Hierarchy of bounds on accessible information and informational power

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Quantum theory imposes fundamental limitations on the amount of information that can be carried by any quantum system. On the one hand, the Holevo bound rules out the possibility of encoding more information in a quantum system than in its classical counterpart, comprised of perfectly distinguishable states. On the other hand, when states are uniformly distributed in the state space, the so-called subentropy lower bound is saturated. How uniform quantum systems are can be naturally quantified by characterizing them as t-designs, with $t = \infty$ corresponding to the uniform distribution. Here we show the existence of a trade-off between the uniformity of a quantum system and the amount of information it can carry. To this aim, we derive a hierarchy of informational bounds as a function of t and prove their tightness for qubits and qutrits. By deriving asymptotic formulas for large dimensions, we also show that the statistics generated by any t-design with $t > 1$ contains no more than a single bit of information, and this amount decreases with t. Holevo and subentropy bounds are recovered as particular cases for $t = 1$ and $t = \infty$, respectively.

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I. INTRODUCTION

Quantum theory imposes fundamental limitations on the amount of information that can be encoded into or extracted from any quantum system. Formally, the former case is referred to as the problem of the accessible information [1–6] of a quantum ensemble; the latter, as the problem of the informational power [7–15] of a quantum measurement. Recently, a duality relation between these two quantities was established [7], which allows us to generally refer to the problem of quantifying the information carried by a quantum system.

On the one hand, the well-known Holevo upper bound [2] rules out the possibility of encoding more information in a quantum system than in its purely classical counterpart, comprised of perfectly distinguishable states. On the other hand, for a genuinely quantum system whose states are uniformly distributed in the state space, the so-called subentropy lower bound [6] is saturated. Therefore, one might conjecture the existence of a general trade-off between the uniformity of a quantum system and the amount of information it can carry.

A natural means to quantify how uniform quantum systems are is provided by their characterization in terms of spherical quantum t-designs [16–21], with $t = 1$ corresponding to an arbitrary quantum measurement and $t = \infty$ corresponding to the completely uniform distribution. Other remarkable examples of t-designs for the case $t = 2$ are symmetric, informationally complete (SIC) quantum measurements [16,20] and complete sets of mutually unbiased bases (MUBs) [18,19,21]. They play a fundamental role in a plethora of applications such as quantum tomography [22], cryptography [23], information locking [24], quantumness of Hilbert space [25,26], entropic uncertainty relations [27–32], and foundations of quantum theory [33–36].

In this work, for any t we derive an upper bound on the information that can be carried by any quantum t-design as a function of the dimension of the system. In this sense, the resulting hierarchy of bounds proves the correctness of the above-mentioned conjecture and formally quantifies it. Furthermore, we show the tightness of our bounds for qubits and qutrits. By deriving asymptotic formulas for large dimensions, we also show that the statistics generated by any t-design with $t > 1$ contains no more than a single bit of information and that this amount decreases with t. The Holevo upper bound [2] and the subentropy lower bound [6] are recovered as particular cases for $t = 1$ and $t = \infty$, respectively.

The paper is structured as follows. First, we introduce quantum t-designs in Sec. II A, discuss the relevant figures of merit in Sec. II B, and provide a way to estimate them in Sec. II C. Then we introduce our main result, namely, a hierarchy of upper bounds on the accessible information and the informational power of t-designs in Sec. III A, we derive close analytic expressions for low values of t and asymptotic formulas for large dimensions in Sec. III B, and we prove tightness for qubits and qutrits in Sec. III C. We conclude by summarizing our results and presenting some outlooks in Sec. IV.

II. FORMALIZATION

A. Spherical quantum t-designs

In this subsection we recall some basic facts [37] from quantum information theory and specialize them to the case of spherical quantum t-design.

Any quantum system is associated with a Hilbert space $\mathcal{H}$, and we denote by $L(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. We only consider finite-dimensional Hilbert spaces.

A *quantum state* ρ is represented by a positive-semidefinite operator in $L(\mathcal{H})$ such that $\mathrm{Tr}[\rho] \leqslant 1$. A *pure state* ψ is such that rank $\psi = 1$ and is denoted in Dirac notation by a vector $|\psi\rangle$ with $\psi = |\psi\rangle\langle\psi|$. Any quantum preparation is represented by an *ensemble* ρ_x, namely, a measurable function ρ_x from reals to states such that $\int_x \mathrm{Tr}[\rho_x] dx = 1$. An *ensemble of pure states* is such that $\rho_x \neq 0$ if and only if ρ_x is a pure state. The *uniform ensemble* is the ensemble of pure states distributed with the uniform (Haar) measure on the unit sphere of $\mathcal{H}$.

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A *quantum effect* π is represented by a positive-semidefinite operator in $L(\mathcal{H})$ such that $\pi \leqslant \mathbb{1}_d$, where $\mathbb{1}_d$ is the d-dimensional identity operator. Any quantum measurement is represented by a positive operator-valued measure (POVM) π_y, namely, a measurable function π_y from reals to effects such that $\int_y \pi_y dy = \mathbb{1}_d$.

For any ensemble ρ_x and POVM π_y, the joint probability density $p_{x,y}$ of input x and outcome y is given by the Born rule, i.e., $p_{x,y} = \text{Tr}[\rho_x \pi_y]$.

Definition I. Spherical quantum t-design. A spherical quantum t-design is an ensemble ρ_x such that

$$\int \frac{\rho_x^{\otimes k}}{\text{Tr}[\rho_x]^{k-1}} dx := \int \frac{\psi_x^{\otimes k}}{||\psi_x||^{2(k-1)}} dx$$

holds for any $k \leqslant t$, where ψ_x is the uniform ensemble.

Lemma I. Let ψ_x be the uniform ensemble. Then one has

$$\int \frac{\psi_x^{\otimes k}}{||\psi_x||^{2(k-1)}} dx = \binom{d-1+k}{k}^{-1} P_{\text{sym}},$$

where P_{sym} is the projector over the symmetric subspace of $\mathcal{H}^{\otimes k}$ and d is the dimension of $\mathcal{H}$.

Proof. See Ref. [18].

Remark I. From Definition I and Lemma I it immediately follows that any POVM is a 1-design up to a normalization factor of d.

Remarkable examples of 2-designs are SIC POVMs [16] and $d+1$ mutually unbiased bases [20].

A concept that is relevant in the following is the so-called index of coincidence [32].

Definition II. Index of coincidence. For any POVM π_y and any unit-trace state ρ, the index of coincidence $C_k(\pi_y,\rho)$ is given by

$$C_k(\pi_y,\rho) := \int \frac{\text{Tr}[\pi_y \rho]^k}{\text{Tr}[\pi_y]^{k-1}} dy.$$

The following result characterizes the index of coincidence of t-designs.

Lemma II. Let $\mathcal{H}$ be a d-dimensional Hilbert space. Let $\pi_y \in \mathcal{L}(\mathcal{H})$ be a t-design POVM. Let $|\psi\rangle \in \mathcal{H}$ be a unit-trace pure state. For any $k \leqslant t$, the index of coincidence $C_k(\pi_y,\psi)$ is independent of π_y and ψ and is given by

$$C_k = d\binom{d-1+k}{k}^{-1}.$$

Proof. By Definition II one has

$$C_k(\pi_y,\psi) = \int \text{Tr}\left[\psi^{\otimes k} \frac{\pi_y^{\otimes k}}{\text{Tr}[\pi_y]^{k-1}}\right] dy.$$

By Lemma I one has

$$C_k(\pi_y,\psi) = d\binom{d-1+k}{k}^{-1} \text{Tr}[\psi^{\otimes k} P_{\text{sym}}].$$

Since $\psi^{\otimes k}$ belongs to the symmetric subspace, the statement immediately follows.

B. Informational measures

In this subsection we recall some basic definitions [38] from classical information theory. Given two probability densities p_x and q_y, the relative entropy $D(p_x||q_y)$, given by

$$D(p_x||q_y) := \int p_x \ln \frac{p_x}{q_x} dx,$$

is a nonsymmetric measure of the distance between the two densities. Given two random variables X and Y distributed according to probability density $p_{x,y}$, the mutual information $I(X;Y)$, given by

$$I(X;Y) := D(p_{x,y}||p_x p_y),$$

is a measure of their correlation. For any ensemble ρ_x and POVM π_y, we denote by $I(\rho_x,\pi_y)$ the mutual information $I(X;Y)$ between random variables X and Y distributed according to $p_{x,y} = \text{Tr}[\rho_x \pi_y]$.

The accessible information [1–4] of an ensemble measures how much information can be extracted from the ensemble.

Definition III. Accessible information. The accessible information $A(\rho_x)$ of an ensemble ρ_x is the supremum over any POVM π_y of the mutual information $I(\rho_x,\pi_y)$, namely,

$$A(\rho_x) := \sup_{\pi_y} I(\rho_x,\pi_y).$$

The informational power [7] of a POVM measures how much information can be extracted by the POVM.

Definition IV. Informational power. The informational power $W(\pi_y)$ of a POVM π_y is the supremum over any ensemble ρ_x of the mutual information $I(\rho_x,\pi_y)$, namely,

$$W(\pi_y) := \sup_{\rho_x} I(\rho_x,\pi_y).$$

The following lemma allows us to recast the upper bounds on the informational power into bounds for the accessible information. Therefore in the following, without loss of generality, we focus on the former problem.

Theorem I. For any POVM π_y, the informational power $W(\pi_y)$ is given by

$$W(\pi_y) = \sup_{\rho} A(\rho^{1/2}\pi_y\rho^{1/2}).$$

Proof. See Refs. [7] and [12].

In the following it is convenient to introduce the shorthand notation $\eta(x) := -x \ln x$.

Theorem II. For any d-dimensional POVM π_y, one has

$$W(\pi_y) \leqslant \ln d - \inf_{\psi_x} \iint ||\psi_x||^2 \,\text{Tr}[\pi_y] \times \eta\left(\frac{\langle\psi_x|\pi_y|\psi_x\rangle}{||\psi_x||^2\,\text{Tr}[\pi_y]}\right) dx dy,$$

where the infimum is over any ensemble ψ_x of pure states.

Proof. A Davies-like theorem applies [7], so it is sufficient to maximize over ensembles ψ_x of pure states. For any such ensemble one has

$$\begin{aligned} I(\psi_x,\pi_y) =& D(\langle\psi_x|\pi_y|\psi_x\rangle||\,\mathrm{Tr}[\rho\pi_y]||\psi_x||^2) \\ =& \iint \langle\psi_x|\pi_y|\psi_x\rangle \ln\frac{\langle\psi_x|\pi_y|\psi_x\rangle}{||\psi_x||^2\,\mathrm{Tr}[\rho\pi_y]}dxdy = \iint \langle\psi_x|\pi_y|\psi_x\rangle\left[\ln\frac{d\langle\psi_x|\pi_y|\psi_x\rangle}{||\psi_x||^2\,\mathrm{Tr}[\pi_y]} - \ln\frac{d\,\mathrm{Tr}[\rho\pi_y]}{\mathrm{Tr}[\pi_y]}\right]dxdy \\ =& D\left(\langle\psi_x|\pi_y|\psi_x\rangle\middle|\middle|\frac{||\psi_x||^2\,\mathrm{Tr}[\pi_y]}{d}\right) - D\left(\mathrm{Tr}[\rho\pi_y]\middle|\middle|\frac{\mathrm{Tr}[\pi_y]}{d}\right) \leqslant D\left(\langle\psi_x|\pi_y|\psi_x\rangle\middle|\middle|\frac{||\psi_x||^2\,\mathrm{Tr}[\pi_y]}{d}\right), \end{aligned}$$

where the final inequality holds due to the positivity of the relative entropy. Then the statement follows by direct inspection.

C. Polynomial interpolation

In this subsection we introduce an optimization technique based on Hermite polynomial interpolation. The following lemma bounds the error made in interpolating a function with a polynomial.

Lemma III. Let a and b be reals and t be a positive integer. Let $f(x)$ be a real function with continuous derivatives up to order $t+1$ on $[a,b]$. Let $\{x_i\}_{i=1}^m$ be reals such that $a \leqslant x_i \leqslant b$ and $x_i < x_{i'}$ for any $i < i'$. Let $\{j_i\}_{i=1}^m$ be positive integers such that $\sum_i j_i = t-1$. Let $p(x)$ be the polynomial of degree t that agrees with $f(x)$ at x_i up to derivative of order $j_i - 1$ for $1 \leqslant i \leqslant m$, namely,

$$p^{(j_i)}(x_i) = f^{(j_i)}(x_i), \quad 0 \leqslant i \leqslant m.$$

For any $x \in [a,b]$ there exists x' such that $\min(x,x_1) < x' < \max(x,x_m)$ and

$$f(x) - p(x) = \frac{f^{(t+1)}(x')}{(t+1)!}\prod_{i=1}^m (x-x_i)^{k_i}.$$

Proof. See Ref. [39].

The following lemma, derived in Ref. [11], provides a polynomial lower bound to a function. We reproduce its proof here for completeness.

Lemma IV. Let a and b be reals and t be a positive integer. Let $f(x)$ be a real function with continuous derivatives up to order $t+1$ on $[a,b]$ such that $f^{(j)}(x) < 0$ for even j and $f^{(j)}(x) > 0$ for odd j, for any $j > 1$ and $x \in [a,b]$. Let $\{x_i\}_{i=1}^{\lfloor t/2\rfloor}$ be reals such that $a < x_i < b$ and $x_i < x_{i'}$ for any $i < i'$. The polynomial $p(x)$ of degree t such that $p(a) = f(a)$ and $p(b) = f(b)$ if t is odd, and

$$p^{(j)}(x_i) = f^{(j)}(x_i), \quad \forall 1 \leqslant i \leqslant \lfloor t/2\rfloor, \quad j = 0,1,$$

is such that $p(x) \leqslant f(x)$ for $x \in [a,b]$.

Proof. See Ref. [11]. Let us distinguish two cases. If t is odd, then

$$(x-a)(x-b)\prod_{i=1}^{\lfloor t/2\rfloor}(x-x_i)^2 \leqslant 0$$

for $x \in [a,b]$. If t is even, then

$$(x-a)\prod_{i=1}^{\lfloor t/2\rfloor}(x-x_i)^2 \geqslant 0$$

for $x \in [a,b]$. Then the statement immediately follows from Lemma III.

III. INFORMATIONAL BOUNDS

A. Main result

The informational power problem is formally the optimization of an entropic function over complex vectors under a normalization constraint, therefore it is unfeasible in the majority of cases. However, in this subsection we recast the informational power problem for t-design POVMs into an unconstrained optimization over $\lfloor t/2\rfloor$ real variables $\{x_i\}$.

Theorem III. The informational power $W(\pi_y)$ of any d-dimensional t-design POVM π_y satisfies

$$W(\pi_y) \leqslant \ln d - d\sum_{k=1}^t a_k \binom{d+k-1}{k}^{-1},$$

where a_k are the coefficients of the polynomial $p(x) := \sum_{k=1}^t a_k x^k$ such that $p(1) = 0$ if t is odd, and

$$p^{(j)}(x_i) = \eta^{(j)}(x_i), \quad \forall 1 \leqslant i \leqslant \lfloor t/2\rfloor, \quad j = 0,1,$$

for some choice of $\{x_i\}_{i=1}^{\lfloor t/2\rfloor}$ such that $0 < x_i < 1$ and $x_i < x_{i'}$ for any $i < i'$.

Proof. By direct inspection one has

$$\eta^{(j)}(x) = (-)^{j-1}(j-2)!x^{-j+1}, \quad \forall j \geqslant 2,$$

then $\eta^{(j)}(x) < 0$ for even j and $\eta^{(j)}(x) > 0$ for odd j, for any $j > 1$ and $x \in [0,1]$. Then by Lemma IV one has that $p(x) \leqslant \eta(x)$ for $x \in [0,1]$, and by Theorem II one has that

$$W(\pi_y) \leqslant \ln d - \inf_{\psi_x}\sum_{k=1}^t a_k \iint \frac{|\langle\psi_x|\pi_y|\psi_x\rangle|^k}{(||\psi_x||^2\,\mathrm{Tr}[\pi_y])^{k-1}}dxdy.$$

By Definition II and Lemma II one has

$$W(\pi_y) \leqslant \ln d - d\sum_{k=1}^t a_k\binom{d+k-1}{k}^{-1}\inf_{\psi_x}\int ||\psi_x||^2 dx,$$

so the statement immediately follows.

Remark II. Since the a_k depend on the choice of $\{x_i\}$, the tightest bound provided by Theorem III is

$$W(\pi_y) \leqslant \ln d - d\sup_{\{x_i\}}\sum_{k=1}^t a_k\binom{d+k-1}{n}^{-1}. \qquad (1)$$

B. Applications

In this subsection we solve the optimization problem in Eq. (1) to derive upper bounds on the informational power of

t-designs as a function of the dimension d, for $t \in [1,5]$ and $t = \infty$, and asymptotic formulas for $d \to \infty$. The case $t = 1$ coincides with the well-known Holevo [2] bound; the case $t = 2$ was derived in Ref. [14]; the case $t = \infty$ coincides with the well-known subentropy bound [6].

Corollary I. Informational power of 1*-designs.* For any 1-design POVM π_y, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leqslant W_1(d)$, with $W_1(d) := \ln d$.

Proof. There is actually no optimization in this case since $\lfloor t/2 \rfloor = 0$ so the set $\{x_i\}_{i=1}^{\lfloor t/2 \rfloor}$ is empty. The statement follows by direct inspection.

Corollary II. Informational power of 2*-designs.* For any 2-design POVM π_y, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leqslant W_2(d)$, with

$$W_2(d) := \ln \frac{2d}{d+1}.$$

Proof. The supremum in Eq. (1) is achieved by $x_1 = 2/(d+1)$. Then the statement follows by direct inspection.

Remark III. The limit for $d \to \infty$ of the upper bound in Corollary II is given by

$$W_2(d) \to \ln 2 = 1 \text{ bit} \simeq 0.693 \text{ nat}.$$

Corollary III. Informational power of 3-designs. For any 3-design POVM π_y, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leqslant W_3(d)$, with

$$W_3(d) := \ln \frac{2d}{d+2} + 2\frac{\ln \frac{d+2}{2}}{d(d+1)}.$$

Proof. The supremum in Eq. (1) is achieved by $x_1 = 2/(d+2)$. Then the statement follows by direct inspection.

Remark IV. The limit for $d \to \infty$ of the upper bound in Corollary III is given by

$$W_3(d) \to \ln 2 = 1 \text{ bit} \simeq 0.693 \text{ nat}.$$

Corollary IV. Informational power of 4*-designs.* For any 4-design POVM π_y, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leqslant W_4(d)$, with

$$W_4(d) := \frac{1}{2} \ln \frac{6d^2}{(d+2)(d+3)} + \frac{(d-3)\sqrt{3d(d+2)}}{6d(d+1)} \times \ln \frac{2d+3-\sqrt{3d(d+2)}}{d+3}.$$

Proof. The supremum in Eq. (1) is achieved by

$$x_{1,2} = \frac{3d+6 \pm \sqrt{3d(d+2)}}{d^2+5d+6}.$$

Then the statement follows by direct inspection.

Remark V. The limit for $d \to \infty$ of the upper bound in Corollary IV is given by

$$W_4(d) \to \frac{\ln 6}{2} + \frac{\ln(2-\sqrt{3})}{2\sqrt{3}} \simeq 0.744 \text{ bit} \simeq 0.516 \text{ nat}.$$

Corollary V. Informational power of 5*-designs.* For any 5-design POVM π_y, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leqslant W_5(d)$, with

$$W_5(d) := \ln d + \frac{(d-1)(d+3)(d^2+2d+4)}{2d(d+1)^2(d+2)} \times \ln \frac{6}{(d+3)(d+4)} + \frac{\sqrt{d+3}(d-1)(d^2-2d-12)}{2\sqrt{3}d(d+1)^{\frac{3}{2}}(d+2)} \times \ln \frac{2d+5-\sqrt{3(d+1)(d+3)}}{d+4}.$$

Proof. The supremum in Eq. (1) is achieved by

$$x_{1,2} = \frac{3d+9 \pm \sqrt{3(d^2+4d+3)}}{d^2+7d+12}.$$

Then the statement follows by direct inspection.

Remark VI. The limit for $d \to \infty$ of the upper bound in Corollary V is given by

$$W_5(d) \to \frac{\ln 6}{2} + \frac{\ln(2-\sqrt{3})}{2\sqrt{3}} \simeq 0.744 \text{ bit} \simeq 0.516 \text{ nat}$$

Corollary VI. For the continuous d-dimensional ∞-design POVM π_y the informational power $W(\pi_y)$ is given by

$$W(\pi_y) = W_\infty(d) := \ln d - \sum_{n=2}^{d} n^{-1}.$$

Proof. By expanding $\eta(x)$ in Taylor series around 1 and applying the binomial theorem one has

$$\eta(x) = 1 - x - \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(n-2)!}{k!(n-k)!}(-x)^k.$$

Then by Theorem II and Lemma II one has

$$W(\pi_y) \leqslant \ln d - d + 1 + d \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(n-2)!(-)^k}{k!(n-k)!} \binom{d-1+k}{k}^{-1}.$$

Then by direct inspection (see, e.g., Ref. [40]) one has

$$W(\pi_y) \leqslant \ln d - \sum_{n=2}^{d} n^{-1}.$$

Since this bound is saturated by any orthonormal ensemble [6], the statement follows.

Remark VII. The limit for $d \to \infty$ of $W_\infty(d)$ in Corollary VI is given by

$$W_\infty(d) \to 1 - \gamma \simeq 0.610 \text{ bit} \simeq 0.423 \text{ nat},$$

where γ represents the Euler-Mascheroni constant.

We conclude this subsection by summarizing the derived bounds in Table I and illustrating them in Fig. 1.

C. Tightness

The bound in Theorem III is of course tight for $t = 1$ for any dimension d, where optimal ensembles are given by any orthonormal basis [2]. In this subsection we prove tightness for

TABLE I. Upper bounds $W_t(d)$ on the informational power $W(\pi_y)$ of any d-dimensional t-design POVM π_y for $t \in [1,5]$ and $t = \infty$, along with their asymptotic formulas.

t	$W_t(d)$	$\lim_{d\to\infty}$
1	$\ln d$	∞
2	$\ln \frac{2d}{d+1}$	$\ln 2$
3	$\ln \frac{2d}{d+2} + 2\frac{\ln \frac{d+2}{2}}{d(d+1)}$	$\ln 2$
4	$\frac{1}{2}\ln \frac{6d^2}{(d+2)(d+3)} + \frac{(d-3)\sqrt{3d(d+2)}}{6d(d+1)} \ln \frac{2d+3-\sqrt{3d(d+2)}}{d+3}$	$\frac{\ln 6}{2} + \frac{\ln(2-\sqrt{3})}{2\sqrt{3}}$
5	$\ln(d) + \frac{(d-1)(d+3)(d^2+2d+4)}{2d(d+1)^2(d+2)} \ln \frac{6}{(d+3)(d+4)} + \frac{\sqrt{d+3}(d-1)(d^2-2d-12)}{2\sqrt{3}d(d+1)^{\frac{3}{2}}(d+2)} \ln \frac{2d+5-\sqrt{3(d+1)(d+3)}}{d+4}$	$\frac{\ln 6}{2} + \frac{\ln(2-\sqrt{3})}{2\sqrt{3}}$
∞	$\ln d - \sum_{n=2}^{d} n^{-1}$	$1-\gamma$

2,3,5-designs in dimension 2 and for 2-designs in dimension 3. For $d = 2$ the Bloch-sphere representation provides a natural isomorphism between two-dimensional POVMs and solids in $\mathbb{R}^3$, so we denote POVMs by the name of the corresponding solid (tetrahedron, octahedron, icosahedron). Formal definitions of each POVM are given in [11] and [12].

The informational powers of the two-dimensional tetrahedral, octahedral, and icosahedral POVMs were derived in Refs. [7,11–14]. By noting that these POVMs are 2-, 3- and 5-designs, respectively, their informational power directly follows from Theorem III.

Corollary VII. The 2-dimensional tetrahedral (SIC) POVM π_y is a 2-design, its informational power is given by

$$W_2(2) = \ln \tfrac{4}{3},$$

and the optimal (antitetrahedral) ensemble ψ_x is such that $\psi_x \pi_x = 0$ for any x.

Proof. Any SIC POVM is a 2-design [17], and the antitetrahedral ensemble saturates the bound in Corollary II.

Corollary VIII. The two-dimensional octahedral (complete mutually unbiased bases) POVM is a 3-design, its informational power is given by

$$W_3(2) = \tfrac{1}{6}\ln 4,$$

and the optimal (antioctahedral) ensemble ψ_x is such that $\psi_x \pi_x = 0$ for any x.

Proof. It follows by direct inspection that the two-dimensional octahedral POVM is a 3-design, and the anti-octahedral ensemble saturates the bound in Corollary III.

Corollary IX. The two-dimensional icosahedral POVM is a 5-design, its informational power is given by

$$W_5(2) = \ln 2 - \frac{5}{12}\ln 5 - \frac{\sqrt{5}}{12}\ln \frac{9-3\sqrt{5}}{6}$$

and the optimal (anti-icosahedral) ensemble ψ_x is such that $\psi_x \pi_x = 0$ for any x.

Proof. It follows by direct inspection that the two-dimensional icosahedral POVM is a 5-design, and the anti-icosahedral ensemble saturates the bound in Corollary V.

In Ref. [13] it was shown that the informational power of group covariant three-dimensional SIC POVMs is given by $W_2(3) = \ln \frac{3}{2}$. Noting that these POVMs are 2-designs, the optimality of this value immediately follows from Corollary II.

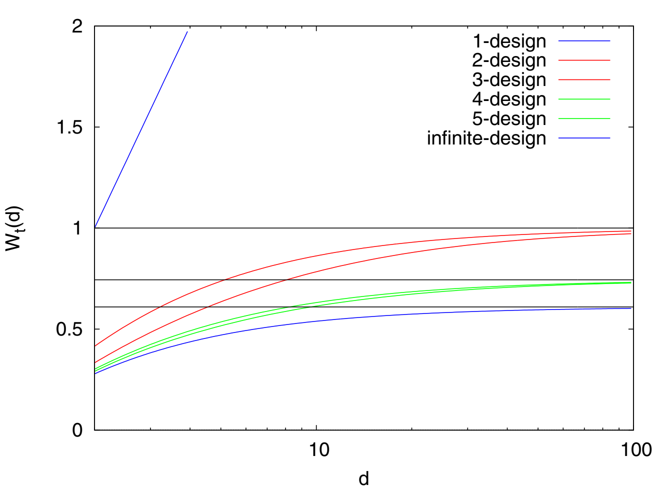

FIG. 1. (Color online) Upper bounds $W_t(d)$ on the informational power $W(\pi_y)$ (in bits) of any quantum t-design POVM π_y as a function of the dimension d (on a log scale). From top to bottom: $t = 1$ (blue line; see key), $t = 2$ and 3 (red lines), $t = 4$ and 5 (green lines), and $t = \infty$ (blue line), as provided by Corollaries I, II, III, IV, V, and VI, respectively. The asymptotes $W_{2,3}(d) \to 1$bit, $W_{4,5}(d) \to 0.744$bit, and $W_\infty(d) \to 0.609$bit are depicted too (horizontal black lines).

IV. CONCLUSION AND OUTLOOK

In this work we have provided in Theorem III an upper bound on the information that can be carried by any quantum t-design for any t, as a function of the dimension of the system, and we have derived in Corollaries I, II, III, IV, V and VI closed analytic expressions for this bound for $t \in [1,5]$ and $t = \infty$. The Holevo upper bound [2] and the subentropy lower bound [6] have been recovered as particular cases for $t = 1$ and $t = \infty$, respectively. In this sense, the resulting hierarchy of bounds represents a trade-off between the uniformity of a quantum system and the amount of information it can carry. By deriving asymptotic formulas for large dimensions, we have also shown that the statistics generated by any t-design contains no more than a single bit of information, and that this amount decreases with t. Furthermore, in Corollaries VII, VIII, and IX we have shown the tightness of our bounds for qubits and qutrits. Finally, as a direct consequence of Theorem I it immediately follows that all the presented upper bounds on the informational power of t-design POVMs holds as upper bounds on the accessible information of t-design ensembles.

Various open problems related to the accessible information and informational power of quantum t-designs were discussed in Refs. [12] and [14]. In view of the results presented here, we may add the following questions to the list. The asymptotic formulas for the bounds on the informational power of 2- and 3-designs (Remarks III and IV), as well as those for 4- and 5-designs (Remarks V and VI), are pairwise identical. Can this be generalized to higher t? Can this phenomenon be given a physical interpretation? Moreover, for all the t-design qubit POVMs π_y we explicitly optimized (Corollaries VII, VIII, and IX), the optimal ensemble ψ_x turned out to be such that $\psi_x \pi_x = 0$ for any x. Is this always the case for qubit t-designs? Finally, closed analytic expressions for the bounds provided by Theorem III for $t \geqslant 6$ require lengthy calculations, therefore their derivation can be made easier by the use of a symbolic calculation package. This will be done in a forthcoming work [15], where their tightness and asymptotic formulas will also be discussed.

Note added in proof. Recently, the author was informed by Wojciech Słomczyński and Anna Szymusiak [41] of a recent result of theirs showing that the bound in Corollary II is saturated by the 64 Hoggar lines' SIC-POVM in dimension 8.

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