Entangling power and operator entanglement of nonunitary quantum evolutions

Fan-Zhen Kong,[1,2] Jun-Long Zhao,[1] Ming Yang,[1,*] and Zhuo-Liang Cao[3,†]
[1] *Key Laboratory of Opto-electronic Information Acquisition and Manipulation, Ministry of Education, School of Physics & Material Science, Anhui University, Hefei 230601, China*
[2] *Department of Computer Science, Jining University, Qufu, Shandong 273155, China*
[3] *School of Electronic and Information Engineering, Hefei Normal University, Hefei 230601, China*
(Received 13 May 2015; published 29 July 2015)

We propose a method to calculate the operator entanglement and entangling power of a noisy nonunitary operation in terms of linear entropy. By decomposing the Kraus operators of noisy evolution as the sum of products of Pauli matrices, we derive the analytical expression of the operator entanglement for a general nonunitary operation. The definition of entangling power is extended from the ideal unitary operation case to the nonunitary case via a Kraus operator representation and the analytical expression of the entangling power for a general nonunitary operation is derived. To demonstrate the effectiveness of the above method, we investigate the properties of operator entanglement and entangling power of nonunitary operations caused by phase damping noise. Our findings imply that the pure phase damping noise has its own operator entanglement and entangling power, which increase exponentially with time and asymptotically approach their respective upper bounds. In addition, when the phase damping noise is added to an ideal operation, such as an ISWAP operation or a controlled-Z operation, it can make the operation's entangling power grow exponentially with the strength of noise, but leave its operator entanglement invariant. In this sense, we can conclude that, for a general operation, operator entanglement is a more intrinsic property than entangling power.

DOI: 10.1103/PhysRevA.92.012127 PACS number(s): 03.65.Ud, 03.65.Yz, 03.65.Fd

I. INTRODUCTION

Quantum entanglement has been thought to be an essential resource for quantum communication and quantum computation. Considerable effort has been made to investigate the properties of entanglement for the quantum state. It has been shown that product states can be transformed to entangled states by nonlocal evolution operations [1,2]. Consequently, it is natural to study the entangling properties of nonlocal unitary evolution operations.

The nonlocal property of quantum evolution has been investigated regarding different aspects, such as entangling power [1,3–11], operator entanglement [5,12–18], entangle capacity [19–24], disorder power [25], and disentangling power [26]. The notion of operator entanglement was introduced by Zanardi [12]. The quantum operator belongs to a Hilbert-Schmidt space, so we can lift all the notions developed so far for the entanglement of quantum states to the level of operators. The entanglement of the quantum operator has been discussed in several works [5,12,14]. Wang *et al.* demonstrated that the entangling power of a general controlled unitary operator acting on two equal-dimensional qudits is proportional to the corresponding operator entanglement if linear entropy is adopted as the common measure for both of them, which implies that the entangling power and operator entanglement are two inequivalent notions in general [5]. Balakrishnan and Sankaranarayanan compared two different measures of the operator entanglement of two-qubit quantum gate, namely, Schmidt strength and linear entropy, and showed that there is a one-to-one relation between two different measures only for the Schmidt number 2 class of gates [13]. Wang and Zanardi investigated the entanglement of unitary operators on $d_1 \times d_2$ quantum systems [14]. Additionally, Ma and Wang indicated that the entangling power and operator entanglement of quantum unitary operations can be calculated by the matrix realignment and partial transpose [15]. The operator entanglement of the geometric quantum phase shift gate was studied by Yang *et al.* [16]. Balakrishnan and Lakshmanan discussed the chaining property for two-qubit operator entanglement measures [17]. Xia *et al.* studied the operator entanglement of the two-qubit joint unitary operation in terms of the Schmidt number [18].

Although there have been many studies on operator entanglement, most of them are based on the ideal unitary operations without considering the influence of environment. Actually, the nonlocal operations are always implemented in an environment such as a generic noisy reservoir or a heat bath that is commonly thought of as counteracting entanglement creation because of its decoherence and mixing-enhancing effects. However, several works showed that two noninteracting qubits can be entangled if the two qubits interact with a common heat bath, which implies that the pure noise induced evolution might have entangling power [27,28]. Zanardi *et al.* defined the partial entangling power of a unitary transformation based on completely positive (CP) maps [1]. Although a bipartite nonunitary evolution process corresponds to a CP map, the partial entangling power and entangling power of nonunitary evolution are two different concepts. One of the states of the input products is fixed in the definition of the partial entangling power, but no state in the input products is fixed for the entangling power of the nonunitary evolution. In fact, there have been several advances in the entangling power for operations in a noise environment. Bandyopadhyay and Lidar studied the entangling capacities of a noisy two-qubit Hamiltonian by phenomenologically considering stochastic noise [29]. Vallejos *et al.* studied the entangling power of quantum Baker maps with controlled-NOT coupling and

*mingyang@ahu.edu.cn
†zlcao@ahu.edu.cn

their numerical evidence indicates that the control subspace becomes an ideal Markovian environment for the target map in the limit of large Hilbert space dimension [30]. However, as we mentioned above, entangling power and operator entanglement are two entirely different notions, though they are proportional under certain conditions [5]. So it is necessary to investigate the operator entanglement of pure noise induced evolution or noisy operation and compare the influences of noise on entangling power and operator entanglement.

In this paper we study the operator entanglement and entangling power of pure noise induced evolutions and noisy operations. The results show that noisy quantum operations are equivalent to some mixed operations and Kraus operators can be regarded as the operational elements of the noisy evolution process [31]. An ideal unitary operation corresponds to a pure corresponding state [32], so a noisy nonunitary operation corresponds to a mixed corresponding state. By decomposing the Kraus operators of noisy evolution as the sum of products of Pauli matrices, we can use the linear entropy of the mixed corresponding state as the entanglement of the noisy evolution process. Based on this formulation, we derive the analytical expressions of the entanglement of some noisy operations, such as phase damping, nonunitary evolutions for realizing the ISWAP gate and controlled-Z gate, and the dynamics of the entanglement of these nonunitary evolutions was studied too. In addition, the entangling power of the above-mentioned nonunitary quantum evolutions was studied in terms of Kraus operators and we compared the influences induced by phase damping noise on entangling power and entanglement of those quantum evolutions. The results showed that phase damping noise does not affect the entanglement of ideal quantum operations; however, the entangling power of the two operations grows exponentially with the increase of the strength of phase damping noise.

This paper is organized as follows. In Sec. II we introduce the linear entropy as an operator entanglement measure and the notion of corresponding state and demonstrate that the corresponding state of nonunitary evolution is a mixed state. In Sec. III we study the mixed corresponding state of two-qubit noisy nonunitary evolutions in terms of the Kraus operator approach and explicitly derive the analytical expressions of operator entanglement and entangling power for two-qubit nonunitary evolutions. In Sec. IV, taking the phase damping noise as an example, we study the operator entanglement and entangling power of pure noise induced evolution and the noisy nonunitary evolutions for realizing an ISWAP gate and a controlled-Z gate. The results are summarized in Sec. V.

II. OPERATOR ENTANGLEMENT AND THE CORRESPONDING STATE

An operator U_A acting on the Hilbert space $\mathcal{H}_A$ of system A belongs to the Hilbert-Schmidt space $\mathcal{H}_{\text{HS}}$, which is isomorphic to $\mathcal{H}_A^{\otimes 2}$ as a Hilbert space. The natural isomorphism $|\Psi(U_A)\rangle$ from the operator to the state space is known to be

$$|\Psi(U_A)\rangle := (U_A \otimes \mathbb{I})|\Phi_0\rangle, \tag{1}$$

where $|\Phi_0\rangle = \frac{1}{\sqrt{d_A}}\sum_{i=1}^{d_A}|i\rangle^A|i\rangle^{A'}$ [32], A' is a copy of system A, d_A is the dimension of system A, and the maximally entangled state $|\Phi_0\rangle$ is a normalized basis in $\mathcal{H}_A$ and $\mathcal{H}'_A$. Here $|\Psi(U_A)\rangle$ is the corresponding state [12,32] of the operator U_A. Moving to the bipartite case, we obtain the corresponding state

$$|\Psi(U_{AB})\rangle := (U_{AB} \otimes \mathbb{I}_{A'B'})|\Phi_0\rangle^{\otimes 2} \quad \left(U_{AB} \in \mathcal{H}_{\text{HS}}^{\otimes 2}\right) \tag{2}$$

of the operator U_{AB} acting on the bipartite composite system [12], where systems A and B have the same dimension d_A.

The linear entropy $E(\rho_{AB}) = 1 - \text{tr}\rho_A^2$ quantifies entanglement in a state (with density operator ρ_{AB}) of the total system, where $\rho_A = \text{tr}_B(\rho_{AB})$ is the reduced density operator of subsystem A. For a product of corresponding states, $E(\rho_{AB}) = 0$, so the linear entropy measures the impurity of the subsystem and thus indirectly measures the entanglement of the composite system. The idea for defining the entanglement of an operator is to lift the notion of entanglement from the state level to the operator level [12]. Generally, linear entropy of the corresponding state $|\Psi(U_{AB})\rangle$ is adopted as the entanglement measure of the bipartite operator U_{AB}.

We start by writing $\varrho = |\Psi(U_{AB})\rangle\langle\Psi(U_{AB})|$, where the reduced density matrix is $\varrho_{AA'} = tr_{BB'}(|\Psi(U_{AB})\rangle\langle\Psi(U_{AB})|)$ and the operator entanglement of U_{AB} is given by [12]

$$E(U_{AB}) = 1 - \text{tr}(\varrho_{AA'})^2, \tag{3}$$

where U_{AB} is an ideal operator that is not affected by noise and its corresponding state $|\Psi(U_{AB})\rangle$ is a pure state. However, the general quantum operations are often implemented under a noise environment and are described by completely positive maps that can be written as the operator sum representation $\mathcal{E}(\rho_{AB}) = \sum_n D_n \rho_{AB} D_n^\dagger$. This allows us to associate $\mathcal{E}$ with the following operator over $\mathcal{H}^{\otimes 4}$ [12]:

$$\sum_n |\Psi(D_n)\rangle\langle\Psi(D_n)| = (\mathcal{E}_{AB} \otimes \mathbb{I}_{A'B'})(|\Phi_0\rangle\langle\Phi_0|)^{\otimes 2}. \tag{4}$$

So we can conclude that the corresponding state of a noisy nonunitary evolution described by the operator sum representation is a mixed state.

III. ENTANGLING POWER AND ENTANGLEMENT OF A NONUNITARY EVOLUTION

Because the corresponding state of a nonunitary evolution is a mixed state, in this section we derive the operator entanglement of the noisy nonunitary evolution by decomposing its Kraus operators as sums of the products of Pauli matrices and the entangling power of the noisy nonunitary evolution.

A. Entanglement of a noisy evolution for a two-qubit system

In general, we can suppose there is no initial correlation between systems and their environments and any noisy nonunitary evolution process for a two-qubit composite system composed of subsystems A and B is given by $\mathcal{E}(\rho_{AB}) = \sum_n D_n \rho_{AB}(0) D_n^\dagger$ with the Kraus operators satisfying $\sum_n D_n^\dagger D_n = I$. Furthermore, the Kraus operators can be decomposed as

$$D_n = \sum_{i,j=0}^{3} \mu_{nij}\sigma_i^A \otimes \sigma_j^B, \tag{5}$$

where $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$ denote the Pauli matrices and μ_{nij} are the decomposition coefficients.

Consequently, the noisy evolution of the two-qubit composite system can be represented as

$$\mathcal{E}(\rho_{AB}) = \sum_{i,j,k,l=0}^{3} \nu_{ijkl}\sigma_i^A \otimes \sigma_j^B \rho_{AB}(0)\sigma_k^A \otimes \sigma_l^B, \quad (6)$$

where $\nu_{ijkl} = \sum_n \mu_{nij}\mu_{nkl}^*$. According to the Jamiolkowski isomorphism theory [32], the corresponding mixed state of this noisy nonunitary evolution process can be given by

$$\sum_n |\Psi(D_n)\rangle\langle\Psi(D_n)| = \sum_{i,j,k,l=0}^{3} \nu_{ijkl}|\Phi_i\rangle|\Phi_j\rangle\langle\Phi_k|\langle\Phi_l|, \quad (7)$$

where $|\Psi(D_n)\rangle$ is the corresponding state of the Kraus operator D_n and $|\Phi_0\rangle$, $|\Phi_1\rangle$, $|\Phi_2\rangle$, and $|\Phi_3\rangle$ form a complete Bell basis. The formula $|\Phi_i\rangle^{AA'} = (\sigma_i^A \otimes \mathrm{I}^{A'})|\Phi_0\rangle^{AA'}$ has been used here. Now we can give the expression of the entanglement for the nonunitary evolution of two-qubit system in terms of the linear entropy of its corresponding mixed state

$$E(\mathcal{E}) = 1 - \mathrm{tr}\left[\mathrm{tr}_{BB'}\left(\sum_{ij,kl=0}^{3} \nu_{ijkl}|\Phi_i\rangle|\Phi_j\rangle\langle\Phi_k|\langle\Phi_l|\right)\right]^2. \quad (8)$$

B. Entangling power of the nonunitary evolution

Entangling power quantifies the entanglement capability of a unitary operator U acting on a bipartite quantum system with state space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and is defined as [1,15]

$$e_p(U) := \overline{E(U|\psi_A\rangle \otimes |\psi_B\rangle)}^{\psi_A,\psi_B}, \quad (9)$$

where the overbar stands for the average over all product states distributed according to a probability density $P(\psi_A,\psi_B)$. It tells us how much entanglement the operator U can produce, on average, when acting on product states. According to this definition, the entangling power of any noisy nonunitary evolution, described by $\mathcal{E}(\rho) = \sum_n D_n\rho(0)D_n^\dagger$, can be given as follows:

$$e_p(\mathcal{E}) := \overline{E\left(\sum_n D_n|\psi_A\rangle \otimes |\psi_B\rangle\right)}^{\psi_A,\psi_B}. \quad (10)$$

If we use the linear entropy as an entanglement measure and the initial product states $|\psi_A\rangle \otimes |\psi_B\rangle$ belong to a uniform distribution, the entangling power of the nonunitary evolution is given by

$$e_p(\mathcal{E}) := 1 - \sum_{m,n} \mathrm{tr}(D_n \otimes D_m\Omega_{p_0}D_n^\dagger \otimes D_m^\dagger S_{13}), \quad (11)$$

where

$$\Omega_{p_0} := \int d\mu(\psi_A,\psi_B)(|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|)^{\otimes 2}$$
$$= \frac{1}{d_A d_B(d_A+1)(d_B+1)}(\mathbb{I} + S_{13})(\mathbb{I} + S_{24}), \quad (12)$$

where $d\mu(\psi_A,\psi_B)$ is the measure over the product state manifold induced by $P(\psi_A,\psi_B)$, $d_{A(B)} = \dim[\mathcal{H}_{A(B)}]$, and S_{ij} $(i,j = 1,\ldots,4)$ denotes the transposition between the ith and jth factors of $\mathcal{H}^{\otimes 2} \cong (C^{d_A} \otimes C^{d_B}) \otimes (C^{d_A} \otimes C^{d_B})$ [1].

IV. ENTANGLING POWER AND OPERATOR ENTANGLEMENT OF NOISY NONUNITARY QUANTUM EVOLUTIONS VIA PHASE DAMPING

After generalizing the operator entanglement and entangling power from unitary operations to nonunitary cases, we will explicitly study the entangling power and operator entanglement of the pure phase damping induced noisy operation and the phase damping induced noisy ISWAP and controlled-Z gates.

A. Entangling power and operator entanglement of the phase damping noise

If we consider two noninteracting qubits that are both coupled to a common phase damping noise environment, the total Hamiltonian (set $\hbar = 1$) of this model can be expressed as $H = H_0 + H_I$, where the free Hamiltonian H_0 of the system and the interaction Hamiltonian H_I are given by

$$H_0 = \frac{1}{2}\omega\sigma_z^A + \frac{1}{2}\omega\sigma_z^B + \sum_j \Omega_j a_j^\dagger a_j, \quad (13)$$

$$H_I = \sum_j \sigma_z^A(\lambda_j^* a_j^\dagger + \lambda_j a_j) + \sum_j \sigma_z^B(\lambda_j^* a_j^\dagger + \lambda_j a_j), \quad (14)$$

where ω is the frequency of the qubits A and B, $a_j^\dagger$ (a_j) is the creation (annihilation) operator of the jth mode of the environment with the frequency Ω_j, λ_j is a coupling constant between the system (A or B) and the jth mode of the environment, and σ_z^A (σ_z^B) denotes the Pauli matrix of system A (B). In the interaction picture the evolution operator of the total system is given by $U = U_0^\dagger e^{-iH_I t}U_0$, where $U_0 = e^{-iH_0 t}$. In the standard product basis, the evolution operator can be expressed in the form

$$U = \begin{pmatrix} \mathscr{O}^\dagger\mathscr{V}\mathscr{O} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathscr{O}^\dagger\mathscr{V}^\dagger\mathscr{O} \end{pmatrix}, \quad (15)$$

where $\mathscr{O} = \exp(-it\sum_j \Omega_j a_j^\dagger a_j)$ and $\mathscr{V} = \exp[-it\sum_j 2(\lambda_j^* a_j^\dagger + \lambda_j a_j)]$. If we assume that there is no initial correlation between systems A and B and their environment, the initial state of the environment is the vacuum state $|\{0\}\rangle \equiv \prod_j \otimes|0_j\rangle = |\cdots 0 \cdots\rangle$. According to the discussion in Ref. [31], the Kraus operator for systems A and B can be defined as follows:

$$D_k = \sum_{\{k_j\}}^{k}{}' \langle\{k_j\}|U|\{0\}\rangle, \quad (16)$$

where $\sum'$ stands for summation under the condition $\sum_j k_j = k$, $|\{k\}\rangle \equiv \prod_j \otimes|k_j\rangle = |k_1 \cdots k_j \cdots\rangle$. Substituting Eq. (15) into Eq. (16), we can get the Kraus operators that describe the phase damping of the composite system. By definition, there is an infinite number of Kraus operators when the two qubits suffer from phase damping noise. Rearranging all Kraus operators $D_{k\neq 0}$ as one group, we can redefine the

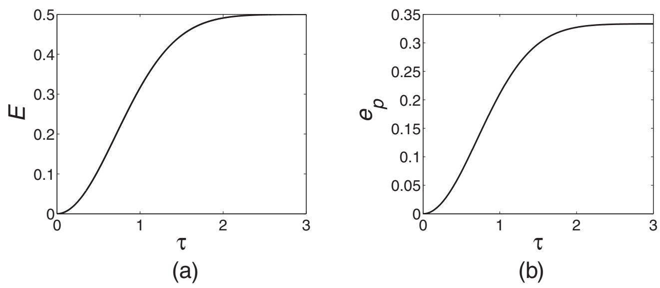

FIG. 1. (a) Operator entanglement of phase damping noise versus τ. (b) Entangling power of phase damping noise versus τ. Here $\tau = \sqrt{\Gamma}t$.

following two Kraus operators:

$$D_0 = \begin{pmatrix} e^{-\Gamma t^2/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\Gamma t^2/2} \end{pmatrix}, \tag{17}$$

$$D_1 = \begin{pmatrix} \sqrt{1-e^{-\Gamma t^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-e^{-\Gamma t^2}} \end{pmatrix}, \tag{18}$$

where $\Gamma = 4\sum_j |\lambda_j|^2$ and $\sum_i D_i^\dagger D_i = I$ is satisfied. By applying Eqs. (5)–(8), we can calculate the entanglement of the pure phase damping induced evolution of two qubits as follows:

$$E(\mathcal{E}) = \tfrac{1}{2} - \tfrac{1}{2}e^{-\Gamma t^2}. \tag{19}$$

By applying Eqs. (11) and (12), the entangling power of the pure phase damping induced evolution can be calculated as follows:

$$e_p(\mathcal{E}) = \tfrac{1}{3} - \tfrac{1}{3}e^{-\Gamma t^2}. \tag{20}$$

Figure 1 shows that both operator entanglement [Fig. 1(a)] and entangling power [Fig. 1(b)] of the pure phase damping induced evolution increase exponentially with time. That is to say, the phase damping noise has its own operator entanglement and entanglement power like other operations and its entangling power and operator entanglement increase gradually with time up to their respective maxima $\frac{1}{3}$ and $\frac{1}{2}$.

B. Entangling power and entanglement of nonunitary evolutions for realizing ideal operations

In this section we study the entangling power and entanglement of evolutions for realizing the ISWAP gate and controlled-Z gate in the presence of phase damping noise. Consider the interaction Hamiltonian between qubits A and B, which is used to implement the ISWAP quantum gate

$$H_s = J\left(\sigma_x^A\sigma_x^B + \sigma_y^A\sigma_y^B\right), \tag{21}$$

where J is the coupling constant between qubits A and B. If the phase damping is taken into consideration, the total Hamiltonian of the composite system and its environment is $H = H_0 + H_s + H_I$, where H_0 is the free Hamiltonian and H_I is the interaction Hamiltonian between the composite system and its environment, which causes the phase damping. The evolution operator of the whole system is $\tilde{U}_s = U_0^\dagger e^{-i(H_I+H_s)t}U_0$ in the interaction picture. Suffering from the phase damping, the evolution of qubits A and B can be described by the following two Kraus operators:

$$D_0^s = \begin{pmatrix} e^{-\Gamma t^2/2} & 0 & 0 & 0 \\ 0 & \cos 2Jt & -i\sin 2Jt & 0 \\ 0 & -i\sin 2Jt & \cos 2Jt & 0 \\ 0 & 0 & 0 & e^{-\Gamma t^2/2} \end{pmatrix}, \tag{22}$$

$$D_1^s = \begin{pmatrix} \sqrt{1-e^{-\Gamma t^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-e^{-\Gamma t^2}} \end{pmatrix}. \tag{23}$$

Therefore, we can calculate the entangling power and entanglement of the evolution for realizing the ISWAP gate in the presence of phase damping noise as follows:

$$\begin{aligned} e_p(\mathcal{E}_s) = \tfrac{1}{3} - \tfrac{1}{9}e^{-\Gamma t^2} - \tfrac{2}{9}\cos^4 2Jt \\ -\tfrac{2}{9}\left(e^{-\Gamma t^2} - 1\right)\cos^2 2Jt, \end{aligned} \tag{24}$$

$$E(\mathcal{E}_s) = \tfrac{3}{4} - \tfrac{1}{4}\cos^4 2Jt - \tfrac{1}{2}e^{-\Gamma t^2}\cos^2 2Jt \tag{25}$$

Whether the phase damping is present or not, Figs. 2 and 3 show that both entangling power and entanglement of the evolution for realizing the ISWAP gate oscillate periodically with time. The phase damping noise leads to a significant increase of entangling power of the evolution for realizing the ISWAP gate [Fig. 2(b)] and its maximum exceeds the upper bound ($\frac{1}{3}$) for a two-qubit unitary operation [1], which means that the entangling power of a noise-assisted nonunitary operation is larger than that of the corresponding unitary one. In addition, the phase damping noise suppresses the oscillation amplitude of both entangling power [Fig. 2(b)] and entanglement [Fig. 3(b)] of the evolution. If the evolution time is accurately controlled, the ideal ISWAP gate can be realized. Surprisingly, whether there is phase damping or not, the entanglement of the evolution is always $\frac{3}{4}$ at the moment when the ideal ISWAP gate is realized. In other words, the phase damping noise does not affect the operator entanglement of

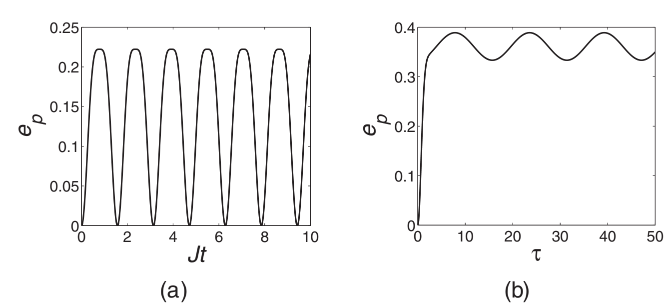

FIG. 2. (a) Entangling power of the evolution for realizing an ISWAP gate versus Jt without noise. (b) Entangling power of the evolution for realizing an ISWAP gate versus τ in the presence of phase damping noise. Here $\tau = \sqrt{\Gamma}t$.

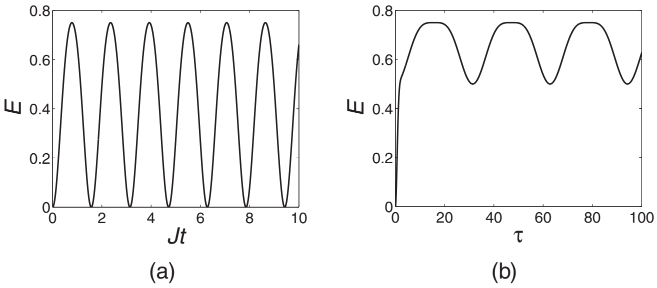

FIG. 3. (a) Entanglement of the evolution for realizing an ISWAP gate versus Jt without noise. (b) Entanglement of the evolution for realizing an ISWAP gate versus τ in the presence of phase damping noise. Here $\tau = \sqrt{\Gamma}t$.

the ISWAP gate. However, at the moment $t = t_0$ when the ideal ISWAP gate is realized, i.e., when $\cos 2Jt_0 = 0$ is satisfied, the entangling power of the evolution for realizing the ISWAP gate becomes $\tilde{e_p}(\text{ISWAP}) = \frac{1}{3} - \frac{1}{9}e^{-\Gamma t_0^2}$, which has an exponential relation with the strength of phase damping noise.

Next we study the entangling power and the entanglement of the evolution for realizing the controlled-Z gate under the phase damping noise circumstances. The total Hamiltonian of the whole system is $H = H_{cz} + H_I$. As shown in Eq. (61) of Ref. [33], the Hamiltonian $H_{cz} = \frac{g}{2}(\sigma_z^A + \sigma_z^B - \sigma_z^A\sigma_z^B)$ is introduced for realizing the controlled-Z gate, with g being a constant. In a similar way, we can get the Kraus operators for qubits A and B:

$$D_0^{cz} = \begin{pmatrix} e^{-igt}e^{-\Gamma t^2/2} & 0 & 0 & 0 \\ 0 & e^{-igt} & 0 & 0 \\ 0 & 0 & e^{-igt} & 0 \\ 0 & 0 & 0 & e^{igt}e^{-\Gamma t^2/2} \end{pmatrix}, \tag{26}$$

$$D_1^{cz} = \begin{pmatrix} e^{-igt}\sqrt{1 - e^{-\Gamma t^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{igt}\sqrt{1 - e^{-\Gamma t^2}} \end{pmatrix}. \tag{27}$$

In the presence of phase damping noise, the entangling power and the entanglement of evolution for realizing the controlled-

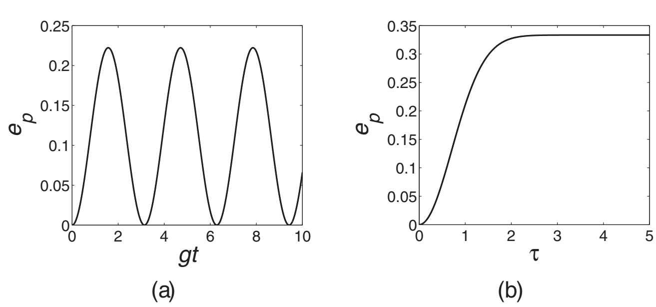

FIG. 4. (a) Entangling power of the evolution for realizing a controlled-Z gate versus gt without noise. (b) Entangling power of the evolution for realizing a controlled-Z gate versus τ in the presence of phase damping noise. Here $\tau = \sqrt{\Gamma}t$.

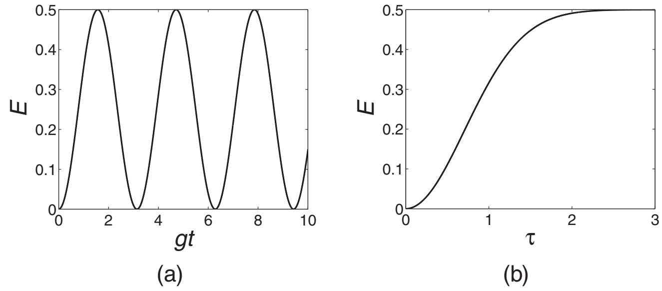

FIG. 5. (a) Entanglement of the evolution for realizing a controlled-Z gate versus gt without noise. (b) Entanglement of the evolution for realizing a controlled-Z gate versus τ in the presence of phase damping noise. Here $\tau = \sqrt{\Gamma}t$.

Z gate can be calculated as follows:

$$e_p(\mathcal{E}_{cz}) = \tfrac{1}{3} - \tfrac{2}{9}e^{-\Gamma t^2} - \tfrac{1}{9}e^{-\Gamma t^2}\cos 2gt, \tag{28}$$

$$E(\mathcal{E}_{cz}) = \tfrac{1}{2} - \tfrac{1}{2}e^{-\Gamma t^2}\cos^2 gt. \tag{29}$$

Figures 4 and 5 show that the phase damping noise suppresses the oscillation behavior of entangling power and entanglement of the evolution for realizing controlled-Z gate. The entangling power [Fig. 4(b)] and entanglement [Fig. 5(b)] of the noisy evolution for realizing the controlled-Z gate asymptotically approach their upper bound without oscillation. Additionally, the entanglement of the evolution for realizing the controlled-Z gate is a constant $(\frac{1}{2})$ in the presence of the phase damping noise at the moment $t = t_0$ (here t_0 is given by $\cos gt_0 = 0$) when the controlled-Z gate is realized. However, the entangling power of the controlled-Z gate $\tilde{e}_{p_{cz}} = \frac{1}{3} - \frac{1}{9}e^{-\Gamma t_0^2}$ grows exponentially with the increase of the strength of phase damping noise.

V. CONCLUSION

We have extended the definitions of operator entanglement and entangling power from the ideal operation case to the general noisy operation case, i.e., a method of how to calculate the operator entanglement and entangling power of a nonunitary operation has been proposed in terms of linear entropy. When an ideal operation is implemented in a noise environment, the operation is no longer a pure operation and it can be regarded as a mixed operation composed of several operation elements, i.e., the Kraus operators describing the noisy evolution. Then we can use the linear entropy of its mixed corresponding state as the entanglement measure of the noisy evolution. Because a noisy nonunitary operation can be described by a mixture of the evolutions induced by the corresponding Kraus operators, the definition of entangling power can be generalized from the ideal unitary operation case to the nonunitary case. So, for a given input product state, the output state of a nonunitary noisy operation can be described by a mixture of those states generated by the corresponding Kraus operator acting on the input product states. Consequently, the entangling power of a nonunitary noisy operation can be measured as the mean entanglement of the output states averaged over all possible input product states.

In this paper we mainly considered operator entanglement and entangling power of nonunitary operations caused by phase damping noise. Our results show that the pure phase damping noise has its own operator entanglement and entangling power. When the phase damping noise is added to an ideal operation, such as the ISWAP operation or the controlled-Z operation, it can increase the entangling power of the operation, but leave its operator entanglement invariant. These results indicates that, for a general operation, operator entanglement is a more intrinsic property than entangling power.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grants No. 11274010, No. 11204002, No. 11374085, and No. 11204061; the Specialized Research Fund for the Doctoral Program of Higher Education (Grants No. 20113401110002 and No. 20123401120003); the "211" Project of Anhui University; the Talent Foundation of Anhui University under Grant No. 33190019; and the personnel department of Anhui Province.

[1] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301(R) (2000).

[2] M. Y. Ye, D. Sun, Y. S. Zhang, and G. C. Guo, Phys. Rev. A **70**, 022326 (2004).

[3] A. T. Rezakhani, Phys. Rev. A **70**, 052313 (2004).

[4] S. Balakrishnan and R. Sankaranarayanan, Phys. Rev. A **78**, 052305 (2008).

[5] X. G. Wang, B. C. Sanders, and D. W. Berry, Phys. Rev. A **67**, 042323 (2003).

[6] S. Balakrishnan and R. Sankaranarayanan, Phys. Rev. A **82**, 034301 (2010).

[7] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, Phys. Rev. A **81**, 062346 (2010).

[8] L. Clarisse, S. Ghosh, S. Severini, and A. Sudbery, Phys. Rev. A **72**, 012314 (2005).

[9] R. F. Abreu and R. O. Vallejos, Phys. Rev. A **73**, 052327 (2006).

[10] Z. Guan, H. He, Y. J. Han, C. F. Li, F. Galve, and G. C. Guo, Phys. Rev. A **89**, 012324 (2014).

[11] H. N. Xiong, X. M. Lu, and X. G. Wang, J. Phys. B **45**, 015501 (2012).

[12] P. Zanardi, Phys. Rev. A **63**, 040304(R) (2001).

[13] S. Balakrishnan and R. Sankaranarayanan, Phys. Rev. A **83**, 062320 (2011).

[14] X. G. Wang and P. Zanardi, Phys. Rev. A **66**, 044303 (2002).

[15] Z. H. Ma and X. G. Wang, Phys. Rev. A **75**, 014304 (2007).

[16] Y. Yang, X. G. Wang, and Z. Sun, Phys. Lett. A **372**, 4369 (2008).

[17] S. Balakrishnan and M. Lakshmanan, Eur. Phys. J. Plus **129**, 231 (2014).

[18] H. Z. Xia, C. Li, Q. Yang, M. Yang, and Z. L. Cao, Braz. J. Phys. **42**, 167 (2012).

[19] M. S. Leifer, L. Henderson, and N. Linden, Phys. Rev. A **67**, 012306 (2003).

[20] X. G. Wang and B. C. Sanders, Phys. Rev. A **68**, 014301 (2003).

[21] W. Dür, G. Vidal, J. I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. **87**, 137901 (2001).

[22] E. T. Campbell, Phys. Rev. A **82**, 042314 (2010).

[23] A. Chefles, Phys. Rev. A **72**, 042332 (2005).

[24] P. Ye and Y. Z. Zheng, Phys. Lett. A **328**, 284 (2004).

[25] F. Galve, F. Plastina, M. G. A. Paris, and R. Zambrini, Phys. Rev. Lett. **110**, 010501 (2013).

[26] L. Clarisse, S. Ghosh, S. Severini, and A. Sudbery, Phys. Lett. A **365**, 400 (2007).

[27] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. **91**, 070402 (2003).

[28] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).

[29] S. Bandyopadhyay and D. A. Lidar, Phys. Rev. A **70**, 010301(R) (2004).

[30] R. O. Vallejos, P. R. del Santoro, and A. M. Ozorio de Almeida, J. Phys. A **39**, 5163 (2006).

[31] Y. X. Liu, S. K. Özdemir, A. Miranowicz, and N. Imoto, Phys. Rev. A **70**, 042308 (2004).

[32] W. Dür, M. Hein, J. I. Cirac, and H. J. Briegel, Phys. Rev. A **72**, 052326 (2005).

[33] A. Z. Chaudhry and J. Gong, Phys. Rev. A **85**, 012315 (2012).