Quantum versus semiclassical description of light interaction with atomic ensembles: Revision of the Maxwell-Bloch equations and single-photon superradiance

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Interaction of atoms with quantum states of light is a long-standing problem that apart from fundamental physics has potential applications for optical quantum-state storage, quantum communication, and quantum information. A fully quantum mechanical treatment of this problem is usually very complicated mathematically. Here we show, however, that quantum mechanical evolution equations describing single-photon emission (absorption) by atomic ensembles can be written in a form equivalent to the semiclassical Maxwell-Bloch equations. This connection allows us to find exact analytical solutions of the fully quantum mechanical problem. We also found that semiclassical Maxwell-Bloch equations should be written in a form different from those commonly used. Namely, the classical limit of the quantum problem gives a propagation equation with the Laplacian operator on the right-hand side rather than with the second-order time derivative.

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I. INTRODUCTION

Collective spontaneous emission from a cloud of N atoms has been a subject of long-standing interest since the pioneering work of Dicke [1]. If a single photon is stored in the atomic cloud (and shared among many atoms) the system is in an entangled state with no macroscopic dipole moment [2–5]. Recent studies have focused on collective and virtual effects in such systems [3–16]. Cooperative emission can provide insights into quantum electrodynamics and is important for various applications of the entangled atomic ensembles and generated quantum states of light for optical quantum-state storage [17], quantum cryptography [18,19], quantum communication [9,20,21], and quantum information [9,11].

Virtual transitions are a fascinating feature of quantum electrodynamics. Apart from the influence on a single atom, virtual transitions modify the evolution of atomic ensembles. Let us consider N two-level (a and b, with $E_a - E_b = \hbar\omega$) atoms that are prepared in a collective state with only one atom excited. The initial excitation is distributed among the atoms with a probability amplitude $\beta(\mathbf{r})$ that depends on the atom position $\mathbf{r}$. If we disregard virtual transitions, then for a dense cloud of volume V evolution of the atomic system in the scalar photon theory is described by an integral equation with the sin kernel [22]

$$\frac{\partial\beta(t,\mathbf{r})}{\partial t} = -\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta(t,\mathbf{r}'), \quad (1)$$

where $\beta(t,\mathbf{r})$ is the probability amplitude to find an atom at position $\mathbf{r}$ excited at time t, γ is the single-atom decay rate, $k_0 = \omega/c$, and the integral is taken over the volume of the atomic sample. However, inclusion of virtual processes yields an equation with the exp kernel [6,7,15,23]

$$\frac{\partial\beta(t,\mathbf{r})}{\partial t} = i\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta(t,\mathbf{r}'). \quad (2)$$

The evolution equation (1) was the subject of investigation several decades ago [22,24,25], while Eq. (2) has been studied in detail recently [6–8,12–15,26,27]. Virtual transitions have interesting effects on collective emission of atoms [12,14,15]. In particular, if the initial atomic state is superradiant, the virtual transitions partially transfer population into slowly decaying states, which results in a trapping of atomic excitation. On the other hand, for slowly decaying states virtual processes yield additional decay channels, which leads to a slow decay of the otherwise trapped states. The collective frequency (Lamb) shift produced by virtual processes is another fascinating subject of recent theoretical [26–32] and experimental investigation [33].

Equations (1) and (2) disregard retardation caused by a finite value of the speed of light and assume that evolution of the system at time t depends only on the state of the system at this moment of time (the local or Markov approximation). This assumption is valid if the atomic system evolves slowly so that during propagation of the signal through the sample the atomic state does not change substantially. However, if the size of the sample is large enough the local approximation breaks down and the system's dynamics becomes nonlocal in time. Now the evolution of the system at time t depends on the history, that is, on the states of atoms in the previous moments of time, and is governed by the equation [34]

$$\frac{\partial\beta(t,\mathbf{r})}{\partial t} = i\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{c},\mathbf{r}'\right). \quad (3)$$

Nonlocal effects yield oscillations with a collective atomic frequency that corresponds to the collective emission and reabsorption of the photon during its propagation through the atomic cloud [35,36]. The collective interaction of light and atoms together with parametric resonance can yield generation of high-frequency coherent radiation by driving the system with low frequency [37] or can be used to control propagation of γ rays on a short (superradiant) time scale [38].

A fully quantum mechanical treatment of photon emission (absorption) by atomic ensembles is a mathematically difficult task. Various techniques have been applied to address this problem. Solving the evolution equation for the state vector is the most common approach. It has been recently applied to study collective spontaneous emission of N multilevel atoms [39–41]. Another technique is the quantum multipath interference approach, which has been used to study superradiant and subradiant emission from entangled atoms [5]. A method based on the equation of motion for the atomic and field operators has been used to investigate cooperative scattering by cold atoms [42] and cooperative fluorescence from a strongly driven dilute atomic cloud [43]. The multiple-scattering expansion has been applied to calculate coherent propagation of photons through the medium [44,45]. Cooperativity in light scattering by cold atoms has also been studied in terms of a master equation for the atomic density matrix [46].

Here we show that the fully quantum mechanical description of the interaction of light with atomic ensembles in the limit of weak excitation (e.g., single-photon superradiance) can be reduced to the solution of propagation equations, which are equivalent to the semiclassical Maxwell-Bloch equations. This reduction substantially simplifies the quantum mechanical problem. Namely, we find that evolution of the electromagnetic field interacting with an ensemble of two-level atoms located at positions $\mathbf{r}_j$ is described by the coupled equations

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E}=-\nabla^2\mathbf{P}+\nabla(\mathrm{div}\mathbf{P}), \tag{4}$$

$$\frac{dS_j(t)}{dt}=-i\omega S_j(t)+\frac{i}{\hbar}\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \tag{5}$$

where

$$\mathbf{P}(t,\mathbf{r})=\vec{\wp}_{ab}\sum_j[S_j(t)+S_j^*(t)]\delta(\mathbf{r}-\mathbf{r}_j), \tag{6}$$

$$S_j(t)=\langle\Psi(t)|\hat{\sigma}_j|\Psi_G\rangle+\langle\Psi_G|\hat{\sigma}_j|\Psi(t)\rangle, \tag{7}$$

$$\mathbf{E}(t,\mathbf{r})=\langle\Psi(t)|\hat{\mathbf{E}}(\mathbf{r})|\Psi_G\rangle+\langle\Psi_G|\hat{\mathbf{E}}(\mathbf{r})|\Psi(t)\rangle, \tag{8}$$

$\hat{\sigma}_j=|b_j\rangle\langle a_j|$ is the lowering operator for atom j, $\hat{\mathbf{E}}(\mathbf{r})$ is the electric-field operator, $\vec{\wp}_{ab}$ is the electric dipole transition matrix element between levels a and b, $\Psi(t)$ is the state of the photon-atom system at time t, and Ψ_G is the ground state [47].

Equations (4)–(8) are somewhat different from the conventional semiclassical Maxwell-Bloch equations, which for weak atomic excitation read

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E}=-\frac{1}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}+\nabla(\mathrm{div}\mathbf{P}), \tag{9}$$

$$\frac{\partial\sigma_j(t)}{\partial t}=-i\omega\sigma_j(t)+\frac{i}{\hbar}\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \tag{10}$$

where

$$\mathbf{P}(t,\mathbf{r})=\vec{\wp}_{ab}\sum_j[\sigma_j(t)+\sigma_j^*(t)]\delta(\mathbf{r}-\mathbf{r}_j) \tag{11}$$

is the medium polarization, σ_j is the coherence of atom j,

$$\sigma_j(t)=\langle\Psi(t)|\hat{\sigma}_j|\Psi(t)\rangle, \tag{12}$$

and $\mathbf{E}(t,\mathbf{r})$ is the average of the electric-field operator

$$\mathbf{E}(t,\mathbf{r})=\langle\Psi(t)|\hat{\mathbf{E}}(\mathbf{r})|\Psi(t)\rangle. \tag{13}$$

Equations (12) and (13) involve averaging over the state vector of the system $\Psi(t)$. Such an average vanishes for the problem of single-photon superradiance and therefore the semiclassical treatment is not applicable. In the present approach we define $S_j(t)$ and $\mathbf{E}(t,\mathbf{r})$ as matrix elements between $\Psi(t)$ and the ground state Ψ_G of the atom-photon system according to Eqs. (7) and (8). Thus defined quantities are no longer equal to zero for the problem of single-photon superradiance and therefore they can properly describe system evolution. The physical meaning of $S_j(t)$ and $\mathbf{E}(t,\mathbf{r})$ is now different. As we show, $S_j(t)$ is related to the probability amplitude to find atom j excited, while $\mathbf{E}(t,\mathbf{r})$ is related to the photon probability amplitude.

One should also note that the right-hand sides of Eqs. (4) and (9) involve different operators, ∇^2 in one case and $\partial^2/\partial t^2$ in the other. Such a difference is not attributed to the single-photon superradiance. We show that a general derivation of the semiclassical propagation equation from the full quantum mechanical Hamiltonian yields Eq. (4). This implies that $\mathbf{E}(t,\mathbf{r})$, defined by Eq. (13) as an average of the electric-field operator, has the meaning of the displacement vector rather than the electric field. Thus, the conventional Maxwell-Bloch equations must be revisited. We address this issue in the next section.

II. REVISION OF THE MAXWELL-BLOCH EQUATIONS

The classical Maxwell equations in a dielectric medium can be written as

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E}=-\frac{1}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2}+\nabla(\mathrm{div}\mathbf{P}), \tag{14}$$

where $\mathbf{P}$ is the vector of medium polarization

$$\mathbf{P}=\sum_j\mathbf{d}_j\delta(\mathbf{r}-\mathbf{r}_j) \tag{15}$$

and $\mathbf{d}_j$ is the dipole moment of atom j located at position $\mathbf{r}_j$. In terms of the electric displacement vector

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}, \tag{16}$$

Eq. (14) reads

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{D}=-\nabla^2\mathbf{P}+\nabla(\mathrm{div}\mathbf{P}). \tag{17}$$

In this section we obtain semiclassical equations for the electromagnetic field interacting with an atomic medium. Namely, we start from microscopic fully quantum mechanical equations for light and atoms and obtain evolution equations for the averaged quantities. The semiclassical limit should give Maxwell-Bloch equations. We show that such a limit indeed yields a Bloch equation for the polarization. However, the propagation equation has the form of Eq. (17) rather than the commonly used Eq. (14).

We consider a medium composed of two-level (a excited and b ground state) atoms with spacing between levels $E_a-E_b=\hbar\omega$. For an atom j the dipole moment operator is $\hat{\mathbf{d}}_j=\vec{\wp}_{ab}(\hat{\sigma}_j+\hat{\sigma}_j^\dagger)$, where $\hat{\sigma}_j=|b_j\rangle\langle a_j|$, $\hat{\sigma}_j^\dagger=|a_j\rangle\langle b_j|$, and

$\vec{\wp}_{ab}$ is the electric dipole transition matrix element between levels a and b, $\vec{\wp}_{ab} = e\langle a_j|\mathbf{r}|b_j\rangle$, which is assumed to be real. The Hamiltonian for the electromagnetic field interacting with atoms reads

$$\hat{H} = \sum_{\mathbf{k},\mu} \hbar\nu_k\left(\hat{a}^\dagger_{\mathbf{k},\mu}\hat{a}_{\mathbf{k},\mu} + \frac{1}{2}\right) + \frac{\hbar\omega}{2}\sum_j \hat{\sigma}_{zj} - \sum_j(\hat{\sigma}_j + \hat{\sigma}^\dagger_j)\vec{\wp}_{ab}\cdot\hat{\mathbf{E}}(\mathbf{r}_j), \quad (18)$$

where $\hat{\sigma}_{zj} = |a_j\rangle\langle a_j| - |b_j\rangle\langle b_j|$, $\hat{\mathbf{E}}(\mathbf{r})$ is the electric-field operator

$$\hat{\mathbf{E}}(\mathbf{r}) = i\sum_{\mathbf{k},\mu} g_k(\vec{\epsilon}_{\mathbf{k},\mu}\hat{a}_{\mathbf{k},\mu}e^{i\mathbf{k}\cdot\mathbf{r}} - \text{H.c.}), \quad (19)$$

$$g_k = \left(\frac{\hbar\nu_k}{2V\varepsilon_0}\right)^{1/2} \quad (20)$$

is a constant that describes the coupling strength between a single atom and the electric field, $\vec{\epsilon}_{\mathbf{k},\mu}$ are unit polarization vectors, $\nu_k = ck$, c is the speed of light in vacuum, and V is the photon volume. Field operators $\hat{a}_{\mathbf{k},\mu}$ and $\hat{a}^\dagger_{\mathbf{k},\mu}$ obey the boson commutation relations

$$[\hat{a}_{\mathbf{k},\mu},\hat{a}_{\mathbf{k}',\mu'}] = [\hat{a}^\dagger_{\mathbf{k},\mu},\hat{a}^\dagger_{\mathbf{k}',\mu'}] = 0, \quad (21)$$

$$[\hat{a}_{\mathbf{k},\mu},\hat{a}^\dagger_{\mathbf{k}',\mu'}] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\mu\mu'}, \quad (22)$$

while for atomic operators the commutation relations are

$$[\hat{\sigma}_j,\hat{\sigma}^\dagger_{j'}] = -\delta_{jj'}\hat{\sigma}_{zj}, \quad (23)$$

$$[\hat{\sigma}_j,\hat{\sigma}_{zj'}] = 2\delta_{jj'}\hat{\sigma}_j. \quad (24)$$

We perform calculations in the Heisenberg picture in which operators are time dependent. The Heisenberg equation of motion for an operator $\hat{A}(t)$ reads

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A},\hat{H}]. \quad (25)$$

Applying this equation for the atomic and field operators, we obtain

$$\frac{d\hat{a}_{\mathbf{k},\mu}(t)}{dt} = -i\nu_k\hat{a}_{\mathbf{k},\mu}(t) + \frac{g_k}{\hbar}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu}) \times \sum_j[\hat{\sigma}_j(t) + \hat{\sigma}^\dagger_j(t)]e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \quad (26)$$

$$\frac{d\hat{\sigma}_j(t)}{dt} = -i\omega\hat{\sigma}_j(t) - \frac{i}{\hbar}\hat{\sigma}_{zj}(t)\vec{\wp}_{ab}\cdot\hat{\mathbf{E}}(t,\mathbf{r}_j), \quad (27)$$

$$\frac{d\hat{\sigma}_{zj}(t)}{dt} = \frac{2i}{\hbar}[\hat{\sigma}^\dagger_j(t) - \hat{\sigma}_j(t)]\vec{\wp}_{ab}\cdot\hat{\mathbf{E}}(t,\mathbf{r}_j). \quad (28)$$

By making conventional averaging of Eqs. (19) and (26) in the Heisenberg picture over the initial state vector $\Psi(0)$ of the photon-atom system, that is, introducing $A(t) = \langle\Psi(0)|\hat{A}(t)|\Psi(0)\rangle$, etc., we obtain equations for the average quantities

$$\frac{da_{\mathbf{k},\mu}(t)}{dt} = -i\nu_k a_{\mathbf{k},\mu}(t) + \frac{g_k}{\hbar}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu}) \times \sum_j[\sigma_j(t) + \sigma^*_j(t)]e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \quad (29)$$

$$\mathbf{E}(t,\mathbf{r}) = i\sum_{\mathbf{k},\mu} g_k[\vec{\epsilon}_{\mathbf{k},\mu}a_{\mathbf{k},\mu}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \text{c.c.}]. \quad (30)$$

In Appendix A we show that Eqs. (29) and (30) yield the propagation equation [see Eq. (A14)]

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = -\nabla^2\mathbf{P} + \nabla(\text{div}\mathbf{P}), \quad (31)$$

where $\mathbf{P}$ is the medium polarization

$$\mathbf{P}(t,\mathbf{r}) = \vec{\wp}_{ab}\sum_j[\sigma_j(t) + \sigma^*_j(t)]\delta(\mathbf{r} - \mathbf{r}_j). \quad (32)$$

Equation (31) has the same form as the classical equation (17) and therefore $\varepsilon_0\mathbf{E}$ should be interpreted as the displacement vector $\mathbf{D}$ rather than the electric field.

In the semiclassical treatment we replace the operator $\hat{\mathbf{E}}(t,\mathbf{r}_j)$ in Eqs. (27) and (28) by its average value and obtain the evolution equation for the operators $\hat{\sigma}_j(t)$ and $\hat{\sigma}_{zj}(t)$,

$$\frac{d\hat{\sigma}_j(t)}{dt} = -i\omega\hat{\sigma}_j(t) - \frac{i}{\hbar}\hat{\sigma}_{zj}(t)\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \quad (33)$$

$$\frac{d\hat{\sigma}_{zj}(t)}{dt} = \frac{2i}{\hbar}[\hat{\sigma}^\dagger_j(t) - \hat{\sigma}_j(t)]\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j). \quad (34)$$

Averaging these equations over the initial state vector yields

$$\frac{d\sigma_j(t)}{dt} = -i\omega\sigma_j(t) - \frac{i}{\hbar}\sigma_{zj}(t)\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \quad (35)$$

$$\frac{d\sigma_{zj}(t)}{dt} = \frac{2i}{\hbar}[\sigma^*_j(t) - \sigma_j(t)]\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \quad (36)$$

which are conventional Bloch equations for atomic evolution.

Often in the literature the semiclassical Maxwell-Bloch equations are written in the form motivated by Eq. (14), namely, the propagation equation is written as

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E} = -\frac{1}{c^2}\frac{\partial^2\mathbf{P}}{\partial t^2} + \nabla(\text{div}\mathbf{P}). \quad (37)$$

Our analysis shows, however, that in the proper treatment of the problem the propagation equation (37) must be replaced with Eq. (31). Equations (37) and (31) give the same answer only if the medium polarization $\mathbf{P}$ obeys the homogeneous wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{P} = 0 \quad (38)$$

describing propagation of $\mathbf{P}$ without dispersion with the speed of light in vacuum c. This is usually not the case for the dielectric medium.

III. DESCRIPTION OF SINGLE-PHOTON SUPERRADIANCE BY MAXWELL-BLOCH EQUATIONS

Next we proceed to analyze single-photon superradiance [2–5] as a problem residing at the quantum-classical interface. Entangled quantum states of atomic ensembles can possess a zero dipole moment. This is, e.g., the case for a state

$$\Psi(0) = \sum_{j=1}^{N} \beta_j |b_1 b_2 \cdots a_j \cdots b_N\rangle|0\rangle \tag{39}$$

for which the system has no photons and a single atom is collectively excited. In Eq. (39) β_j is the probability amplitude to find atom j excited. For such a state the dipole moment $\langle\Psi(0)|\hat{\mathbf{d}}_j|\Psi(0)\rangle$ vanishes for every atom and thus the medium has no initial polarization. Nevertheless, the state (39) decays, emitting a single photon in the quantum mechanical description. On the other hand, the semiclassical treatment by means of the Maxwell-Bloch equations implies that atoms decay only if polarization is nonzero. Thus, the semiclassical treatment fails to describe the evolution of the state (39).

Here we show that for weak atomic excitation, quantum mechanical equations describing the evolution of the system can be written in the form of the Maxwell-Bloch equations (31), (32), and (35), however, functions in these equations will have a physical meaning different from the displacement vector and medium polarization $\mathbf{P}$.

In the limit of weak excitation one can approximately replace $\hat{\sigma}_{zj} \approx -1$ in Eq. (27). Then Eq. (27) becomes linear and decouples from Eq. (28)

$$\frac{d\hat{\sigma}_j(t)}{dt} = -i\omega\hat{\sigma}_j(t) + \frac{i}{\hbar}\vec{\wp}_{ab}\cdot\hat{\mathbf{E}}(t,\mathbf{r}_j). \tag{40}$$

For the problem of single-photon superradiance, e.g., when we are interested in the evolution of the state (39), averaging Eqs. (26) and (40) over the initial state vector gives identically zero. Yet one can reduce the quantum mechanical equations to the form of the Maxwell-Bloch equations by averaging differently. Namely, instead of taking the matrix element $\langle\Psi(0)|\cdots|\Psi(0)\rangle$ from both sides of Eqs. (26) and (40) we take the matrix element between Ψ_G and $\Psi(0)$, where Ψ_G is the ground state of the atom-photon system. Introducing

$$A_{\mathbf{k},\mu}(t) = \langle\Psi(0)|\hat{a}_{\mathbf{k},\mu}(t)|\Psi_G\rangle + \langle\Psi_G|\hat{a}_{\mathbf{k},\mu}(t)|\Psi(0)\rangle, \tag{41}$$

$$S_j(t) = \langle\Psi(0)|\hat{\sigma}_j(t)|\Psi_G\rangle + \langle\Psi_G|\hat{\sigma}_j(t)|\Psi(0)\rangle, \tag{42}$$

$$\mathbf{E}(t,\mathbf{r}) = \langle\Psi(0)|\hat{\mathbf{E}}(t,\mathbf{r})|\Psi_G\rangle + \langle\Psi_G|\hat{\mathbf{E}}(t,\mathbf{r})|\Psi(0)\rangle, \tag{43}$$

Eqs. (19), (26), and (40) yield

$$\frac{dA_{\mathbf{k},\mu}(t)}{dt} = -i\nu_k A_{\mathbf{k},\mu}(t) + \frac{g_k}{\hbar}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu}) \times \sum_j [S_j(t) + S_j^*(t)]e^{-i\mathbf{k}\cdot\mathbf{r}_j}, \tag{44}$$

$$\mathbf{E}(t,\mathbf{r}) = i\sum_{\mathbf{k},\mu} g_k[\vec{\epsilon}_{\mathbf{k},\mu}A_{\mathbf{k},\mu}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \text{c.c.}], \tag{45}$$

$$\frac{dS_j(t)}{dt} = -i\omega S_j(t) + \frac{i}{\hbar}\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j). \tag{46}$$

Equations (44) and (45) have the same form as Eqs. (29) and (30). However, now $a_{\mathbf{k},\mu}(t)$ and $\sigma_j(t)$ are replaced with $A_{\mathbf{k},\mu}(t)$ and $S_j(t)$, respectively, which are nonzero for the problem of single-photon superradiance. Therefore, single-photon superradiance can also be described by the Maxwell-Bloch equations

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = -\nabla^2\mathbf{P} + \nabla(\text{div}\mathbf{P}), \tag{47}$$

$$\frac{dS_j(t)}{dt} = -i\omega S_j(t) + \frac{i}{\hbar}\vec{\wp}_{ab}\cdot\mathbf{E}(t,\mathbf{r}_j), \tag{48}$$

where

$$\mathbf{P}(t,\mathbf{r}) = \vec{\wp}_{ab}\sum_j [S_j(t) + S_j^*(t)]\delta(\mathbf{r}-\mathbf{r}_j). \tag{49}$$

However, interpretation of $\mathbf{E}$ and $\mathbf{P}$ is now different. In the present formulation, $S_j(t)$ is no longer coherence of the atom j and $\varepsilon_0\mathbf{E}$ is not a displacement vector. To understand the physical meaning of $\mathbf{E}(t,\mathbf{r})$ and $S_j(t)$ we write them in a different form using the Schrödinger picture. Taking into account that

$$\hat{\sigma}_j(t) = \exp\left(\frac{i\hat{H}t}{\hbar}\right)\hat{\sigma}_j\exp\left(-\frac{i\hat{H}t}{\hbar}\right), \tag{50}$$

$$\hat{a}_{\mathbf{k},\mu}(t) = \exp\left(\frac{i\hat{H}t}{\hbar}\right)\hat{a}_{\mathbf{k},\mu}\exp\left(-\frac{i\hat{H}t}{\hbar}\right), \tag{51}$$

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right)|\Psi_G\rangle = |\Psi_G\rangle, \tag{52}$$

$$\exp\left(-\frac{i\hat{H}t}{\hbar}\right)|\Psi(0)\rangle = |\Psi(t)\rangle, \tag{53}$$

we obtain

$$S_j(t) = \langle\Psi(t)|\hat{\sigma}_j|\Psi_G\rangle + \langle\Psi_G|\hat{\sigma}_j|\Psi(t)\rangle \tag{54}$$

and

$$\mathbf{E}(t,\mathbf{r}) = \langle\Psi(t)|\hat{\mathbf{E}}(\mathbf{r})|\Psi_G\rangle + \langle\Psi_G|\hat{\mathbf{E}}(\mathbf{r})|\Psi(t)\rangle. \tag{55}$$

Thus, $S_j(t)$ and $\mathbf{E}(t,\mathbf{r})$ are expressed in terms of the matrix elements of the operators $\hat{\sigma}_j$ and $\hat{\mathbf{E}}(\mathbf{r})$ between the ground state of the system Ψ_G and the state of the system $\Psi(t)$ at the moment of time t.

As an example, let us consider evolution of the initial state (39), which is given by the state vector

$$\Psi(t) = \sum_{j=1}^{N}\beta_j(t)|b_1b_2\cdots a_j\cdots b_N\rangle|0\rangle + \sum_{\mathbf{k},\mu}\gamma_{\mathbf{k},\mu}(t)|b_1b_2\cdots b_N\rangle|1_{\mathbf{k},\mu}\rangle + \cdots. \tag{56}$$

If we disregard virtual processes then the ground state of the Hamiltonian (18) is

$$\Psi_G = |b_1b_2\cdots b_N\rangle|0\rangle \tag{57}$$

and therefore

$$\hat{\sigma}_j|\Psi_G\rangle = 0, \tag{58}$$

$$\hat{a}_{\mathbf{k},\mu}|\Psi_G\rangle = 0. \tag{59}$$

Then, for the state vector (56), Eq. (54) gives

$$S_j(t) = \beta_j(t), \tag{60}$$

which is the probability amplitude that atom j is excited at time t and there are no photons. On the other hand, Eq. (55) yields that

$$\mathbf{E}(t,\mathbf{r}) = i\sum_{\mathbf{k},\mu} g_k[\vec{\epsilon}_{\mathbf{k},\mu}\gamma_{\mathbf{k},\mu}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \text{c.c.}] \tag{61}$$

is expressed in terms of the probability amplitudes $\gamma_{\mathbf{k},\mu}(t)$ that a single photon with wave vector $\mathbf{k}$ and polarization μ is emitted at time t and all atoms are in the ground state. Thus, in the present treatment, the Maxwell-Bloch equations are written for the probability amplitudes.

Our approach can be extended for the case of weak excitation of atomic ensembles by several photons if we choose the matrix elements in Eqs. (41)–(43) differently. The selection criterion is that the matrix elements should not be equal to zero for the initial state of the atom-photon system and provide sufficient information about the system evolution. For a few-photon excitation the evolution equations will be the same as Eqs. (47)–(49), however, the initial conditions will now be different and depend on the choice of the matrix elements in Eqs. (41)–(43).

Reduction of the quantum mechanical evolution equations to the Maxwell-Bloch semiclassical form substantially simplifies the problem and allows us to write down analytical solutions for evolution of the entangled quantum mechanical systems. Maxwell-Bloch equations have been extensively studied in the literature. In next section we provide their analytical solutions in the slowly varying amplitude approximation. Such solutions are known for particular cases, but we will give them in a general form valid for arbitrary initial conditions.

IV. EXACT ANALYTICAL SOLUTIONS OF THE MAXWELL-BLOCH EQUATIONS FOR WEAK ATOMIC EXCITATION

We assume a continuous distribution of atoms with uniform density n and consider propagation of a pulse along the z axis. In the slowly varying amplitude approximation the functions entering the Maxwell-Bloch equations can be written as a product of $e^{-i\omega t+i\omega z/c}$ and slowly varying envelopes that obey the first-order differential equations

$$\left(c\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\Omega(t,z) = i\Omega_a^2\rho_{ab}(t,z), \tag{62}$$

$$\dot{\rho}_{aa} = -\gamma\rho_{aa} - i(\Omega^*\rho_{ab} - \text{c.c.}), \tag{63}$$

$$\dot{\rho}_{ab} = -\frac{\gamma}{2}\rho_{ab} + i\Omega(\rho_{bb} - \rho_{aa}), \tag{64}$$

$$\rho_{bb} + \rho_{aa} = 1, \tag{65}$$

where $\Omega = \vec{\wp}_{ab}\cdot\mathbf{E}/\hbar$ is the Rabi frequency, γ is the single-atom spontaneous decay rate,

$$\Omega_a = \sqrt{\frac{3n\lambda_{ab}^2\gamma c}{8\pi}} \tag{66}$$

is a collective atomic frequency proportional to the square root of atomic density n, and λ_{ab} is the wavelength of the a-b transition. In such an approximation the Laplacian operator of $\mathbf{P}$ in Eq. (47) is equivalent to the second-order time derivative of $\mathbf{P}$.

Assuming weak atomic excitation $\rho_{aa} \ll 1$ and omitting the single-atom decay rate, we obtain two coupled equations for $\Omega(t,z)$ and $\rho_{ab}(t,z)$,

$$\left(c\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\Omega = i\Omega_a^2\rho_{ab}, \tag{67}$$

$$\dot{\rho}_{ab} = i\Omega. \tag{68}$$

The physical meaning of $\Omega(t,z)$ and $\rho_{ab}(t,z)$ in Eqs. (67) and (68) depends on the particular problem we are solving. If the state of the system has nonzero polarization (dipole moment) then ρ_{ab} is atomic coherence and Ω is the Rabi frequency describing the slowly varying electric-field amplitude. On the other hand, if we are interested in evolution of entangled states with vanishing polarization then the quantities in Eqs. (67) and (68) are related to the probability amplitudes that are not equal to zero.

Here we solve Eqs. (67) and (68) with the initial condition $t=0$: $\Omega = \Omega(0,z)$ and $\rho_{ab} = \rho_{ab}(0,z)$ using the Laplace transform method. We obtain the following exact analytical solution in terms of the Bessel functions $J_0(x)$ and $J_1(x)$ (see Appendix B for details):

$$\rho_{ab}(t,z) = \rho_{ab}(0,z) + \frac{1}{c}\int_{z-ct}^{z} dz'\left[i\Omega(0,z')J_0\left(\frac{2\Omega_a}{c}\sqrt{(z-z')[z'-(z-ct)]}\right) - \Omega_a\rho_{ab}(0,z')\sqrt{\frac{z'-(z-ct)}{z-z'}}J_1\left(\frac{2\Omega_a}{c}\sqrt{(z-z')[z'-(z-ct)]}\right)\right], \tag{69}$$

$$\Omega(t,z) = \Omega(0,z-ct) + \frac{\Omega_a}{c}\int_{z-ct}^{z} dz'\left[i\Omega_a\rho_{ab}(0,z')\, J_0\left(\frac{2\Omega_a}{c}\sqrt{(z-z')[z'-(z-ct)]}\right) - \Omega(0,z')\sqrt{\frac{z-z'}{z'-(z-ct)}}J_1\left(\frac{2\Omega_a}{c}\sqrt{(z-z')[z'-(z-ct)]}\right)\right]. \tag{70}$$

The solution (70) with $\rho_{ab}(0,z) = 0$ describing superradiant forward scattering was investigated in Ref. [48], which studied implications of superradiance and slow light effects for quantum memories.

Equations (69) and (70) reduce to a simple answer for special initial conditions. For example, for uniform initial excitation of the atomic medium $\rho_{ab}(0,z) = \text{const}$ and $\Omega(0,z) = 0$ we find that the system undergoes oscillations with collective atomic frequency Ω_a,

$$\Omega(t,z) = i\Omega_a \rho_{ab}(0)\sin(\Omega_a t), \tag{71}$$

$$\rho_{ab}(t,z) = \rho_{ab}(0)\cos(\Omega_a t). \tag{72}$$

Such collective oscillations of the field envelope were first predicted by Burnham and Chiao for a sample of resonant medium [35]. They play an important role in the light amplification mechanism of the quantum amplification by superradiant emission of radiation device (QASER), which does not need the population of atoms in the excited state and generates high-frequency coherent light by means of superradiant collective resonance [37].

For the initial δ-function pulse $\Omega(0,z) = A\delta(z)$ and $\rho_{ab}(0,z) = 0$ we obtain

$$\Omega(t,z) = A\delta(z-ct) - \frac{A\Omega_a}{c}\sqrt{\frac{z}{ct-z}} \times J_1\left(\frac{2\Omega_a}{c}\sqrt{z(ct-z)}\right)\theta(ct-z), \tag{73}$$

$$\rho_{ab}(t,z) = i\frac{A}{c}J_0\left(\frac{2\Omega_a}{c}\sqrt{z(ct-z)}\right)\theta(ct-z). \tag{74}$$

The solution (73) appears in a problem of scattering of short synchrotron radiation pulses by a nuclear resonant medium [49–53], which is essential to Mössbauer spectroscopy. The Bessel function in Eq. (73) leads to the so-called dynamical beats, which are experimentally well established [54]. They can be understood as interference beating of the two wings of the spectrum of the incident white radiation that develops a hole near the resonant frequency during pulse propagation [55]. The solution (74) was also obtained in Ref. [56], which studied superradiant decay upon coherent excitation of helium atoms inside helium plasma by short laser pulses.

For the initial δ-function excitation of the medium $\rho_{ab}(0,z) = B\delta(z)$ and $\Omega(0,z) = 0$ Eqs. (69) and (70) yield

$$\Omega(t,z) = i\frac{B\Omega_a^2}{c}J_0\left(\frac{2\Omega_a}{c}\sqrt{z(ct-z)}\right)\theta(ct-z), \tag{75}$$

$$\rho_{ab}(t,z) = B\delta(z) - \frac{B\Omega_a}{c}\sqrt{\frac{ct-z}{z}} \times J_1\left(\frac{2\Omega_a}{c}\sqrt{z(ct-z)}\right)\theta(ct-z). \tag{76}$$

The solution (75) was obtained in Ref. [57], which studied propagation of small-area pulses of coherent light through a resonant medium. Such a solution describes propagation of the leading portion of a step-function pulse.

V. CONCLUSION

We found that semiclassical Maxwell-Bloch equations should be written in a form different from those commonly used. Namely, the right-hand side of the propagation equation must involve the ∇^2 operator rather than $\partial^2/\partial t^2$. That is, the classical limit of the quantum problem gives propagation equation (4) rather than Eq. (9). Physically this means that the quantum mechanical average of the electric-field operator $\hat{\mathbf{E}}(\mathbf{r})$ gives the displacement vector rather than the electric field in the medium. Such a difference becomes important for pulses that cannot be treated in the slowly varying amplitude approximation.

We studied the collective interaction of light with an ensemble of two-level atoms using the fully quantum mechanical description and analyzed the problem of single-photon superradiance. This is a long-standing problem that is usually treated in ways that are very complicated mathematically. We showed, however, that in the weak excitation limit one can reduce quantum mechanical evolution equations to a form identical to the semiclassical Maxwell-Bloch equations but with a different physical interpretation. We obtained analytical solutions of these equations in the slowly varying amplitude approximation for general initial conditions. Such solutions describe collective emission of atomic ensembles as well as propagation of light pulses interacting with the atomic medium.

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APPENDIX A: DERIVATION OF THE PROPAGATION EQUATION

Here we derive the propagation equation for $\mathbf{E}(t,\mathbf{r})$ starting from Eqs. (29) and (30). Changing the function

$$a_{\mathbf{k},\mu}(t) = \tilde{a}_{\mathbf{k},\mu}(t)e^{-i\nu_k t}, \tag{A1}$$

we obtain the system of equations

$$\frac{d\tilde{a}_{\mathbf{k},\mu}(t)}{dt} = \frac{g_k}{\hbar}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\sum_j[\sigma_j(t)+\sigma_j^*(t)]e^{i\nu_k t - i\mathbf{k}\cdot\mathbf{r}_j}, \tag{A2}$$

$$\mathbf{E}(t,\mathbf{r}) = i\sum_{\mathbf{k},\mu} g_k[\vec{\epsilon}_{\mathbf{k},\mu}\tilde{a}_{\mathbf{k},\mu}(t)e^{i\mathbf{k}\cdot\mathbf{r}-i\nu_k t} - \text{c.c.}]. \tag{A3}$$

Applying the operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ to both sides of Eq. (A3) yields

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = \frac{i}{c^2}\sum_{\mathbf{k},\mu} g_k\vec{\epsilon}_{\mathbf{k},\mu}\left(\frac{d^2\tilde{a}_{\mathbf{k},\mu}}{dt^2}e^{i\mathbf{k}\cdot\mathbf{r}-i\nu_k t} - 2i\nu_k\frac{d\tilde{a}_{\mathbf{k},\mu}}{dt}e^{i\mathbf{k}\cdot\mathbf{r}-i\nu_k t} - \text{c.c.}\right). \tag{A4}$$

Substituting the time derivative of $\tilde{a}_{\mathbf{k},\mu}$ from Eq. (A2) we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = \frac{1}{\hbar c^2}\sum_j\sum_{\mathbf{k},\mu} g_k^2(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}\left[\left(i\frac{d}{dt}(\sigma_j+\sigma_j^*) + \nu_k(\sigma_j+\sigma_j^*)\right)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)} + \text{c.c.}\right] \tag{A5}$$

or

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = \frac{2}{\hbar c^2}\sum_j\sum_{\mathbf{k},\mu} g_k^2(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}\left[-\frac{d}{dt}(\sigma_j+\sigma_j^*)\sin[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] + \nu_k(\sigma_j+\sigma_j^*)\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)]\right]. \tag{A6}$$

Since $\sin[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)]$ is an odd function of $\mathbf{k}$ and $g_{\mathbf{k},\mu}^2$ is an even function then

$$\sum_{\mathbf{k},\mu} g_k^2(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}\sin[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] = 0 \tag{A7}$$

and the previous equation reduces to

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = \frac{2}{\hbar c^2}\sum_j(\sigma_j+\sigma_j^*)\sum_{\mathbf{k},\mu}\nu_k g_k^2(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)]. \tag{A8}$$

Taking into account the expression for the atom-field coupling constant (20) we find

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = \frac{1}{V\varepsilon_0}\sum_j(\sigma_j+\sigma_j^*)\sum_{\mathbf{k},\mu}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}k^2\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)]. \tag{A9}$$

Summation over two polarizations yields

$$\sum_{\mu}(\vec{\wp}_{ab}\cdot\vec{\epsilon}_{\mathbf{k},\mu})\vec{\epsilon}_{\mathbf{k},\mu}k^2 = \vec{\wp}_{ab}k^2 - \mathbf{k}(\vec{\wp}_{ab}\cdot\mathbf{k}). \tag{A10}$$

Taking into account that

$$[\vec{\wp}_{ab}k^2 - \mathbf{k}(\vec{\wp}_{ab}\cdot\mathbf{k})]\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] = -\vec{\wp}_{ab}\nabla^2\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] + \nabla(\text{div}\{\vec{\wp}_{ab}\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)]\}), \tag{A11}$$

replacing the sum over $\mathbf{k}$ by an integral, and using

$$\sum_{\mathbf{k}}\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] = \frac{V}{(2\pi)^3}\int d\mathbf{k}\cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_j)] = V\delta(\mathbf{r}-\mathbf{r}_j), \tag{A12}$$

we finally obtain

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(t,\mathbf{r}) = -\vec{\wp}_{ab}\nabla^2\sum_j(\sigma_j+\sigma_j^*)\delta(\mathbf{r}-\mathbf{r}_j) + \sum_j(\sigma_j+\sigma_j^*)\nabla\{\text{div}[\vec{\wp}_{ab}\delta(\mathbf{r}-\mathbf{r}_j)]\}. \tag{A13}$$

Equation (A13) can be written as

$$\varepsilon_0\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\mathbf{E}(t,\mathbf{r})=-\nabla^2\mathbf{P}+\nabla(\mathrm{div}\mathbf{P}),\tag{A14}$$

where

$$\mathbf{P}=\vec{\wp}_{ab}\sum_j(\sigma_j+\sigma_j^*)\delta(\mathbf{r}-\mathbf{r}_j)\tag{A15}$$

is the medium polarization.

APPENDIX B: SOLUTION OF EVOLUTION EQUATIONS

Here we solve the evolution equations

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Omega=i\eta\rho_{ab},\tag{B1}$$

$$\dot{\rho}_{ab}=-\gamma_{\mathrm{coh}}\rho_{ab}+i\Omega,\tag{B2}$$

where

$$\eta=\frac{\Omega_a^2}{c}.\tag{B3}$$

To obtain a more general result we will keep the decoherence term in Eq. (B2), however, for simplicity of derivations we assume that initially there is no atomic coherence, that is, $\rho_{ab}(0,z)=0$, while there is an electric-field pulse in the medium $\Omega(0,z)$. Then Eq. (B2) yields

$$\rho_{ab}(t,z)=ie^{-\gamma_{\mathrm{coh}}t}\int_0^t\Omega(t',z)e^{\gamma_{\mathrm{coh}}t'}dt'.\tag{B4}$$

Substituting Eq. (B4) into Eq. (B1) gives

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Omega(t,z)+\eta e^{-\gamma_{\mathrm{coh}}t}\int_0^t\Omega(t',z)e^{\gamma_{\mathrm{coh}}t'}dt'=0.\tag{B5}$$

Introducing the function

$$\Omega(t,z)=F(t,z)e^{-\gamma_{\mathrm{coh}}t},\tag{B6}$$

we obtain

$$\frac{\partial}{\partial z}F(t,z)+\frac{1}{c}\frac{\partial}{\partial t}F(t,z)-\frac{\gamma_{\mathrm{coh}}}{c}F(t,z)+\eta\int_0^tF(t',z)dt'=0.\tag{B7}$$

Making the Laplace transform $\hat{L}$ of this equation over time we have

$$\frac{\partial}{\partial z}F(s,z)+\frac{s}{c}F(s,z)-\frac{F(0,z)}{c}-\frac{\gamma_{\mathrm{coh}}}{c}F(s,z)+\eta\frac{F(s,z)}{s}=0,\tag{B8}$$

where $F(0,z)=\Omega(0,z)$ is the pulse at $t=0$ and $F(s,z)=\hat{L}[F(t,z)]$. For an infinitely long medium the solution of Eq. (B8) is

$$F(s,z)=\frac{1}{c}\int_{-\infty}^{z}\exp\left[\left(\frac{s}{c}+\frac{\eta}{s}-\frac{\gamma_{\mathrm{coh}}}{c}\right)(z'-z)\right]F(0,z')dz'.\tag{B9}$$

Taking the inverse Laplace transform and using

$$\hat{L}^{-1}\left[\exp\left(-\alpha s-\frac{\beta}{s}\right)\right]=-\sqrt{\frac{\beta}{t-\alpha}}J_1[2\sqrt{\beta(t-\alpha)}]\theta(t-\alpha)+\delta(t-\alpha),\tag{B10}$$

where $J_1(x)$ is the Bessel function, we find

$$F(t,z)=\int_{z-ct}^{z}\left[\delta[z'-(z-ct)]-\sqrt{\frac{\eta(z-z')/c}{z'-(z-ct)}}J_1\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-(z-ct)]}\right)\right]\exp\left[-\frac{\gamma_{\mathrm{coh}}}{c}(z'-z)\right]F(0,z')dz'.\tag{B11}$$

Integration of the term with the δ function finally yields

$$\Omega(t,z)=\Omega(0,z-ct)-\int_{z-ct}^{z}\sqrt{\frac{\eta(z-z')/c}{z'-(z-ct)}}J_1\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-(z-ct)]}\right)\exp\left(-\frac{\gamma_{\mathrm{coh}}}{c}[z'-(z-ct)]\right)\Omega(0,z')dz'.\tag{B12}$$

The first term in this equation is the initial pulse propagating with the speed of light c through the sample. The second term is the response of the atomic medium.

To calculate the coherence ρ_{ab} one can use Eq. (B1), which gives

$$\rho_{ab}=\frac{1}{i\eta}\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Omega.\tag{B13}$$

Instead of t and z it is convenient to introduce the variables $\xi=z-ct$ and z. Using these variables, Eqs. (B12) and (B13) read

$$\Omega(\xi,z)=\Omega(0,\xi)-\int_{\xi}^{z}\sqrt{\frac{\eta(z-z')/c}{z'-\xi}}J_1\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-\xi]}\right)\exp\left(-\frac{\gamma_{\mathrm{coh}}}{c}[z'-\xi]\right)\Omega(0,z')dz',\tag{B14}$$

$$\rho_{ab}=\frac{1}{i\eta}\frac{\partial\Omega}{\partial z}.\tag{B15}$$

Using

$$\frac{\partial}{\partial z}\left[\sqrt{\frac{\eta}{c}(z-z')}J_1\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-\xi]}\right)\right]=\frac{\eta}{c}\sqrt{z'-\xi}\,J_0\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-\xi]}\right), \tag{B16}$$

we obtain the expression for evolution of the atomic coherence

$$\rho_{ab}(t,z)=\frac{i}{c}\int_{z-ct}^{z}J_0\left(2\sqrt{\frac{\eta}{c}(z-z')[z'-(z-ct)]}\right)\exp\left(-\frac{\gamma_{\text{coh}}}{c}[z'-(z-ct)]\right)\Omega(0,z')dz', \tag{B17}$$

where $\Omega(0,z)$ is the pulse at $t=0$.

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