

Gaussian intrinsic entanglement: An entanglement quantifier based on secret correlations

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Intrinsic entanglement (IE) is a quantity which aims at quantifying bipartite entanglement carried by a quantum state as an optimal amount of the intrinsic information that can be extracted from the state by measurement. We investigate in detail the properties of a Gaussian version of IE, the so-called Gaussian intrinsic entanglement (GIE). We show explicitly how GIE simplifies to the mutual information of a distribution of outcomes of measurements on a conditional state obtained by a measurement on a purifying subsystem of the analyzed state, which is first minimized over all measurements on the purifying subsystem and then maximized over all measurements on the conditional state. By constructing for any separable Gaussian state a purification and a measurement on the purifying subsystem which projects the purification onto a product state, we prove that GIE vanishes on all Gaussian separable states. Via realization of quantum operations by teleportation, we further show that GIE is nonincreasing under Gaussian local trace-preserving operations and classical communication. For pure Gaussian states and a reduction of the continuous-variable GHZ state, we calculate GIE analytically and we show that it is always equal to the Gaussian Rényi-2 entanglement. We also extend the analysis of IE to a non-Gaussian case by deriving an analytical lower bound on IE for a particular form of the non-Gaussian continuous-variable Werner state. Our results indicate that mapping of entanglement onto intrinsic information is capable of transmitting also quantitative properties of entanglement and that this property can be used for introduction of a quantifier of Gaussian entanglement which is a compromise between computable and physically meaningful entanglement quantifiers.

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I. INTRODUCTION

Since the dawn of quantum information theory, its development has been guided by the findings of classical information theory. Indeed, some key quantum information concepts, including early entanglement distillation protocols [1], quantum error correction [2], and some fundamental quantum information inequalities [3], appeared initially as nontrivial translations of their classical counterparts into the language of quantum states. Naturally, the further independent development of quantum information theory has led to the emergence of concepts with no analogy in classical theory. This category includes, for instance, bound entanglement [4], entanglement distribution by separable states [5], and superactivation of entanglement [6]. It is not surprising then that the opposite effect occurred when quantum information started to enrich classical information theory with new concepts such as bound information [7,8], secrecy distribution by nonsecret correlations [9], and a classical analogy to superactivation [10].

Classical analogies of quantum phenomena are almost exclusively cryptographic analogies of some properties of quantum entanglement. Entanglement is the key resource in quantum information and it is synonymous with correlations among two or more quantum systems which cannot be prepared by local operations and classical communication (LOCC). The cryptographic parallels of entanglement properties are carried by classical probability distributions containing so-called secret correlations [11,12]. The correlations are a fundamental resource in cryptography and appear in the scenario when two honest parties, Alice and Bob, and an adversary Eve, share three correlated random variables A , B , and E obeying a probability distribution $P(A, B, E)$. The distribution carries secret correlations if it is impossible for Alice and Bob to create the distribution by local operations and

public communication [13]. Owing to the apparent similarity with entanglement, secret correlations can therefore be viewed as a classical analogy to entanglement [14]. In fact, secret correlations and quantum entanglement are not just analogs but are directly linked as the latter can be mapped onto the former as follows [15]. A third adversary party Eve, seemingly missing in a quantum state ρ_{AB} , is associated with all information which could potentially be carried by a third system E , i.e., the global state $|\Psi\rangle_{ABE}$ of the tripartite system is a purification of the state ρ_{AB} ($\text{Tr}_E|\Psi\rangle_{ABE}\langle\Psi| = \rho_{AB}$). A given quantum state ρ_{AB} can then be mapped onto a probability distribution $P(A, B, E)$ by performing measurements Π_A , Π_B , and Π_E on subsystems A , B , and E of the purification as [15]

$$P(A, B, E) = \text{Tr}(\Pi_A \otimes \Pi_B \otimes \Pi_E |\Psi\rangle_{ABE} \langle\Psi|). \quad (1)$$

The presence of secret correlations in the obtained distribution can be certified with the help of the so-called intrinsic conditional information defined as [16]

$$I(A; B \downarrow E) = \inf_{E \rightarrow \tilde{E}} [I(A; B | \tilde{E})]. \quad (2)$$

Here,

$$I(A; B | E) = H(A, E) + H(B, E) - H(A, B, E) - H(E) \quad (3)$$

is the mutual information between A and B conditioned on E , where $H(X)$ is the Shannon entropy [17], and the minimization is performed over all channels $E \rightarrow \tilde{E}$ characterized by a conditional probability distribution $P(\tilde{E}|E)$. The intrinsic information gives a lower bound to the information of formation [12] quantifying the amount of secret bits [18] needed for preparation of the distribution, and an upper bound to the rate at which a secret key can be distilled from the

distribution [16] in the secret-key agreement protocol [11]. More importantly, the distribution $P(A, B, E)$ contains secret correlations if and only if $I(A; B \downarrow E) > 0$ [9,12]. Moving back to the mapping (1) one can then show using intrinsic information (2) that provided that the state ρ_{AB} is entangled one can always find measurements Π_j such that the obtained distribution contains secret correlations [7]. Moreover, the multipartite form of the mapping (1) is even capable of mapping more subtle properties of entanglement such as its boundedness [8].

So far, the mapping (1) has been investigated only from the point of view of the ability to transmit qualitative properties of quantum states onto classical probability distributions. A natural step forward would therefore be to elucidate whether the mapping can also preserve the quantitative properties of input states. Specifically, it would be of interest to know whether there is a function of a probability distribution $P(A, B, E)$ associated with a quantum state ρ_{AB} via mapping (1) which does not increase under any LOCC operation on the state. This would mean that the composition of the mapping and the function preserve the fundamental property that entanglement does not increase under LOCC operations. This is, however, important from a practical point of view because such a function then can be used to quantify entanglement [19].

An interesting attempt to quantify entanglement with the mapping (1) has been put forward by Gisin and Wolf [7]. They introduced the following optimized intrinsic information:

$$\mu(\rho_{AB}) = \inf_{\{\Pi_E, |\Psi\rangle\}} \left\{ \sup_{\{\Pi_A, \Pi_B\}} [I(A; B \downarrow E)] \right\}, \quad (4)$$

where the supremum is taken over all projective measurements $\{\Pi_A = |A\rangle\langle A|\}$ and $\{\Pi_B = |B\rangle\langle B|\}$ on subsystems A and B , respectively, and the infimum is taken over all purifications $|\Psi\rangle$ of the state ρ_{AB} and all positive operator-valued measures (POVM) $\{\Pi_E\}$ on subsystem E . Further, in Ref. [7] it was shown that the quantity (4) possesses some properties of an entanglement measure such as equality to the von Neumann entropy on pure states and convexity, and it was also calculated analytically for two-qubit Werner states. The quantity (4) is particularly interesting because unlike most of the other entanglement measures it is intimately related with a meaningful protocol: it is an upper bound in the secret-key agreement protocol [11]. What is more, it may even characterize secret correlations distillable to a secret key provided that the conjectured bipartite nondistillable secret correlations with a strictly positive intrinsic information (the so-called bipartite bound information [7]) do not exist. Despite this fact, the other properties of entanglement measures have not been investigated for the quantity (4) but it inspired the introduction of a different measure called squashed entanglement [20]. In particular, the key questions of whether the quantity (4) is nonincreasing under LOCC operations and whether it can be calculated also for other quantum states remain open.

To find answers to the latter questions can be a hard or even intractable task owing to the apparent complexity of the quantity (4). Nevertheless, the quantity (4) can still inspire the introduction of a closely related quantity for which the proof of monotonicity under LOCC operations as well as its computation can be considerably easier. The quantity in

question is the so-called *intrinsic entanglement* (IE) defined as [21]

$$E_{\downarrow}(\rho_{AB}) = \sup_{\{\Pi_A, \Pi_B\}} \left\{ \inf_{\{\Pi_E, |\Psi\rangle\}} [I(A; B \downarrow E)] \right\}. \quad (5)$$

In comparison with the quantity (4), the order of optimization in the definition of IE is reversed and hence $E_{\downarrow} \leq \mu$ due to the max-min inequality [22]. In fact, the two quantities may coincide if the intrinsic information (2) together with the sets $\{\Pi_A, \Pi_B\}$ and $\{\Pi_E, |\Psi\rangle\}$ possess the strong max-min property [22] which guarantees that the order of optimization in Eq. (5) can be commuted. Reference [21] further deals with a Gaussian version of IE, the so-called Gaussian intrinsic entanglement (GIE). The GIE is defined as in Eq. (5), where all channels $E \rightarrow \tilde{E}$ in Eq. (2), and all quantum states ρ_{AB} and $|\Psi\rangle$, and measurements $\{\Pi_j\}$, $j = A, B, E$, are assumed to be Gaussian. It is further shown that GIE simplifies considerably to the optimized mutual information of a distribution of outcomes of Gaussian measurements on subsystems A and B of a conditional state obtained by a Gaussian measurement on subsystem E of a Gaussian purification of the state ρ_{AB} . Next, it is proved that GIE vanishes if and only if the state ρ_{AB} is separable and that it does not increase under Gaussian local trace-preserving operations and classical communication (GLTPOCC). Finally, some analytical formulas are obtained for GIE as well as IE. First, GIE is calculated analytically for pure Gaussian states as well as for a two-mode reduction of the three-mode CV GHZ state [23] and it is shown that it always coincides with the Gaussian Rényi-2 (GR2) entanglement [24]. Second, an analytical lower bound on IE is derived for a subset of the set of the non-Gaussian continuous-variable Werner states [25], which is given by convex mixtures of the two-mode squeezed vacuum state and the vacuum state.

This paper accompanies the original paper on GIE [21]. It contains details of proofs of the properties of GIE presented in Ref. [21]. Additionally, we also provide two results not mentioned in Ref. [21]. First, we show that the monotonicity of GIE under GLTPOCC implies the invariance of GIE with respect to Gaussian local unitaries. Second, we prove that if we allow for non-Gaussian measurements $\{\Pi_A, \Pi_B\}$ in the definition of GIE we get a quantity which is on pure Gaussian states equal to the entropy of entanglement in analogy with the quantifier (4) which is also equal to the entropy of entanglement for pure states [7].

The paper is organized as follows. Section II contains a brief introduction into the formalism of Gaussian states. In Sec. III, we show explicitly that for GIE the channel $E \rightarrow \tilde{E}$ in Eq. (2) can be integrated into Eve's measurement. The next Sec. IV contains a proof that in the definition of GIE (5) we can use a fixed purification and the minimization over all Gaussian purifications can be omitted. Section V then presents the construction of a Gaussian measurement which projects a Gaussian purification of a separable Gaussian state onto a product state and Sec. VI is dedicated to a detailed proof of the monotonicity of GIE under GLTPOCC operations. Derivation of an analytical expression for GIE and proof of its equality to GR2 entanglement is given for pure Gaussian states in Sec. VII and for the two-mode reduction of the three-mode CV GHZ state in Sec. VIII. Finally, in Sec. IX

we derive an analytical lower bound on IE for a subclass of the non-Gaussian continuous-variable Werner states. Section X contains conclusions.

II. GAUSSIAN STATES

In this paper, we consider quantum systems with infinite-dimensional Hilbert state spaces which can be physically implemented by modes of the electromagnetic field. A system of n modes can be conveniently described by a vector of quadratures $\xi = (x_1, p_1, \dots, x_n, p_n)^T$ whose components obey the canonical commutation rules $[\xi_j, \xi_k] = i(\Omega_n)_{jk}$ with

$$\Omega_n = \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (6)$$

being the so-called symplectic matrix. According to definition, Gaussian states are quantum states of modes, which possess a Gaussian Wigner function. An n -mode Gaussian state ρ is therefore fully characterized by a vector of first moments $\langle \xi \rangle = \text{Tr}(\xi \rho)$, and by a covariance matrix (CM) γ with entries $\gamma_{jk} = \langle \{\Delta \xi_j, \Delta \xi_k\} \rangle$, where $\Delta \xi_j = \xi_j - \langle \xi_j \rangle$ and $\{A, B\} \equiv AB + BA$ is the anticommutator. The quantity GIE analyzed in this paper depends only on the elements of the CM and thus the vector of the first moments $\langle \xi \rangle$ is from now assumed to be zero for simplicity. We use Gaussian unitary operations which are for n modes represented at the level of CMs by a real $2n \times 2n$ symplectic matrix S fulfilling $S\Omega_n S^T = \Omega_n$. Recall, also, that any CM γ can be symplectically diagonalized, i.e., there exists a symplectic matrix S that brings γ to the Williamson normal form [26]

$$S\gamma S^T = \text{diag}(\nu_1, \nu_1, \dots, \nu_n, \nu_n), \quad (7)$$

where $\nu_1 \geq \dots \geq \nu_n \geq 1$ are the symplectic eigenvalues of γ .

As for measurements, we restrict ourselves to Gaussian measurements which can be implemented by appending auxiliary vacuum modes, using passive and active linear optics (phase shifters, squeezers, and beam splitters) and homodyne detections. Any such measurement on n modes is described by the following POVM [27]:

$$\Pi(d) = \frac{1}{(2\pi)^n} D(d) \Pi_0 D^\dagger(d), \quad (8)$$

where the seed element Π_0 is a normalized density matrix of a generally mixed n -mode Gaussian state with zero first moments and CM Γ , $D(d) = \exp(-i d^T \Omega_n \xi)$ is the displacement operator, and $d = (d_1^{(x)}, d_1^{(p)}, \dots, d_n^{(x)}, d_n^{(p)})^T \in \mathbb{R}_{2n}$ is a vector of measurement outcomes. From the normalization condition $\text{Tr}[\Pi_0] = 1$ it follows that the POVM (8) satisfies the completeness condition

$$\int_{\mathbb{R}_{2n}} \Pi(d) d^{2n} d = \mathbb{1}, \quad (9)$$

where $d^{2n} d = \prod_{l=1}^n dd_l^{(x)} dd_l^{(p)}$.

In the present analysis of IE [Eq. (5)], we assume that the state $\rho_{AB} \equiv \rho_{A_1 \dots A_N B_1 \dots B_M}$ is an $(N+M)$ -mode Gaussian state of N modes A_1, A_2, \dots, A_N and M modes B_1, B_2, \dots, B_M , which is described by the CM γ_{AB} . Further, we also assume that $|\tilde{\Psi}\rangle_{ABE}$ is an $(N+M+K)$ -mode Gaussian purification of the state ρ_{AB} , which contains K purifying modes E_1, E_2, \dots, E_K ,

and which is described by the CM $\tilde{\gamma}_\pi$. By performing Gaussian measurements (8) with covariance matrices (CMs) Γ_A, Γ_B , and Γ_E on subsystems A, B , and E of the purification $|\tilde{\Psi}\rangle_{ABE}$, the mapping (1) yields a zero-mean Gaussian distribution $P(d_A, d_B, d_E)$ of measurement outcomes d_A, d_B , and d_E , which is given by the formula

$$P(d_A, d_B, d_E) = \frac{e^{-d^T \Sigma^{-1} d}}{\pi^{N+M+K} \sqrt{\det \Sigma}}, \quad (10)$$

where $d = (d_A^T, d_B^T, d_E^T)^T$ and

$$\Sigma = \begin{pmatrix} \gamma_{AB} + \Gamma_A \oplus \Gamma_B & \tilde{\gamma}_{ABE} \\ \tilde{\gamma}_{ABE}^T & \tilde{\gamma}_E + \Gamma_E \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \beta^T & \delta \end{pmatrix} \quad (11)$$

is the classical covariance matrix (CCM) [28] of the distribution expressed with respect to $AB|E$ splitting. Here, γ_{AB} , $\tilde{\gamma}_{ABE}$, and $\tilde{\gamma}_E$ are blocks of the CM $\tilde{\gamma}_\pi$ of the purification $|\tilde{\Psi}\rangle_{ABE}$, when expressed with respect to the same splitting, i.e.,

$$\tilde{\gamma}_\pi = \begin{pmatrix} \gamma_{AB} & \tilde{\gamma}_{ABE} \\ \tilde{\gamma}_{ABE}^T & \tilde{\gamma}_E \end{pmatrix}. \quad (12)$$

In what follows, we analyze a Gaussian version of the quantifier (5), where the role of the distribution $P(A, B, E)$ is played by the Gaussian distribution (10).

III. PROOF THAT ANY GAUSSIAN CHANNEL CAN BE INTEGRATED INTO EVE'S MEASUREMENT

At the beginning we show that the quantity IE [Eq. (5)] greatly simplifies in the Gaussian scenario. First, we prove that any Gaussian channel $E \rightarrow \tilde{E}$ appearing in Eq. (2) can be always incorporated into Eve's measurement.

The proof goes as follows. We assume that the channel $E \rightarrow \tilde{E}$ in Eq. (2) is a Gaussian channel $d_E \rightarrow \tilde{d}_E$ mapping a $2K \times 1$ column vector d_E onto an $L \times 1$ column vector \tilde{d}_E , where d_E contains measurement outcomes of a measurement on Eve's K modes of an $(N+M+K)$ -mode purification $|\tilde{\Psi}\rangle_{ABE}$ of the state ρ_{AB} . Such a channel is described by a linear transformation

$$\tilde{d}_E = X d_E + y, \quad (13)$$

where X is a fixed real $L \times 2K$ matrix and $y = (y_1, y_2, \dots, y_L)^T$ is an $L \times 1$ random column vector distributed with a zero mean Gaussian distribution characterized by an $L \times L$ CCM Y with elements $Y_{ij} = \langle \{y_i, y_j\} \rangle$, $i, j = 1, \dots, L$. The input to the channel is a vector d_E of Eve's measurement outcomes, which is distributed according to a zero mean Gaussian distribution with a fixed CCM $\delta = \tilde{\gamma}_E + \Gamma_E$ given in Eq. (11). The channel is therefore fully characterized by a joint Gaussian distribution $P(d_E, \tilde{d}_E)$ with zero mean and a CCM of the form

$$\chi = \begin{pmatrix} \delta & \delta X^T \\ X \delta & X \delta X^T + Y \end{pmatrix}. \quad (14)$$

The input Gaussian distribution $P(d_A, d_B, d_E)$ [Eq. (10)] is then transformed by the channel as

$$\tilde{P}(d_A, d_B, \tilde{d}_E) = \int P(\tilde{d}_E | d_E) P(d_A, d_B, d_E) d^{2K} d_E, \quad (15)$$

where

$$P(\tilde{d}_E|d_E) = \frac{P(d_E, \tilde{d}_E)}{P(d_E)} = \frac{e^{-(\tilde{d}_E - X d_E)^T Y^{-1} (\tilde{d}_E - X d_E)}}{\sqrt{\pi^L \det Y}} \quad (16)$$

is a conditional Gaussian probability distribution of the channel. We now substitute into the right-hand side (RHS) of Eq. (15) for the distribution $P(d_A, d_B, d_E)$ from Eq. (10), which gives the output distribution (15) in the form

$$\tilde{P}(d_A, d_B, \tilde{d}_E) = \frac{e^{-\tilde{d}^T \tilde{\Sigma}^{-1} \tilde{d}}}{\pi^{N+M+\frac{L}{2}} \sqrt{\det \tilde{\Sigma}}}, \quad (17)$$

where $\tilde{d} = (d_A^T, d_B^T, \tilde{d}_E^T)^T$ and

$$\tilde{\Sigma} = \begin{pmatrix} \alpha & \beta X^T \\ X \beta^T & X \delta X^T + Y \end{pmatrix}, \quad (18)$$

where the matrices α , β , and δ are defined in Eq. (11).

The figure of merit considered in this paper is the conditional mutual information $I(A; B|E)$ of the output distribution (17) which coincides with the standard mutual information $I(A; B)$ of the corresponding conditional distribution $\tilde{P}(d_A, d_B|\tilde{d}_E)$ [21]. The latter distribution is Gaussian and the mutual information depends on its CCM which is given by the Schur complement [29] of the CCM (18),

$$\sigma_{AB} = \alpha - \beta X^T (X \delta X^T + Y)^{-1} X \beta^T, \quad (19)$$

where the inverse is to be understood generally as the pseudoinverse.

Now, we prove that for any channel (13) there is a measurement on Eve's modes characterized by a CM $\tilde{\Gamma}_E$ such that

$$\sigma_{AB} = \alpha - \beta(\tilde{\gamma}_E + \tilde{\Gamma}_E)^{-1} \beta^T. \quad (20)$$

As a result, without loss of generality, we can omit the minimization appearing in Eq. (2) and the intrinsic conditional information $I(A; B \downarrow E)$ in the definition (5) can thus be replaced with the standard conditional mutual information $I(A; B|E)$ [Eq. (3)].

Our proof utilizes the singular value decomposition [29] of the matrix X ,

$$X = U S \mathcal{V}^T, \quad (21)$$

where U is an $L \times L$ real orthogonal matrix, \mathcal{V} is a $2K \times 2K$ real orthogonal matrix, and S is an $L \times 2K$ rectangular diagonal matrix of the form

$$S = \begin{pmatrix} s_Q & \mathbb{O}_{Q \times (2K-Q)} \\ \mathbb{O}_{(L-Q) \times Q} & \mathbb{O}_{(L-Q) \times (2K-Q)} \end{pmatrix}, \quad (22)$$

where $\mathbb{O}_{I \times J}$ is an $I \times J$ zero matrix, $s_Q = \text{diag}(\zeta_1, \zeta_2, \dots, \zeta_Q)$ is a $Q \times Q$ diagonal matrix with the strictly positive sin-

gular values $\zeta_1 \geq \zeta_2 \geq \dots \geq \zeta_Q > 0$ on the diagonal, and $Q = \text{rank} X$. Inserting Eq. (21) into (19) one obtains

$$\sigma_{AB} = \alpha - \beta \mathcal{V} S^T (S \mathcal{V}^T \delta \mathcal{V} S^T + U^T Y U)^{-1} S \mathcal{V}^T \beta^T. \quad (23)$$

Making use of Eq. (22) one can further express the $L \times L$ matrix $S \mathcal{V}^T \delta \mathcal{V} S^T$, appearing in the round brackets in Eq. (23), as

$$S \mathcal{V}^T \delta \mathcal{V} S^T = \tau_Q \omega_Q \tau_Q, \quad (24)$$

where τ_Q is an $L \times L$ matrix of the form

$$\tau_Q = s_Q \oplus \mathbb{1}_{L-Q} \quad (25)$$

and

$$\omega_Q = (\mathcal{V}^T \delta \mathcal{V})_Q \oplus \mathbb{O}_{(L-Q) \times (L-Q)}, \quad (26)$$

where $(\mathcal{V}^T \delta \mathcal{V})_Q$ is the first $Q \times Q$ block of the matrix $\mathcal{V}^T \delta \mathcal{V}$ and $\mathbb{1}_I$ is an $I \times I$ identity matrix. Substitution for $S \mathcal{V}^T \delta \mathcal{V} S^T$ in Eq. (23) from (24) and utilizing the formula

$$\tau_Q^{-1} S = \begin{pmatrix} \mathbb{1}_Q & \mathbb{O}_{Q \times (2K-Q)} \\ \mathbb{O}_{(L-Q) \times Q} & \mathbb{O}_{(L-Q) \times (2K-Q)} \end{pmatrix} \equiv \mathcal{I} \quad (27)$$

further yields the matrix (23) in the form

$$\sigma_{AB} = \alpha - \beta \mathcal{V} \mathcal{I}^T (\omega_Q + Y_{U, s_Q})^{-1} \mathcal{I} \mathcal{V}^T \beta^T, \quad (28)$$

where

$$Y_{U, s_Q} = \tau_Q^{-1} U^T Y U \tau_Q^{-1} \quad (29)$$

is an $L \times L$ positive-semidefinite matrix. Substitution for the matrix \mathcal{I} from Eq. (27) into (28) further yields

$$\sigma_{AB} = \alpha - \beta \mathcal{V} \mathcal{W} \mathcal{V}^T \beta^T, \quad (30)$$

with

$$\mathcal{W} = \begin{pmatrix} \omega_Q & \mathbb{O} \\ \mathbb{O}^T & \tilde{\mathbb{O}} \end{pmatrix} \quad (31)$$

being a $2K \times 2K$ matrix, where we have defined $\mathbb{O} \equiv \mathbb{O}_{Q \times (2K-Q)}$, $\tilde{\mathbb{O}} = \mathbb{O}_{(2K-Q) \times (2K-Q)}$, and

$$w_Q \equiv [(\omega_Q + Y_{U, s_Q})^{-1}]_Q \quad (32)$$

is the first $Q \times Q$ block of the matrix $(\omega_Q + Y_{U, s_Q})^{-1}$. If we express the matrix (29) in the block form

$$Y_{U, s_Q} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}, \quad (33)$$

where \mathcal{A} is a $Q \times Q$ block, \mathcal{C} is a $Q \times (L-Q)$ block, and \mathcal{B} is an $(L-Q) \times (L-Q)$ block, we can write

$$w_Q = [(\mathcal{V}^T \delta \mathcal{V})_Q + \mathcal{A} - \mathcal{C} \mathcal{B}^{-1} \mathcal{C}^T]^{-1}, \quad (34)$$

where we have used the blockwise inversion [29]

$$\begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathcal{A} - \mathcal{C} \mathcal{B}^{-1} \mathcal{C}^T)^{-1} & \mathcal{A}^{-1} \mathcal{C} (\mathcal{C}^T \mathcal{A}^{-1} \mathcal{C} - \mathcal{B})^{-1} \\ (\mathcal{C}^T \mathcal{A}^{-1} \mathcal{C} - \mathcal{B})^{-1} \mathcal{C}^T \mathcal{A}^{-1} & (\mathcal{B} - \mathcal{C}^T \mathcal{A}^{-1} \mathcal{C})^{-1} \end{pmatrix}. \quad (35)$$

Repeated use of the formula [29]

$$(\mathcal{A} - \mathcal{C} \mathcal{B}^{-1} \mathcal{C}^T)^{-1} = \mathcal{A}^{-1} + \mathcal{A}^{-1} \mathcal{C} (\mathcal{B} - \mathcal{C}^T \mathcal{A}^{-1} \mathcal{C})^{-1} \mathcal{C}^T \mathcal{A}^{-1} \quad (36)$$

further reveals that the $2K \times 2K$ matrix \mathcal{W} given in Eq. (31) can be obtained as a limit of the $2K \times 2K$ matrix

$$\mathcal{W}_x = \left[\mathcal{V}^T \delta \mathcal{V} + \begin{pmatrix} \mathcal{A} - \mathcal{C}\mathcal{B}^{-1}\mathcal{C}^T & \mathbb{O} \\ \mathbb{O}^T & x\mathbb{1} \end{pmatrix} \right]^{-1}, \quad (37)$$

when $x \rightarrow +\infty$, $x \geq 0$, and $\mathbb{1} \equiv \mathbb{1}_{2K-Q}$. The Schur complement (30) is then obtained from the matrix

$$\sigma_{AB,x} \equiv \alpha - \beta \mathcal{V} \mathcal{W}_x \mathcal{V}^T \beta^T \quad (38)$$

in the limit for $x \rightarrow +\infty$. By substitution we get immediately

$$\sigma_{AB,x} \equiv \alpha - \beta \mathcal{Z}_x^{-1} \beta^T, \quad (39)$$

with

$$\mathcal{Z}_x = \bar{\gamma}_E + \Gamma_E + \mathcal{V} \begin{pmatrix} \mathcal{A} - \mathcal{C}\mathcal{B}^{-1}\mathcal{C}^T & \mathbb{O} \\ \mathbb{O}^T & x\mathbb{1} \end{pmatrix} \mathcal{V}^T, \quad (40)$$

where we have used the equality $\delta = \bar{\gamma}_E + \Gamma_E$. The last matrix in the latter equation is positive semidefinite and therefore the matrix

$$\tilde{\Gamma}_E^x = \Gamma_E + \mathcal{V} \begin{pmatrix} \mathcal{A} - \mathcal{C}\mathcal{B}^{-1}\mathcal{C}^T & \mathbb{O} \\ \mathbb{O}^T & x\mathbb{1} \end{pmatrix} \mathcal{V}^T \quad (41)$$

represents a legitimate CM of a Gaussian quantum state. Consequently, a Gaussian measurement (8) on Eve's system described by a CM Γ_E followed by a Gaussian channel (13) characterized by the matrices X and Y on the outcomes of the measurement can be replaced with another Gaussian measurement with the CM $\tilde{\Gamma}_E = \tilde{\Gamma}_E^{x \rightarrow +\infty}$ which concludes the proof.

IV. PROOF THAT MINIMIZATION OVER PURIFICATIONS CAN BE OMITTED

The next step of simplification of IE [Eq. (5)] is the proof that in the Gaussian scenario, without loss of generality, we can use in Eq. (5) a fixed minimal purification [30] of the state ρ_{AB} , i.e., a purification containing minimum possible number of purifying modes. Moreover, we also show that the minimization over all Gaussian purifications can be integrated into a minimization over Eve's measurement.

According to the assumption, the state ρ_{AB} is an $(N+M)$ -mode Gaussian state, where subsystem A consists of N modes and subsystem B consists of M modes. The minimal purification of such a state is an $(N+M+R)$ -mode pure Gaussian state $|\Psi\rangle_{ABE}$ satisfying $\text{Tr}_E |\Psi\rangle_{ABE} \langle \Psi| = \rho_{AB}$, for which the purifying subsystem E consists of $R \leq N+M$ modes, where R is the number of symplectic eigenvalues of the CM γ_{AB} of the state ρ_{AB} , that are strictly greater than one [30]. When expressed with respect to the $AB|E$ splitting the CM ($\equiv \gamma_\pi$) of the minimal purification reads as

$$\gamma_\pi = \begin{pmatrix} \gamma_{AB} & \gamma_{ABE} \\ \gamma_{ABE}^T & \gamma_E \end{pmatrix}, \quad (42)$$

where

$$\gamma_E = \bigoplus_{i=1}^R v_i \mathbb{1}_2, \quad \gamma_{ABE} = S^{-1} \left(\bigoplus_{i=1}^R \sqrt{v_i^2 - 1} \sigma_z \right). \quad (43)$$

Here, $\sigma_z = \text{diag}(1, -1)$ is the Pauli diagonal matrix and S is the symplectic matrix that brings the CM γ_{AB} to the Williamson

normal form (7), where $n = N+M$ and $v_1 \geq v_2 \geq \dots \geq v_R > v_{R+1} = \dots = v_{N+M} = 1$.

In Eq. (5), we consider the minimization over all Gaussian purifications of the investigated Gaussian state ρ_{AB} . For any such purification $|\tilde{\Psi}\rangle_{ABE}$ with K -mode purifying subsystem E , there is a Gaussian unitary transformation $U_E(\bar{S}_E)$ on Eve's modes which connects the purification $|\tilde{\Psi}\rangle_{ABE}$ with the minimal purification $|\Psi\rangle_{ABE}$ by the formula [31,32]

$$|\tilde{\Psi}\rangle_{ABE} = U_E^\dagger(\bar{S}_E) |\Psi\rangle_{ABE} |\{0\}\rangle_{E_{R+1} \dots E_K}. \quad (44)$$

Here, $|\{0\}\rangle_{E_{R+1} \dots E_K} \equiv \bigotimes_{i=1}^{K-R} |0\rangle_{E_{R+i}}$ is the product of $K-R$ ancillary vacuum modes that Eve can use, and the operator $U_E(\bar{S}_E)$ symplectically diagonalizes reduced state $\bar{\rho}_E = \text{Tr}_{AB}(|\tilde{\Psi}\rangle_{ABE} \langle \tilde{\Psi}|)$ of Eve's subsystem E . Denoting the CM of the purification $|\tilde{\Psi}\rangle_{ABE}$ as $\tilde{\gamma}_\pi$ one can express the transformation (44) on the level of CMs in the form

$$\tilde{\gamma}_\pi = [\mathbb{1}_{AB} \oplus \bar{S}_E^{-1}] \gamma_\pi \oplus \mathbb{1}_{E_{R+1} \dots E_K} [\mathbb{1}_{AB} \oplus (\bar{S}_E^T)^{-1}], \quad (45)$$

where $\mathbb{1}_{AB}$ is a $2(N+M) \times 2(N+M)$ identity matrix, $\mathbb{1}_{E_{R+1} \dots E_K}$ is a $2(K-R) \times 2(K-R)$ identity matrix, and \bar{S}_E is the $2K \times 2K$ symplectic matrix symplectically diagonalizing the local CM $\tilde{\gamma}_E$ of Eve's subsystem, i.e.,

$$\bar{S}_E \tilde{\gamma}_E \bar{S}_E^T = \gamma_E \oplus \mathbb{1}_{E_{R+1} \dots E_K}, \quad (46)$$

where γ_E is the diagonal $2R \times 2R$ CM of the reduced state of subsystem E of the minimal purification $|\Psi\rangle_{ABE}$ given in Eq. (43). Expressing now the CMs γ_π and $\tilde{\gamma}_\pi$ with respect to the $A|B|E$ splitting,

$$\tilde{\gamma}_\pi = \begin{pmatrix} \gamma_A & \omega_{AB} & \tilde{\gamma}_{AE} \\ \omega_{AB}^T & \gamma_B & \tilde{\gamma}_{BE} \\ \tilde{\gamma}_{AE}^T & \tilde{\gamma}_{BE}^T & \tilde{\gamma}_E \end{pmatrix} \quad (47)$$

and

$$\gamma_\pi = \begin{pmatrix} \gamma_A & \omega_{AB} & \gamma_{AE} \\ \omega_{AB}^T & \gamma_B & \gamma_{BE} \\ \gamma_{AE}^T & \gamma_{BE}^T & \gamma_E \end{pmatrix}, \quad (48)$$

one gets from Eq. (45) for the $2(N+M) \times 2K$ block $(\tilde{\gamma}_{AE}^T, \tilde{\gamma}_{BE}^T)^T$ the expression

$$\begin{pmatrix} \tilde{\gamma}_{AE} \\ \tilde{\gamma}_{BE} \end{pmatrix} = \begin{pmatrix} \gamma_{AE} & \mathbb{O}_{2N \times 2(K-R)} \\ \gamma_{BE} & \mathbb{O}_{2M \times 2(K-R)} \end{pmatrix} (\bar{S}_E^T)^{-1}. \quad (49)$$

Further, by inverting Eq. (46) we can also express the CM $\tilde{\gamma}_E$ as

$$\tilde{\gamma}_E = \bar{S}_E^{-1} (\gamma_E \oplus \mathbb{1}_{E_{R+1} \dots E_K}) (\bar{S}_E^T)^{-1}. \quad (50)$$

As any Gaussian channel on Eve's measurement outcomes can be integrated into Eve's measurement the CCM relevant to the optimization of the conditional mutual information is given by the Schur complement

$$\bar{\sigma}_{AB} = \alpha - \begin{pmatrix} \tilde{\gamma}_{AE} \\ \tilde{\gamma}_{BE} \end{pmatrix} (\tilde{\gamma}_E + \tilde{\Gamma}_E)^{-1} \begin{pmatrix} \tilde{\gamma}_{AE} \\ \tilde{\gamma}_{BE} \end{pmatrix}^T \quad (51)$$

of the CCM

$$\bar{\Sigma} = \tilde{\gamma}_\pi + \Gamma_A \oplus \Gamma_B \oplus \tilde{\Gamma}_E \equiv \begin{pmatrix} \alpha & \beta \\ \beta^T & \delta \end{pmatrix}, \quad (52)$$

where Γ_A , Γ_B , and $\bar{\Gamma}_E$ are CMs of the measurements on Alice's, Bob's, and Eve's subsystems and the last 2×2 block matrix is the expression of the matrix $\bar{\Sigma}$ with respect to $AB|E$ splitting. Inserting from Eq. (49) into Eq. (51) one gets after some algebra

$$\bar{\sigma}_{AB} = \alpha - \begin{pmatrix} \gamma_{AE} \\ \gamma_{BE} \end{pmatrix} \mathcal{T}_R \begin{pmatrix} \gamma_{AE} \\ \gamma_{BE} \end{pmatrix}^T, \quad (53)$$

where \mathcal{T}_R is the first $2R \times 2R$ diagonal block of the $2K \times 2K$ matrix

$$\mathcal{T} = [\gamma_E \oplus \mathbb{1}_{E_{R+1} \dots E_K} + \bar{\Gamma}_E(\bar{S}_E)]^{-1}, \quad (54)$$

and $\bar{\Gamma}_E(\bar{S}_E) = \bar{S}_E \bar{\Gamma}_E \bar{S}_E^T$ is a $2K \times 2K$ CM of another of Eve's measurements. If we express finally the latter matrix in the block form

$$\bar{\Gamma}_E(\bar{S}_E) = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{C}^T & \bar{B} \end{pmatrix}, \quad (55)$$

with \bar{A} being a $2R \times 2R$ matrix and \bar{B} being a $2(K-R) \times 2(K-R)$ matrix we can express the block \mathcal{T}_R using formula (35) as

$$\mathcal{T}_R = [\gamma_E + \bar{A} - \bar{C}(\bar{B} + \mathbb{1})^{-1}\bar{C}^T]^{-1}. \quad (56)$$

The matrix

$$\Gamma_E \equiv \bar{A} - \bar{C}(\bar{B} + \mathbb{1})^{-1}\bar{C}^T \quad (57)$$

can be viewed as a CM of an R -mode conditional Gaussian state obtained by projecting the last $K-R$ modes of a K -mode Gaussian state with CM (55) onto a coherent state and therefore Γ_E is a legitimate CM of a physical quantum state. Consequently, one finally gets for the matrix (51) the following equation:

$$\bar{\sigma}_{AB} = \alpha - \begin{pmatrix} \gamma_{AE} \\ \gamma_{BE} \end{pmatrix} (\gamma_E + \Gamma_E)^{-1} \begin{pmatrix} \gamma_{AE} \\ \gamma_{BE} \end{pmatrix}^T = \sigma_{AB}. \quad (58)$$

Thus, for any Gaussian purification and any Gaussian measurement on subsystem E , the matrix (51) can be obtained from the minimal purification with CM (42) and a Gaussian measurement with CM (57) on Eve's part of the purification. Hence, when calculating the quantity defined in Eq. (5) in the Gaussian scenario we can consider only a fixed minimal purification and we can omit the minimization with respect to all Gaussian purifications, which accomplishes the proof.

Having found simplifications of IE in the Gaussian scenario, we are now in the position to incorporate them into the definition (5). Let us consider a Gaussian state ρ_{AB} and its minimal purification with CM (42) which has been mapped by Gaussian measurements with CMs Γ_A , Γ_B , and Γ_E onto the Gaussian distribution of the form (10). As we have already said, the intrinsic information in Eq. (5) can be replaced with the standard conditional mutual information (3), which coincides with the standard mutual information [$\equiv I_c(A; B)$] of the corresponding conditional distribution. The latter distribution possesses the CCM in the form of the Schur complement [29]

$$\sigma_{AB} = \gamma_{AB} + \Gamma_A \oplus \Gamma_B - \gamma_{ABE} (\gamma_E + \Gamma_E)^{-1} \gamma_{ABE}^T, \quad (59)$$

where γ_{AB} , γ_{ABE} , and γ_E are submatrices of the CM γ_π of the minimal purification of the state ρ_{AB} , which are defined

in Eq. (43). From the formula for mutual information of a bivariate Gaussian distribution [33], it follows further that $I_c(A; B) = f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E)$, where

$$f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E) = \frac{1}{2} \ln \left(\frac{\det \sigma_A \det \sigma_B}{\det \sigma_{AB}} \right) \quad (60)$$

with $\sigma_{A,B}$ being local submatrices of CCM (59). If we use now the definition of IE [Eq. (5)], and we take into account that we can omit minimization over all purifications, we arrive finally at the following formula for GIE [21]:

$$E_\downarrow^G(\rho_{AB}) = \sup_{\Gamma_A, \Gamma_B} \inf_{\Gamma_E} f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E). \quad (61)$$

Before going further, let us note one consequence stemming from the fact that for any purification with K purifying modes described by the CM (47) and any measurement on the modes with CM $\bar{\Gamma}_E$ we can find a measurement with CM (57) on the minimal purification giving the same matrix (51). This implies that for any two purifications containing a generally different and not necessarily minimal number of purifying modes, we can find measurements on the purifying subsystems which yield the same matrix (51). To show this, consider two purifications with CMs $\bar{\gamma}_\pi$ and $\bar{\gamma}'_\pi$ which contain K and K' purifying modes, respectively, where $K \leq K'$. By using Eq. (45) for both the CMs $\bar{\gamma}_\pi$ and $\bar{\gamma}'_\pi$ one finds that they are connected by a similar equation

$$\bar{\gamma}'_\pi = [\mathbb{1}_{AB} \oplus \mathcal{S}_E^{-1}] \bar{\gamma}_\pi \oplus \mathbb{1}_{2(K'-K)} [\mathbb{1}_{AB} \oplus (\mathcal{S}_E^T)^{-1}]. \quad (62)$$

Here, $\mathbb{1}_{2(K'-K)} = \mathbb{1}_{E_{K+1} \dots E_{K'}}$ and the symplectic matrix $\mathcal{S}_E = [\bar{S}_E^{-1} \oplus \mathbb{1}_{2(K'-K)}] S'_E$ satisfies $\mathcal{S}_E \gamma'_E \mathcal{S}_E^T = \bar{\gamma}_E \oplus \mathbb{1}_{2(K'-K)}$ and it consists of symplectic matrices \bar{S}_E and S'_E which symplectically diagonalize the local CMs $\bar{\gamma}_E$ and γ'_E of CMs $\bar{\gamma}_\pi$ and $\bar{\gamma}'_\pi$, respectively, corresponding to subsystem E . Making use of the formula (62), we can now repeat the procedure leading from Eq. (45) to (58) to show that for the purification with CM $\bar{\gamma}'_\pi$ and an arbitrary measurement with CM $\bar{\Gamma}'_E$ on subsystem E there always exists a measurement with CM $\bar{\Gamma}_E$ on the purification with CM $\bar{\gamma}_\pi$ for which it holds that $\sigma'_{AB} = \bar{\sigma}_{AB}$. If we perform, on the other hand, on the subsystem E of the purification with CM $\bar{\gamma}'_\pi$ the measurement with CM $\bar{\Gamma}'_E \equiv \mathcal{S}_E^{-1} [\bar{\Gamma}_E \oplus \mathbb{1}_{2(K'-K)}] (\mathcal{S}_E^T)^{-1}$, one finds easily that the matrix σ'_{AB} is equal to the matrix $\bar{\sigma}_{AB}$ corresponding to the purification with CM $\bar{\gamma}_\pi$ and the measurement with CM $\bar{\Gamma}_E$. Therefore, without loss of generality, we can consider in the formula (61) an arbitrary fixed purification, i.e., a fixed purification containing an arbitrary number of purifying modes, and we can restrict ourselves to minimizing only over all Gaussian measurements on the purifying modes. This property proves to be useful in the proof of the monotonicity of the GIE under GLTPOCC, which is given later.

V. GAUSSIAN MEASUREMENT PROJECTING PURIFICATION OF A SEPARABLE GAUSSIAN STATE ONTO A PRODUCT STATE

A basic property of any entanglement measure is that it vanishes on all separable states [34]. In Ref. [7], it was shown that for any separable state, whatever measurements are performed by Alice and Bob there is always Eve's measurement such that the conditional mutual information (3)

of the probability distribution (1) vanishes. Inspired by the proof of the latter statement, we show here that also the GIE is zero for all separable Gaussian states.

The vanishing of the GIE on separable Gaussian states is a direct consequence of the fact that for any separable Gaussian state ρ_{AB}^{sep} there is a Gaussian measurement on the purifying system E of the minimal purification of the state, that projects modes A and B onto a pure product state. Indeed, by performing such a measurement on subsystem E of the minimal purification of a separable state ρ_{AB}^{sep} , one finds that the conditional distribution $P(d_A, d_B | d_E)$ factorizes as $P(d_A, d_B | d_E) = P(d_A | d_E)P(d_B | d_E)$ for any measurement on subsystems A and B . Consequently, the conditional mutual information (3) and therefore also GIE are equal to zero.

It remains to find the measurement mentioned above. The sought measurement can be constructed after consideration of a measurement on another purification created using the separability criterion [35]. According to the separability criterion, a Gaussian state with CM γ_{AB}^{sep} is separable if and only if there exist pure-state CMs $\gamma_{A,B}^{\text{sep}}$ such that the matrix $Q \equiv \gamma_{AB}^{\text{sep}} - \gamma_A^{\text{sep}} \oplus \gamma_B^{\text{sep}} \geq 0$. If V denotes the orthogonal matrix diagonalizing the matrix Q , i.e., $V^T Q V = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_P, 0, \dots, 0)$, where $\lambda_i, i = 1, \dots, P$, are the strictly positive eigenvalues of the matrix Q , the state ρ_{AB}^{sep} can be expressed as

$$\rho_{AB}^{\text{sep}} = \int p(\mathbf{r}) \bigotimes_{j=A,B} D_j[(\mathcal{V}\mathbf{r})_j] |\gamma_j^p\rangle_j \langle \gamma_j^p| D_j^\dagger[(\mathcal{V}\mathbf{r})_j] \Pi_{l=1}^P dr_l. \quad (63)$$

Here, $D_j(d_j)$ stands for the J_j -mode displacement operator performing phase-space displacement of the subsystem j by $d_j = (d_{j_1}^{(x)}, d_{j_1}^{(p)}, \dots, d_{j_{j_j}}^{(x)}, d_{j_{j_j}}^{(p)})^T$ with $\xi_j = (x_{j_1}, p_{j_1}, \dots, x_{j_{j_j}}, p_{j_{j_j}})^T$ being the vector of the quadratures of the subsystem, $p(\mathbf{r}) \equiv \Pi_{i=1}^P \exp(-r_i^2/\lambda_i)/\sqrt{\pi\lambda_i}$, $|\gamma_{A,B}^p\rangle_{A,B}$ are pure states with CMs $\gamma_{A,B}^p$ and zero displacements, \mathcal{V} is the $2(N+M) \times P$ matrix composed of the first P columns of the matrix V , $\mathbf{r} = (r_1, r_2, \dots, r_P)^T$, $(\mathcal{V}\mathbf{r})_A = [(\mathcal{V}\mathbf{r})_1, (\mathcal{V}\mathbf{r})_2, \dots, (\mathcal{V}\mathbf{r})_{2N}]^T$, and $(\mathcal{V}\mathbf{r})_B = [(\mathcal{V}\mathbf{r})_{2N+1}, (\mathcal{V}\mathbf{r})_{2N+2}, \dots, (\mathcal{V}\mathbf{r})_{2(N+M)}]^T$. Now, we construct a new $(N+M+P)$ -mode purification by encoding the displacements $r_j, j = 1, \dots, P$, into the eigenvectors $|r_j\rangle_{E_j}$ of position quadratures of P purifying modes E_1, E_2, \dots, E_P as

$$|\tilde{\Psi}\rangle_{ABE} = \int \sqrt{p(\mathbf{r})} \bigotimes_{j=A,B} D_j[(\mathcal{V}\mathbf{r})_j] |\gamma_j^p\rangle_j |\mathbf{r}\rangle_E \Pi_{l=1}^P dr_l, \quad (64)$$

where $|\mathbf{r}\rangle_E \equiv |r_1\rangle_{E_1} |r_2\rangle_{E_2} \dots |r_P\rangle_{E_P}$. By measuring position quadratures on all modes of the subsystem E with the outcome \mathbf{r}' one gets the following product conditional state:

$$D_A[(\mathcal{V}\mathbf{r}')_A] |\gamma_A^p\rangle_A D_B[(\mathcal{V}\mathbf{r}')_B] |\gamma_B^p\rangle_B. \quad (65)$$

At this point, we have shown that there is a Gaussian measurement that can be performed on the P modes of the $(N+M+P)$ -mode pure state $|\tilde{\Psi}\rangle_{ABE}$ that leaves Alice's and Bob's modes separable. As the $(N+M+R)$ -mode minimal purification $|\Psi\rangle_{ABE}$ and the $(N+M+P)$ -mode purification $|\tilde{\Psi}\rangle_{ABE}$ both possess the same reduced state ρ_{AB}^{sep} , there is a

Gaussian unitary transformation $U_E(\tilde{S}_E)$ which transforms the purification (64) into the minimal purification as [31,32]

$$U_E(\tilde{S}_E) |\tilde{\Psi}\rangle_{ABE} = |\Psi\rangle_{ABE} |\{0\}\rangle_{E_{R+1}\dots E_P}. \quad (66)$$

Here, $|\{0\}\rangle_{E_{R+1}\dots E_P} \equiv |0\rangle_{E_{R+1}} |0\rangle_{E_{R+2}} \dots |0\rangle_{E_P}$ is the product of $P-R$ vacuum states and $R \leq P$ is the number of symplectic eigenvalues of the CM γ_{AB}^{sep} that are strictly greater than one. The operator $U_E(\tilde{S}_E)$ on the P modes E_1, E_2, \dots, E_P corresponds to the symplectic transformation \tilde{S}_E symplectically diagonalizing the $2P \times 2P$ CM $\tilde{\gamma}_E$ of the subsystem E of the purification (64), i.e., $\tilde{S}_E \tilde{\gamma}_E \tilde{S}_E^T = \text{diag}(\nu_1, \nu_1, \dots, \nu_R, \nu_R, 1, 1, \dots, 1, 1)$, where $\nu_1, \nu_2, \dots, \nu_R$ are symplectic eigenvalues of the CM γ_{AB}^{sep} which are strictly greater than one. Thus, by appending $P-R$ vacuum states $|0\rangle_{E_j}, j = R+1, R+2, \dots, P$, to the minimal purification $|\Psi\rangle_{ABE}$, applying the Gaussian unitary $U_E^\dagger(\tilde{S}_E)$ to the subsystem E , and projecting the subsystem onto the position eigenstate $|\mathbf{r}'\rangle_E$, we get the product state (65). Simple algebra reveals that this measurement can be rewritten as a projection of R modes E_1, E_2, \dots, E_R of the minimal purification $|\Psi\rangle_{ABE}$ onto an unnormalized (and generally unnormalizable) Gaussian state

$$\Pi'_0 = \langle \{0\} | U_E(\tilde{S}_E) |\mathbf{r} = 0\rangle_E \langle \mathbf{r} = 0 | U_E^\dagger(\tilde{S}_E) |\{0\}\rangle, \quad (67)$$

displaced by some factor dependent on the elements of the vector \mathbf{r}' and symplectic matrix \tilde{S}_E , where in Eq. (67) we have omitted the subscripts of the state $|\{0\}\rangle_{E_{R+1}\dots E_P}$ for brevity. Now, let us define a normalized R -mode zero mean Gaussian state

$$\Pi_0 = \frac{\langle \{0\} | U_E(\tilde{S}_E) |\mathbf{s}\rangle_E^{(x)} \langle \mathbf{s} | U_E^\dagger(\tilde{S}_E) |\{0\}\rangle}{\text{Tr}[\langle \{0\} | U_E(\tilde{S}_E) |\mathbf{s}\rangle_E^{(x)} \langle \mathbf{s} | U_E^\dagger(\tilde{S}_E) |\{0\}\rangle]}, \quad (68)$$

which is obtained by replacing the P -mode position eigenvector $|\mathbf{r} = 0\rangle_E$ in the state in Eq. (67) with a P -mode zero mean position squeezed vacuum state $|\mathbf{s}\rangle_E^{(x)} \equiv |s_1\rangle_{E_1}^{(x)} |s_2\rangle_{E_2}^{(x)} \dots |s_P\rangle_{E_P}^{(x)}$, where $|s_j\rangle_{E_j}^{(x)}$ is the zero mean position squeezed vacuum state of mode E_j with the squeezing parameter s_j . It is now obvious that by performing a Gaussian measurement $\Pi_E(d_E) = D_E(d_E) \Pi_0 D_E^\dagger(d_E) / (2\pi)^R$ on the subsystem E of the minimal purification $|\Psi\rangle_{ABE}$, we project the purification onto a product state

$$D_A(d'_A) |\gamma_A^p\rangle_A D_B(d'_B) |\gamma_B^p\rangle_B \quad (69)$$

in the limit of infinite squeezing parameters s_j . The vectors of displacements d'_A and d'_B are linear combinations of the elements of the vector d_E of the measurement outcomes [36]. We have therefore found that for any separable Gaussian state of two subsystems A and B , there is a Gaussian measurement (8) on the purifying part of the state that projects the minimal purification onto a product of pure local states of subsystems A and B as we set out to prove.

VI. MONOTONICITY OF GIE UNDER GAUSSIAN LOCAL TRACE-PRESERVING OPERATIONS AND CLASSICAL COMMUNICATION

The most important property of any good entanglement measure is its monotonicity [19], which means that the measure

does not increase under LOCC operations. Specifically, a good Gaussian entanglement measure should not increase under (generally probabilistic) Gaussian local operations and classical communication (GLOCC) [37]. In this section, we prove that the GIE defined in Eq. (61) is nonincreasing under the subset of GLOCC given by GLTPOCC. This means that if the operation ($\equiv \mathcal{E}$) transforms the input Gaussian state ρ_{AB}^T onto a state $\rho_{AB}^\mathcal{E}$, then

$$E_\downarrow^G(\rho_{AB}^T) \geq E_\downarrow^G(\rho_{AB}^\mathcal{E}). \quad (70)$$

It was shown in the previous section that for two different purifications having in general a differing number of modes, one can always find measurements on Eve's modes of either purification that yield the same matrix (59). Therefore, for CMs γ_π and Γ_E in Eq. (59) we can consider a CM of an arbitrary (not necessarily minimal) purification and a CM of a measurement on Eve's modes of this purification. In the following paragraph, we prove the monotonicity of GIE under GLTPOCC by using a suitable nonminimal purification of the output state $\rho_{AB}^\mathcal{E}$.

A trace-preserving operation \mathcal{E} transforms the input state ρ_{AB}^T to a state

$$\rho_{AB}^\mathcal{E} = \text{Tr}_{\text{in}}[\chi (\rho_{AB}^T)^T \otimes \mathbb{1}_{\text{out}}], \quad (71)$$

where χ is a positive-semidefinite operator representing the operation [38] on the tensor product $\mathcal{H}_{AB} \otimes \mathcal{H}_{\text{out}}$ of the input Hilbert space \mathcal{H}_{AB} and the output Hilbert space \mathcal{H}_{out} , $\mathbb{1}_{\text{out}}$ is the identity operator on the output Hilbert space, and Tr_{in} is the trace over the input Hilbert space. The map preserves the trace of the input state, i.e., $\text{Tr}_{\text{in}}[\rho_{AB}^T] = \text{Tr}_{\text{out}}[\rho_{AB}^\mathcal{E}]$, which imposes the following constraint on the state χ :

$$\text{Tr}_{\text{out}}[\chi] = \mathbb{1}_{\text{in}}, \quad (72)$$

where Tr_{out} is the trace over the output Hilbert space and $\mathbb{1}_{\text{in}}$ is the identity operator on the input Hilbert space.

Let us denote for the state ρ_{AB}^T its minimal purification $|\Psi\rangle_{ABE_\rho}$ with the CM γ_π^T . Let us further denote the measurements on subsystems A , B , and E that achieve the optimum in Eq. (61) as $\Pi_A(d_A)$, $\Pi_B(d_B)$, and $\Pi_{E_\rho}(d_{E_\rho})$ and the corresponding CMs as Γ_A^T , Γ_B^T , and Γ_E^T , respectively. That is,

$$E_\downarrow^G(\rho_{AB}^T) = f(\gamma_\pi^T, \Gamma_A^T, \Gamma_B^T, \Gamma_E^T). \quad (73)$$

Likewise, the purification of the state $\rho_{AB}^\mathcal{E}$ is denoted as $|\Psi^\mathcal{E}\rangle_{ABE}$ and it has the CM $\gamma_\pi^\mathcal{E}$. The measurements on subsystems A , B , and E , which achieve the optimum in Eq. (61) are denoted as $\Pi_A^\mathcal{E}(d_A)$, $\Pi_B^\mathcal{E}(d_B)$, and $\Pi_E^\mathcal{E}(d_E)$ and they have the CMs $\Gamma_A^\mathcal{E}$, $\Gamma_B^\mathcal{E}$, and $\Gamma_E^\mathcal{E}$, respectively. That is,

$$E_\downarrow^G(\rho_{AB}^\mathcal{E}) = f(\gamma_\pi^\mathcal{E}, \Gamma_A^\mathcal{E}, \Gamma_B^\mathcal{E}, \Gamma_E^\mathcal{E}). \quad (74)$$

To prove the inequality (70) we will now find a suitable nonminimal purification of the state (71). The purification can be constructed using the trick that any Gaussian operation on a known state can be implemented via teleportation [36,39]. First, we prepare an $(N + M + N_{\text{out}} + M_{\text{out}})$ -mode state $\chi_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$ of N -mode subsystem A_{in} , M -mode subsystem B_{in} , N_{out} -mode subsystem A_{out} , and M_{out} -mode subsystem B_{out} , which represents the operation \mathcal{E} . Next, subsystems

A and B of the input state ρ_{AB}^T are teleported by a standard continuous-variable teleportation [40], where the state $\chi_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$ serves as a quantum channel. The sender performs Bell measurements on pairs of subsystems (A, A_{in}) and (B, B_{in}) and sends the outcomes of the measurements to the receiver who appropriately displaces his subsystems A_{out} and B_{out} . As a result, he obtains the output state $\rho_{A_{\text{out}}B_{\text{out}}}^\mathcal{E}$ of the operation \mathcal{E} . Let us now consider a pure state

$$|\Phi\rangle = |\Psi\rangle_{ABE_\rho} |\chi\rangle_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}E_\chi} \quad (75)$$

formed as a product of the minimal purification $|\Psi\rangle_{ABE_\rho}$ (with CM γ_π^T) of the input state ρ_{AB}^T and a suitable purification $|\chi\rangle_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}E_\chi}$ of the state $\chi_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$, which will be specified later. Now, we perform Bell measurements on the pairs of subsystems (A, A_{in}) and (B, B_{in}) . A Bell measurement on a pair of modes (j, j_{in}) , $j = A_1, \dots, A_N, B_1, \dots, B_M$, is formally described by the set of rank-one operators $\{|\beta_j\rangle_{j_{\text{in}}}\langle\beta_j|\}_{\beta_j \in \mathbb{C}}$, where β_j is the measurement outcome, \mathbb{C} is the set of complex numbers, and [41]

$$|\beta_j\rangle_{j_{\text{in}}} = \sum_{n=0}^{\infty} \mathcal{D}_j(\beta_j) |n\rangle_j |n\rangle_{j_{\text{in}}}, \quad (76)$$

where $\mathcal{D}_j(\beta_j) = \exp(\beta_j a_j^\dagger - \beta_j^* a_j) = D_j[\sqrt{2}(\text{Re}\beta_j, \text{Im}\beta_j)^T]$ is the displacement operator on mode j , a_j (a_j^\dagger) is the annihilation (creation) operator of the mode, and $|n\rangle$, $n = 0, 1, \dots$, are the Fock states. If we now perform the Bell measurements on pairs of modes $(A_1, A_{1\text{in}}), \dots, (A_N, A_{N\text{in}})$ and $(B_1, B_{1\text{in}}), \dots, (B_M, B_{M\text{in}})$ of the state (75) followed by compensation of the displacements exactly as in the implementation of a generic Gaussian operation by teleportation [39], we obtain a pure state of the form

$$\begin{aligned} & |\Psi^\mathcal{E}\rangle_{A_{\text{out}}B_{\text{out}}E_\rho E_\chi} \\ &= \frac{1}{\sqrt{p_0}} {}_{AA_{\text{in}}}\langle\{\tilde{0}\}| {}_{BB_{\text{in}}}\langle\{\tilde{0}\}| \Psi\rangle_{ABE_\rho} |\chi\rangle_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}E_\chi}, \end{aligned} \quad (77)$$

where $\sqrt{p_0}$ is the normalization factor, and where we have defined $|\{\tilde{0}\}\rangle_{j_{\text{in}}} \equiv |\beta_{j1}=0\rangle_{j_1 j_{1\text{in}}} \dots |\beta_{jJ_j}=0\rangle_{j_j j_j j_{j\text{in}}}$, $j = A, B$, where $J_A = N$ and $J_B = M$ is the number of modes of subsystems A and B , respectively. The state (77) satisfies $\text{Tr}_{E_\rho E_\chi}(|\Psi^\mathcal{E}\rangle_{A_{\text{out}}B_{\text{out}}E_\rho E_\chi} \langle\Psi^\mathcal{E}|) = \rho_{A_{\text{out}}B_{\text{out}}}^\mathcal{E}$ and therefore it is the sought suitable purification of the state $\rho_{AB}^\mathcal{E}$. Consequently, the prescription

$$\begin{aligned} P(d_A, d_B, d_{E_\rho}, d_{E_\chi}) &= \text{Tr}[|\Psi^\mathcal{E}\rangle \langle\Psi^\mathcal{E}| \Pi_{A_{\text{out}}}^\mathcal{E}(d_A) \otimes \Pi_{B_{\text{out}}}^\mathcal{E}(d_B) \\ &\quad \otimes \Pi_{E_\rho E_\chi}^\mathcal{E}(d_{E_\rho}, d_{E_\chi})] \end{aligned} \quad (78)$$

defines the optimal distribution whose conditional mutual information equals to $E_\downarrow^G(\rho_{AB}^\mathcal{E})$ where $\Pi_{A_{\text{out}}}^\mathcal{E}(d_A)$, $\Pi_{B_{\text{out}}}^\mathcal{E}(d_B)$, and $\Pi_{E_\rho E_\chi}^\mathcal{E}(d_{E_\rho}, d_{E_\chi})$ are optimal measurements with CMs $\Gamma_A^\mathcal{E}$, $\Gamma_B^\mathcal{E}$, and $\Gamma_E^\mathcal{E}$. Here, a different symbol $\Pi_{E_\rho E_\chi}^\mathcal{E}(d_{E_\rho}, d_{E_\chi})$ for the optimal measurement $\Pi_E^\mathcal{E}(d_E)$ has been used to express the fact that it acts on two purifying subsystems E_ρ and E_χ . Here and in what follows, we also omit the indices of the purification (77) for brevity.

Now, we will construct a suitable purification $|\chi\rangle_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}E_\chi}$ of the state $\chi_{A_{\text{in}}B_{\text{in}}A_{\text{out}}B_{\text{out}}}$ representing the Gaussian map \mathcal{E} . As the map can be created by local operations and classical communication, the corresponding

Gaussian state $\chi_{A_{in}B_{in}A_{out}B_{out}}$ is separable across $A_{in}A_{out}|B_{in}B_{out}$ splitting [36]. For the $2(N + N_{out} + M + M_{out})$ -dimensional CM $\gamma_{A_{in}A_{out}B_{in}B_{out}}^\chi$ of the state there therefore exist local $2(N + N_{out})$ -dimensional CM $\gamma_{A_{in}A_{out}}^\chi$ and $2(M + M_{out})$ -dimensional CM $\gamma_{B_{in}B_{out}}^\chi$ corresponding to generally mixed Gaussian states $\chi_{A_{in}A_{out}}$ and $\chi_{B_{in}B_{out}}$ of the subsystems A and B such that [35]

$$O \equiv \gamma_{A_{in}A_{out}B_{in}B_{out}}^\chi - \gamma_{A_{in}A_{out}}^\chi \oplus \gamma_{B_{in}B_{out}}^\chi \geq 0. \quad (79)$$

Repeating the algorithm leading to Eq. (63) for the case of the state $\chi_{A_{in}B_{in}A_{out}B_{out}}$, we then arrive at the following expression of the state:

$$\chi_{A_{in}B_{in}A_{out}B_{out}} = \int q(\mathbf{r}) D(\mathscr{W}\mathbf{r}) (\chi_{A_{in}A_{out}} \otimes \chi_{B_{in}B_{out}}) D^\dagger(\mathscr{W}\mathbf{r}) d^{P'}\mathbf{r}. \quad (80)$$

Here, $d^{P'}\mathbf{r} \equiv \prod_{i=1}^{P'} dr_i$, $q(\mathbf{r}) \equiv \prod_{i=1}^{P'} \exp(-r_i^2/o_i)/\sqrt{\pi o_i}$ with $o_i, i = 1, 2, \dots, P'$, being all strictly positive eigenvalues of the matrix O , \mathscr{W} is the $2(N + N_{out} + M + M_{out}) \times P'$ matrix composed of the first P' columns of the matrix W which diagonalizes the matrix O as $W^T O W = \text{diag}(o_1, o_2, \dots, o_{P'}, 0, \dots, 0)$, and $\mathbf{r} = (r_1, r_2, \dots, r_{P'})^T$. Further, for CM $\gamma_{j_{in}j_{out}}^\chi$, $j = A, B$, there always exist pure-state CMs $\gamma_{j_{in}j_{out}}^{\chi, P}$ such that [42]

$$T_j \equiv \gamma_{j_{in}j_{out}}^\chi - \gamma_{j_{in}j_{out}}^{\chi, P} \geq 0, \quad j = A, B. \quad (81)$$

Denoting as R_j , $j = A, B$, the orthogonal matrix bringing the matrix T_j to the diagonal form, i.e., $R_j^T T_j R_j = \text{diag}(t_1^j, t_2^j, \dots, t_{P_j}^j, 0, \dots, 0)$, where $t_l^j, l = 1, 2, \dots, P_j$, are the strictly positive eigenvalues of the matrix T_j , we can further express the local Gaussian states $\chi_{j_{in}j_{out}}$ as

$$\chi_{j_{in}j_{out}} = \int q_j(\mathbf{r}_j) D_j(\mathscr{R}_j \mathbf{r}_j) |\gamma_{j_{in}j_{out}}^{\chi, P}\rangle_{j_{in}j_{out}} \langle \gamma_{j_{in}j_{out}}^{\chi, P} | D_j^\dagger(\mathscr{R}_j \mathbf{r}_j) d^{P_j} \mathbf{r}_j. \quad (82)$$

Here, $d^{P_j} \mathbf{r}_j \equiv \prod_{l=1}^{P_j} dr_{jl}$, $q_j(\mathbf{r}_j) \equiv \prod_{i=1}^{P_j} \exp(-r_{ji}^2/t_i^j)/\sqrt{\pi t_i^j}$, \mathscr{R}_A is the $2(N + N_{out}) \times P_A$ matrix composed of the first P_A columns of the matrix R_A , \mathscr{R}_B is the $2(M + M_{out}) \times P_B$ matrix composed of the first P_B columns of the matrix R_B , and $\mathbf{r}_j = (r_{j1}, r_{j2}, \dots, r_{jP_j})^T$. Inserting now into Eq. (80) for the states $\chi_{A_{in}A_{out}}$ and $\chi_{B_{in}B_{out}}$ from Eq. (82) we get

$$\begin{aligned} \chi_{A_{in}B_{in}A_{out}B_{out}} &= \int q(\mathbf{r}) D(\mathscr{W}\mathbf{r}) \\ &\times \left[\bigotimes_{j=A, B} q_j(\mathbf{r}_j) D_j(\mathscr{R}_j \mathbf{r}_j) |\gamma_{j_{in}j_{out}}^{\chi, P}\rangle_{j_{in}j_{out}} \langle \gamma_{j_{in}j_{out}}^{\chi, P} | D_j^\dagger(\mathscr{R}_j \mathbf{r}_j) \right] \\ &\times D^\dagger(\mathscr{W}\mathbf{r}) d^{P_A} \mathbf{r}_A d^{P_B} \mathbf{r}_B d^{P'} \mathbf{r}. \end{aligned} \quad (83)$$

By encoding the vectors of displacements \mathbf{r} , \mathbf{r}_A , and \mathbf{r}_B into eigenvectors $|\mathbf{r}\rangle_{E_O}$, $|\mathbf{r}_A\rangle_{E_A}$, and $|\mathbf{r}_B\rangle_{E_B}$ of position quadratures of Eve's $(P' + P_A + P_B)$ -mode subsystem $E_\chi \equiv (E_O E_A E_B)$, we obtain finally the sought purification

$$\begin{aligned} |\chi\rangle &= \int \sqrt{q(\mathbf{r}) q_A(\mathbf{r}_A) q_B(\mathbf{r}_B)} D(\mathscr{W}\mathbf{r}) \bigotimes_{j=A, B} D_j(\mathscr{R}_j \mathbf{r}_j) |\gamma_{j_{in}j_{out}}^{\chi, P}\rangle_{j_{in}j_{out}} \\ &\otimes |\mathbf{r}\rangle_{E_O} |\mathbf{r}_A\rangle_{E_A} |\mathbf{r}_B\rangle_{E_B} d^{P_A} \mathbf{r}_A d^{P_B} \mathbf{r}_B d^{P'} \mathbf{r}, \end{aligned} \quad (84)$$

where we have omitted the indices $A_{in}B_{in}A_{out}B_{out}E_\chi$ of the purification $|\chi\rangle$ for brevity.

A specific feature of the state (84) is that by a simple measurement on the purifying subsystem E_χ we can project the state onto a displaced product state $\chi_{A_{in}A_{out}} \otimes \chi_{B_{in}B_{out}}$ of the subsystems (A_{in}, A_{out}) and (B_{in}, B_{out}) . More precisely, consider the following measurement on Eve's subsystem E_χ :

$$\tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}') = |\mathbf{r}'\rangle_{E_O} \langle \mathbf{r}'| \otimes \mathbb{1}_{E_A} \otimes \mathbb{1}_{E_B}, \quad (85)$$

which describes the projection of subsystem E_O onto a P' -mode position eigenvector $|\mathbf{r}'\rangle_{E_O}$ and projection of subsystems E_A and E_B onto maximally mixed states, which gives Eve no information on the state of the two subsystems. Recall that the latter measurements on subsystems E_A and E_B can be seen as Gaussian measurements (8) with seed elements given by thermal states in the limit of infinite temperature. By performing the measurement (85) on the subsystem E_χ of the purification (84), we then arrive using Eq. (82) at the conditional state of the form

$$\text{Tr}_{E_\chi} [|\chi\rangle \langle \chi| \tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')] = q(\mathbf{r}') \bigotimes_{j=A, B} \chi_{j_{in}j_{out}} [(\mathscr{W}\mathbf{r}')_{j_{in}j_{out}}], \quad (86)$$

which is the desired product state with respect to the $A_{in}A_{out}|B_{in}B_{out}$ splitting. Here,

$$\begin{aligned} \chi_{j_{in}j_{out}} [(\mathscr{W}\mathbf{r}')_{j_{in}j_{out}}] &\equiv D_{j_{in}j_{out}} [(\mathscr{W}\mathbf{r}')_{j_{in}j_{out}}] \chi_{j_{in}j_{out}} \\ &\times D_{j_{in}j_{out}}^\dagger [(\mathscr{W}\mathbf{r}')_{j_{in}j_{out}}], \end{aligned} \quad (87)$$

where

$$\begin{aligned} (\mathscr{W}\mathbf{r}')_{A_{in}A_{out}} &\equiv [(\mathscr{W}\mathbf{r}')_{A_{in}1}, (\mathscr{W}\mathbf{r}')_{A_{in}2}, \dots, (\mathscr{W}\mathbf{r}')_{A_{in}2N}, \\ &(\mathscr{W}\mathbf{r}')_{A_{out}1}, (\mathscr{W}\mathbf{r}')_{A_{out}2}, \dots, (\mathscr{W}\mathbf{r}')_{A_{out}2N_{out}}]^T, \\ (\mathscr{W}\mathbf{r}')_{B_{in}B_{out}} &\equiv [(\mathscr{W}\mathbf{r}')_{B_{in}1}, (\mathscr{W}\mathbf{r}')_{B_{in}2}, \dots, (\mathscr{W}\mathbf{r}')_{B_{in}2M}, \\ &(\mathscr{W}\mathbf{r}')_{B_{out}1}, (\mathscr{W}\mathbf{r}')_{B_{out}2}, \dots, (\mathscr{W}\mathbf{r}')_{B_{out}2M_{out}}]^T. \end{aligned} \quad (88)$$

Before going further, let us note that any trace-preserving Gaussian operation \mathcal{E} is represented by an unphysical (infinitely squeezed) density matrix χ . This is because the matrix is obtained by an action of the operation \mathcal{E} on one part of an unphysical maximally entangled state $|\tilde{\Phi}\rangle$ [38]. The unphysical states can nevertheless be dealt with rigorously in the context of positive forms [43] or by using the limiting procedure proposed in Ref. [36]. The latter approach consists of the replacement of the state $|\tilde{\Phi}\rangle$ by its physical approximation $|\tilde{\Phi}(r)\rangle$ given by a tensor product of identical two-mode squeezed vacuum states with squeezing parameter r . The operation \mathcal{E} is then represented by a quantum state $\chi(r)$ obtained by action of the operation on one part of the state $|\tilde{\Phi}(r)\rangle$, which is a physical approximation of the exact state χ . For a quantum operation \mathcal{E} , which can be prepared by local operations and classical communication, the density matrix $\chi(r)$ is separable and hence the above formulas remain valid also for quantum state $\chi(r)$. The sought exact result is recovered and the limiting procedure is thus accomplished by taking the limit $r \rightarrow \infty$ at the end of our calculations.

Returning to the monotonicity proof, consider now the probability density

$$\tilde{P}(d_A, d_B, d_{E_\rho}, \mathbf{r}') = \text{Tr} [|\Psi^\mathcal{E}\rangle \langle \Psi^\mathcal{E}| \Pi_{A_{\text{out}}}^\mathcal{E}(d_A) \otimes \Pi_{B_{\text{out}}}^\mathcal{E}(d_B) \otimes \Pi_{E_\rho}(d_{E_\rho}) \otimes \tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')], \quad (89)$$

which is obtained from the probability density (78) by replacing the optimal measurement $\Pi_{E_\rho E_\chi}^\mathcal{E}(d_{E_\rho}, d_{E_\chi})$ with a product measurement $\Pi_{E_\rho}(d_{E_\rho}) \otimes \tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')$. Here, $\Pi_{E_\rho}(d_{E_\rho})$ is the optimal measurement with the CM $\Gamma_E^\mathcal{I}$ on the minimal purification $|\Psi\rangle_{ABE_\rho}$ of the state $\rho_{AB}^\mathcal{E}$ and $\tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')$ is the measurement (85) with the CM $\tilde{\Gamma}_{E_\chi}^\mathcal{E}$, which projects the purification (84) onto the product state (86). At given CMs $\gamma_\pi^\mathcal{E}$, $\Gamma_A^\mathcal{E}$, and $\Gamma_B^\mathcal{E}$ the product measurement with the CM $\Gamma_E^\mathcal{I} \oplus \tilde{\Gamma}_{E_\chi}^\mathcal{E}$ does not generally minimize the function $f(\gamma_\pi^\mathcal{E}, \Gamma_A^\mathcal{E}, \Gamma_B^\mathcal{E}, \Gamma_E^\mathcal{I})$ with respect to the CM $\Gamma_E^\mathcal{I}$ and hence the function f corresponding to the distribution $\tilde{P}(d_A, d_B, d_{E_\rho}, \mathbf{r}')$ [Eq. (89)] satisfies

$$E_\downarrow^G(\rho_{AB}^\mathcal{E}) \leq f(\gamma_\pi^\mathcal{E}, \Gamma_A^\mathcal{E}, \Gamma_B^\mathcal{E}, \Gamma_E^\mathcal{I} \oplus \tilde{\Gamma}_{E_\chi}^\mathcal{E}). \quad (90)$$

What is more, one can show that there exist Gaussian measurements $\tilde{\Pi}_A(\tilde{d}_A)$ and $\tilde{\Pi}_B(\tilde{d}_B)$ on the subsystems A and B of the normalized conditional state

$$\rho_{AB|E_\rho}(d_{E_\rho}) = \frac{\text{Tr}_{E_\rho} [|\Psi\rangle_{ABE_\rho} \langle \Psi| \Pi_{E_\rho}(d_{E_\rho})]}{P(d_{E_\rho})} \quad (91)$$

obtained by the optimal measurement $\Pi_{E_\rho}(d_{E_\rho})$ on subsystem E_ρ of the minimal purification $|\Psi\rangle_{ABE_\rho}$, which are characterized by the CMs $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$, such that the conditional distribution

$$\tilde{p}(\tilde{d}_A, \tilde{d}_B | d_{E_\rho}) = \text{Tr} [\rho_{AB|E_\rho}(d_{E_\rho}) \tilde{\Pi}_A(\tilde{d}_A) \otimes \tilde{\Pi}_B(\tilde{d}_B)] \quad (92)$$

yields the function f defined in Eq. (60), which is greater or equal than the function on the RHS of inequality (90), i.e.,

$$f(\gamma_\pi^\mathcal{E}, \Gamma_A^\mathcal{E}, \Gamma_B^\mathcal{E}, \Gamma_E^\mathcal{I} \oplus \tilde{\Gamma}_{E_\chi}^\mathcal{E}) \leq f(\gamma_\pi^\mathcal{I}, \tilde{\Gamma}_A, \tilde{\Gamma}_B, \Gamma_E^\mathcal{I}). \quad (93)$$

This can be shown as follows. The function on the RHS of inequality (90) is the mutual information of the conditional Gaussian distribution $\tilde{P}(d_A, d_B | d_{E_\rho}, \mathbf{r}') = \tilde{P}(d_A, d_B, d_{E_\rho}, \mathbf{r}') / \tilde{P}(d_{E_\rho}, \mathbf{r}')$, where the distribution $\tilde{P}(d_A, d_B, d_{E_\rho}, \mathbf{r}')$ is given in Eq. (89). The conditional distribution is the distribution of outcomes of Gaussian measurements with CMs $\Gamma_A^\mathcal{E}$ and $\Gamma_B^\mathcal{E}$ on subsystems A and B of the conditional state ($\equiv \rho_{AB|E}^\mathcal{E}$) obtained by Gaussian measurement $\Pi_{E_\rho}(d_{E_\rho}) \otimes \tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')$ (with CM $\Gamma_E^\mathcal{I} \oplus \tilde{\Gamma}_{E_\chi}^\mathcal{E}$) on the purification (77), where the state $|\chi\rangle$ is given in Eq. (84). Substituting from Eqs. (77), (84), and (86) into the explicit expression for the (unnormalized) conditional state

$$\tilde{\rho}_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E} = \text{Tr}_{E_\rho E_\chi} [|\Psi^\mathcal{E}\rangle \langle \Psi^\mathcal{E}| \Pi_{E_\rho}(d_{E_\rho}) \otimes \tilde{\Pi}_{E_\chi}^\mathcal{E}(\mathbf{r}')], \quad (94)$$

one arrives after some algebra at the conditional state in the form

$$\tilde{\rho}_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E} = \frac{P(d_{E_\rho})q(\mathbf{r}')}{p_0} \text{Tr}_{AA_{\text{in}}BB_{\text{in}}} \left\{ \rho_{AB|E_\rho}(d_{E_\rho}) \otimes \bigotimes_{j=A,B} \chi_{j_{\text{in}}j_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{j_{\text{in}}j_{\text{out}}}] \right\}$$

$$\times \bigotimes_{k=A,B} \{ |\tilde{0}\rangle_{kk_{\text{in}}} \langle \tilde{0}| \}, \quad (95)$$

where the state $\rho_{AB|E_\rho}(d_{E_\rho})$ is defined in Eq. (91). Here and in what follows, we do not write explicitly in some places the dependence of the conditional states on the measurement outcomes for brevity.

Expressing the operator $|\{\tilde{0}\}\rangle_{j_{\text{in}}}\langle\{\tilde{0}\}|$ on the RHS of the latter equation using Eq. (76) and carrying out the trace over the subsystems A and B , we further get

$$\tilde{\rho}_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E} = \frac{P(d_{E_\rho})q(\mathbf{r}')}{p_0} \times \text{Tr}_{A_{\text{in}}B_{\text{in}}} \left\{ \bigotimes_{j=A,B} \chi_{j_{\text{in}}j_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{j_{\text{in}}j_{\text{out}}}] \times \rho_{A_{\text{in}}B_{\text{in}}|E_\rho}^T(d_{E_\rho}) \otimes \mathbb{1}_{A_{\text{out}}B_{\text{out}}} \right\}. \quad (96)$$

Let us assume now that the considered separable operation \mathcal{E} is GLTPOCC, i.e., it can be decomposed into Gaussian local trace-preserving operations on subsystems A and B , and the addition of classical Gaussian noise. The density matrices $\chi_{j_{\text{in}}j_{\text{out}}}$, $j = A, B$, representing the local operations then satisfy the trace-preservation constraints (72), i.e.,

$$\text{Tr}_{j_{\text{out}}} [\chi_{j_{\text{in}}j_{\text{out}}}] = \mathbb{1}_{j_{\text{in}}}, \quad j = A, B \quad (97)$$

which imply fulfillment of the trace-preservation constraints for the states (87):

$$\text{Tr}_{j_{\text{out}}} \{ \chi_{j_{\text{in}}j_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{j_{\text{in}}j_{\text{out}}}] \} = \mathbb{1}_{j_{\text{in}}}, \quad j = A, B. \quad (98)$$

As a consequence, one finds the trace of the conditional state (96) to be

$$\text{Tr}_{A_{\text{out}}B_{\text{out}}} [\tilde{\rho}_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E}] = \tilde{P}(d_{E_\rho}, \mathbf{r}') = \frac{P(d_{E_\rho})q(\mathbf{r}')}{p_0}, \quad (99)$$

and therefore the normalized conditional state reads as

$$\rho_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E} = \text{Tr}_{A_{\text{in}}B_{\text{in}}} \left\{ \bigotimes_{j=A,B} \chi_{j_{\text{in}}j_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{j_{\text{in}}j_{\text{out}}}] \times \rho_{A_{\text{in}}B_{\text{in}}|E_\rho}^T(d_{E_\rho}) \otimes \mathbb{1}_{A_{\text{out}}B_{\text{out}}} \right\}. \quad (100)$$

If we further substitute here for the operators $\chi_{j_{\text{in}}j_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{j_{\text{in}}j_{\text{out}}}]$ from Eq. (87) and we use the relation $D^T(d) = D(-\Lambda d)$, where T stands for the transposition in Fock basis and the diagonal matrix $\Lambda \equiv \text{diag}(1, -1, 1, -1, \dots, 1, -1)$ realizes the transposition operation on the CM level, we get the conditional state (100) in the form

$$\rho_{A_{\text{out}}B_{\text{out}}|E}^\mathcal{E} = D_{A_{\text{out}}B_{\text{out}}} [(\mathcal{W}\mathbf{r}')_{A_{\text{out}}B_{\text{out}}}] (\mathcal{E}_A \otimes \mathcal{E}_B) (\rho'_{A_{\text{in}}B_{\text{in}}|E_\rho}) \times D_{A_{\text{out}}B_{\text{out}}}^\dagger [(\mathcal{W}\mathbf{r}')_{A_{\text{out}}B_{\text{out}}}]. \quad (101)$$

Here,

$$\rho'_{A_{\text{in}}B_{\text{in}}|E_\rho} \equiv D_{A_{\text{in}}B_{\text{in}}} [-\Lambda (\mathcal{W}\mathbf{r}')_{A_{\text{in}}B_{\text{in}}}] \rho_{A_{\text{in}}B_{\text{in}}|E_\rho}(d_{E_\rho}) \times D_{A_{\text{in}}B_{\text{in}}}^\dagger [-\Lambda (\mathcal{W}\mathbf{r}')_{A_{\text{in}}B_{\text{in}}}], \quad (102)$$

and \mathcal{E}_j , $j = A, B$, is the local Gaussian trace-preserving operation represented by the density matrix $\chi_{j_{in}j_{out}}$, i.e.,

$$(\mathcal{E}_A \otimes \mathcal{E}_B)(\rho'_{A_{in}B_{in}|E_\rho}) \\ = \text{Tr}_{A_{in}B_{in}} \left\{ \chi_{A_{in}A_{out}} \otimes \chi_{B_{in}B_{out}} (\rho'_{A_{in}B_{in}|E_\rho})^{T_{in}} \otimes \mathbb{1}_{A_{out}B_{out}} \right\}. \quad (103)$$

We have already said that the RHS of Eq. (90) is the mutual information of the conditional distribution

$$\tilde{P}(d_A, d_B | d_{E_\rho}, \mathbf{r}') = \text{Tr}_{A_{out}B_{out}} \left[\rho_{A_{out}B_{out}|E}^\mathcal{E} \Pi_{A_{out}}^\mathcal{E}(d_A) \otimes \Pi_{B_{out}}^\mathcal{E}(d_B) \right] \quad (104)$$

of the outcomes of Gaussian measurements $\Pi_{A_{out}}^\mathcal{E}(d_A)$ and $\Pi_{B_{out}}^\mathcal{E}(d_B)$ (characterized by CMs $\Gamma_A^\mathcal{E}$ and $\Gamma_B^\mathcal{E}$) on the conditional state (101). Substituting into the RHS of the latter equation for the conditional state from Eq. (101), one finds after some algebra that

$$\tilde{P}(d_A, d_B | d_{E_\rho}, \mathbf{r}') = \tilde{P}[d_A - (\mathcal{W}\mathbf{r}')_{A_{out}}, d_B - (\mathcal{W}\mathbf{r}')_{B_{out}} | d_{E_\rho}, \mathbf{r}'], \quad (105)$$

where

$$(\mathcal{W}\mathbf{r}')_{j_{out}} \equiv [(\mathcal{W}\mathbf{r}')_{j_{out}1}, (\mathcal{W}\mathbf{r}')_{j_{out}2}, \dots, (\mathcal{W}\mathbf{r}')_{j_{out}2J_{j_{out}}}]^T,$$

where $J_{A_{out}} = N_{out}$ and $J_{B_{out}} = M_{out}$, and

$$\tilde{P}(d_A, d_B | d_{E_\rho}, \mathbf{r}') \\ = \text{Tr}_{A_{out}B_{out}} \left[(\mathcal{E}_A \otimes \mathcal{E}_B)(\rho'_{A_{in}B_{in}|E_\rho}) \Pi_{A_{out}}^\mathcal{E}(d_A) \otimes \Pi_{B_{out}}^\mathcal{E}(d_B) \right]. \quad (106)$$

The mutual information of the distribution in Eq. (105) does not depend on the displacements $-(\mathcal{W}\mathbf{r}')_{j_{out}}$, $j = A, B$, and hence it is equal to the mutual information of the distribution (106). The tensor product $\mathcal{E}_A \otimes \mathcal{E}_B$ of Gaussian local trace-preserving operations \mathcal{E}_j , $j = A, B$, appearing on the RHS of Eq. (106), transforms the $(N + M)$ -mode Gaussian state $\rho'_{A_{in}B_{in}|E_\rho}$ [Eq. (102)] onto an $(N_{out} + M_{out})$ -mode Gaussian state. More precisely, the operation \mathcal{E}_j , $j = A, B$, transforms $J_{j_{in}}$ modes $j_{in}1, j_{in}2, \dots, j_{in}J_{j_{in}}$ of the state (102) onto $J_{j_{out}}$ output modes $j_{out}1, j_{out}2, \dots, j_{out}J_{j_{out}}$, where $J_{A_{in}} = N$ and $J_{B_{in}} = M$. As each operation \mathcal{E}_j is Gaussian and trace preserving, it can be realized in three steps encompassing (1) a Gaussian unitary interaction U_j between the $J_{j_{in}}$ input modes and $J_{j_{anc}}$ ancillary modes in vacuum states, where $J_{A_{anc}} = N_{anc}$ and $J_{B_{anc}} = M_{anc}$, followed by (2) discarding of $J_{j_{disc}} \equiv J_{j_{in}} + J_{j_{anc}} - J_{j_{out}}$ modes, and (3) addition of classical Gaussian noise [44,45]. The noise can be created by a random displacement of the output state in phase space distributed according to a zero mean Gaussian distribution. The addition of the zero mean Gaussian noise acts only on the level of the CMs where it is represented by the addition of a positive-semidefinite matrix F_j to the CM of the output state. Similarly, the measurement $\Pi_{j_{out}}^\mathcal{E}(d_j)$ on the subsystem is on the level of the CM represented by the addition of a CM $\Gamma_j^\mathcal{E}$ to the CM of the measured state. Denoting as γ_2 the $2(N_{out} + M_{out})$ -dimensional CM of the state obtained by propagation of the input state $\rho'_{A_{in}B_{in}|E_\rho}$ through steps (1) and (2) of the implementation of the operations \mathcal{E}_A and \mathcal{E}_B , the CCM of the distribution (106) reads as

$$\gamma_2 + F_A \oplus F_B + \Gamma_A^\mathcal{E} \oplus \Gamma_B^\mathcal{E} = \gamma_2 + (\Gamma_A^\mathcal{E} + F_A) \oplus (\Gamma_B^\mathcal{E} + F_B). \quad (107)$$

Therefore, the addition of local classical Gaussian noise into subsystems A_{out} and B_{out} followed by the local Gaussian measurements $\Pi_{A_{out}}^\mathcal{E}(d_A)$ and $\Pi_{B_{out}}^\mathcal{E}(d_B)$ on the subsystems can be viewed just as more noisy local Gaussian measurements $\tilde{\Pi}_{A_{out}}^\mathcal{E}(d_A)$ and $\tilde{\Pi}_{B_{out}}^\mathcal{E}(d_B)$ characterized by the CM $\tilde{\Gamma}_A^\mathcal{E} \equiv \Gamma_A^\mathcal{E} + F_A$ and $\tilde{\Gamma}_B^\mathcal{E} \equiv \Gamma_B^\mathcal{E} + F_B$. Consequently, the conditional distribution (106) can be expressed as

$$\tilde{P}(d_A, d_B | d_{E_\rho}, \mathbf{r}') \\ = \text{Tr}_{A_{out}B_{out}} \left[(\tilde{\mathcal{E}}_A \otimes \tilde{\mathcal{E}}_B)(\rho'_{A_{in}B_{in}|E_\rho}) \tilde{\Pi}_{A_{out}}^\mathcal{E}(d_A) \otimes \tilde{\Pi}_{B_{out}}^\mathcal{E}(d_B) \right], \quad (108)$$

where $\tilde{\mathcal{E}}_A$ and $\tilde{\mathcal{E}}_B$ are local Gaussian trace-preserving operations which can be implemented using steps (1) and (2) but which do not require addition of classical noise. If we now express the latter two operations via local Gaussian unitary transformations U_A and U_B on a larger system consisting of N -mode subsystem A_{in} , M -mode subsystem B_{in} , N_{anc} auxiliary vacuum modes denoted as a subsystem A_{anc} , and M_{anc} auxiliary vacuum modes denoted as a subsystem B_{anc} , the distribution (108) attains the form

$$\tilde{P}(d_A, d_B) \\ = \text{Tr}_{A_{out}B_{out}} \text{Tr}_{A_{disc}B_{disc}} \left[(U_A \otimes U_B) \rho'_{A_{in}B_{in}|E_\rho} \otimes |\{0\}\rangle_{A_{anc}} \langle\{0}| \right. \\ \otimes |\{0}\rangle_{B_{anc}} \langle\{0}| (U_A^\dagger \otimes U_B^\dagger) \tilde{\Pi}_{A_{out}}^\mathcal{E}(d_A) \\ \left. \otimes \tilde{\Pi}_{B_{out}}^\mathcal{E}(d_B) \otimes \mathbb{1}_{A_{disc}} \otimes \mathbb{1}_{B_{disc}} \right], \quad (109)$$

where here and in what follows we omit the dependence of the distribution on the variables d_{E_ρ} and \mathbf{r}' for brevity. Here, $\text{Tr}_{j_{disc}}$, $j = A, B$, is the trace over the discarded $J_{j_{disc}}$ -mode subsystem j_{disc} ($J_{A_{disc}} = N + N_{anc} - N_{out}$ and $J_{B_{disc}} = M + M_{anc} - M_{out}$), $|\{0\}\rangle_{j_{anc}}$ is the tensor product of $J_{j_{anc}}$ vacuum states, and $\mathbb{1}_{j_{disc}}$ is the identity operator on the space of the discarded subsystem j_{disc} . Next, the linearity of the Gaussian unitary transformation $U_A \otimes U_B$ allows us to transform the displacement $D_{A_{in}B_{in}}[-\Lambda(\mathcal{W}\mathbf{r}')_{A_{in}B_{in}}]$ in Eq. (102) through the transformation which will result, together with utilization of the invariance of the trace under cyclic permutations, in a displacement of the measurement outcomes d_A and d_B . However, as we have already said, such a displacement is irrelevant from the point of view of mutual information and hence we can replace in what follows the displaced state $\rho'_{A_{in}B_{in}|E_\rho}$ on the RHS of Eq. (109) with the undisplaced state $\rho_{A_{in}B_{in}|E_\rho}(d_{E_\rho})$ defined in Eq. (91). Further, the distribution (109) can be seen as the reduction

$$\tilde{P}(d_A, d_B) = \int \mathfrak{P}(d_A, d'_A, d_B, d'_B) d^{2J_{A_{disc}}} d'_A d^{2J_{B_{disc}}} d'_B \quad (110)$$

of the following distribution:

$$\mathfrak{P}(d_A, d'_A, d_B, d'_B) \\ = \text{Tr}_{A_{out}B_{out}} \text{Tr}_{A_{disc}B_{disc}} \left[(U_A \otimes U_B) \rho_{A_{in}B_{in}|E_\rho} \otimes |\{0\}\rangle_{A_{anc}} \langle\{0}| \right. \\ \otimes |\{0}\rangle_{B_{anc}} \langle\{0}| (U_A^\dagger \otimes U_B^\dagger) \tilde{\Pi}_{A_{out}}^\mathcal{E}(d_A) \\ \left. \otimes \tilde{\Pi}_{B_{out}}^\mathcal{E}(d_B) \otimes \Pi_{A_{disc}}(d'_A) \otimes \Pi_{B_{disc}}(d'_B) \right], \quad (111)$$

where $\Pi_{j_{disc}}(d'_j)$, $j = A, B$, is a Gaussian measurement on the discarded subsystem j_{disc} with the measurement outcome d'_j and where we have omitted the dependence of

the state $\rho_{A_{\text{in}}B_{\text{in}}|E_\rho}$ on the measurement outcome d_{E_ρ} for simplicity. As discarding variables cannot increase the mutual information [46], one obtains that the mutual information $I(A; B)$ of the distribution (109) and the mutual information $I(A, A'; B, B')$ of the distribution (111) satisfy the inequality $I(A; B) \leq I(A, A'; B, B')$. Now, making use of the invariance of the trace under cyclic permutations and the equality $U^\dagger(\mathcal{S})\Pi(d)U(\mathcal{S}) = \Pi_{\mathcal{S}}(\mathcal{S}^{-1}d)$, where $\Pi_{\mathcal{S}}(d)$ is a component of a Gaussian POVM with the seed element $U^\dagger(\mathcal{S})\Pi_0U(\mathcal{S})$ and $U(\mathcal{S})$ is a Gaussian unitary transformation corresponding to the symplectic matrix \mathcal{S} , we can write down the distribution (111) as

$$\mathfrak{P}(d_A, d'_A, d_B, d'_B) = \mathcal{P}[(\mathcal{S}_A^{-1}\Delta_A)^T, (\mathcal{S}_B^{-1}\Delta_B)^T].$$

Here, \mathcal{S}_A and \mathcal{S}_B denote the symplectic matrices corresponding to the local Gaussian unitaries U_A and U_B , respectively, $\Delta_j = (d_j^T, d_j'^T)^T$, $j = A, B$, and

$$\begin{aligned} \mathcal{P}(d_A, d'_A, d_B, d'_B) &= \text{Tr}_{A_{\text{in}}A_{\text{anc}}B_{\text{in}}B_{\text{anc}}}[\rho_{A_{\text{in}}B_{\text{in}}|E_\rho} \\ &\otimes |\{0\}\rangle_{A_{\text{anc}}B_{\text{anc}}}\langle\{0\}| \Pi'_{A_{\text{in}}A_{\text{anc}}}(d_A, d'_A) \otimes \Pi'_{B_{\text{in}}B_{\text{anc}}}(d_B, d'_B)], \end{aligned} \quad (112)$$

where $|\{0\}\rangle_{A_{\text{anc}}B_{\text{anc}}} = |\{0\}\rangle_{A_{\text{anc}}} \otimes |\{0\}\rangle_{B_{\text{anc}}}$ and $\Pi'_{j_{\text{in}}j_{\text{anc}}}(d_j, d'_j)$, $j = A, B$, is the Gaussian measurement on the subsystem $(j_{\text{in}}, j_{\text{anc}})$ with the seed element $\Pi'_{0j_{\text{in}}j_{\text{anc}}} \equiv U_j^\dagger \bar{\Pi}_{0j_{\text{out}}}^\varepsilon \otimes \Pi_{0j_{\text{disc}}} U_j$, where $\bar{\Pi}_{0j_{\text{out}}}^\varepsilon$ and $\Pi_{0j_{\text{disc}}}$ are the seed elements of the Gaussian measurements $\bar{\Pi}_{j_{\text{out}}}^\varepsilon(d_j)$ and $\Pi_{j_{\text{disc}}}(d'_j)$, respectively, which appear on the RHS of Eq. (111). From the invariance of the mutual information under local symplectic transformations, it then follows that the mutual information of the distribution (111) and the distribution (112) are equal and hence we can further work with the distribution (112).

Let us denote now the CM of the conditional state $\rho_{A_{\text{in}}B_{\text{in}}|E_\rho}$ as γ_{AB}^c and the CMs of the measurements $\Pi'_{A_{\text{in}}A_{\text{anc}}}(d_A, d'_A)$ and $\Pi'_{B_{\text{in}}B_{\text{anc}}}(d_B, d'_B)$ as Γ'_A and Γ'_B , respectively. The mutual information of the distribution (112) then attains the form [33]

$$I(A, A'; B, B') = \frac{1}{2} \ln \left(\frac{\det \sigma'_A \det \sigma'_B}{\det \sigma'_{AB}} \right), \quad (113)$$

where

$$\sigma'_{AB} = \gamma_{AB}^c \oplus \mathbb{1}_{\text{anc}} + \Gamma'_A \oplus \Gamma'_B, \quad (114)$$

with σ'_j being the CM of the reduced state of the subsystem $(j_{\text{in}}, j_{\text{anc}})$, $j = A, B$, and $\mathbb{1}_{\text{anc}}$ is the $2(N_{\text{anc}} + M_{\text{anc}})$ -dimensional identity matrix describing the CM of the vacuum state $|\{0\}\rangle_{A_{\text{anc}}B_{\text{anc}}}$. Further, it is convenient to express the CMs Γ'_A and Γ'_B with respect to in|anc splitting as

$$\Gamma'_A = \begin{pmatrix} A_{\text{in}} & C_A \\ C_A^T & A_{\text{anc}} \end{pmatrix}, \quad \Gamma'_B = \begin{pmatrix} B_{\text{in}} & C_B \\ C_B^T & B_{\text{anc}} \end{pmatrix}. \quad (115)$$

Consider now the determinant formula [29]

$$\det(M) = \det(\mathcal{D})\det(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}), \quad (116)$$

which is valid for any $(n+m) \times (n+m)$ matrix

$$M = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad (117)$$

where \mathcal{A} , \mathcal{B} , and \mathcal{C} are, respectively, $n \times n$, $n \times m$, and $m \times n$ matrices and \mathcal{D} is an $m \times m$ invertible matrix. Applying the formula to the RHS of Eq. (113) we can bring it after some algebra into the form

$$I(A, A'; B, B') = \frac{1}{2} \ln \left(\frac{\det \mu_A \det \mu_B}{\det \mu_{AB}} \right), \quad (118)$$

where

$$\mu_{AB} = \gamma_{AB}^c + \tilde{\Gamma}_A \oplus \tilde{\Gamma}_B \quad (119)$$

and $\mu_{A,B}$ are CMs of the reduced states of the subsystems A and B . Here,

$$\tilde{\Gamma}_A = A_{\text{in}} - C_A(A_{\text{anc}} + \mathbb{1}_{A_{\text{anc}}})^{-1}C_A^T \quad (120)$$

is the N -mode CM,

$$\tilde{\Gamma}_B = B_{\text{in}} - C_B(B_{\text{anc}} + \mathbb{1}_{B_{\text{anc}}})^{-1}C_B^T \quad (121)$$

is the M -mode CM, $\mathbb{1}_{A_{\text{anc}}}$ is the $2N_{\text{anc}} \times 2N_{\text{anc}}$ identity matrix, and $\mathbb{1}_{B_{\text{anc}}}$ is the $2M_{\text{anc}} \times 2M_{\text{anc}}$ identity matrix. Hence, we can interpret the mutual information $I(A, A'; B, B')$ as the mutual information of a new conditional Gaussian probability density $\tilde{p}(\tilde{d}_A, \tilde{d}_B | d_{E_\rho})$ given in Eq. (92), which is obtained by the Gaussian measurements $\tilde{\Pi}_A(\tilde{d}_A)$ and $\tilde{\Pi}_B(\tilde{d}_B)$ with CMs $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ on the conditional state $\rho_{A_{\text{in}}B_{\text{in}}|E_\rho}$ defined in Eq. (91). If we now take into account the fact that the CM γ_{AB}^c of the state reads as

$$\gamma_{AB}^c = \gamma_{AB} - \gamma_{ABE}^T (\gamma_E^T + \Gamma_E^T)^{-1} (\gamma_{ABE}^T)^T, \quad (122)$$

where γ_{ABE}^T and γ_E^T are the respective blocks of the CM γ_π^T , we find that the mutual information (118) is equal to

$$I(A, A'; B, B') = f(\gamma_\pi^T, \tilde{\Gamma}_A, \tilde{\Gamma}_B, \Gamma_E^T), \quad (123)$$

and thus the inequality $I(A; B) \leq I(A, A'; B, B')$ translates into the inequality (93) as we wanted to prove.

Finally, as at given CMs γ_π^T and Γ_E^T , the CMs $\tilde{\Gamma}_A$ and $\tilde{\Gamma}_B$ given in Eqs. (120) and (121) generally do not maximize the function $f(\gamma_\pi^T, \Gamma_A, \Gamma_B, \Gamma_E^T)$ with respect to CMs Γ_A and Γ_B one gets

$$f(\gamma_\pi^T, \tilde{\Gamma}_A, \tilde{\Gamma}_B, \Gamma_E^T) \leq f(\gamma_\pi^T, \Gamma_A, \Gamma_B, \Gamma_E^T) = E_\downarrow^G(\rho_{AB}^T), \quad (124)$$

where Γ_A^T and Γ_B^T are CMs of the optimal measurements $\Pi_A(d_A)$ and $\Pi_B(d_B)$ which maximize f and the equality follows from Eq. (73).

In summary, combining inequalities (90), (93), and (124), the monotonicity of GIE [Eq. (61)] under GLTPOCC can be expressed by the following chain of inequalities:

$$\begin{aligned} E_\downarrow^G(\rho_{AB}^\varepsilon) &\leq f(\gamma_\pi^\varepsilon, \Gamma_A^\varepsilon, \Gamma_B^\varepsilon, \Gamma_E^\varepsilon \oplus \tilde{\Gamma}_{E_x}^\varepsilon) \\ &\leq f(\gamma_\pi^T, \tilde{\Gamma}_A, \tilde{\Gamma}_B, \Gamma_E^T) \leq E_\downarrow^G(\rho_{AB}^T), \end{aligned} \quad (125)$$

which accomplishes the monotonicity proof.

Before moving to an explicit evaluation of GIE, let us note that an important subset of GLTPOCC operations is the class of Gaussian local unitary operations ($\equiv U_A \otimes U_B$) which transform the input Gaussian state ρ_{AB}^T to $\rho_{AB}^U \equiv (U_A \otimes U_B)\rho_{AB}^T(U_A^\dagger \otimes U_B^\dagger)$. Inequality (125) and the reversibility of unitary operations then implies the invariance of GIE with respect to the local Gaussian unitary operations $E_\downarrow^G(\rho_{AB}^U) = E_\downarrow^G(\rho_{AB}^T)$. When calculating GIE, we can therefore assume

without any loss of generality that the CM γ_{AB} of the considered state is in the standard form [47]

$$\gamma_{AB} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}, \quad (126)$$

where $c_1 \geq |c_2| \geq 0$, which can greatly simplify our calculations.

VII. GIE FOR PURE STATES

As a first example, we calculate GIE for the class of pure Gaussian states ρ_p with CM γ_{AB}^p . For these states, any purification is a product state with respect to the $AB|E$ splitting and therefore the block γ_{ABE} in CCM (42) is a matrix of zeros. This implies that the Schur complement (59) reads as $\sigma_{AB} = \gamma_{AB}^p + \Gamma_A \oplus \Gamma_B$ and the GIE coincides with the Gaussian classical mutual information ($\equiv \mathcal{I}_c^G$) of a quantum state ρ_p [48,49]:

$$E_{\downarrow}^G(\rho_p) = \mathcal{I}_c^G(\rho_p) \equiv \sup_{\Gamma_A, \Gamma_B} \frac{1}{2} \ln \left(\frac{\det \sigma_A \det \sigma_B}{\det \sigma_{AB}} \right). \quad (127)$$

From the results of Ref. [49] it then follows that the supremum is attained by double homodyne detection which gives [21]

$$E_{\downarrow}^G(\rho_p) = \frac{1}{2} \ln(\det \gamma_A) = \ln[\cosh(2\tilde{r})], \quad (128)$$

where γ_A is the CM of the reduced state ρ_A of mode A of the state ρ_p and $\tilde{r} \geq 0$ is the squeezing parameter characterizing the latter state, which is defined by the equation $\cosh(2\tilde{r}) = \sqrt{\det \gamma_A}$. Interestingly, the RHS of Eq. (128) is equal to the Gaussian Rényi-2 (GR2) entropy $S_2(\rho_A)$ which is nothing but the GR2 entanglement $E_2^G(\rho_p)$ [24]. This means that for all pure Gaussian states it holds that $E_{\downarrow}^G = E_2^G$. Comparing, on the other hand, GIE with the entropy of entanglement $E(\rho_p) = S(\rho_A)$ [1,50], where [51]

$$S(\rho_A) = \cosh^2(\tilde{r}) \ln[\cosh^2(\tilde{r})] - \sinh^2(\tilde{r}) \ln[\sinh^2(\tilde{r})] \quad (129)$$

is the marginal von Neumann entropy, one finds that the inequality $E \geq E_{\downarrow}^G$ is satisfied for all pure Gaussian states [21]. However, the equality to the entropy of entanglement is restored for true IE E_{\downarrow} [Eq. (5)], which admits also non-Gaussian measurements on modes A and B . Namely, $E_{\downarrow}(\rho_p) = \mathcal{I}_c(\rho_p) \equiv \sup_{\Pi_A \otimes \Pi_B} I(A; B)$, where the RHS is the classical mutual information of a quantum state ρ_p [48] with $I(A; B)$ being the mutual information of a distribution of outcomes of generally non-Gaussian measurements Π_A and Π_B on modes A and B of the state ρ_p . The quantity $\mathcal{I}_c(\rho_p)$ is invariant with respect to local unitaries and thus ρ_p can be replaced by the locally unitarily equivalent two-mode squeezed vacuum (TMSV) state $\rho_p(\lambda) = |\psi(\lambda)\rangle\langle\psi(\lambda)|$, where

$$|\psi(\lambda)\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle_{AB} \quad (130)$$

with $\lambda = \tanh \tilde{r}$. Non-Gaussian local photon counting on modes A and B of the state $|\psi(\lambda)\rangle$ then yields a probability distribution with $I(A; B) = S(\rho_A)$ [49], which is the highest mutual information one can achieve [52]. Thus, we find that $E_{\downarrow} = E$ holds for all pure Gaussian states as required. A

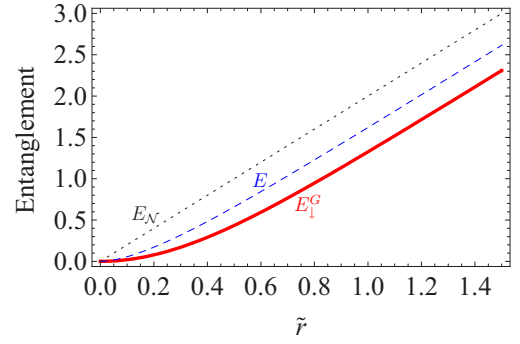


FIG. 1. (Color online) GIE E_{\downarrow}^G (solid red curve), entropy of entanglement E (dashed blue curve), and logarithmic negativity E_N (dotted black curve) for pure Gaussian states versus the squeezing parameter \tilde{r} .

comparison of GIE (128), entropy of entanglement (129), and logarithmic negativity [53,54] $E_N(\rho_p) = 2\tilde{r}$ [55] as functions of the squeezing parameter \tilde{r} is depicted in Fig. 1.

VIII. GIE FOR A TWO-MODE REDUCTION OF THE THREE-MODE CV GHZ STATE

Despite the complexity of optimization in Eq. (61) it is possible to calculate GIE analytically for some mixed two-mode Gaussian states. In what follows, we illustrate this by calculating GIE for a two-mode Gaussian state ($\equiv \rho_{AB}^{\text{GHZ}}$) with CM

$$\gamma_{AB}^{\text{GHZ}} = \begin{pmatrix} \alpha & \kappa \\ \kappa & \alpha \end{pmatrix}, \quad (131)$$

which is a reduction of the three-mode CV GHZ state [23] having CM

$$\gamma_{ABE}^{\text{GHZ}} = \begin{pmatrix} \alpha & \kappa & \kappa \\ \kappa & \alpha & \kappa \\ \kappa & \kappa & \alpha \end{pmatrix}. \quad (132)$$

Here, $\alpha = \text{diag}(x_+, x_-)$ and $\kappa = (x_- - x_+) \sigma_z$, where $x_{\pm} = (e^{\pm 2r} + 2e^{\mp 2r})/3$ and $r \geq 0$ is a squeezing parameter. This calculation will be accomplished in two steps. First, we will calculate an easier computable upper bound [$\equiv U(\rho_{AB}^{\text{GHZ}})$] on $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}})$. In the second step, we will show, that for homodyne detections on modes A, B with CMs $\Gamma_A^{x'}$ and $\Gamma_B^{x'}$ homodyne detection on mode E with CM $\Gamma_E^{x'}$ minimizes the mutual information (60), i.e., $f(\gamma_{\pi}, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'}) = \inf_{\Gamma_E^{x'}} f(\gamma_{\pi}, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'})$, and simultaneously the upper bound $U(\rho_{AB}^{\text{GHZ}})$ is saturated, i.e.,

$$U(\rho_{AB}^{\text{GHZ}}) = f(\gamma_{\pi}, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'}), \quad (133)$$

where γ_{π} denotes the CM of the purification of the state ρ_{AB}^{GHZ} . The quantity $f(\gamma_{\pi}, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'})$ is thus the largest possible minimal mutual information with respect to all Gaussian measurements on mode E , which finally yields

$$E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) = f(\gamma_{\pi}, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'}). \quad (134)$$

Let us start by noting that from the max-min inequality [22] it follows that GIE satisfies inequality $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) \leq U(\rho_{AB}^{\text{GHZ}})$,

where

$$U(\rho_{AB}^{\text{GHZ}}) \equiv \inf_{\Gamma_E} \sup_{\Gamma_A, \Gamma_B} f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E). \quad (135)$$

Next, consider the quantity

$$\mathcal{I}_c^G(\rho_{AB|E}) = \sup_{\Gamma_A, \Gamma_B} f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E), \quad (136)$$

which is the Gaussian classical mutual information of the conditional quantum state $\rho_{AB|E}$ of modes A and B after a measurement with CM Γ_E on mode E of the purification with CM γ_π [49]. Let us take as the CM γ_π the CM (132), $\gamma_\pi = \gamma_{ABE}^{\text{GHZ}}$, and denote as $\gamma_{AB|E}$ the CM of the conditional state $\rho_{AB|E}$. As the CM γ_π is symmetric under exchange of any pair of modes, the CM $\gamma_{AB|E}$ is also symmetric for any CM Γ_E . To calculate the expression on the RHS of Eq. (136) it is convenient first to express the CM $\gamma_{AB|E}$ in the standard form (126) where $a = b$ due to the symmetry, i.e.,

$$\gamma_{AB|E}^{\text{st}} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & a & 0 \\ 0 & c_2 & 0 & a \end{pmatrix}. \quad (137)$$

The mutual information $f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E)$ is then given by Eq. (60) where $\sigma_{AB} = \gamma_{AB|E}^{\text{st}} + \Gamma_A \oplus \Gamma_B$. Further, in Ref. [49] it was shown that for symmetric states with CM (137) the optimal measurements on modes A and B are always symmetric with CMs of the form $\Gamma_A = \Gamma_B = \text{diag}(e^{-2t}, e^{2t})$, $t \geq 0$. From Eqs. (60) and (137), it then follows that

$$f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E) = -\ln \sqrt{h}, \quad (138)$$

where

$$h = \left[1 - \frac{c_1^2}{(a + e^{-2t})^2} \right] \left[1 - \frac{c_2^2}{(a + e^{2t})^2} \right]. \quad (139)$$

In order to maximize the function (138) with respect to CMs Γ_A and Γ_B , we have to minimize the function on the RHS of Eq. (139) with respect to $t \geq 0$. This can be done by the following chain of inequalities:

$$\begin{aligned} & \left[1 - \frac{c_1^2}{(a + e^{-2t})^2} \right] \left[1 - \frac{c_2^2}{(a + e^{2t})^2} \right] \\ & \geq \left[1 - \frac{c_1^2}{(a + e^{-2t})^2} \right] \left[1 - \frac{c_1^2}{(a + e^{2t})^2} \right] \\ & = 1 - \frac{c_1^2}{a^2} + \frac{c_1^2}{a^2[1 + a^2 + 2a \cosh(2t)]} \\ & \quad \times \left[2 + \frac{a^2 c_1^2 - (a^2 - 1)^2}{1 + a^2 + 2a \cosh(2t)} \right] \\ & \geq 1 - \frac{c_1^2}{a^2}. \end{aligned} \quad (140)$$

Here, the first inequality is a consequence of inequality $c_1 \geq |c_2|$ and the second inequality is fulfilled if

$$(2a + 1)^2 \geq a^2(a^2 - c_1^2). \quad (141)$$

Importantly, the lower bound $1 - c_1^2/a^2$ in inequalities (140) is tight because it can be achieved in the limit $t \rightarrow +\infty$ which corresponds to the homodyne detection of x quadratures on

both modes A and B . We have thus arrived to the finding that, for all symmetric states with CM (137) for which the parameters a and c_1 satisfy inequality (141), the optimal measurement in Gaussian classical mutual information (136) is double homodyne detection of x quadratures. Hence, one gets

$$\mathcal{I}_c^G(\rho_{AB|E}) = \frac{1}{2} \ln \frac{a^2}{a^2 - c_1^2}. \quad (142)$$

Before going further, let us note that the inequality (141) has been derived in Ref. [49] as a condition under which, for two-mode squeezed thermal states which possess CMs (137) with $c_2 = -c_1$, the optimal measurement in (136) is double homodyne detection. The present analysis thus extends the result of Ref. [49] to all symmetric states satisfying condition (141).

Moving to the derivation of the upper bound (135) it is first convenient to find a simpler condition under which the state ρ_{AB}^{GHZ} with CM (137) satisfies inequality (141). For this purpose, we first rewrite inequality (141) into an equivalent form

$$2 + \frac{1}{a} - s \geq 0, \quad (143)$$

where we have introduced $s \equiv \sqrt{a^2 - c_1^2}$. Since a is a symplectic eigenvalue of the local state of mode A , it satisfies the inequality $a \geq 1 > 0$ and therefore $1/a > 0$. Consequently, for CMs (137) for which $s \leq 2$ the inequality (143) is always satisfied. Let us denote now as a_{max} the maximal value of the parameter a of the CM (137) over all CMs Γ_E of Eve's measurements. From the obvious inequality $a \geq s$, it then follows that if

$$a_{\text{max}} \leq 2, \quad (144)$$

then $s \leq a \leq a_{\text{max}} \leq 2$, and inequality (143) is therefore always satisfied. By calculating a_{max} for the state ρ_{AB}^{GHZ} and using inequality (144), we can find easily a region of the squeezing parameter r for which the Gaussian classical mutual information (136) is given by formula (142).

To calculate the quantity a_{max} we first calculate the local symplectic eigenvalue a of CM (137). The CM describes a conditional quantum state obtained by a Gaussian measurement with CM Γ_E on mode E of the purification of the state ρ_{AB}^{GHZ} with CM (132). We further decompose the latter CM as

$$\gamma_{ABE}^{\text{GHZ}} = S_{ABE}(\gamma_{AE}^{\text{TMSV}} \oplus \gamma_B^{\text{sq}})S_{ABE}^T, \quad (145)$$

where

$$\gamma_{AE}^{\text{TMSV}} = \begin{pmatrix} v \mathbb{1}_2 & \sqrt{v^2 - 1} \sigma_z \\ \sqrt{v^2 - 1} \sigma_z & v \mathbb{1}_2 \end{pmatrix} \quad (146)$$

is the CM of pure two-mode squeezed vacuum state with

$$v = \sqrt{x_+ x_-} = \frac{1}{3} \sqrt{5 + 4 \cosh(4r)}, \quad (147)$$

$\gamma_B^{\text{sq}} = \text{diag}(e^{-2r}, e^{2r})$, and $S_{ABE} = (U_{AB} \oplus \mathbb{1}_E)(S_A \oplus \mathbb{1}_B \oplus S_E)$, where $S_A = S_E^{-1} = \text{diag}(\sqrt{x_-/x_+}, \sqrt{x_+/x_-})$ and

$$U_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix}. \quad (148)$$

The decomposition (145) expresses the simple fact that the CV GHZ state can be obtained by the mixing of mode A of

the TMSV state with CM (146) transformed by the squeezing operation described by the matrix $S_A \oplus S_E$ with the squeezed state in mode B with CM γ_B^{sq} on a balanced beam splitter described by the matrix U_{AB} [31]. The conditional state $\rho_{AB|E}$ is then obtained by performing a Gaussian measurement with CM Γ_E on mode E of the purification. Since the maximization of a is carried out over all CMs Γ_E , we can integrate the squeezing transformation S_E into the CM Γ_E and can therefore drop the matrix S_E from any further considerations. Let us express now the CM of Eve's measurement as $\Gamma_E = U(\varphi)\text{diag}(V_x, V_p)U^T(\varphi)$, where

$$U(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \quad (149)$$

where $\varphi \in [0, \pi)$, $V_x \geq V_p \geq 0$, and $V_x V_p \geq 1$. By performing the Gaussian measurement with CM Γ_E on mode E of the TMSV state with CM (146), mode A collapses into the Gaussian state with CM $\gamma_A^{\text{cond}} = U^T(\varphi)\text{diag}(\mathcal{V}_x, \mathcal{V}_p)U(\varphi)$, where

$$\mathcal{V}_x = \frac{vV_x + 1}{v + V_x}, \quad \mathcal{V}_p = \frac{vV_p + 1}{v + V_p}. \quad (150)$$

Hence, at given v the quantities \mathcal{V}_x and \mathcal{V}_p will lie in the subset \mathcal{M} of the $(\mathcal{V}_p, \mathcal{V}_x)$ plane characterized by the inequalities $1/v \leq \mathcal{V}_p \leq v$, $1/\mathcal{V}_p \leq \mathcal{V}_x \leq v$, and $\mathcal{V}_x \geq \mathcal{V}_p$. In other words, if $\mathcal{V}_p \in [1/v, 1]$, then $\mathcal{V}_x \in [1/\mathcal{V}_p, v]$, whereas if $\mathcal{V}_p \in (1, v]$, then $\mathcal{V}_x \in [\mathcal{V}_p, v]$.

Let us now return back to the derivation of the local symplectic eigenvalue a . After the measurement on mode E of the TMSV state, mode A collapses into a Gaussian state with CM γ_A^{cond} which is subsequently transformed by the squeezing operation described by the matrix S_A and then mixed with the squeezed state with CM γ_B^{sq} on a balanced beam splitter characterized by the matrix U_{AB} . This gives the conditional state $\rho_{AB|E}$ with CM

$$\gamma_{AB|E} = U_{AB}(S_A \gamma_A^{\text{cond}} S_A^T \oplus \gamma_B^{\text{sq}})U_{AB}^T. \quad (151)$$

Expressing further the latter CM in block form with respect to $A|B$ splitting,

$$\gamma_{AB|E} = \begin{pmatrix} A & C \\ C & A \end{pmatrix}, \quad (152)$$

one can calculate the entry a of the CM (137) from the formula $a = \sqrt{\det A}$ in the form

$$a = \frac{\sqrt{1 + \mathcal{V}_x \mathcal{V}_p + 2[\mathcal{V}_+ \cosh(2q) + \mathcal{V}_- \sinh(2q) \cos(2\varphi)]}}{2}, \quad (153)$$

where $\mathcal{V}_\pm = (\mathcal{V}_x \pm \mathcal{V}_p)/2$ and $q = r + \ln(\sqrt{x_-/x_+})/2$. As the inequality $\mathcal{V}_- \geq 0$ holds a is maximized if $\varphi = 0$. Further, the extremal equations $\partial a/\partial \mathcal{V}_x = 0$ and $\partial a/\partial \mathcal{V}_p = 0$ have no solution in the interior of the set \mathcal{M} and therefore the maximum lies on the boundary of the set. On the boundary, the local symplectic eigenvalue a attains the maximum

$$a_{\max} = \frac{\sqrt{1 + v^2 + 2v \cosh(2q)}}{2} = v \quad (154)$$

for $\mathcal{V}_x = \mathcal{V}_p = v$. Next, making use of the explicit expression for the symplectic eigenvalue v [Eq. (147)] and the inequality

(144), one finds after some algebra that the inequality (144) is fulfilled if the squeezing parameter r satisfies the inequality

$$r \leq r_{\text{th}} \equiv \frac{1}{4} \text{arccosh} \left(\frac{31}{4} \right) \doteq 0.684. \quad (155)$$

Consequently, for the class of two-mode Gaussian states ρ_{AB}^{GHZ} for which r satisfies inequality (155), the Gaussian classical mutual information (136) of the conditional state $\rho_{AB|E}$ is for any Gaussian measurement on mode E given by the formula (142). Later in this section we show explicitly that the latter statement in fact holds for all $r \geq 0$. This is because for derivation of the inequality (155), we used the inequality (144) which is stronger than the original inequality (143), and therefore the threshold squeezing for which the latter inequality is satisfied is larger than r_{th} . By minimizing the left-hand side (LHS) of inequality (143) over all CMs Γ_E one finds that the LHS has a lower bound of the form

$$2 + \frac{1}{a} - s \geq 2 + \frac{1}{\sqrt{x_+ x_-}} - \frac{x_-}{e^r \sqrt{x_+}}, \quad (156)$$

where the parameters x_\pm are defined below Eq. (132). Further, the RHS of the latter inequality is a monotonously decreasing function of the squeezing parameter r which approaches the value $2 - 2/\sqrt{3}$ in the limit of $r \rightarrow +\infty$. Hence, one finally gets the following lower bound:

$$2 + \frac{1}{a} - s \geq 2 - \frac{2}{\sqrt{3}} \doteq 0.845 \quad (157)$$

for the LHS of the inequality (143) and therefore the inequality is indeed satisfied for any $r \geq 0$. Since the minimization of the LHS of the inequality (143) is very similar to the minimization needed for calculation of the upper bound (135), it is more convenient first to carry out the latter minimization. Explicit minimization of the LHS of the inequality (143) is postponed until near the end of the present section.

In the last step of the calculation of the upper bound $U(\rho_{AB}^{\text{GHZ}})$ [Eq. (135)], we perform minimization on the RHS of the following equation:

$$U(\rho_{AB}^{\text{GHZ}}) = \inf_{\Gamma_E} \left[\frac{1}{2} \ln \left(\frac{a^2}{a^2 - c_1^2} \right) \right] \quad (158)$$

over all single-mode CMs Γ_E . This amounts to the minimization of the ratio c_1/a , where a is given in Eq. (153). The parameter c_1 appearing in CM (137) can be calculated as a larger eigenvalue of the matrix QCQ^T :

$$c_1 = \frac{\text{Tr}(QCQ^T) + \sqrt{[\text{Tr}(QCQ^T)]^2 - 4 \det C}}{2}, \quad (159)$$

where Q symplectically diagonalizes the matrix A , i.e., $QAQ^T = a\mathbb{1}_2$, and where we have used the equality $\det(QCQ^T) = \det C$. If we calculate explicitly the CM (151) we get after some algebra

$$\det C = \frac{1}{2}(1 + \mathcal{V}_x \mathcal{V}_p) - a^2, \quad (160)$$

and the utilization of the expression $Q = \text{diag}(\sqrt{\lambda_2/\lambda_1}, \sqrt{\lambda_1/\lambda_2})U(\theta)S_A^{-1}$, where $U(\theta)S_A^{-1}A(S_A^T)^{-1}U^T(\theta) = \text{diag}(\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2$, yields

$$\text{Tr}(QCQ^T) = a\text{Tr}(CA^{-1}) = \frac{(\mathcal{V}_x \mathcal{V}_p - 1)}{2a}. \quad (161)$$

Substituting now from Eqs. (160) and (161) into Eq. (159), one finds the ratio c_1/a to be minimized in the form

$$\frac{c_1}{a} = \frac{\mathcal{K}}{a^2} + \sqrt{\left(\frac{\mathcal{K}}{a^2} - 1\right)^2 - \frac{1}{a^2}} \equiv g \quad (162)$$

with $\mathcal{K} = (\mathcal{V}_x \mathcal{V}_p - 1)/4$.

The minimal value of the ratio (162) is easily found by a direct substitution for $r = 0$ which corresponds to the vacuum density matrix ρ_{AB}^{GHZ} . In this case, one has $\nu = 1$ which implies $\mathcal{V}_x = \mathcal{V}_p = 1$ and therefore $\mathcal{K} = 0$ which gives $g = \sqrt{(a^2 - 1)/a^2}$. As for $r = 0$ one further gets $q = 0$ and we see from Eq. (153) that $a = 1$ and thus $g = 0$. Consequently, for $r = 0$ the upper bound (135) vanishes, $U(\rho_{AB}^{\text{GHZ}}) = 0$, and therefore $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) = 0$ which is in accordance with our previous finding that GIE vanishes on all separable states.

For $r > 0$ the minimization of g [Eq. (162)], with respect to the variables φ , \mathcal{V}_x , and \mathcal{V}_p is best performed if we introduce new variables $\tau = \sqrt{\mathcal{V}_x \mathcal{V}_p}$ and $z = \sqrt{\mathcal{V}_x/\mathcal{V}_p}$, where $\tau \in [1, \nu]$ and $z \in [1, \nu/\tau]$. Then, the task is to minimize g in the subset \mathcal{O} of the three-dimensional space of the variables φ , τ , and z characterized by the intervals $\varphi \in [0, \pi]$, $\tau \in [1, \nu]$, and $z \in [1, \nu/\tau]$. Note that here and in what follows, we admit for the sake of simplicity also phase $\varphi = \pi$, although it is not necessary because the function g is π periodic. Calculating now the extremal equations $\partial g/\partial \varphi = 0$ and $\partial g/\partial z = 0$ and taking into account inequality $c_1 \geq 0$ and inequality $a^2 - c_1^2 \geq 1$ which has to be satisfied for any CM of a physical quantum state [56], one finds that the equations are equivalent to the extremal equations $\partial a/\partial \varphi = 0$ and $\partial a/\partial z = 0$. The first extremal equation $\partial a/\partial \varphi = 0$ is satisfied if either $\varphi = 0$, $\pi/2, \pi$ or $z = 1$. Since for $\varphi = \pi/2$ the second equation $\partial a/\partial z = 0$ has no solution z in the interval $[1, \nu]$ and all points with $\varphi = 0, \pi$ or $z = 1$ lie on the boundary of the set \mathcal{O} the function g has no stationary points in the interior of the set \mathcal{O} . A detailed analysis of the behavior of the function g on the boundary of the set \mathcal{O} reveals that the candidates for extremes will lie on the following parts of the boundary:

(1) The segment ($\tau = \nu$, $z = 1$, $\varphi \in [0, \pi]$) and the curves ($\tau \in [1, \nu]$, $z = \nu/\tau$, $\varphi = 0$) and ($\tau \in [1, \nu]$, $z = \nu/\tau$, $\varphi = \pi$), where

$$U_1 \equiv \frac{1}{2} \ln \left(\frac{1}{1 - g^2} \right) = \ln \left(\frac{e^r x_+}{\sqrt{x_-}} \right) \quad (163)$$

in all three cases. The value U_1 can be obtained in various ways including homodyne detection of quadrature p_E on mode E , i.e., $\Gamma_E = \Gamma_E^p \equiv \Gamma_p^{t \rightarrow +\infty}$, where $\Gamma_p^t \equiv \text{diag}(e^{2t}, e^{-2t})$, or by tracing out mode E .

(2) The segment ($\tau = 1$, $z = 1$, $\varphi \in [0, \pi]$) corresponding to heterodyne detection on mode E , i.e., $\Gamma_E = \mathbb{1}_2$, where

$$U_2 = \ln \left(\frac{e^r \sqrt{\frac{x_-}{x_+}} + e^{-r} \sqrt{\frac{x_+}{x_-}}}{2} \right). \quad (164)$$

(3) In the point $\tau = 1$, $z = \nu$, and $\varphi = \pi/2$ which corresponds to homodyne detection of quadrature x_E on mode E , i.e., $\Gamma_E = \Gamma_E^x \equiv \Gamma_x^{t \rightarrow +\infty}$, where $\Gamma_x^t \equiv \text{diag}(e^{-2t}, e^{2t})$, and

where

$$U_3 = \ln \left(\frac{x_-}{e^r \sqrt{x_+}} \right). \quad (165)$$

It remains to find the smallest of the three quantities U_1 , U_2 , and U_3 . For this purpose it is convenient to express them as $U_j = \ln[\cosh(p_j)]$, $j = 1, 2, 3$, where $p_1 = \ln(e^r \sqrt{x_-})$, $p_2 = \ln(e^r \sqrt{x_-/x_+})$, and $p_3 = \ln(e^r/\sqrt{x_+})$. As for $r > 0$ it holds that $\nu > 1$, we have $p_1 - p_3 = \ln \nu > 0$ and therefore $p_1 > p_3$ which implies $U_1 > U_3$. Similarly, one gets $p_2 - p_3 = \ln \sqrt{\nu} > 0$ and therefore $p_2 > p_3$ which gives finally $U_2 > U_3$. Consequently, the sought upper bound (135) is equal to U_3 , i.e.,

$$U(\rho_{AB}^{\text{GHZ}}) = \ln \left(\frac{x_-}{e^r \sqrt{x_+}} \right) \quad (166)$$

and is achieved by triple homodyne detection of x quadratures.

In the final step of evaluation of the GIE we find for some fixed measurements with CMs Γ_A and Γ_B on modes A and B of the purification with CM (132) an infimum over all CMs Γ_E which saturates the upper bound (166), $\inf_{\Gamma_E} f(\gamma_\pi, \Gamma_A, \Gamma_B, \Gamma_E) = U(\rho_{AB}^{\text{GHZ}})$. This means that this is the largest infimum and hence GIE is equal to the upper bound (166). Let us denote as $\Gamma_j^{x'} = S^{-1} \Gamma_j^x (S^T)^{-1}$, $j = A, B$, where the CM Γ_j^x describes homodyne detection of quadrature x on mode j and the single-mode symplectic matrix S brings the CM (152) to the standard form (137), i.e., $(S \oplus S) \gamma_{AB|E} (S^T \oplus S^T) = \gamma_{AB|E}^{\text{st}}$. Then, $\mathcal{I}_c^G(\rho_{AB|E}) = f(\gamma_\pi, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E)$ and as we have shown above $\inf_{\Gamma_E} f(\gamma_\pi, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E) = f(\gamma_\pi, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E^{x'}) = U(\rho_{AB}^{\text{GHZ}})$, where $\Gamma_E^{x'} = S_E \Gamma_E^x S_E^T$. Thus, for measurements with CMs $\Gamma_A^{x'}$ and $\Gamma_B^{x'}$ on modes A and B of the purification with CM (132), the measurement on mode E with CM $\Gamma_E^{x'}$ gives the minimal mutual information $f(\gamma_\pi, \Gamma_A^{x'}, \Gamma_B^{x'}, \Gamma_E)$ which is at the same time largest with respect to the CMs Γ_A and Γ_B as it saturates the upper bound (166). Consequently,

$$E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) = U(\rho_{AB}^{\text{GHZ}}) = \ln \left(\frac{x_-}{e^r \sqrt{x_+}} \right) \quad (167)$$

as we wanted to prove.

In the course of the derivation of the formula (167), we have used the equality (142) which was shown to be valid for all CMs Γ_E when the inequality (155) is fulfilled. Hence, the analytical expression of GIE in Eq. (167) is also valid for all states ρ_{AB}^{GHZ} for which $r \leq 0.684$. However, by repeating the previous minimization of the ratio $g = c_1/a$ [Eq. (162)], in the subset \mathcal{O} for function $1/a - s$ on the LHS of inequality (143), we find that the inequality (143) and therefore also the formula (167) holds for all $r \geq 0$.

In order to show this, consider first the case when $r = 0$. From the previous results, it then follows that $a = 1$ and $c_1 = 0$ which implies fulfillment of the inequality (143). For $r > 0$ we can proceed as follows. Note first that the minimization of $1/a$, which is the first part of the function $1/a - s$, has already been done by maximization of a . This gave the minimum $1/a_{\text{max}} = 1/\nu = 1/\sqrt{x_+ x_-}$ which is attained if Eve projects her mode onto an infinitely hot thermal state which is equivalent to dropping of mode E . Now, if it happens that the function s defined below Eq. (143) attains its maximum

($\equiv s_{\max}$) also when Eve drops her mode, then $1/a_{\max} - s_{\max}$ represents the sought lower bound for the function $1/a - s$. If we derive the function s with respect to φ and z and we use the expressions (153) and (162), we arrive after some algebra at the following expressions:

$$\frac{\partial s}{\partial x} = -2 \frac{(\tau^2 - 1)(a^2 - c_1^2)}{4a^2 c_1 - (\tau^2 - 1)a} \frac{\partial a}{\partial x}, \quad x = \varphi, z. \quad (168)$$

Consequently, for $\tau > 1$ the extremal equations $\partial s/\partial \varphi = 0$ and $\partial s/\partial z = 0$ are equivalent to the equations $\partial a/\partial \varphi = 0$ and $\partial a/\partial z = 0$. However, as it was shown before, the latter equations have no solution in the interior of the set \mathcal{O} and thus the extremes will lie on the boundary of the set \mathcal{O} . On the boundary plane $z = 1$, $\varphi \in [0, \pi]$, and $\tau \in [1, \nu]$ the function s is independent of φ and it monotonously increases with τ attaining the maximum

$$s_{\max} = \frac{x_-}{e^r \sqrt{x_+}} \quad (169)$$

at $\tau = \nu$ which corresponds to dropping Eve's mode E . The second boundary plane $\tau = 1$, $\varphi \in [0, \pi]$, and $z \in [1, \nu]$ corresponds to pure-state Gaussian measurements on mode E which yield pure conditional states $\rho_{AB|E}$ for which $s = 1$. On the boundary planes $\varphi = 0$ and π , $\tau \in [1, \nu]$, and $z \in [1, \nu/\tau]$ the extremal equation $\partial a/\partial z = 0$ does not have any solution for $z \in [1, \nu/\tau]$ and therefore the extremes of s will lie on the boundary of the plane. Likewise, for the last boundary surface $z = \nu/\tau$, $\varphi \in [0, \pi]$, and $\tau \in [1, \nu]$ the extremal equations $\partial s/\partial \varphi = 0$ and $\partial s/\partial \tau = 0$ have no solution in the interior of the surface and therefore also in this case the extremes will be on the boundary. We have already calculated the extremes of s on the boundary curves of the surface except for the curves $z = \nu/\tau$, $\varphi = 0, \pi$, and $\tau \in [1, \nu]$, where s attains the maximum (169) for $\tau = \nu$. In summary, there are two extremes of the function s on the set \mathcal{O} . One is equal to $s = 1$ and it is localized on the boundary plane $\tau = 1$, and the other one is equal to s_{\max} [Eq. (169)], which lies on the segment $\tau = \nu$, $z = 1$, and $\varphi \in [0, \pi]$ which corresponds to dropping Eve's mode E . Since one can easily show that $s_{\max} \geq 1$ we finally find that the function s attains the maximum value (169) exactly in the same points where the function a is also maximized. Thus, the function $1/a - s$ on the LHS of inequality (143) has the lower bound given in inequality (156) which is further restricted from below as in inequality (157). From that it follows, finally, that the inequality (143) and hence also the formula (167) for GIE of the state ρ_{AB}^{GHZ} are indeed satisfied for all $r \geq 0$ as we wanted to prove.

It might again be of interest to compare GIE for state ρ_{AB}^{GHZ} with the GR2 entanglement. For a generally mixed two-mode Gaussian state ρ_{AB} with CM γ_{AB} the GR2 entanglement is defined as [24]

$$E_2(\rho_{AB}) = \inf_{\substack{\theta_{AB} \leq \gamma_{AB} \\ \det \theta_{AB} = 1}} \frac{1}{2} \ln(\det \theta_A), \quad (170)$$

where the minimization is carried over all pure two-mode Gaussian states with CM θ_{AB} smaller than γ_{AB} . The considered state ρ_{AB}^{GHZ} is a reduced state of a pure three-mode state and therefore it belongs to the class of Gaussian states with minimal partial uncertainty [57] for which GR2 entanglement can be

expressed analytically [24]. Making use of the fact that the state ρ_{AB}^{GHZ} is a reduction of the fully symmetric state with CM (132) with local symplectic eigenvalue $\nu = \sqrt{x_+ x_-}$ [Eq. (147)], GR2 entanglement reads explicitly as

$$E_2(\rho_{AB}^{\text{GHZ}}) = \frac{1}{2} \ln g' \quad (171)$$

with

$$g' = \begin{cases} 1, & \text{if } \nu = 1 \\ \frac{\zeta}{8\nu^2}, & \text{if } \nu > 1 \end{cases} \quad (172)$$

where

$$\zeta = 3\nu^4 + 6\nu^2 - 1 - \sqrt{(\nu^2 - 1)^3(9\nu^2 - 1)}. \quad (173)$$

Consider first the case $\nu = 1$. From Eqs. (171) and (172), it then follows that $E_2(\rho_{AB}^{\text{GHZ}}) = 0$. Equation (147) further reveals that the equality $\nu = 1$ is equivalent with the equality $r = 0$ which implies $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) = 0$ and thus GIE coincides with GR2 entanglement. Moving to the case $\nu > 1$ we see that GR2 entanglement is equal to the RHS of Eq. (171) where $g' = \zeta/(8\nu^2)$, whereas from Eq. (167) it follows that $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}}) = (\ln \tilde{g})/2$, where $\tilde{g} \equiv x_-^2/(e^{2r} x_+)$. Expressing now $e^{\pm 2r}$ using Eq. (147) one gets

$$e^{\pm 2r} = \frac{\sqrt{9\nu^2 - 1} \pm 3\sqrt{\nu^2 - 1}}{2\sqrt{2}}, \quad (174)$$

which further gives

$$x_{\pm} = \frac{e^{\pm 2r} + 2e^{\mp 2r}}{3} = \frac{\sqrt{9\nu^2 - 1} \mp \sqrt{\nu^2 - 1}}{2\sqrt{2}}. \quad (175)$$

If we now rewrite the quantity \tilde{g} as $\tilde{g} = x_-^2(2\nu^2 - x_-^2)/\nu^2$ and substitute to the RHS for x_- from Eq. (175) we finally find that $\tilde{g} = \zeta/(8\nu^2) = g'$. In this way, we have arrived at a surprising result: GIE also coincides with the GR2 entanglement for a one-parametric family of mixed two-mode Gaussian states ρ_{AB}^{GHZ} , i.e., $E_2(\rho_{AB}^{\text{GHZ}}) = E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}})$. A comparison of $E_{\downarrow}^G(\rho_{AB}^{\text{GHZ}})$ [Eq. (167)] with other entanglement measures is depicted in Fig. 2.

The results presented in this section lay the foundations for further exploration of GIE which is deferred for further research. This may include analytical or numerical evaluation of GIE for other two-mode Gaussian states with a three-mode purification or states with some symmetry such as two-mode

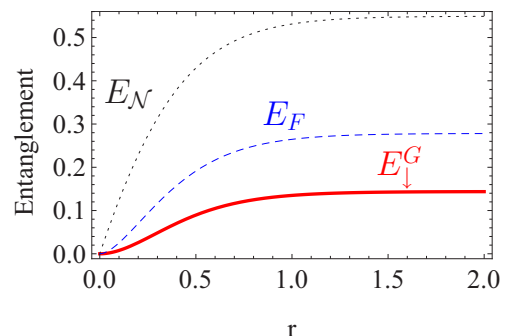


FIG. 2. (Color online) GIE E_{\downarrow}^G (solid red curve), entanglement of formation E_F (dashed blue curve), and logarithmic negativity E_N (dotted black curve) versus the squeezing parameter r for CM (131).

squeezed thermal states with standard-form CM (126), where $a = b$ and $c_2 = -c_1$. With these results in hand, we can also begin to explore the exciting question of the relation of two seemingly very different quantities: GIE and GR2 entanglement.

IX. LOWER BOUND ON IE FOR THE CONTINUOUS-VARIABLE NON-GAUSSIAN WERNER STATE

So far, we have investigated the properties of IE [Eq. (5)] only in the Gaussian scenario. Owing to the relative simplicity of Gaussian states and measurements we were able to calculate IE analytically for some nontrivial mixed Gaussian states and there in principle do not seem to be any obstacles preventing its evaluation, at least numerically, for other two-mode Gaussian states. A natural question that then arises is whether IE can be calculated also for some non-Gaussian states. It is apparent that this case will be much more complicated. Indeed, the calculation of IE for non-Gaussian states involves optimization over all general non-Gaussian measurements and purifications and therefore one is led to the apprehension that it will be infeasible, both analytically and numerically. In this section, we show that despite this complexity a nontrivial analytical lower bound on IE can be found even in the case of some mixed two-mode non-Gaussian states.

The states which we have in mind form the following two-parametric subfamily of the continuous-variable Werner states [25]

$$\rho_0 = p|\psi(\lambda)\rangle_{AB}\langle\psi(\lambda)| + (1-p)|00\rangle_{AB}\langle 00|, \quad (176)$$

where $0 \leq p \leq 1$, which is just a mixture of a two-mode squeezed vacuum state (130) with the vacuum. Making use of the partial transposition separability criterion [58], one can show easily [25] that for $p > 0$ the state (176) is entangled. For calculation of IE we first need to find a purification of the state (176), which can be taken in the form

$$|\Psi\rangle_{ABE} = \sqrt{p}|\psi(\lambda)\rangle_{AB}|0\rangle_E + \sqrt{1-p}|00\rangle_{AB}|1\rangle_E, \quad (177)$$

where Eve's purifying system is obviously a two-level quantum system (qubit) with basis vectors $|0\rangle_E$ and $|1\rangle_E$. As the definition (5) of IE involves minimization with respect to all purifications of the state (176), we need to know the form of an arbitrary purification which can be expressed as

$$\begin{aligned} |\Psi'\rangle_{ABE'} &= (\mathbb{1}_{AB} \otimes V)|\Psi\rangle_{ABE} \\ &= \sqrt{p}|\psi(\lambda)\rangle_{AB}V|0\rangle_E + \sqrt{1-p}|00\rangle_{AB}V|1\rangle_E, \end{aligned} \quad (178)$$

where V is an isometry from a qubit Hilbert space \mathcal{H}_E to a Hilbert space $\mathcal{H}_{E'}$ of another purifying system E' and $\mathbb{1}_{AB}$ is the identity operator on modes A and B . Instead of calculating the full IE for the state (176), here we will calculate its lower bound

$$\mathcal{L}_\downarrow(\rho_0) = \inf_{\{\Pi_E, |\Psi\rangle\}} [I(A; B \downarrow E)] \quad (179)$$

for fixed photon-counting measurements on modes A and B . Assume, therefore, that the projective measurements $\{|m\rangle_A \langle m|, m = 0, 1, \dots\}$ and $\{|n\rangle_B \langle n|, n = 0, 1, \dots\}$ are carried

out on modes A and B of the purification (178), whereas the subsystem E' is exposed to some generalized measurement $\{\Pi_{E'}(k)\}$. The outcomes of the measurements are then distributed according to the probability distribution

$$p(m, n, k) = \begin{cases} p_E(k) - \lambda^2 p \Pi_{00}(k), & \text{if } m = n = 0 \\ p(1 - \lambda^2) \lambda^{2m} \delta_{mn} \Pi_{00}(k), & \text{otherwise} \end{cases} \quad (180)$$

where

$$\begin{aligned} p_E(k) &= p \Pi_{00}(k) + \sqrt{p(1-p)(1-\lambda^2)} [\Pi_{10}(k) + \Pi_{01}(k)] \\ &\quad + (1-p) \Pi_{11}(k) \end{aligned} \quad (181)$$

is the probability distribution of measurement outcome k , where

$$\Pi_{ij}(k) \equiv \langle i | V^\dagger \Pi_{E'}(k) V | j \rangle, \quad i, j = 0, 1. \quad (182)$$

By calculating the entropies $H(A, B, E)$, $H(A, E)$, and $H(B, E)$ for the distribution (180) and the marginal distributions $p_{AE}(m, k) \equiv \sum_{n=0}^{\infty} p(m, n, k)$ and $p_{BE}(n, k) \equiv \sum_{m=0}^{\infty} p(m, n, k)$, we further observe that $H(A, B, E) = H(A, E) = H(B, E)$ and the conditional mutual information (3) simplifies to

$$I(A; B | E) = H(A) - I(A; E), \quad (183)$$

where $I(A; E) = H(A) + H(E) - H(A, E)$ is the mutual information of the marginal distribution $p_{AE}(m, k)$.

Moving to the minimizations in Eq. (179), we see from Eq. (183) that it boils down to the maximization of the mutual information $I(A; E)$ over all channels $E \rightarrow \tilde{E}$, isometries V , and measurements $\{\Pi_{E'}(k)\}$ on purifying subsystem E' . Since sending a random variable E over a channel $P(\tilde{E}|E)$ cannot increase the mutual information, i.e., $I(A; \tilde{E}) \leq I(A; E)$, it is best for Eve to not apply any channel to her measurement outcomes. Further, as the operators $V^\dagger \Pi_{E'}(k) V$ appearing in Eq. (182) are Hermitian, positive semidefinite, and sum to a qubit identity operator, they comprise a qubit generalized measurement. Therefore, in Eq. (179) we can omit minimization with respect to all purifications and we can minimize only over single-qubit measurements on the fixed purification (177). The latter minimization can be carried out with the help of the following upper bound on the classical mutual information [52]

$$I(A; E) \leq \min\{\mathcal{S}(\rho_A), \mathcal{S}(\rho_E), \mathcal{I}_q(\rho_{AE})\}, \quad (184)$$

where $\mathcal{S}(\rho_A)$ and $\mathcal{S}(\rho_E)$ are marginal von Neumann entropies of the reduced states ρ_A and ρ_E , respectively, of subsystems A and E of the state (177) and $\mathcal{I}_q(\rho_{AE}) = \mathcal{S}(\rho_A) + \mathcal{S}(\rho_E) - \mathcal{S}(\rho_{AE})$ is the quantum mutual information of the reduced state ρ_{AE} of the subsystem (AE) . From the purity of the state (177) it further follows that $\mathcal{S}(\rho_{AE}) = \mathcal{S}(\rho_B)$, whereas the symmetry of the state (176) under the exchange of modes A and B implies $\mathcal{S}(\rho_A) = \mathcal{S}(\rho_B)$. As a consequence, we get $\mathcal{I}_q(\rho_{AE}) = \mathcal{S}(\rho_E)$, and for finding of the minimum on the RHS of the inequality (184) we have to compare the marginal entropies $\mathcal{S}(\rho_A)$ and $\mathcal{S}(\rho_E)$. Using once again the purity argument we get $\mathcal{S}(\rho_E) = \mathcal{S}(\rho_0)$ and therefore we need to compare $\mathcal{S}(\rho_A)$ with $\mathcal{S}(\rho_0)$. In Ref. [60] it was already shown with the help of the majorization theory [59] that $\mathcal{S}(\rho_A) \geq \mathcal{S}(\rho_0)$ and the

entropy $\mathcal{S}(\rho_0)$ has been calculated in the form

$$\mathcal{S}(\rho_0) = - \sum_{i=1}^2 e_i \ln e_i, \quad (185)$$

where

$$e_{1,2} = \frac{1 \pm \sqrt{1 - 4p(1-p)\lambda^2}}{2} \quad (186)$$

are the eigenvalues of the state (176). Therefore, from Eq. (184) it follows that the mutual information $I(A; E)$ has an upper bound equal to $\mathcal{S}(\rho_E) = \mathcal{S}(\rho_0)$ [Eq. (185)], which is achieved by a measurement of the qubit E in the eigenbasis of the reduced state

$$\rho_E = p|0\rangle_E\langle 0| + \sqrt{p(1-p)(1-\lambda^2)}(|0\rangle_E\langle 1| + |1\rangle_E\langle 0|) + (1-p)|1\rangle_E\langle 1|. \quad (187)$$

Consequently, we get finally from Eqs. (179) and (183) the analytical form of the lower bound on IE:

$$\mathcal{L}_\downarrow(\rho_0) = H(A) - \mathcal{S}(\rho_E), \quad (188)$$

where $\mathcal{S}(\rho_E)$ is given by the RHS of Eq. (185) and $H(A)$ is the Shannon entropy of the photon-number distribution in mode A of the state (176) [60]:

$$H(A) = \mathcal{S}(\rho_A) = - \left\{ \ln(1 - p\lambda^2) + p\lambda^2 \ln \left[\frac{p(1 - \lambda^2)}{1 - p\lambda^2} \right] + \frac{2p\lambda^2 \ln \lambda}{1 - \lambda^2} \right\}. \quad (189)$$

The lower bound (188) is depicted by a solid red curve in Fig. 3. For comparison, we have plotted into the figure also cases when Eve just drops her qubit E or she measures it in the $\{|0\rangle, |1\rangle\}$ and $\{|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}\}$ bases.

In the previous text, we have performed minimization on the RHS of Eq. (5) for a particular fixed measurement on modes A and B of the purification (177), which was given by photon counting. In order to calculate the true IE, we would have to carry out the minimization for arbitrary local projective measurements on modes A and B and then we would have to perform maximization over the measurements. Our derivation given above thus yields only a lower bound on IE the actual

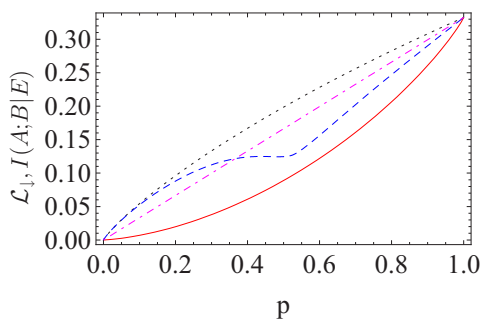


FIG. 3. (Color online) Lower bound \mathcal{L}_\downarrow [Eq. (188)] (solid red curve) and $I(A; B|E)$ versus the parameter p for measurement in the $\{|0\rangle, |1\rangle\}$ basis (dashed-dotted magenta curve) and the $\{|\pm\rangle\}$ basis (dashed blue curve) and when Eve's qubit E is dropped (dotted black curve) for $\lambda = 0.3$.

value of which can in fact be larger and may not be reached by photon counting. However, photon counting on modes A and B of the state (176) gives $I(A; B) = \mathcal{S}(\rho_A)$ [60] which is the highest classical mutual information one can get by locally measuring the state. This leads us to the conjecture that this measurement is in fact optimal and therefore the lower bound (188) coincides with IE. The proof or disproof of this conjecture as well as further analysis of IE for other non-Gaussian states is already beyond the scope of this paper and will be given elsewhere.

X. CONCLUSIONS

In this paper, we gave a detailed analysis of the properties of GIE, which is a quantifier of bipartite Gaussian entanglement introduced in Ref. [21]. The GIE is a Gaussian version of a more general quantity IE which is a lower bound to the ‘‘classical measure of entanglement’’ [7] obtained by commuting the order of optimization in the definition of IE.

Initially, we have shown that the assumption of Gaussianity of all channels, states, and measurements greatly simplifies IE. First, we have proved that the classical channel on Eve's measurement outcomes can be integrated into her measurement. In the next step, we have demonstrated that in the definition of IE we can use an arbitrary fixed purification of a considered state and that we can omit the minimization over all purifications. As a result of these simplifications, the GIE boils down to the optimized mutual information of a distribution of outcomes of Gaussian measurements on subsystems A and B of a conditional state obtained by a Gaussian measurement on subsystem E of a Gaussian purification of the considered state.

Next, the simple form of GIE enabled us to show that it satisfies some properties of a Gaussian entanglement measure. For this purpose, we have constructed for any Gaussian separable state a Gaussian purification and a Gaussian measurement on the purifying part E , which projects the state onto a product of states of subsystems A and B . This allowed us to prove two important properties of GIE. First, making use of the result we have shown that if a Gaussian state is separable, then GIE vanishes. Second, combining the result with the realization of LOCC operations by teleportation with a separable shared state we have arrived to an important observation that GIE does not increase under the GLTPOCC. In particular, the monotonicity property implies that GIE is invariant with respect to all local Gaussian unitary operations.

Finally, we have calculated analytically GIE for two simple classes of two-mode Gaussian states. For pure Gaussian states, GIE is equal to the GR2 entanglement [24] whereas equality to the entropy of entanglement is established provided that Alice and Bob are allowed to perform non-Gaussian measurements. An analytical formula for GIE has been also derived for one-parametric family of two-mode reductions of the three-mode CV GHZ state, which was also found to be equal to the GR2 entanglement. Last but not least, we have also extended our analysis of the proposed entanglement quantifier to a non-Gaussian case by calculating a lower bound on IE for a particular subset of a set of two-mode continuous-variable Werner states.

The results obtained in this paper raise several questions which remain open for further research. First, it is imperative to know, whether GIE is monotonic under all (including trace-decreasing) GLOCC operations. If answered in the affirmative, we could call GIE a Gaussian entanglement measure. Another important question concerns computability of GIE on other Gaussian states. Knowing GIE for other Gaussian states, one can then further investigate a rather surprising finding that GIE and GR2 entanglement are equal on some Gaussian states. A proof showing the equality of the two quantities on all bipartite Gaussian states would link GR2 entanglement with the secret-key agreement protocol [11] and, what is more, this would also mean that GIE possesses all the properties of GR2

entanglement including, e.g., monogamy. Finally, GIE is a faithful quantity [21] which is nonzero on all entangled states and therefore it opens a possibility to quantify the amount of entanglement in Gaussian bound entangled states [35].

We hope that the results presented here will further stimulate research in the field of the computable and physically meaningful entanglement measures.

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