 1, that is $P(A_i) = 1$.

To set the classical bound of the Bell inequality (1) to zero, we must ensure that

$$\sum_{i,j=1}^m M_{ij}^{(2)} (a_i b_j + a_i c_j + b_i c_j) + \sum_{i,j,k=1}^m M_{ijk}^{(3)} a_i b_j c_k \leq 0 \quad (3)$$

for all deterministic strategies. Equation (3) gives 2^{3m} linear constraints for the Bell coefficients. Due to the permutational symmetry in Eq. (2), two strategies which may be derived from each other by swapping the strategies of any two participants (e.g., by swapping the values of a_i and b_i) lead to the same constraint, which makes it possible to reduce the number of constraints. Also, Eq. (3) is trivially fulfilled for any strategy assigning nonzero values for only one of the participants. We note that it follows from Eq. (3) that $M_{ij}^{(2)} \leq 0$. We may get this from strategy $a_i = 1$, $b_j = 1$, while all other a and b and all c values are zero.

B. Quantum bound

Now let us consider the quantum case. The maximum quantum violation of a two-outcome Bell inequality, i.e., the one presented in Eq. (1), is always attained by von Neumann measurements [40]. Moreover, it is sufficient to restrict ourselves to pure states $|\psi\rangle$, that is, $\hat{\rho} = |\psi\rangle\langle\psi|$. Then

$$P(A_i B_j) = \langle\psi|\hat{A}_i \otimes \hat{B}_j \otimes \hat{I}|\psi\rangle, \\ P(A_i C_j) = \langle\psi|\hat{A}_i \otimes \hat{I} \otimes \hat{C}_j|\psi\rangle, \\ P(B_j C_k) = \langle\psi|\hat{I} \otimes \hat{B}_j \otimes \hat{C}_k|\psi\rangle, \\ P(A_i B_j C_k) = \langle\psi|\hat{A}_i \otimes \hat{B}_j \otimes \hat{C}_k|\psi\rangle, \quad (4)$$

where \hat{A}_i , \hat{B}_j , and \hat{C}_k are the measurement operators of Alice, Bob, and Cecil, respectively, projecting onto the subspace corresponding to outcome +1 in the subspace of the participant concerned, and \hat{I} is the unity operator in that subspace. Along this study, we will restrict ourselves to three-qubit states; hence, the measurement operators \hat{A}_i , \hat{B}_j , and \hat{C}_k are in fact projectors in the qubit space.

C. Quantum case with limited detection efficiency

Let us consider the quantum case when all participants detect their particles with the same limited detection efficiency η . As a side remark, we note that interesting results have been obtained in the asymmetric case, that is, when the parties feature different efficiencies [22,41] or when measurements corresponding to the same party have different efficiencies [42]. In our symmetric scenario, the participants agree to output -1 in case of no detection. In this case, we get the joint probabilities of detecting outcome +1 by two and by all three participants if we multiply the probabilities of Eq. (4) by η^2 and by η^3 , respectively, that is, by the probability of the detection of the particles concerned. Then the condition for the violation of the Bell inequality in Eq. (1) can be written as

$$\langle \psi | \hat{\mathcal{M}}_\eta | \psi \rangle \equiv \eta^2 \mathcal{M}^{(2)} + \eta^3 \mathcal{M}^{(3)} > 0, \tag{5}$$

where

$$\begin{aligned} \mathcal{M}^{(2)} \equiv & \sum_{i,j=1}^m M_{ij}^{(2)} (\langle \psi | \hat{A}_i \otimes \hat{B}_j \otimes \hat{I} | \psi \rangle \\ & + \langle \psi | \hat{A}_i \otimes \hat{I} \otimes \hat{C}_j | \psi \rangle + \langle \psi | \hat{I} \otimes \hat{B}_j \otimes \hat{C}_k | \psi \rangle), \end{aligned} \tag{6}$$

$$\mathcal{M}^{(3)} \equiv \sum_{i,j,k=1}^m M_{ijk}^{(3)} \langle \psi | \hat{A}_i \otimes \hat{B}_j \otimes \hat{C}_k | \psi \rangle, \tag{7}$$

and $\hat{\mathcal{M}}_\eta$ is the effective Bell operator at η efficiency. As we have shown earlier, $M_{ij}^{(2)} \leq 0$, therefore $\mathcal{M}^{(2)} \leq 0$. Therefore, if η is very small, according to Eq. (5), there is no Bell violation. The critical detector efficiency above which the violation may be detected is

$$\eta_{\text{crit}} = -\frac{\mathcal{M}^{(2)}}{\mathcal{M}^{(3)}}. \tag{8}$$

To find the Bell inequality which minimizes η_{crit} in case of a particular choice of the state and the measurement operators is a problem of standard linear programming. To ensure that the classical bound is zero, the set of linear constraints given by Eq. (1) must be satisfied. As the Bell coefficients may be multiplied by any positive number, we may fix the norm by fixing the value of $\mathcal{M}^{(2)}$. We may choose any negative number. In particular, let us choose $\mathcal{M}^{(2)} = -1$. This provides an additional linear constraint. Then we must maximize $\mathcal{M}^{(3)}$, which is a linear expression for the Bell coefficients. The symmetries according to Eq. (2) are further linear constraints to be enforced, but instead of doing that we may restrict ourselves to coefficients $M_{ijk}^{(3)}$, with $i \leq j \leq k$ and $M_{ij}^{(2)}$, with $i \leq j$, and rewrite the constraints and the expression to be maximized in terms of these independent parameters. This way we get a much smaller problem to solve.

TABLE I. Table for critical detection efficiencies using the W state. The numbers in brackets refer to the Bell inequalities. In the first column ijk refers to the respective number of settings i , j , and k for the parties Alice, Bob, and Cecil. Detection efficiency thresholds are rounded up to five digits.

Settings	η_{crit}	Equation
222	0.83747	(13)
223	0.6	(18)
333	0.6	(14)
444	0.509036	(15)
666	0.502417	(19)
888	0.501338	(19)

Let the set of measurement operators be the same for all parties, and let us confine ourselves to real measurement operators. This particular restriction was also proved to be useful in other studies for exploring nonlocality of the W state [33]. In this case the operator $\hat{A}_i = \hat{B}_i = \hat{C}_i$ can be characterized by a single real variable Φ_i :

$$\begin{aligned} \hat{A}_i |0\rangle &= \frac{1}{2}(1 - \cos \Phi_i) |0\rangle - \frac{1}{2} \sin \Phi_i |1\rangle \equiv c_i^- |0\rangle + s_i |1\rangle, \\ \hat{A}_i |1\rangle &= -\frac{1}{2} \sin \Phi_i |0\rangle + \frac{1}{2}(1 + \cos \Phi_i) |1\rangle \equiv s_i |0\rangle + c_i^+ |1\rangle. \end{aligned} \tag{9}$$

If $\Phi_i = 0$, the measurement gives the value +1 with probability 1 for the $|1\rangle$ state.

Let the quantum state be also symmetric in terms of the permutations of the parties. One such state is the three-qubit GHZ state [37], which case has been already investigated thoroughly [24,39]. In this paper our primary concern is the three-qubit W state [34] but we also study generic symmetric three-qubit states. In the following section we focus on the W state (Sec. III) and then we move on to investigate the more general case in Sec. IV. Our main results concerning the found detection efficiency thresholds are summarized in Tables I and II for the W state and the generic three-qubit states, respectively.

III. DETECTION EFFICIENCIES USING THE W STATE

The W state is defined by [34]

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{10}$$

TABLE II. Table for critical detection efficiencies using symmetric three-qubit states. The numbers in brackets refer to the Bell inequalities. In the first column ijk refers to the number of i , j , and k settings for the parties Alice, Bob, and Cecil. Detection efficiency thresholds are rounded up to five digits.

Settings	η_{crit}	Equation
222	0.6	(13)
223	0.6	(18)
333	0.51678	(27)
444	0.5	(28)

where we have used the shorthand notation

$$|\alpha\beta\gamma\rangle \equiv |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle. \quad (11)$$

Now, by using Eqs. (9)–(11), it is straightforward to calculate the quantum conditional probabilities appearing in Eqs. (6) and (7):

$$\begin{aligned} \langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{I}|W\rangle &= \frac{1}{3}(2s_i s_j + c_i^- c_j^+ + c_i^+ c_j^- + c_i^- c_j^-), \\ \langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{A}_k|W\rangle &= \frac{2}{3}(c_i^- s_j s_k + s_i c_j^- s_k + s_i s_j c_k^-) \\ &\quad + \frac{1}{3}(c_i^- c_j^- c_k^+ + c_i^- c_j^+ c_k^- + c_i^+ c_j^- c_k^+). \end{aligned} \quad (12)$$

If the number of measurement settings per party is small, we can scan the space of measurement angles with an even step size, and solve the linear programming problem for each set of angles. In each case the optimal Bell inequality we arrive at has to be a tight one in the symmetrized probability space. We refer to [35] for the framework of symmetric Bell inequalities and to further studies which make use of this framework [36], reducing considerably the complexity of the problem. There is a finite number of such inequalities, so we get the same solution for a whole range of angles. Therefore, if our step size is not too large, we will certainly get the Bell inequality that gives the smallest critical efficiency with the $|W\rangle$ state. Then for the known inequality we may calculate the optimum measurement angles. Also, due to the tightness, the Bell coefficients can always be normalized such that they are integer numbers.

Next we list our results for different numbers of settings, where the numerical study was carried out up to eight settings per party.

A. W state, $m = 2$

For two measurement settings per party we got the following Bell coefficients:

$$M_{11}^{(2)} = -1 \quad M_{111}^{(3)} = 2 \quad M_{112}^{(3)} = 1 \quad M_{122}^{(3)} = -1. \quad (13)$$

Here we only show the values of the independent Bell coefficients, that is, $M_{ij}^{(2)}$ with $i \leq j$ and $M_{ijk}^{(3)}$, with $i \leq j \leq k$. The values of the coefficients that cannot be derived from the coefficients given above by some permutation of the parties, e.g., $M_{12}^{(2)}$, are zero. This inequality is equivalent to the inequality 22 in the list of Sliwa [43]. The optimum angles for this inequality are $\Phi_1 = 2.28059$ and $\Phi_2 = 0.33432$, and the critical efficiency is $\eta_{\text{crit}} = 0.83747$. We will see later that the $|W\rangle$ state is not the best choice for this inequality.

B. W state, $m = 3, 4, 5$

For $m = 3$ and 4 we have got the inequalities with the smallest η_{crit} if we have chosen small measurement angles. In the case of $m = 3$, the nonzero independent Bell coefficients of this inequality are

$$\begin{aligned} M_{11}^{(2)} &= -6, & M_{23}^{(2)} &= -3, & M_{123}^{(3)} &= 3, & M_{223}^{(3)} &= 2, \\ M_{233}^{(3)} &= 2, \end{aligned} \quad (14)$$

while for $m = 4$ we have got

$$\begin{aligned} M_{12}^{(2)} &= -6, & M_{34}^{(2)} &= -2, & M_{112}^{(3)} &= 6, & M_{114}^{(3)} &= -6, \\ M_{122}^{(3)} &= 6, & M_{123}^{(3)} &= 3, & M_{124}^{(3)} &= 3, & M_{134}^{(3)} &= -1, \\ M_{223}^{(3)} &= -6, & M_{234}^{(3)} &= -1, & M_{334}^{(3)} &= 2, & M_{344}^{(3)} &= 2. \end{aligned} \quad (15)$$

For both inequalities the optimal angles for all measurement settings approach zero near the threshold efficiency. This observation allows us to make some analytical considerations.

Let x be small, and let us consider the measurement angles proportional to this small number, that is, $\Phi_i \equiv \phi_i x$. Then, Eq. (12) may be approximated as

$$\langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{I}|W\rangle \approx \frac{1}{6}[1 - \cos x(\phi_i + \phi_j)] + \frac{x^4}{48}\phi_i^2\phi_j^2, \quad (16)$$

$$\langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{A}_k|W\rangle \approx \frac{x^4}{48}(\phi_i\phi_j + \phi_i\phi_k + \phi_j\phi_k)^2. \quad (17)$$

We have neglected terms of sixth and higher order in x . We have used Eq. (9) defining the quantities appearing in Eq. (12), which may be approximated at leading order as $s_i \approx -\phi_i x/2$, $c_i^+ \approx 1$ and $c_i^- \approx \phi_i^2 x^2/4$. Also, it is easy to see that $2s_i s_j + c_i^- c_j^+ + c_i^+ c_j^- = [1 - \cos(\Phi_i + \Phi_j)]/2$. Due to Eq. (17), $\mathcal{M}^{(3)}$ [see Eq. (7)] is fourth order in x . Then, according to Eq. (8), we may only get a finite value for η_{crit} if $\mathcal{M}^{(2)}$ defined in Eq. (6) is also fourth order in x . This is true if whenever the $M_{ij}^{(2)}$ Bell coefficient is not zero the corresponding measurement angles satisfy $\phi_i + \phi_j = 0$.

We may get the Bell inequalities of Eqs. (14) and (15) by solving the linear programming problem using the small angles limit, that is, Eqs. (16) and (17), when calculating $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$, and dropping the overall factor x^4 . In case of $m = 3$ [Eq. (14)], we take $\phi_1 = 0$ and $\phi_2 = -\phi_3 = 1$ (that is, we choose $x = \Phi_2$). This way, there are no free parameters left. With this choice $M_{11}^{(2)}$ and $M_{23}^{(2)}$ may take a nonzero value, as $\Phi_1 = -\Phi_1 = 0$ and $\Phi_2 = -\Phi_3 = x$. Indeed, these are the nonzero $M_{ij}^{(2)}$ coefficients in Eq. (14). Actually, the solution of the linear programming problem in this case is not unique, there are other Bell inequalities leading to the same η_{crit} . We have shown the one having the smallest number of nonzero Bell coefficients. Now, from Eq. (8) we can easily calculate the value of η_{crit} . Using the measurement angles defined above, $\langle W|\hat{A}_1 \otimes \hat{A}_1 \otimes \hat{I}|W\rangle \approx 0$ and $\langle W|\hat{A}_2 \otimes \hat{A}_3 \otimes \hat{I}|W\rangle \approx x^4/48$ [see Eq. (16)]. Furthermore, $\langle W|\hat{A}_1 \otimes \hat{A}_2 \otimes \hat{A}_3|W\rangle \approx \langle W|\hat{A}_2 \otimes \hat{A}_2 \otimes \hat{A}_3|W\rangle = \langle W|\hat{A}_2 \otimes \hat{A}_3 \otimes \hat{A}_3|W\rangle \approx x^4/48$. Also, due to the permutational symmetry of the state $|W\rangle$, the matrix elements are the same for all permutations of the operators. Therefore, by substituting the values for the measurement angles and the Bell coefficients into Eq. (6), we get $\mathcal{M}^{(2)} = -18x^3/48$. Similarly, from Eq. (7), we arrive at $\mathcal{M}^{(3)} = 30x^3/48$. Therefore, $\eta_{\text{crit}} = 3/5 = 0.6$. We have noted that this is not the only Bell inequality with the same threshold efficiency. The reason is that the quantum value does not depend on $M_{11}^{(2)}$, $M_{111}^{(3)}$, as the matrix elements they are multiplied with are zero, being $\Phi_1 = 0$. The requirement of the zero classical value does not define uniquely these coefficients.

In the case of $m = 4$, similarly to Eq. (14) for $m = 3$, Eq. (15) can also be derived by using the small angles limit.

Now, we choose the measurement angles $\Phi_1 = -\Phi_2 = x$ and $\Phi_3 = -\Phi_4 = \lambda x$. Now we have a single parameter λ . It is enough to consider $|\lambda| \leq 1$. We get the required inequality if we choose any value for λ between 0.21 and 0.78. We show in the Appendix that the optimum is $\lambda = 0.466715$, which is a root of a fifth-order equation, and then $\eta_{\text{crit}} = 0.509036$.

We have also derived the optimal $m = 5$ Bell inequality similarly to the smaller ones in Sec. III B, with measurement angles $\Phi_1 = 0$, $\Phi_2 = -\Phi_3 = x$, and $\Phi_4 = -\Phi_5 = \lambda x$. It turned out to be equivalent to the $m = 4$ case, so we got no improvement on the critical efficiency.

C. W state, $m_A = 3$ and $m_B = m_C = 2$

If we do not require permutational symmetry, we may create a Bell inequality with the same $\eta_{\text{crit}} = 0.6$ as for the $m_A = m_B = m_C = 3$ case using the $|W\rangle$ state with only two measurement settings for Bob and Cecil. As before, Alice's measurement settings \hat{A}_1, \hat{A}_2 , and \hat{A}_3 are characterized by $\Phi_1^A = 0, \Phi_2^A = x$, and $\Phi_3^A = -x$, respectively. However, for Bob and Cecil the measurement angles will be chosen as $\Phi_1^B = 0, \Phi_2^B = x$ and $\Phi_1^C = 0, \Phi_2^C = -x$, respectively. Here we used the upper indices to distinguish between the parties. The asymmetric inequality will have the same quantum value as the symmetric one for any η , if the sums of the Bell coefficients multiplying matrix elements that have the same numerical value are the same for both inequalities. At the same time we must ensure that the classical bound is also the same, that is, zero. In the case of a known symmetric inequality, these requirements define a set of linear constraints for the coefficients of the asymmetric one. It is a problem of linear programming to decide whether these constraints can be satisfied or not. In the present case the problem is solvable; the simplest Bell inequality we have got, after dividing each coefficient by a factor of 6, is

$$\begin{aligned} S \equiv & -P(11|A_1B_1) - P(11|A_1C_1) - P(11|B_1C_1) \\ & -P(11|A_3B_2) - P(11|A_2C_2) - P(11|B_2C_2) \\ & +P(111|A_1B_2C_2) + P(111|A_2B_1C_2) + P(111|A_3B_2C_1) \\ & +P(111|A_2B_2C_2) + P(111|A_3B_2C_2) \leq 0. \end{aligned} \quad (18)$$

We have also tried to derive asymmetric Bell inequalities with a smaller number of measurement settings for some of the parties from the $m = 4$ case given by Eq. (15), and also from the inequalities we will show later, but we have found no solution for the problem involved.

D. W state, $m \geq 6$

For $m = 6$, using $\Phi_1 = -\Phi_2 = x, \Phi_3 = -\Phi_4 = \mu x$, and $\Phi_5 = -\Phi_6 = \nu x$, we got a new inequality, with $\eta_{\text{crit}} = 0.502417$, marginally better than before. Now the optimal choice for the parameters is $\mu = 0.495815$ and $\nu = 0.295435$ (see the Appendix). The nonzero independent Bell coefficients of this $m = 6$ inequality are

$$\begin{aligned} M_{12}^{(2)} = -18, & \quad M_{34}^{(2)} = -18, & \quad M_{56}^{(2)} = -18, & \quad M_{112}^{(3)} = 18, \\ M_{114}^{(3)} = -18, & \quad M_{122}^{(3)} = 18, & \quad M_{123}^{(3)} = 9, & \quad M_{124}^{(3)} = 9, \\ M_{136}^{(3)} = -9, & \quad M_{156}^{(3)} = 8, & \quad M_{223}^{(3)} = -18, & \quad M_{245}^{(3)} = -9, \end{aligned}$$

$$\begin{aligned} M_{256}^{(3)} = 4, & \quad M_{334}^{(3)} = 18, & \quad M_{336}^{(3)} = -18, & \quad M_{344}^{(3)} = 18, \\ M_{345}^{(3)} = 9, & \quad M_{346}^{(3)} = 9, & \quad M_{356}^{(3)} = 1, & \quad M_{445}^{(3)} = -18, \\ M_{456}^{(3)} = 5, & \quad M_{556}^{(3)} = 4, & \quad M_{566}^{(3)} = 8. \end{aligned} \quad (19)$$

For $m = 7$ we have got no further improvement.

For $m = 8$ there are three free parameters. The angles are given as $\Phi_1 = -\Phi_2 = x, \Phi_3 = -\Phi_4 = \rho x, \Phi_5 = -\Phi_6 = \sigma x$, and $\Phi_7 = -\Phi_8 = \tau x$. The optimal choice of the parameters is $\rho = 0.498442, \sigma = 0.306395$, and $\tau = 0.169989$. Then $\eta_{\text{crit}} = 0.501338$. The nonzero independent coefficients are

$$\begin{aligned} M_{12}^{(2)} = -6, & \quad M_{34}^{(2)} = -6, & \quad M_{56}^{(2)} = -6, & \quad M_{78}^{(2)} = -6, \\ M_{112}^{(3)} = 6, & \quad M_{114}^{(3)} = -6, & \quad M_{122}^{(3)} = 6, & \quad M_{123}^{(3)} = 3, \\ M_{124}^{(3)} = 3, & \quad M_{136}^{(3)} = -3, & \quad M_{223}^{(3)} = -6, & \quad M_{245}^{(3)} = -3, \\ M_{334}^{(3)} = 6, & \quad M_{336}^{(3)} = -6, & \quad M_{344}^{(3)} = 6, & \quad M_{345}^{(3)} = 3, \\ M_{346}^{(3)} = 3, & \quad M_{358}^{(3)} = -3, & \quad M_{378}^{(3)} = 2, & \quad M_{445}^{(3)} = -6, \\ M_{467}^{(3)} = -3, & \quad M_{478}^{(3)} = 2, & \quad M_{556}^{(3)} = 6, & \quad M_{558}^{(3)} = -6, \\ M_{566}^{(3)} = 6, & \quad M_{567}^{(3)} = 3, & \quad M_{568}^{(3)} = 3, & \quad M_{578}^{(3)} = 1, \\ M_{667}^{(3)} = -6, & \quad M_{678}^{(3)} = 1, & \quad M_{778}^{(3)} = 2, & \quad M_{788}^{(3)} = 2. \end{aligned} \quad (20)$$

We have not tried any larger numbers of settings; the number of constraints is too large. We may have got further improvement, but we do not expect we could go below 0.5 with the critical efficiency.

We summarized critical detection efficiencies we found in this paper for the three-qubit W state in Table I.

IV. DETECTION EFFICIENCIES FOR SYMMETRIC THREE-QUBIT STATES

By considering a more general symmetric state we have been able to reach $\eta_{\text{crit}} = 0.5$ exactly already with $m = 4$. But we found improvement even for $m = 3$. The state considered is

$$|\psi\rangle = \cos\alpha|W\rangle + \sin\alpha|111\rangle. \quad (21)$$

This state is also symmetric for the permutations of the parties; therefore, the matrix elements of the tensor products of single party operators will not depend on the order of those operators. In the above state we find that the weight of $|111\rangle$ goes to zero as the threshold efficiency is approached. Like before, the measurement angles also vanish at η_{crit} .

Now, besides the matrix elements calculated with the $|W\rangle$ state [see Eqs. (12), (16), and (17)] for the conditional probabilities of Eq. (4) appearing in Eqs. (6) and (7) we also need

$$\begin{aligned} \langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{I}|111\rangle &= \frac{1}{\sqrt{3}}s_i s_j \approx \frac{x^2}{4\sqrt{3}}\phi_i \phi_j, \\ \langle W|\hat{A}_i \otimes \hat{A}_j \otimes \hat{A}_k|111\rangle &= \frac{x^2}{4\sqrt{3}}(c_i^+ s_j s_k + s_i c_j^+ s_k + s_i s_j c_k^+) \\ &\approx \frac{x^2}{4\sqrt{3}}(\phi_i \phi_j + \phi_i \phi_k + \phi_j \phi_k), \end{aligned}$$

$$\begin{aligned}\langle 111|\hat{A}_i \otimes \hat{A}_j \otimes \hat{I}|111\rangle &= c_i^+ c_j^+ \approx 1, \\ \langle 111|\hat{A}_i \otimes \hat{A}_j \otimes \hat{A}_k|111\rangle &= c_i^+ c_j^+ c_k^+ \approx 1.\end{aligned}\quad (22)$$

These matrix elements, including their limits for small angles, may be calculated similarly to the ones given in Eqs. (12), (16), and (17). We have also used the same notations. For small angles we have kept only the leading-order terms. Equation (22) shows that the matrix elements $\langle W|\hat{\mathcal{M}}_\eta|111\rangle$ and $\langle 111|\hat{\mathcal{M}}_\eta|111\rangle$ of the effective Bell operator [see Eq. (5)] for small measurement angles, that is, for small x , will be second and zeroth order in x , respectively. With $|\psi\rangle$ given in Eq. (21) we can write

$$\begin{aligned}\langle \psi|\hat{\mathcal{M}}_\eta|\psi\rangle &= \cos^2 \alpha \langle W|\hat{\mathcal{M}}_\eta|W\rangle + 2 \sin \alpha \cos \alpha \langle W|\hat{\mathcal{M}}_\eta|111\rangle \\ &\quad + \sin^2 \alpha \langle 111|\hat{\mathcal{M}}_\eta|111\rangle.\end{aligned}\quad (23)$$

Let us make the same restriction as before, namely, let $M_{ij}^{(2)} = 0$, whenever the corresponding measurement angles do not satisfy $\phi_1 + \phi_2 = 1$, which makes sure that $\langle w|\hat{\mathcal{M}}_\eta|W\rangle$ is fourth order in x . Then $\langle \psi|\hat{\mathcal{M}}_\eta|\psi\rangle$ is also fourth order, if the mixing angle α is taken proportional with x^2 , that is, $\alpha = ax^2$. For small x we may write

$$\begin{aligned}\langle \psi|\hat{\mathcal{M}}_\eta|\psi\rangle &\approx \langle W|\hat{\mathcal{M}}_\eta|W\rangle + 2x^2 \langle W|\hat{\mathcal{M}}_\eta|111\rangle a \\ &\quad + x^4 \langle 111|\hat{\mathcal{M}}_\eta|111\rangle a^2.\end{aligned}\quad (24)$$

With this choice, all matrix elements appearing in $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$ according to Eqs. (6) and (7) are fourth order in x , and we may derive the Bell inequalities with the smallest critical efficiency using linear programming exactly the same way as we have done with the $|W\rangle$ state. There is one extra parameter a characterizing the mixing angle. From Eq. (24) it is easy to determine the optimum choice for this parameter. The equation defines a parabola as a function of a , and its maximum value is given as

$$a = -\langle W|\hat{\mathcal{M}}_\eta|111\rangle/x^2 \langle 111|\hat{\mathcal{M}}_\eta|111\rangle.\quad (25)$$

We note that $\langle 111|\hat{\mathcal{M}}_\eta|111\rangle \leq 0$ for small x , which follows from the condition that the classical bound is zero, and that all values of matrix elements involved are approximately one [see Eq. (22)]. Then the quantum value with the optimum a may be written as

$$\langle \psi|\hat{\mathcal{M}}_\eta|\psi\rangle \approx \langle W|\hat{\mathcal{M}}_\eta|W\rangle - \frac{\langle W|\hat{\mathcal{M}}_\eta|111\rangle}{\langle 111|\hat{\mathcal{M}}_\eta|111\rangle}.\quad (26)$$

The optimum value of a depends on the Bell coefficients to be determined, so what we can do is to try some initial values for a , determine the Bell inequality with linear programming, calculate the optimum a for this inequality, then repeat these steps until convergence, which typically means just a few iterations.

Let us first start with the smallest number of settings considered.

A. Symmetric state, $m = 2$

Choosing the parameter a according to Eq. (25) in the state $|\psi\rangle$ in Eq. (21), noting that $\alpha = ax^2$, we may get $\eta_{\text{crit}} = 0.6$ in the limit of small measurement angles with

$m = 2$ measurement settings per party. If we choose $\Phi_1 = 0$ and $\Phi_2 = x$, we get the same Bell inequality as we got with the $|W\rangle$ state, which we have shown in Eq. (13). The marginally small admixture of the $|111\rangle$ state lowered the value of η_{crit} from 0.83747 to 0.6, with considerably different measurement angles. The inequality is the same as the three-party one given by Larsson and Semitecolos [28], and which is number 22 on the list of Sliwa [43]. However, in [28] the state they considered is the $|000\rangle$ state with a very small admixture of the $|W\rangle$ state, that is, their state approaches a separable state at the threshold efficiency. Also, in their case, the second measurement angle is zero, and not the first one. Surprisingly, their very different solution does lead to the same $\eta_{\text{crit}} = 0.6$. We have calculated the maximum violation of the inequality numerically for several detector efficiencies above η_{crit} . It turned out that it is always enough to consider permutationally symmetric real states and to take the same real measurement operators for each party. Therefore, the state can be written as a linear combination of $|W\rangle$, $|111\rangle$, and $|000\rangle$ (the fourth independent real symmetric state can always be eliminated by an appropriate choice of the local coordinates).

The maximum violation as a function of the detector efficiency is shown in Fig. 1. Near the threshold efficiency the maximum violation scales as the third power of $\Delta\eta = \eta - \eta_{\text{crit}}$. The optimum state approaches the $|W\rangle$ state, while the coefficients of the $|111\rangle$ and the $|000\rangle$ states are proportional to $\Delta\eta$ and $\Delta\eta^{3/2}$, respectively. If we take the coefficient of the $|000\rangle$ state to be exactly zero, the maximum violation remains basically the same. Near the threshold the difference is negligible, and it is just a little more than 3% around $\eta = 0.9$. Therefore, the optimum solution may be reproduced almost exactly with the state we have considered in the present paper. Near η_{crit} the measurement angles Φ_1 and Φ_2 scale as $\Delta\eta^{3/2}$ and $\Delta\eta^{1/2}$, respectively. It is the first angle that tends to zero faster. If we take this angle to be exactly zero, as we have done in this paper, the scaling behavior of the maximum violation will not change, but its value will be smaller by a factor approaching 6.25 near η_{crit} , and by a factor of 1.33 at $\eta = 1$ (see Fig. 1). If we take the basis used in [28], given by the $|000\rangle$ and the $|W\rangle$ states, the threshold efficiency remains

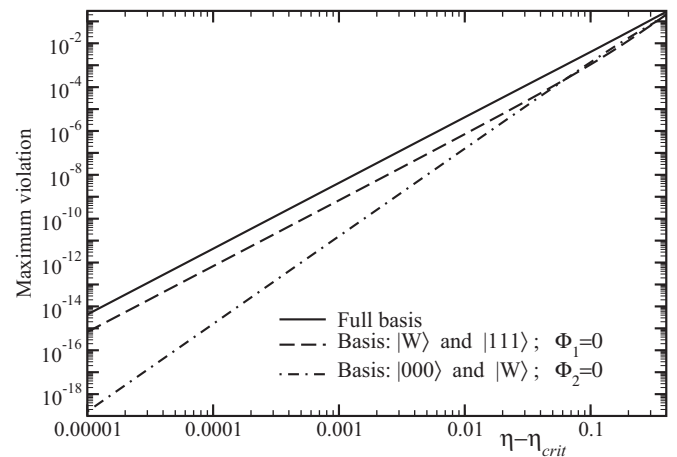


FIG. 1. The maximum violation as a function of the detector efficiency for the Bell inequality with two settings per party.

0.6, but near η_{crit} we get much smaller violations: it will scale as the fourth power of $\Delta\eta$. This time Φ_2 goes to zero faster than Φ_1 . If we take $\Phi_2 = 0$, it will hardly affect the violation near η_{crit} , while it will reduce it by about 30% at $\eta = 1$. The result is shown in Fig. 1. We may conclude that for this inequality our solution is much closer to the optimal arrangement than the one of Larsson and Semitecolos [28]. However, their approach may directly be generalized to a larger number of parties.

B. Symmetric state, $m = 3$

The independent Bell coefficients we got for $m = 3$ are

$$\begin{aligned} M_{11}^{(2)} &= -2, & M_{23}^{(2)} &= -1, & M_{111}^{(3)} &= 4, & M_{112}^{(3)} &= 1, \\ M_{113}^{(3)} &= 1, & M_{122}^{(3)} &= -2, & M_{123}^{(3)} &= 1, & M_{133}^{(3)} &= -2, \\ M_{223}^{(3)} &= 1, & M_{233}^{(3)} &= 1. \end{aligned} \quad (27)$$

For this Bell inequality $\eta_{\text{crit}} = (19 + \sqrt{937})/96 \approx 0.516776$ (see the Appendix), significantly smaller than the 0.6 value we got with the $|W\rangle$ state for $m = 3$.

C. Symmetric state, $m \geq 4$

For $m = 4$ the coefficients are

$$\begin{aligned} M_{12}^{(2)} &= -2, & M_{34}^{(2)} &= -2, & M_{112}^{(3)} &= 2, & M_{114}^{(3)} &= -2, \\ M_{122}^{(3)} &= 2, & M_{123}^{(3)} &= 1, & M_{124}^{(3)} &= 1, & M_{133}^{(3)} &= -2, \\ M_{134}^{(3)} &= 1, & M_{223}^{(3)} &= -2, & M_{234}^{(3)} &= 1, & M_{244}^{(3)} &= -2, \\ M_{334}^{(3)} &= 2, & M_{344}^{(3)} &= 2. \end{aligned} \quad (28)$$

In the Appendix we show that η_{crit} is exactly $1/2$ for this inequality. We have tried $m = 5$ and 6 , but we have got no improvement, so for three participants we could not find a Bell inequality for which the critical efficiency goes below $1/2$.

We summarized critical detection efficiencies we found in this paper for the symmetric three-qubit states in Table II.

V. SUMMARY

We have shown that the required detection efficiencies to demonstrate a loophole-free Bell violation can be significantly lowered if three parties are involved (instead of the usual two-party scenario). Before, no practical three-party Bell tests featuring efficiencies lower than 60% were known to the best of our knowledge. This value has been attained by Larsson and Semitecolos in 2001 in a three-party two-setting Bell scenario [28]. We beat this limit using a W state and three measurements per party. Moreover, for eight settings we reach the value of 50.13%. On the other hand, using a coherent mixture of the W state with a product state $|111\rangle$ allows us to obtain $\eta_{\text{crit}} = 50\%$ even with four settings. We conjecture that $\eta_{\text{crit}} = 50\%$ cannot be beaten in either way.

It is left as an open question if one of our inequalities could be generalized beyond three parties similarly to the family of Bell inequalities by Larsson and Semitecolos [28].

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APPENDIX: DETAILED CALCULATION OF CRITICAL DETECTION EFFICIENCIES

In the Appendix we calculate the critical detector efficiencies for the Bell inequalities given in the main text. According to Eqs. (5)–(7), and taking into account the permutational symmetry of the states considered, the matrix element of the effective Bell operator may be written as

$$\begin{aligned} &\langle \psi | \hat{\mathcal{M}}_\eta | \psi \rangle \\ &= \eta^2 \sum_{j=1}^m \sum_{i=j}^m M_{ij}^{(2)} \pi_{ij0} \langle \psi | \hat{A}_i \otimes \hat{B}_j \otimes \hat{I} | \psi \rangle + \\ &\quad \times \eta^3 \sum_{k=1}^m \sum_{j=k}^m \sum_{i=j}^m M_{ijk}^{(3)} \pi_{ijk} \langle \psi | \hat{A}_i \otimes \hat{B}_j \otimes \hat{C}_k | \psi \rangle, \end{aligned} \quad (A1)$$

where π_{ijk} is the number of permutations of indices i, j , and k , that is, $\pi_{ijk} = 6$ if all three are different, $\pi_{ijk} = 3$ if two indices agree, and $\pi_{ijk} = 1$ if $i = j = k$.

The condition for the violation of the Bell inequality by the results of the measurements performed on the $|W\rangle$ state is

$$\langle W | \hat{\mathcal{M}}_\eta | W \rangle > 0, \quad (A2)$$

and the values of the matrix elements necessary to evaluate $\langle W | \hat{\mathcal{M}}_\eta | W \rangle$ for small measurement angles are given by Eqs. (16) and (17). It makes the calculations simpler if we notice that these matrix elements do not change if we reverse the signs of the measurement angles concerned simultaneously. Also, if one of the measurement angles is Φ and another one is $-\Phi$, then the three particle matrix element will not depend on the third angle. These statements are also true for the matrix elements shown in Eq. (22) in the limit of small angles, which we will need when we consider the state defined by Eq. (21).

We have already shown that for the $m = 3$ inequality given by Eq. (14) $\eta_{\text{crit}} = 0.6$.

Now let us consider the $m = 4$ case given by Eq. (15). The measurement angles to be taken now are $\Phi_i = \phi_i x$, with $\phi_1 = -\phi_2 = 1$ and $\phi_3 = -\phi_4 = \lambda$. By using Eqs. (16), (17), (A1), and (A2), straightforward calculation leads us to

$$-3 - \lambda^4 + \eta[6 - 3(1 - 2\lambda)^2] > 0 \quad (A3)$$

for the condition of the quantum violation. Here we have simplified the expression by a factor of $x^2\eta^2/(48 \times 12)$. At $\eta = \eta_{\text{crit}}$ the left-hand side of the equation is zero, therefore $\eta_{\text{crit}} = 3(1 + 4\lambda - 4\lambda^2)/(3 + \lambda^4)$. It has its minimum value if λ satisfies $2\lambda^5 - 3\lambda^4 - \lambda^3 - 6\lambda + 3 = 0$. The appropriate root calculated numerically is $\lambda = 0.466715$, which leads to $\eta_{\text{crit}} = 0.509036$.

For the the $m = 6$ case shown in Eq. (19) we can follow the same steps as above. Now the measurement angles are given by $\phi_1 = -\phi_2 = 1$, $\phi_3 = -\phi_4 = \mu$, and $\phi_5 = -\phi_6 = \nu$. With these angles we get for the condition of quantum violation, after a simplification by a factor of $x^2\eta^2/(48 \times 36)$,

$$\begin{aligned} & -3(1 + \mu^4 + \nu^4) + \eta[6 + 6\mu^4 + 4\nu^4 - 3(1 - 2\mu)^2 \\ & - 3(\mu^2 - 2\mu\nu)^2 - 3(\mu - \nu - \mu\nu)^2] > 0. \end{aligned} \quad (\text{A4})$$

Again, at $\eta = \eta_{\text{crit}}$ the left-hand side of the equation is zero, and we must choose the parameters μ and ν such that η_{crit} is minimal. We get three equations for the three unknown values, and if we solve those equations numerically we get $\mu = 0.495815$, $\nu = 0.295435$, and $\eta_{\text{crit}} = 0.502417$.

For the inequality with $m = 8$ given by Eq. (20) the expression corresponding to Eq. (A4) is

$$\begin{aligned} & -3(1 + \rho^4 + \sigma^4 + \tau^4) + \eta[6 + 6\rho^4 + 6\sigma^4 + 4\tau^4 \\ & - 3(1 - 2\sigma)^2 - 3(\rho^2 - 2\rho\sigma)^2 - 3(\sigma^2 - 2\sigma\tau)^2 \\ & - 3(\rho - \sigma - \rho\sigma)^2 - 3(\rho\sigma - \rho\tau - \sigma\tau)^2] > 0. \end{aligned} \quad (\text{A5})$$

Here we have followed the same steps as for $m = 6$ taking measurement angles $\phi_1 = -\phi_2 = 1$, $\phi_3 = -\phi_4 = \rho$, $\phi_5 = -\phi_6 = \sigma$, and $\phi_7 = -\phi_8 = \tau$. From the equation we get numerically $\eta_{\text{crit}} = 0.501338$ with $\rho = 0.498442$, $\sigma = 0.306395$, and $\tau = 0.169989$.

Now let the state be the one shown in Eq. (21). From Eq. (26), if we choose the optimal mixing angle, the condition for quantum violation is

$$\langle W|\hat{\mathcal{M}}_\eta|W\rangle - \frac{\langle W|\hat{\mathcal{M}}_\eta|111\rangle}{\langle 111|\hat{\mathcal{M}}_\eta|111\rangle} > 0. \quad (\text{A6})$$

The matrix elements of the Bell operator may be calculated from Eq. (A1), which is also valid if the state vectors are different in the bra and the ket positions, provided both are permutationally symmetric. The matrix elements of the two and three particle operators appearing in the right-hand side of the equation are given in Eqs. (16), (17), and (22). We are concerned with the small angles limit.

First, let us take the $m = 2$ inequality of Eq. (13). With the choice of $\phi_1 = 0$ and $\phi_2 = 1$, we get $\langle W|\hat{\mathcal{M}}_\eta|W\rangle = -3\eta^3x^4/48$, $\langle W|\hat{\mathcal{M}}_\eta|111\rangle = -3\eta^3x^2/4\sqrt{3}$, and $\langle 111|\hat{\mathcal{M}}_\eta|111\rangle = -3\eta^2 + 2\eta^3$ for the matrix elements

of the effective Bell operator. By substituting these values into Eq. (A6), and taking into account that the left-hand side of the equation is zero at $\eta = \eta_{\text{crit}}$, it is easy to see that $\eta_{\text{crit}} = 3/5 = 0.6$.

We may take the same steps for $m = 3$. The inequality is shown by Eq. (27), and the measurement angles are given by $\phi_1 = 0$, $\phi_2 = 1$, and $\phi_3 = -1$. Then the matrix elements of the effective Bell operator are $\langle W|\hat{\mathcal{M}}_\eta|W\rangle = -6\eta^2x^4/48$, $\langle W|\hat{\mathcal{M}}_\eta|111\rangle = -\sqrt{3}\eta^2x^2(2\eta - 1/2)$, and $\langle 111|\hat{\mathcal{M}}_\eta|111\rangle = -\eta^2(12 - 5\eta)$. Then the condition that the left-hand side of Eq. (A6) is zero at $\eta = \eta_{\text{crit}}$ leads to the equation $48\eta_{\text{crit}}^2 - 19\eta_{\text{crit}} - 3 = 0$, whose appropriate root is $\eta_{\text{crit}} = (19 + \sqrt{937})/96 \approx 0.516776$.

In the case of the $m = 4$ inequality of Eq. (28) the measurement angles are given by $\phi_1 = 1$, $\phi_2 = -1$, $\phi_3 = \lambda$, and $\phi_4 = -\lambda$. From these it follows that $\langle W|\hat{\mathcal{M}}_\eta|W\rangle = \eta^2x^4[-1 - \lambda^4 + \eta(1 + 4\lambda - 8\lambda^2 - 4\lambda^3 + \lambda^4)]/4$, $\langle W|\hat{\mathcal{M}}_\eta|111\rangle = -\sqrt{3}\eta^2x^2(1 + \lambda^2)(1 - 3\eta)$, and $\langle 111|\hat{\mathcal{M}}_\eta|111\rangle = -24\eta^2(1 - \eta)$. If we substitute these values into Eq. (A6), we can get

$$\frac{\eta^2}{8(1 - \eta)} \left[r \left(\eta - \frac{1}{2} \right)^2 + p \left(\eta - \frac{1}{2} \right) - q \right] > 0, \quad (\text{A7})$$

where

$$\begin{aligned} r & \equiv 4\lambda^4 + 8\lambda^3 + 34\lambda^2 - 8\lambda + 7 \\ & = 5\lambda^4 + 2(\lambda + 1)^4 + 6\lambda^2 + 4(2\lambda - 1)^2 + 1 > 0, \\ p & \equiv 5\lambda^4 + 6\lambda^2 + 5 > 0, \\ q & \equiv \frac{(\lambda^2 + 4\lambda - 1)^2}{4} \geq 0. \end{aligned} \quad (\text{A8})$$

In Eq. (A7) the prefactor is positive for $0 < \eta < 1$. As r is strictly positive, the inequality is satisfied above the upper root of the second-order expression. Below that the expression is negative for all $\eta \geq 0$, as one can easily see. Therefore, we get the critical efficiency as $(\eta_{\text{crit}} - 1/2) = (\sqrt{p^2 + 4rq} - p)/2r$. As $r > 0$, $p > 0$, and $q \geq 0$, the smallest possible value the right-hand side may take is zero, when we choose λ such that $q = 0$, that is, $\lambda = -2 \pm \sqrt{5}$. With this optimal choice $\eta_{\text{crit}} = 1/2$.

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