

**Weak-value amplification as an optimal metrological protocol**

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The implementation of weak-value amplification requires the pre- and postselection of states of a quantum system, followed by the observation of the response of the meter, which interacts weakly with the system. Data acquisition from the meter is conditioned to successful postselection events. Here we derive an optimal postselection procedure for estimating the coupling constant between system and meter and show that it leads both to weak-value amplification and to the saturation of the quantum Fisher information, under conditions fulfilled by all previously reported experiments on the amplification of weak signals. For most of the preselected states, full information on the coupling constant can be extracted from the meter data set alone, while for a small fraction of the space of preselected states, it must be obtained from the postselection statistics.

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**I. INTRODUCTION**

The notion of weak-value amplification (WVA), introduced in the pioneer work of Aharonov, Albert, and Vaidman [1], has been frequently associated with the possibility of amplifying weak signals, as small birefringence effects [2,3], the spin Hall effect of light [4], tiny deflections of light produced by moving mirrors in optical setups [5–8], slow-velocity measurements [9], small phase-shift time delays [10–12], tiny optical angular rotations [13,14], or the measurement of small frequency changes in the optical domain [15].

As shown in Ref. [1], and further elaborated in Ref. [16], WVA may also lead to exotic results, like obtaining for the measurement of a component of a spin  $\frac{1}{2}$  a value as high as  $100\hbar$  [1], or having a displacement in the average photon number of a field in a cavity much larger than one, after the passage of a single atom [17]. This last example has played a pioneer role in the development of the theory of quantum random walks [18].

The procedure for attaining an amplification of the weak value, which has as essential ingredient a conditional measurement procedure, can be divided into two steps: (i) the system to be measured, prepared in a preselected initial state, first interacts weakly with a meter, through a bilinear coupling—quantified by a coupling constant  $g$ —between observables  $\hat{A}$  of the system and  $\hat{M}$  of the meter, and then is postselected in a predetermined state, usually taken as almost orthogonal to the initial state of the system; (ii) the weak value (real part or imaginary part) is determined by observation of the meter, whenever the postselection in the predetermined state is successful. In this procedure, the amplification of the weak value is not deterministic. It is assumed that the width of the initial distribution of eigenvalues of the pointer is much larger than the spectrum of eigenvalues of  $A$ , and that the interaction between system and meter takes place during a short time interval, so that the free evolution of system plus meter can be safely neglected.

The possibility of amplifying very weak signals via WVA leads quite naturally to the question as to whether such measurements may be used to enhance metrological protocols that aim to estimate the coupling constant  $g$ . However, such procedures may lead not only to amplification of the signal, but also to the mitigation of the number of experimental data

(statistics) that may be used to estimate  $g$ . This has led to debate on the possible advantages of weak measurements over the standard quantum-measurement procedure [19–25] and to proposals for improving the WVA method [26–29]. Proper treatment of this problem requires the machinery of quantum metrology, which establishes general bounds for the uncertainty in the estimation of parameters [30–34], defined by the mean-square estimation error and expressed in terms of the corresponding quantum Fisher information. From Refs. [20,23,32] and the above discussion, it is clear that the amount of information on  $g$  cannot be superior to that quantified by the corresponding quantum Fisher information. This is an upper bound valid for any kind of measurement, including probabilistic measurements, like weak-value amplification and abstention procedures [35]. Moreover, as shown in Ref. [32], this bound is saturated by projective measurements. This implies that one can always find a standard measurement procedure that is at least as good as a weak measurement in estimating a given parameter. In spite of this, practical advantages of weak measurements have been pointed out [19,24].

Here we address the formalism of WVA itself and propose an optimized postselection procedure, which actually saturates the quantum Fisher information corresponding to the estimation of  $g$ , in the weak-coupling limit, and can be applied to all previously reported experiments involving amplification of weak signals. This procedure leads to a postselected state that is not, in general, quasi-orthogonal to the initial state, as opposed to the usual approach. We also show that proper handling of the conditions of weak coupling between quantum system and meter involves a limiting procedure concerning two small quantities, the coupling constant and the overlap between initial and postselected states, which when tackled properly leads to results at variance with previously published analyses. These results imply that WVA, even though relying on a reduced data set, may lead, under proper choice of the detection procedure, to the same information on the parameter to be estimated as optimal quantum measurement protocols.

This paper is organized as follows: Section II reviews basic concepts in quantum metrology and their application to a typical WVA Hamiltonian. In Sec. III, we discuss a general quantum metrological approach to parameter estimation with postselection, which takes into account the degradation of

information in the meter due to the loss of statistical data, and implies that the information on the parameter is shared between the meter and the postselection statistics. In Sec. IV, which concentrates the main results of this paper, we analyze the limiting situation of weak coupling and derive the best postselection procedure, which saturates the quantum Fisher information up to corrections of second order in the coupling constant  $g$ . In the same section, we also show that, depending on the ratio between  $g$  and the overlap of initial and postselected states, the information may get concentrated on the meter or on the postselection statistics. These results are applied to a two-level system in Sec. V, which also contains a discussion on how our results help to improve WVA experiments. Our conclusions are summarized in Sec. VI. In order to make the paper more readable, we have moved detailed calculations to the appendixes.

## II. QUANTUM METROLOGICAL LIMITS

We consider now the ultimate precision limit in the estimation of  $g$  for the WVA Hamiltonian. We assume that the meter  $\mathcal{M}$  and the quantum system  $\mathcal{A}$  couple through the interaction  $\hat{H}_I(t) = \hbar g \delta(t - t_0) \hat{A} \hat{M}$ , with  $g$  being positive and dimensionless, without loss of generality. The delta function accounts for the assumption that the interaction takes place around time  $t_0$  and within a short interval of time compared to the free evolution of the total system  $\mathcal{A} + \mathcal{M}$ . The corresponding evolution operator is  $\hat{U}(g) = \exp[-(i/\hbar) \int \hat{H}_I(t) dt] = \exp(-ig \hat{A} \hat{M})$ .  $\mathcal{A}$  and  $\mathcal{M}$  are initially prepared in the state  $|\Psi_i\rangle = |\psi_i\rangle \otimes |\phi_i\rangle$ , where  $|\psi_i\rangle$  is the initial quantum state of  $\mathcal{A}$  and  $|\phi_i\rangle$  is the initial state of  $\mathcal{M}$ .

If one estimates the value of a general parameter  $x$  through  $\nu$  repeated measurements on the system that carries information about it, then the minimum reachable uncertainty on unbiased estimatives of the parameter is determined by the Cramér–Rao limit [36–38],

$$\delta x \geq 1/\sqrt{\nu F(x)}. \quad (1)$$

Here  $\delta x = \langle (x - x_{\text{est}})^2 \rangle^{1/2}$  is the mean-square estimation error, the average is taken over all possible experimental results, and  $x_{\text{est}}$  is an estimate of the parameter  $x$ , based on the observed data.  $F(x)$  is the *Fisher information*, defined by

$$F(x) = \sum_k \frac{1}{P_k(x)} \left[ \frac{dP_k(x)}{dx} \right]^2, \quad (2)$$

where  $P_k(x)$  is the probability distribution of obtaining an experimental result  $k$ , assuming that the value of the parameter is  $x$ . The Fisher information  $F(x)$  depends, through  $P_k(x)$ , on the state of the system and on the measurement performed on it.

The maximization of  $F(x)$  over all possible measurements leads to the *quantum Fisher information* [30–32]  $\mathcal{F}(x)$ , which depends only on the  $x$ -dependent state of the system, and yields in Eq. (1) the minimum possible value of  $\delta x$ . For a pure state  $|\Psi(x)\rangle$ , it is given by [30]

$$\mathcal{F}(x) = 4 \left[ \frac{d\langle \Psi(x) |}{dx} \frac{d|\Psi(x)\rangle}{dx} - \left| \frac{d\langle \Psi(x) |}{dx} |\Psi(x)\rangle \right|^2 \right]. \quad (3)$$

This quantity yields the maximum amount of information about the parameter  $x$  retrievable via measurements on the state  $|\Psi(x)\rangle$ .

If  $|\Psi(x)\rangle = \exp(-ix \hat{H}) |\Psi_i\rangle$ , with  $\hat{H}$  independent of  $x$ , it follows from Eq. (3) that

$$\mathcal{F}(x) = 4[\langle \Psi_i | \hat{H}^2 | \Psi_i \rangle - \langle \Psi_i | \hat{H} | \Psi_i \rangle^2]. \quad (4)$$

For  $|\Psi_i\rangle = |\psi_i\rangle \otimes |\phi_i\rangle$  and  $\hat{U}(g) = \exp(-ig \hat{A} \hat{M})$ ,

$$\mathcal{F}(g) = 4[\langle \hat{A}^2 \rangle \langle \hat{M}^2 \rangle - \langle \hat{A} \rangle^2 \langle \hat{M} \rangle^2], \quad (5)$$

where from now on the averages of operators corresponding to  $\mathcal{A}$  and  $\mathcal{M}$  are taken respectively in the states  $|\psi_i\rangle$  and  $|\phi_i\rangle$ . We compare now this expression to the one corresponding to the WVA protocol.

## III. PARAMETER ESTIMATION WITH POSTSELECTION

For the evolution corresponding to  $\hat{U}(g)$ , the probability of detecting system  $\mathcal{A}$  in the state  $|\psi_f\rangle$  immediately after  $t_0$  is

$$p_f(g) = \|\langle \psi_f | \hat{U}(g) | \Psi_i \rangle\|^2. \quad (6)$$

If  $\mathcal{A}$  is detected in the state  $|\psi_f\rangle$ ,  $\mathcal{M}$  is left in the normalized state

$$|\phi_f(g)\rangle = \langle \psi_f | \hat{U}(g) | \Psi_i \rangle |\phi_i\rangle / \sqrt{p_f(g)}. \quad (7)$$

The original WVA strategy involves measuring the meter  $\mathcal{M}$  only when the system  $\mathcal{A}$  is postselected in  $|\psi_f\rangle$ . The postselection statistics—described by the postselection probability  $p_f(g)$  in the asymptotic limit  $\nu \rightarrow \infty$ —is ignored in the estimation of  $g$ . Full consideration of the postselection procedure should take it into account. As shown in Ref. [21,23], this can be described through a set of generalized measurement operators  $\{|\psi_f\rangle\langle\psi_f| \otimes \hat{E}_j, (\hat{1}_{\mathcal{A}} - |\psi_f\rangle\langle\psi_f|) \otimes \hat{1}_{\mathcal{M}}\}$ ,  $j = 1, 2, \dots, n$ , where the set  $\{\hat{E}_j\}$ , with  $\sum_{j=1}^n \hat{E}_j = \hat{1}_{\mathcal{M}}$ , acts on the states of  $\mathcal{M}$ . The corresponding Fisher information for the estimate of the coupling constant  $g$ , as defined by Eq. (2), is [21,23]

$$F_{ps}(g) = F_m(g) + F_{p_f}(g), \quad (8)$$

where

$$F_m(g) = p_f(g) \sum_{j=1}^n \frac{1}{P_j(g)} \left[ \frac{dP_j(g)}{dg} \right]^2, \quad (9)$$

with  $P_j(g) = \langle \phi_f(g) | \hat{E}_j | \phi_f(g) \rangle$ , and

$$F_{p_f}(g) = \frac{1}{p_f(g)[1 - p_f(g)]} \left[ \frac{dp_f(g)}{dg} \right]^2. \quad (10)$$

The function  $F_m(g)$  is the Fisher information associated with measurements on the state of the meter after postselection times the probability  $p_f(g)$  that the postselection succeeds. It quantifies the *performance* of the estimate of the original WVA procedure and takes into account both the enhancement provided by the postselection, through the state (7), and the degradation due to the loss of statistical data, through the probability  $p_f(g)$ .

The term  $F_{p_f}(g)$  stands for the information on  $g$  encoded in  $p_f(g)$ . It quantifies the amount of information on  $g$  acquired from  $p_f(g)$  itself. The total Fisher information  $F_{ps}(g)$  is obtained with the best unbiased estimative of  $g$  that considers

all available data in the experiment, when the meter is monitored only if the postselection of the system is successful.

The optimal measurement on the meter is independent of whether the postselection statistics is considered for the estimation of  $g$ . An optimal set of measurements for the meter can be built with the eigenvectors of the symmetric logarithmic derivative operator associated with the state (7) [30,32]. Details are given in Appendix A.

The maximal value  $\mathcal{F}_m(g)$  of  $F_m(g)$  over all positive-operator valued measures (POVMs) acting in the Hilbert space of the meter is obtained by inserting  $|\phi_f(g)\rangle$  into Eq. (3), with  $x \equiv g$ , and multiplying the result by  $p_f(g)$ , yielding

$$\mathcal{F}_m(g) = 4[|\langle \hat{Q}(g)^\dagger \hat{Q}(g) \rangle - |\langle \hat{Q}(g)^\dagger \hat{O}(g) \rangle|^2 / p_f(g)], \quad (11)$$

where  $\hat{O}(g) = \langle \psi_f | e^{-ig\hat{A}\hat{M}} | \psi_i \rangle$  and  $\hat{Q}(g) = \langle \psi_f | \hat{A}\hat{M} e^{-ig\hat{A}\hat{M}} | \psi_i \rangle$  are operators that act in the Hilbert space of the meter. Note that  $\mathcal{F}_m(g)$  is a functional of  $|\psi_i\rangle$  and of  $|\psi_f\rangle$ , the postselected state. We define  $\mathcal{F}_{ps}(g) \equiv \mathcal{F}_m(g) + F_{p_f}(g)$ . Since in all reported WVA experiments only the meter is measured, a challenging question is whether the quantum Fisher information given in Eq. (5) can be attained by  $\mathcal{F}_m(g)$  alone. If not, can this be accomplished by  $\mathcal{F}_{ps}(g)$ ?

In the next section we examine these questions, as we specialize these results to the weak-coupling regime.

#### IV. WEAK-COUPLING REGIME WITH BALANCED METERS

We solve the problem of maximizing  $\mathcal{F}_{ps}(g)$  over the state  $|\psi_f\rangle$  in the weak-coupling limit, with the condition  $\langle \hat{M} \rangle = 0$  (balanced meter). Then  $\Delta \equiv \langle \hat{M}^2 \rangle^{1/2}$  is the standard deviation of the initial distribution of eigenvalues of  $\hat{M}$ , and the quantum Fisher information becomes  $\mathcal{F}(g) = 4\langle \hat{A}^2 \rangle \Delta^2$ . Under these conditions, and for separable initial states, we show that it is always possible to find a postselected state  $|\psi_f\rangle$ , such that  $\mathcal{F}_{ps}(g)$  reaches, up to first order in  $g$ , the quantum Fisher information  $\mathcal{F}(g)$ . Those conditions are fulfilled in all experiments reported so far that aimed to amplify weak signals [2–8,15].

We discuss first the situation where  $\mathcal{F}_m(g)$  alone reaches  $\mathcal{F}(g)$ . For  $g$  sufficiently small, we show in Appendixes A and B that  $\mathcal{F}_m(g) = 4\Delta^2 |\langle \psi_i | \hat{A} | \psi_f \rangle|^2 [1 + O(g)]$ . This implies that the ansatz  $|\psi_f^{\text{opt}}\rangle = \hat{A} |\psi_i\rangle / \langle \hat{A}^2 \rangle^{1/2}$  leads to  $\mathcal{F}_m(g) \rightarrow \mathcal{F}_m^{\text{opt}}(g) = \mathcal{F}(g) + O(g^2)$ , where the superscript opt specifies quantities corresponding to the above postselection. Therefore,  $\mathcal{F}_m^{\text{opt}}(g)$  coincides with the quantum Fisher information, up to first order in  $g$ . For this optimal postselection (which does not have any dependence on the initial state of the meter, other than the balanced-meter requirement), the correction must be necessarily nonpositive, independently of the sign of  $g$ , which excludes corrections of  $O(g)$ .

Quantifying the meaning of “ $g$  sufficiently small” requires careful consideration of the relative magnitudes of  $\delta \equiv \langle \psi_f^{\text{opt}} | \psi_i \rangle = \langle \hat{A} \rangle / \langle \hat{A}^2 \rangle^{1/2}$  and  $g$ , since  $|\delta|$  may become much smaller than one, for some initial states, as discussed in the following. We shall show however that, except for very small

values of  $|\delta|$ , as compared to  $g$ , the information from the meter is enough to saturate the quantum Fisher information  $\mathcal{F}(g)$ .

It is worthwhile to note that, for any  $\hat{A}$  and  $|\psi_i\rangle$ ,  $\hat{A}|\psi_i\rangle = \langle \hat{A} \rangle |\psi_i\rangle + [\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2]^{1/2} |\psi_i\rangle_\perp$ , where  $|\psi_i\rangle_\perp$  is orthogonal to  $|\psi_i\rangle$ . Therefore,  $|\psi_f^{\text{opt}}\rangle$  is not, in general, necessarily quasi-orthogonal to the initial state, which is typically assumed in the WVA literature to be the ideal postselected state. Indeed, depending on  $|\psi_i\rangle$ ,  $|\psi_f^{\text{opt}}\rangle$  may vary continuously from a state parallel to the initial state (when  $|\psi_i\rangle$  is an eigenstate of  $\hat{A}$ ) to a state orthogonal to  $|\psi_i\rangle$  (when  $\langle \hat{A} \rangle = 0$ ).

The weak value of  $\hat{A}$  is defined as  $A_w = \langle \psi_f | \hat{A} | \psi_i \rangle / \langle \psi_f | \psi_i \rangle$  [1]. For  $|\psi_f\rangle = |\psi_f^{\text{opt}}\rangle$ , this becomes  $A_w = \langle \hat{A}^2 \rangle / \langle \hat{A} \rangle$  which, for any state that is not an eigenstate of  $\hat{A}$ , is larger than the average value of  $\hat{A}$ , i.e., there is amplification of the signal. In general, the magnitude of  $A_w$  depends on the smallness of the absolute value of the scalar product of the pre- and postselected states of the system. The limit of amplification, in order that the weak value be well defined, was discussed earlier in the literature [16,39]. In particular, it is required that  $g|A_w|\Delta \ll 1$ . Indeed, the signal obtained from the measurement of the meter is de-amplified as one tries to get very close to the orthogonal postselection, which has been named the *inverted region* [39]. The optimal postselected state  $|\psi_f^{\text{opt}}\rangle$ , which leads to the best precision in the estimation of  $g$ , does not provide the largest possible amplification established by Ref. [16], since  $|\delta|$  is not necessarily much smaller than one.

We show now that, in two limiting cases, the information on  $g$  gets concentrated either in  $\mathcal{F}_m^{\text{opt}}$  or  $F_{p_f}^{\text{opt}}$ . We assume in the following balanced meters and the postselection of  $|\psi_f^{\text{opt}}\rangle$ . We have then, as shown in Appendix C:

(a) If  $|\delta| \ll g\langle \hat{A}^2 \rangle^{1/2}\Delta$ , or equivalently  $g|A_w|\Delta \gg 1$ , then  $F_{p_f}^{\text{opt}}(g) = \mathcal{F}(g)[1 + O(\varepsilon)]$ , and  $\mathcal{F}_m^{\text{opt}}(g) = O(\varepsilon)$ , where  $\varepsilon = \text{Max}\{|\delta|, g^2, (\delta/g)^2\}$ . In this case, full information on  $g$  can be obtained from the statistics of successful postselection detections in  $|\psi_f^{\text{opt}}\rangle$ . One should note, however, that this is the region of parameters for which the usual weak-value theory breaks down [39]. Furthermore, this condition holds in a small region of overlap  $\delta$ , since the convergence of the expansion in  $g$  requires that  $g\langle \hat{A}^2 \rangle^{1/2}\Delta \ll 1$  [39]. Typical experimental values of  $g\langle \hat{A}^2 \rangle^{1/2}\Delta$  range from  $10^{-3}$  [4,9] to  $10^{-8}$  [5].

(b) If  $|\delta| \gg g\langle \hat{A}^2 \rangle^{1/2}\Delta$ , or equivalently  $g|A_w|\Delta \ll 1$  (regime of validity of the weak-value theory),  $\mathcal{F}_m^{\text{opt}}(g) = \mathcal{F}(g)[1 + O(g^2)]$ , and  $F_{p_f}^{\text{opt}}(g) = O(g^2)$ . Therefore, full information on  $g$  is now obtained by considering just the best measurement on the pointer after postselection.

One should note that condition (a) includes the region of initial states where  $|\psi_f^{\text{opt}}\rangle$  becomes orthogonal to  $|\psi_i\rangle$ . This implies, surprisingly, that even though exact orthogonality is avoided in typical WVA treatments, it actually leads to saturation of the quantum Fisher information, with the information on  $g$  fully concentrated in the statistics of successful postselection events. Then, measurements on the meter, the ones considered in most WVA analysis, yield no information on  $g$ . This is contrary to the conclusions attained in previous work, such as, for instance, in Ref. [21], where the limit  $g \rightarrow 0$  is taken irrespectively of the value of  $\delta$ , thus leading to the misleading conclusion that the information on  $g$  is always concentrated on the meter.

All these results were obtained for postselection in  $|\psi_f\rangle = |\psi_f^{\text{opt}}\rangle$ . As shown in Appendix D, the choice  $|\psi_f\rangle = |\psi_i\rangle$  also leads to  $\mathcal{F}_{ps}(g) = \mathcal{F}(g) + O(g^2)$ . In this case, the information obtained from the statistics of successful postselection becomes important in a broader region of initial states. One should note, however, that in this case  $A_w = \langle\psi_i|\hat{A}|\psi_i\rangle$ , and therefore there is no weak-value amplification. This shows that, even in a postselection procedure, WVA is not needed in order to increase the precision in the estimation of  $g$ .

The example discussed in the following illustrates the results of this section and shows that experiments already realized [4,5,13] could be improved by using the present approach.

### V. WEAK-VALUE AMPLIFICATION OF A SPIN

Consider a two-level system, which interacts with the meter in such a way that  $\hat{A} = \hat{\sigma}_z$ . We parametrize the pre- and postselected states by the angles  $\{\theta_i, \theta_f, \phi_i, \phi_f\}$ , so that  $|\psi_i\rangle = \cos(\theta_i/2)|0\rangle + e^{i\phi_i}\sin(\theta_i/2)|1\rangle$  and  $|\psi_f\rangle = \cos(\theta_f/2)|0\rangle + e^{i\phi_f}\sin(\theta_f/2)|1\rangle$ , where  $|0\rangle$  and  $|1\rangle$  are the eigenvectors of  $\hat{\sigma}_z$  corresponding to the eigenvalues  $+1$  and  $-1$ , respectively. Since  $\hat{A}^2 = \hat{1}$ , the small-coupling condition requires that  $g\Delta \ll 1$ . Also, Eq. (5) with  $\langle\hat{M}\rangle = 0$  implies that  $\mathcal{F}(g) = 4\Delta^2$ , which does not depend on the initial state of the system.

According to the previous discussion, the bound  $\mathcal{F}(g)$  can be approached when  $|\psi_f\rangle = \hat{\sigma}_z|\psi_i\rangle$ ,  $\langle\hat{M}\rangle = 0$ , and if terms of  $O(g^2)$  are neglected. For this postselected state,  $\theta_f = \theta_i = \theta$  and  $\phi \equiv \phi_f - \phi_i = \pi$  ( $\delta = \cos\theta$ ). Notice that  $\mathcal{F}_m^{\text{opt}}(g)$ ,  $F_{p_f}(g)$ , and  $\mathcal{F}_{ps}^{\text{opt}}(g)$  are invariant under the transformation  $\theta \rightarrow \pi - \theta$ .

For  $\theta \rightarrow 0$ , the probability of postselection is approximately equal to one, and all the information on  $g$  is in the meter—consistent with case (b) in Sec. IV.

As  $\theta$  approaches  $\pi/2$ , one gets into the region where  $|\delta| \ll g\Delta$ , so that, according to condition (a), all the information on  $g$  gets concentrated on the statistics of postselection. Outside this region, the quantum Fisher information is also saturated by  $\mathcal{F}_{ps}(g)$ , but it quickly concentrates on the meter.

Figure 1 illustrates this balance of information between meter and postselection statistics by plotting the contributions of  $\mathcal{F}_m^{\text{opt}}(g)/\mathcal{F}(g)$  (dashed line) and  $F_{p_f}^{\text{opt}}(g)/\mathcal{F}(g)$  (dotted-dashed line) as a function of  $\theta_i$ , for  $g\Delta = 0.1$ . We have assumed that the initial state of the meter is a pure state with a Gaussian distribution of the eigenvalues of  $\hat{M}$ , with width  $\Delta = \langle\hat{M}^2\rangle^{1/2}$ . The corresponding analytical results are derived in Appendix E. For  $|\delta| \ll g\Delta$  (case (a) in Sec. IV), the major contribution to the quantum Fisher information comes from  $F_{p_f}^{\text{opt}}(g)$ . As  $|\delta|$  increases, the contribution  $\mathcal{F}_m^{\text{opt}}(g)$  becomes more relevant [case (b) in Sec. IV]. The point where the two contributions coincide is very close to  $|\delta| = g\Delta$ , or equivalently  $|A_w| = (g\Delta)^{-1}$ , as one should expect from the above analysis.

Figure 2 illustrates the behavior of  $\mathcal{F}_m(g)/\mathcal{F}(g)$  and  $\mathcal{F}_{ps}(g)/\mathcal{F}(g)$  (inset), as a function of  $\theta_f$ , for several values of  $g\Delta$ . As discussed before,  $\theta_f = \theta_i$  corresponds to an optimal postselection procedure. Figure 2 displays a dip in  $\mathcal{F}_m(g)$ , which corresponds to the region where  $F_{p_f}(g)$  must be taken

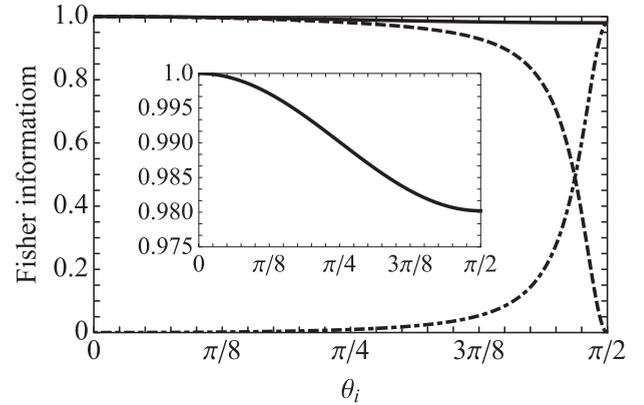


FIG. 1. Bounds for the information from the meter,  $\mathcal{F}_m^{\text{opt}}(g)$  (dashed line) and from the probability of postselection,  $F_{p_f}^{\text{opt}}(g)$  (dotted-dashed line), normalized by the quantum Fisher information, as a function of  $\theta_i$ , for  $g\Delta = 0.1$ . The full line, also displayed in the inset, is the sum of the two contributions. It does not reach the value one, since saturation of  $\mathcal{F}(g)$  is up to errors of  $O(g^2)$ .

into account. This dip becomes wider as  $g\Delta$  increases, a behavior that is analytically described in the Appendix. The inset shows that the maximum of  $\mathcal{F}_{ps}(g)$  is reached for  $\theta_f = \theta_i \approx \pi/3$ , for the values of  $g\Delta$  considered. For  $g\Delta \ll 1$ , the meter information dominates over practically the whole range of values of  $\theta_f$ , except for a very narrow dip, which is compensated by a corresponding sharp increase of the information in  $F_{p_f}(g)$ , so that the full postselection Fisher information  $\mathcal{F}_{ps}(g)$ , displayed in the inset, has a smooth behavior.

Postselection in the initial state also leads to the saturation of  $\mathcal{F}(g)$ . In this case, as shown in the Appendix, one has simply  $\mathcal{F}_m(g)/\mathcal{F}(g) = \cos^2\theta + O(g^2\Delta^2)$  and  $F_{p_f}(g)/\mathcal{F}(g) = \sin^2\theta + O(g^2\Delta^2)$ . It is evident that, for this postselection, information from the postselection statistics must be considered over a broader range of initial states, as compared to the previous case. However, as mentioned before, there is no weak-value amplification in this case.

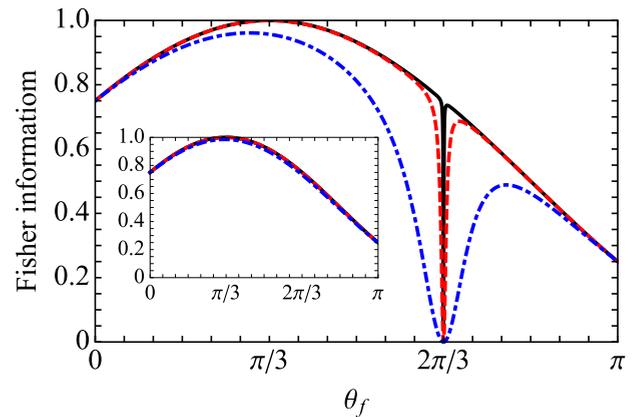


FIG. 2. (Color online)  $\mathcal{F}_m(g)/\mathcal{F}(g)$  and  $\mathcal{F}_{ps}(g)/\mathcal{F}(g)$  (inset), as a function of  $\theta_f$  for  $\theta_i = \pi/3$  and  $\phi = \pi$ . The inset shows the full postselected Fisher information  $\mathcal{F}_{ps}(g)$ , showing that  $F_{p_f}(g)$  fills the dip of  $\mathcal{F}_m(g)$ .  $g\Delta = 10^{-1}$  (dotted-dashed blue line),  $10^{-2}$  (dashed red line),  $10^{-3}$  (full black line).

These results have a direct impact on experiments relying on postselection and weak interactions between meter and system, like those reported in Refs. [4,5,13]. For instance, in Ref. [13] the authors estimate  $g$  with  $\hat{A} = \hat{\sigma}_z$  through measurements on a pointer conditioned to the postselection of a state in the equatorial plane. The initial state is determined by the angle  $\theta_i$ . As  $\theta_i \rightarrow \pi/2$ , the postselected state becomes orthogonal to the initial state, and the authors comment that, in this region, the data are not reliable. Also, the precision in their estimate depends on the initial state. Our approach indicates that the best postselection should be in the state  $\hat{\sigma}_z|\psi_i\rangle$ , the resulting maximal precision being then independent of  $\theta_i$ . Furthermore, as we have shown, maximal precision could be obtained for the initial state in the equatorial plane from the postselection statistics alone.

## VI. CONCLUSION

We have shown that the information on the coupling constant between system and meter, obtained through a postselection procedure that leads to weak-value amplification, saturates the quantum Fisher information in the weak-coupling regime. As the post- and preselected states get orthogonal to each other, the information on the parameter gets transferred from the meter to the postselection statistics, which then plays the dominant role in the estimation protocol, albeit restricted to a small fraction of the space of preselected states. One should note that measurement procedures that discard less information than the one described in this paper could at best increase the information about the parameter by terms of  $O(g^2)$ , which are negligible in the weak-coupling regime. These results imply that the ability to amplify signals of the meter, through postselection, does not spoil the estimation precision, even though it relies on the reduced data set corresponding to the conditional observation of the meter.

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## APPENDIX A: OPTIMAL MEASUREMENT ON THE METER

An optimal set of measurements for the meter can be built with the eigenvectors of the symmetric logarithmic derivative operator associated with the state (7) [30–32]. In the case of a pure state,  $\hat{\rho}(g) = |\phi_f(g)\rangle\langle\phi_f(g)|$ , the symmetric logarithmic derivative operator,  $\hat{L}(g)$ , has an analytic expression:

$$\hat{L}(g) = 2 \frac{d\hat{\rho}}{dg}. \quad (\text{A1})$$

For the case where the meter information matters [case (b) in Sec. IV], and  $|\psi_f\rangle = |\psi^{\text{opt}}\rangle$ , one can compute the zeroth-order projectors from the eigenvectors of Eq. (A1):

$$\hat{E}_{\pm} = |\phi_{\pm}\rangle\langle\phi_{\pm}|, \quad (\text{A2})$$

$$|\phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\phi_i\rangle \pm i|\phi_{i\perp}\rangle), \quad (\text{A3})$$

where, as before,  $|\phi_i\rangle$  is the initial state of the meter, and

$$|\phi_{i\perp}\rangle = \frac{\hat{M}|\phi_i\rangle}{\sqrt{\langle\hat{M}^2\rangle}} \quad (\text{once } \langle\hat{M}\rangle = 0) \quad (\text{A4})$$

is a state orthogonal to  $|\phi_i\rangle$ .

Notice that the above solution is not unique; see, for instance, the discussion in Ref. [40]; and also it might be a difficult task to implement it experimentally.

As a side example, consider the (canonical) case where the meter has a Gaussian distribution of the eigenvalues of the observable  $\hat{M}$ , with zero mean and variance  $\Delta$ . We show in the following that the measurement of the conjugate observable  $\hat{W}$  (such that  $[\hat{M}, \hat{W}] = i$ ) is the best measurement in this case, providing the quantum Fisher information of the meter ( $\mathcal{F}_m$ ), which coincides with the quantum Fisher information for the entire process ( $\mathcal{F}$ ), up to first order in the coupling constant [still in case (b)]. The measurement of  $\hat{W}$  shifts its mean by a quantity proportional to the real part of  $A_w$ , while the variance remains the same [1,41]:

$$\langle\hat{W}\rangle_f \approx -g \text{Re}\{A_w\} = -g \langle\hat{A}^2\rangle / \langle\hat{A}\rangle, \quad (\text{A5})$$

$$(\Delta\hat{W})_f = 1/(2\Delta), \quad (\text{A6})$$

which implies that the meter state is transformed according to

$$\begin{aligned} \phi_i(w) &= \sqrt{\frac{2\Delta^2}{\pi}} \exp\{-\Delta^2 w^2\}, \\ \mapsto \phi_f(w) &= \sqrt{\frac{2\Delta^2}{\pi}} \exp\{-\Delta^2(w + g\langle\hat{A}^2\rangle/\langle\hat{A}\rangle)^2\}. \end{aligned} \quad (\text{A7})$$

Therefore, the Fisher information associated with the post-selected measurements of  $w$ , occurring with probabilities  $P_w(g) = |\phi_f(w)|^2$ , will be given by

$$\begin{aligned} F_m(g) &= p_f \int dw \frac{(\partial_g P_w)^2}{P_w} \\ &= p_f \left( 4\Delta^2 \frac{\langle\hat{A}^2\rangle^2}{\langle\hat{A}\rangle^2} \right) \approx 4\langle\hat{A}^2\rangle\Delta^2, \end{aligned} \quad (\text{A8})$$

recalling that, in the WVA regime,  $p_f \approx |\langle\psi_f^{\text{opt}}|\psi_i\rangle|^2 = \langle\hat{A}\rangle^2/\langle\hat{A}^2\rangle$ .

Equation (A8) coincides with the expression for  $\mathcal{F}(g)$ , as defined in Sec. IV.

## APPENDIX B: EXPRESSIONS FOR $\mathcal{F}_m(g)$ AND $F_{ps}(g)$

We assume here, as in the main text, that system  $\mathcal{A}$  and meter  $\mathcal{M}$  are initially prepared in the separable state  $|\Psi_i\rangle = |\psi_i\rangle \otimes |\phi_i\rangle$ . The maximization of  $F_m(g)$ , and also of  $F_{ps}(g)$ , over all possible POVMs  $\{\hat{E}_j\}$  yields the information  $\mathcal{F}_m(g)$  and, respectively,  $\mathcal{F}_{ps}(g)$ .  $\mathcal{F}_m(g)$  can be obtained through the expression of the quantum Fisher information of the state  $|\phi_f(g)\rangle$  defined by Eq. (7) of the main text. After a

straightforward calculation, we get

$$\begin{aligned} \mathcal{F}_m(g) &= 4p_f(g) \left[ \frac{d\langle\phi_f(g)\rangle}{dg} \frac{d|\phi_f(g)|}{dg} - \left| \frac{d\langle\phi_f(g)\rangle}{dg} |\phi_f(g)\rangle \right|^2 \right] \\ &= 4 \left[ \langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle - \frac{|\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle|^2}{p_f(g)} \right], \end{aligned} \quad (\text{B1})$$

$$F_{p_f}(g) = \frac{4\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle\}}{p_f(g)} + \frac{4\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle\}}{1-p_f(g)}, \quad (\text{B2})$$

where

$$\hat{Q}(g) = \langle\psi_f|\hat{A}\hat{M}e^{-ig\hat{A}\hat{M}}|\psi_i\rangle, \quad (\text{B3})$$

$$\hat{O}(g) = \langle\psi_f|e^{-ig\hat{A}\hat{M}}|\psi_i\rangle, \quad (\text{B4})$$

$$p_f(g) = \langle\hat{O}(g)^\dagger\hat{O}(g)\rangle, \quad (\text{B5})$$

and we have used the notation  $\langle\hat{X}\rangle$  to denote the average (over the initial state) in the Hilbert space where the operator  $\hat{X}$  acts. The coupling constant  $g$  is assumed to be dimensionless.

The above expressions imply that

$$\begin{aligned} \mathcal{F}_{ps}(g) &= \mathcal{F}_m(g) + F_{p_f}(g) \\ &= 4 \left[ \langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle - \frac{\text{Re}^2\{\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle\}}{p_f(g)} + \frac{\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle\}}{1-p_f(g)} \right]. \end{aligned} \quad (\text{B6})$$

Here we present the expansions for the relevant quantities used in the main text. We adopt the notation  $\langle\hat{A}^n\rangle_{fi} \equiv \langle\psi_f|\hat{A}^n|\psi_i\rangle$ , assuming that  $\langle\hat{A}^n\rangle_{fi}$  is bounded, and define  $\delta \equiv \langle\psi_f|\psi_i\rangle$  as real and positive (without loss of generality). Then we have

$$\begin{aligned} \langle\hat{O}^\dagger(g)\hat{O}(g)\rangle &= p_f(g) = \delta^2 + 2g\delta\text{Im}\{\langle\hat{A}_{fi}\rangle\langle\hat{M}\rangle + g^2[|\langle\hat{A}_{fi}\rangle|^2 - \delta\text{Re}\{(\hat{A}^2)_{fi}\}]\langle\hat{M}^2\rangle - g^3/3\text{Im}\{(\hat{A}^3)_{fi} + 3(\hat{A}^2)_{fi}^*\hat{A}_{fi}\}\langle\hat{M}^3\rangle \\ &\quad + g^4/12[\text{Re}\{(\hat{A}^4)_{fi} - 4\hat{A}_{fi}^*(\hat{A}^3)_{fi}\} + 3|(\hat{A}^2)_{fi}|^2]\langle\hat{M}^4\rangle + O(g^5), \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle &= |\langle\hat{A}_{fi}\rangle|^2\langle\hat{M}^2\rangle - 2g\text{Im}\{(\hat{A}^2)_{fi}^*\hat{A}_{fi}\}\langle\hat{M}^3\rangle + g^2[|(\hat{A}^2)_{fi}|^2 - \text{Re}\{(\hat{A}^3)_{fi}^*\hat{A}_{fi}\}]\langle\hat{M}^4\rangle - g^3/3[\text{Im}\{(\hat{A}^4)_{fi}\hat{A}_{fi}^* \\ &\quad - 3(\hat{A}^3)_{fi}(\hat{A}^2)_{fi}^*\}]\langle\hat{M}^5\rangle + g^4/12[\text{Re}\{(\hat{A}^5)_{fi}\hat{A}_{fi}^* - 4(\hat{A}^4)_{fi}(\hat{A}^2)_{fi}^*\} + 3|(\hat{A}^3)_{fi}|^2]\langle\hat{M}^6\rangle + O(g^5), \end{aligned} \quad (\text{B8})$$

$$\begin{aligned} \langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle &= \delta A_{fi}^*\langle\hat{M}\rangle + ig[\delta(\hat{A}^2)_{fi}^* - |\langle\hat{A}_{fi}\rangle|^2]\langle\hat{M}^2\rangle + g^2/2[2(\hat{A}^2)_{fi}^*\hat{A}_{fi} - (\hat{A}^2)_{fi}\hat{A}_{fi}^* - (\hat{A}^3)_{fi}^*\delta]\langle\hat{M}^3\rangle \\ &\quad - ig^3/6[\delta(\hat{A}^4)_{fi}^* - 3(\hat{A}^3)_{fi}^*\hat{A}_{fi} + 3|(\hat{A}^2)_{fi}|^2 - \hat{A}_{fi}^*(\hat{A}^3)_{fi}]\langle\hat{M}^4\rangle + g^4/24[\delta(\hat{A}^5)_{fi}^* - 4(\hat{A}^4)_{fi}^*\hat{A}_{fi} \\ &\quad + 6(\hat{A}^3)_{fi}^*(\hat{A}^2)_{fi} - 4(\hat{A}^2)_{fi}^*(\hat{A}^3)_{fi} + \hat{A}_{fi}^*(\hat{A}^4)_{fi}]\langle\hat{M}^5\rangle + O(g^5). \end{aligned} \quad (\text{B9})$$

We assume here that the above expansions converge, for  $g$  sufficiently small and, as in the main text, that the initial state of the meter satisfies the condition that we call *balanced meter*:

$$\langle\hat{M}\rangle = 0. \quad (\text{B10})$$

### APPENDIX C: ANALYSIS OF $\mathcal{F}_m(g)$ AND $F_{p_f}(g)$ FOR A FIXED POSTSELECTED STATE $|\psi_f\rangle$ AS A FUNCTIONAL OF THE INITIAL STATE $|\psi_i\rangle$

In analyzing the behavior of (B1), (B2), and (B6) we should consider the dependence on  $\delta$  of  $p_f(g)$ . It is easy to show that, by using the balanced-meter condition, we may write

$$p_f(g) = \delta^2 + g^2Z(\delta) + R(g,\delta), \quad (\text{C1})$$

where  $R(g,\delta)$  is of order  $g^3$  and  $Z(\delta) = [|\langle\hat{A}_{fi}\rangle|^2 - \delta\text{Re}\{(\hat{A}^2)_{fi}\}]\langle\hat{M}^2\rangle$ , which we assume to be different from zero. Thus,

$$F_{p_f}(g) = \frac{4g^2Z(\delta)^2 + O(g^3)}{\delta^2 + g^2Z(\delta) + O(g^3)} + \frac{4g^2Z(\delta)^2 + O(g^3)}{1 - \delta^2 - g^2Z(\delta) + O(g^3)}, \quad (\text{C2})$$

$$\begin{aligned} \mathcal{F}_m(g) &= 4|\langle\hat{A}_{fi}\rangle|^2\langle\hat{M}^2\rangle - 8g\text{Im}\{(\hat{A}^2)_{fi}^*\hat{A}_{fi}\}\langle\hat{M}^3\rangle \\ &\quad + 4g^2[|(\hat{A}^2)_{fi}|^2 - \text{Re}\{(\hat{A}^3)_{fi}^*\hat{A}_{fi}\}]\langle\hat{M}^4\rangle + O(g^3) \\ &\quad - \frac{4g^2[Z(\delta)^2 + \delta^2\text{Im}^2\{(\hat{A}^2)_{fi}\}]\langle\hat{M}^2\rangle^2 + O(g^3)}{\delta^2 + g^2Z(\delta) + O(g^3)}. \end{aligned} \quad (\text{C3})$$

Comparison of the magnitude of  $\delta^2$  and  $1 - \delta^2$  with the two terms of  $g^2Z(\delta)$ —the largest terms in the denominators of  $\mathcal{F}_m(g)$  and  $F_{p_f}(g)$ —leads to the following limiting cases:

(i) If  $\delta^2 \gg g^2|\langle\hat{A}_{fi}\rangle|^2\Delta^2$  (assuming  $|\langle\hat{A}_{fi}\rangle| \neq 0$ ) and  $1 - \delta^2 \gg \max\{g^2|\langle\hat{A}_{fi}\rangle|^2\Delta^2, g^2|\text{Re}\{(\hat{A}^2)_{fi}\}|\Delta^2\}$ , where  $\Delta \equiv \langle\hat{M}^2\rangle^{1/2}$  is the standard deviation of the meter eigenvalues distribution, then  $F_{p_f} \sim O(g^2)$  and

$$\mathcal{F}_m(g) = 4|\langle\psi_f|\hat{A}|\psi_i\rangle|^2\Delta^2 + O(g). \quad (\text{C4})$$

(ii) In the small region  $\delta^2 \lesssim g^2|\langle\hat{A}_{fi}\rangle|^2\Delta^2$  ( $|\langle\hat{A}_{fi}\rangle| \neq 0$ ) we should have  $Z(\delta) \approx |\langle\hat{A}_{fi}\rangle|^2\langle\hat{M}^2\rangle + O(\delta)$  and  $R(g,\delta) \approx O(g^3)$ .

Hence  $F_{p_f}(g)$  may be well approximated by

$$F_{p_f}(g) = \frac{4g^2|\hat{A}_{fi}|^4\Delta^4}{\delta^2 + g^2|\hat{A}_{fi}|^2\Delta^2} + O(g), \quad (\text{C5})$$

which has the maximum value  $4|\hat{A}_{fi}|^2\Delta^2$  at  $\delta = 0$  (postselected state orthogonal to the initial state). Analogously, we may neglect the term  $\delta^2\text{Im}^2\{(\hat{A}^2)_{fi}\}\Delta^4$  in Eq. (C3), which will contribute in order  $g^4$  (at most) in the numerator, and write

$$\mathcal{F}_m(g) = 4\frac{\delta^2|\hat{A}_{fi}|^2\Delta^2}{\delta^2 + g^2|\hat{A}_{fi}|^2\Delta^2} + O(g). \quad (\text{C6})$$

Equations (C5) and (C6) explain the sharp dip in  $\mathcal{F}_m$  captured in Fig. 2 of the main text, around the point where  $|\psi_f\rangle$  gets orthogonal to  $|\psi_i\rangle$ , and show that in the same region  $F_{p_f}$  has a sharp peak, so that their sum recovers a smooth function, as shown in the same figure.

(iii) The situation when  $1 - \delta^2 \lesssim \max\{g^2|\hat{A}_{fi}|^2\Delta^2, g^2|\text{Re}\{(\hat{A}^2)_{fi}\}|\Delta^2\}$  is more subtle. We analyze it in the following, for specific choices of the postselected state.

#### APPENDIX D: ANALYSIS OF $\mathcal{F}_{ps}(g)$ , $\mathcal{F}_m(g)$ , AND $F_{p_f}(g)$ FOR THE POSTSELECTED STATE $|\psi_f\rangle = |\psi_f^{\text{opt}}\rangle$ AS A FUNCTIONAL OF THE INITIAL STATE $|\psi_i\rangle$

We consider now that the postselected state of the system  $\mathcal{A}$  is chosen as  $|\psi_f\rangle = |\psi_f^{\text{opt}}\rangle = \hat{A}|\psi_i\rangle/\langle\hat{A}^2\rangle^{1/2}$ . As before, we assume the initial state of the meter to be such that  $\langle\hat{M}\rangle = 0$ .

We focus on the physically relevant quantities, which are the information extracted from the meter,  $\mathcal{F}_m(g)$ , given by Eq. (B1) and from the postselection probability,  $F_{p_f}(g)$ , as expressed by Eq. (B2). Assuming the convergence of the expansions around  $g = 0$ , one has

$$\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle = \langle\hat{A}^2\rangle\langle\hat{M}^2\rangle - g^2\left[1 - \frac{\langle\hat{A}^3\rangle^2}{\langle\hat{A}^2\rangle\langle\hat{A}^4\rangle}\right]\langle\hat{M}^4\rangle\langle\hat{A}^4\rangle + O(g^4), \quad (\text{D1})$$

$$\frac{\text{Re}^2\{\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle\}}{p_f(g)} = \frac{\frac{1}{4}\left(1 - \delta\frac{\langle\hat{A}^4\rangle}{\langle\hat{A}^2\rangle^{1/2}\langle\hat{A}^3\rangle}\right)^2\langle\hat{A}^3\rangle^2\langle\hat{M}^3\rangle^2g^4 + O(g^6)}{\delta^2 + \left(1 - \delta\frac{\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^{3/2}}\right)\langle\hat{A}^2\rangle\langle\hat{M}^2\rangleg^2 + O(g^4)}, \quad (\text{D2})$$

$$\frac{\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle\}}{p_f(g)} = \frac{\left[1 - \delta\frac{\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^{3/2}}\right]^2\langle\hat{M}^2\rangle^2\langle\hat{A}^2\rangle^2g^2 + O(g^4)}{\delta^2 + \left(1 - \delta\frac{\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^{3/2}}\right)\langle\hat{A}^2\rangle\langle\hat{M}^2\rangleg^2 + O(g^4)}, \quad (\text{D3})$$

$$\frac{\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle\}}{1 - p_f(g)} = \frac{\left[1 - \delta\frac{\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^{3/2}}\right]^2\langle\hat{M}^2\rangle^2\langle\hat{A}^2\rangle^2g^2 + O(g^4)}{1 - \delta^2 - \left(1 - \delta\frac{\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^{3/2}}\right)\langle\hat{A}^2\rangle\langle\hat{M}^2\rangleg^2 + O(g^4)}, \quad (\text{D4})$$

where  $\delta \equiv \langle\psi_f^{\text{opt}}|\psi_i\rangle = \langle\hat{A}\rangle/\langle\hat{A}^2\rangle^{1/2}$ .

We show now that  $\mathcal{F}_{ps}(g)$  saturates the quantum Fisher information, up to terms of first order in  $g$ . We notice from Eq. (D1) that the first term on the right-hand side of Eq. (B6) already saturates the quantum Fisher information  $\mathcal{F}$ , up to first order in  $g$ . From Eq. (D2), the second term on the right-hand side of Eq. (B6) is at most of  $O(g^2)$ . Furthermore, the third term on the right-hand side of Eq. (B6) is always positive, and therefore must be of  $O(g^2)$ , since  $\mathcal{F}_{ps}$  cannot be larger than  $\mathcal{F}$ . Therefore,

$$\mathcal{F}_{ps}(g) = 4\langle\hat{A}^2\rangle\Delta^2 + O(g^2). \quad (\text{D5})$$

We analyze now two limiting cases, for which the information on  $g$  is obtained either from the meter or from the postselection statistics.

(a) For  $|\delta| = \langle\hat{A}\rangle/\langle\hat{A}^2\rangle^{1/2} \ll g\langle\hat{A}^2\rangle^{1/2}\Delta$ , the contribution from Eq. (D2) is of  $O(g^2)$ , as well as that from Eq. (D4). On the other hand, Eq. (D3) contributes with  $O(g^0)$  in this limit. In this regime, and assuming also that  $\delta\langle\hat{A}^3\rangle/\langle\hat{A}^2\rangle^{3/2} \ll 1$ , Eq. (D3) can be written as

$$\frac{\text{Im}^2\{\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle\}}{p_f(g)} = \frac{g^2\langle\hat{A}^2\rangle^2\Delta^4}{\delta^2 + g^2\langle\hat{A}^2\rangle\Delta^2} + O(\text{Max}\{\delta, g^2\}) \simeq \langle\hat{A}^2\rangle\Delta^2 + O[\text{Max}\{\delta, g^2, (\delta/g)^2\}], \quad (\text{D6})$$

such that we end up with

$$\mathcal{F}_m(g) = O[\text{Max}\{\delta, g^2, (\delta/g)^2\}], \quad (\text{D7})$$

$$F_{p_f}(g) = 4\langle\hat{A}^2\rangle\Delta^2 + O[\text{Max}\{\delta, g^2, (\delta/g)^2\}]. \quad (\text{D8})$$

This expression coincides, up to first order in  $g$ , with the quantum Fisher information. Therefore, in this limit, the information on  $g$  is obtained solely from the postselection statistics.

(b) For  $|\delta| \gg g\langle\hat{A}^2\rangle^{1/2}\Delta$ , the contribution from Eq. (D2) is of  $O(g^4)$  and that of Eq. (D3) is of  $O(g^2)$ .

We show now that the contribution from Eq. (D4) is of  $O(g^2)$ . This is trivially true if  $1 - \delta^2$  is not much smaller than one: then, it will be the dominating term in the denominator, and the right-hand side of Eq. (D4) will be of  $O(g^2)$ . We show in the following that this still holds if  $1 - \delta^2 \ll 1$ . In the limit  $\delta \rightarrow 1$ ,  $|\psi_i\rangle \rightarrow |a\rangle$ , where  $|a\rangle$  is some eigenstate of  $\hat{A}$  with eigenvalue  $a$ . Therefore, for small values of  $1 - \delta^2$ ,  $|\psi_i\rangle$  should be of the form

$$|\psi_i^\epsilon\rangle = \frac{|a\rangle + \epsilon|b\rangle}{\sqrt{1 + \epsilon^2}}, \quad (\text{D9})$$

where  $\hat{A}|a\rangle = a|a\rangle$ ,  $\langle b|a\rangle = 0$ , and  $\epsilon$  is chosen as real (without loss of generality), with  $\epsilon \ll 1$ . This implies that  $\delta^2 = \langle\psi_i^\epsilon|\hat{A}|\psi_i^\epsilon\rangle^2/\langle\psi_i^\epsilon|\hat{A}^2|\psi_i^\epsilon\rangle = 1 - O(\epsilon^2)$ , that is,  $1 - \delta^2 = O(\epsilon^2)$ . Also, the term of  $O(g^2)$  in the denominator of Eq. (D4) can be written as

$$\begin{aligned} g^2Z(\delta) &\equiv (1 - \delta\langle\hat{A}^3\rangle/\langle\hat{A}^2\rangle^{3/2})\langle\hat{A}^2\rangle\langle\hat{M}^2\rangleg^2 \\ &= \frac{g^2\epsilon^2\langle\hat{M}^2\rangle[2a^2\langle\hat{A}^2\rangle_{bb} - a\langle\hat{A}^3\rangle_{bb} - a^3\hat{A}_{bb} + O(\epsilon^2)]}{(1 + \epsilon^2)[a^2 + \epsilon^2\langle\hat{A}^2\rangle_{bb}]} \\ &= O(\epsilon^2g^2), \end{aligned} \quad (\text{D10})$$

so that

$$\frac{g^2|Z(\delta)|}{1-\delta^2} = O(g^2), \quad (\text{D11})$$

implying that, in the denominator of Eq. (D4), the term  $1 - \delta^2$  always dominates over  $g^2Z(\delta)$ . Furthermore, in the same region  $1 - \delta^2 \ll 1$ , the numerator of Eq. (D4) is of  $O(\epsilon^4 g^2)$ , so that Eq. (D4) is indeed of  $O(g^2)$ . From Eqs. (B1) and (B2), one has then, for  $|\delta| \gg g\langle\hat{A}^2\rangle^{1/2}\Delta$ ,

$$\mathcal{F}_m(g) = 4\langle\hat{A}^2\rangle\Delta^2 + O(g^2), \quad (\text{D12})$$

and

$$F_{p_f}(g) = O(g^2). \quad (\text{D13})$$

Therefore, in this limit, the information on  $g$  stems from the meter alone.

#### APPENDIX E: ANALYSIS OF $\mathcal{F}_{ps}(g)$ , $\mathcal{F}_m(g)$ , AND $F_{p_f}(g)$ FOR THE POSTSELECTED STATE $|\psi_f\rangle = |\psi_i\rangle$ AS A FUNCTIONAL OF THE INITIAL STATE $|\psi_i\rangle$

The Fisher information  $\mathcal{F}_{ps}(g)$ ,  $\mathcal{F}_m(g)$ , and  $F_{p_f}(g)$  can be analyzed for  $|\psi_f\rangle = |\psi_i\rangle$ , in the limit of weak coupling, for  $\langle\hat{M}\rangle = 0$ , by using the expansions (B7)–(B9). For  $|\psi_f\rangle = |\psi_i\rangle$ , these expansions yield, respectively,

$$\langle\hat{Q}^\dagger(g)\hat{Q}(g)\rangle = \langle\hat{A}\rangle^2\langle\hat{M}^2\rangle + g^2(\langle\hat{A}^2\rangle^2 - \langle\hat{A}^3\rangle\langle\hat{A}\rangle)\langle\hat{M}^4\rangle + O(g^4), \quad (\text{E1})$$

$$\langle\hat{Q}^\dagger(g)\hat{O}(g)\rangle = ig(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2)\langle\hat{M}^2\rangle + \frac{g^2}{2}(\langle\hat{A}^2\rangle\langle\hat{A}\rangle - \langle\hat{A}^3\rangle)\langle\hat{M}^3\rangle + O(g^3), \quad (\text{E2})$$

$$p_f(g) = 1 - g^2(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2)\langle\hat{M}^2\rangle + O(g^4). \quad (\text{E3})$$

One gets, up to order  $g^2$ :

$$\begin{aligned} \mathcal{F}_m(g) &= 4\langle\hat{A}\rangle^2\langle\hat{M}^2\rangle + g^2\langle\hat{M}^4\rangle\langle\hat{A}^2\rangle^2\left(1 - \frac{\langle\hat{A}\rangle\langle\hat{A}^3\rangle}{\langle\hat{A}^2\rangle^2}\right) \\ &\quad - \frac{4\left(1 - \frac{\langle\hat{A}^2\rangle}{\langle\hat{A}\rangle^2}\right)^2\langle\hat{A}^2\rangle^2\langle\hat{M}^2\rangle^2g^2 + O(g^4)}{1 + \left(\frac{\langle\hat{A}^2\rangle}{\langle\hat{A}\rangle^2} - 1\right)\langle\hat{A}^2\rangle\langle\hat{M}^2\rangleg^2 + O(g^4)} + O(g^4), \end{aligned} \quad (\text{E4})$$

$$F_{p_f}(g) = \frac{4g^2\langle\hat{M}^2\rangle^2\langle\hat{A}^2\rangle^2\left(1 - \frac{\langle\hat{A}^2\rangle}{\langle\hat{A}\rangle^2}\right)^2 + O(g^4)}{g^2\langle\hat{M}^2\rangle\langle\hat{A}^2\rangle\left(1 - \frac{\langle\hat{A}^2\rangle}{\langle\hat{A}\rangle^2}\right) + O(g^4)} + O(g^2). \quad (\text{E5})$$

Therefore,

$$\mathcal{F}_m(g) = 4\langle\hat{A}\rangle^2\Delta^2 + O(g^2), \quad (\text{E6})$$

and

$$F_{p_f}(g) = 4(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2)\Delta^2 + O(g^2), \quad (\text{E7})$$

so that the total quantum Fisher information corresponding to the postselection strategy is

$$\mathcal{F}_{ps}(g) = 4\langle\hat{A}^2\rangle\Delta^2 + O(g^2), \quad (\text{E8})$$

which shows that the postselection in the state  $|\psi_f\rangle = |\psi_i\rangle$  also saturates the quantum Fisher information, up to terms of first order in  $g$ . One should note, however, that in this case the repartition of information between the meter and the postselection statistics differs markedly from that corresponding to the postselection strategy previously discussed: in particular, even though the probability of postselection is close to one, the information on  $g$  is obtained from the postselection statistics alone if the initial state of the system is such that  $\langle\hat{A}\rangle = 0$ .

#### APPENDIX F: EXACT EXPRESSIONS FOR $\mathcal{F}_{ps}(g)$ , $\mathcal{F}_m(g)$ , AND $F_{p_f}(g)$ FOR A TWO-LEVEL SYSTEM AND A GAUSSIAN METER

We take  $U(g) = e^{-ig\hat{\sigma}_3\hat{M}}$ , where  $\hat{\sigma}_3$  is a Pauli operator of the two-level system and  $\hat{M}$  is an operator of the meter. We assume a balanced meter with an initial Gaussian probability distribution of eigenvalues of  $\hat{M}$ , with a width given by  $\Delta = (\langle\hat{M}^2\rangle)^{1/2}$ . The initial and postselected states are parametrized as  $|\psi_i\rangle = \cos(\theta_i/2)|0\rangle + \exp(i\phi_i)\sin(\theta_i/2)|1\rangle$ ,  $|\psi_f\rangle = \cos(\theta_f/2)|0\rangle + e^{i\phi_f}\sin(\theta_f/2)|1\rangle$ , where  $|0\rangle$  and  $|1\rangle$  are eigenvectors of  $\hat{\sigma}_3$  corresponding respectively to the eigenvalues  $+1$  and  $-1$ . We may easily obtain analytic expressions for the quantities in Eqs. (11), (B5), and (B2). Defining  $A = \cos(\theta_i/2)\cos(\theta_f/2)$  and  $B = e^{i\phi}\sin(\theta_i/2)\sin(\theta_f/2)$ , where  $\phi \equiv \phi_f - \phi_i$ , one has

$$p_f(g) = A^2 + |B|^2 + 2A\text{Re}\{B\}e^{-2g^2\Delta^2}, \quad (\text{F1})$$

$$\begin{aligned} \mathcal{F}_m(g) &= 4\Delta^2\left\{A^2 + |B|^2 - 2A\text{Re}\{B\}e^{-2g^2\Delta^2}(1 - 4g^2\Delta^2) \right. \\ &\quad \left. - \frac{16A^2\text{Re}^2\{B\}g^2\Delta^2e^{-4\Delta^2g^2}}{A^2 + |B|^2 + 2A\text{Re}\{B\}e^{-2g^2\Delta^2}}\right\}, \end{aligned} \quad (\text{F2})$$

$$F_{p_f}(g) = \frac{64g^2\Delta^4A^2\text{Re}^2\{B\}e^{-4g^2\Delta^2}}{p_f(g)[1 - p_f(g)]}. \quad (\text{F3})$$

##### 1. Analytical results for the postselection state $|\psi_f\rangle = |\psi_f^{\text{opt}}\rangle$

In this case, one should take  $\theta_i = \theta_f = \theta$  and  $\phi = \pi$ , so that  $A = \cos^2(\theta/2)$ ,  $B = -\sin^2(\theta/2)$ . From Eqs. (F2) and (F3), one gets

$$\begin{aligned} \frac{\mathcal{F}_m(g)}{\mathcal{F}} &= \frac{1}{2}(1 + \cos^2\theta) + \frac{1}{2}(1 - 4g^2\Delta^2)e^{-2g^2\Delta^2}\sin^2\theta \\ &\quad - \frac{2g^2\Delta^2e^{-4g^2\Delta^2}\sin^4\theta}{1 + \cos^2\theta - e^{-2g^2\Delta^2}\sin^2\theta}, \end{aligned} \quad (\text{F4})$$

$$\frac{F_{p_f}(g)}{\mathcal{F}} = \frac{4g^2\Delta^2e^{-4g^2\Delta^2}\sin^2\theta}{(1 + \cos^2\theta - e^{-2g^2\Delta^2}\sin^2\theta)(1 + e^{-2g^2\Delta^2})}. \quad (\text{F5})$$

For  $g\Delta \ll 1$ , expand Eqs. (F4) and (F5) and obtain

$$\frac{\mathcal{F}_m(g)}{\mathcal{F}} = \frac{\delta^2}{\delta^2 + (1 - \delta^2)g^2\Delta^2}[1 + O(g^2)], \quad (\text{F6})$$

$$\frac{F_{p_f}(g)}{\mathcal{F}} = \frac{(1 - \delta^2)g^2\Delta^2}{\delta^2 + (1 - 2\delta^2)g^2\Delta^2}[1 + O(g^2)], \quad (\text{F7})$$

where  $\delta^2 = \cos^2 \theta$ . We notice that  $F_{p_f}$  contributes to the Fisher information only in a very small region  $\Delta\theta \approx O(g\Delta)$  near the equatorial plane in the Bloch sphere. Outside this region  $\mathcal{F}_m(g)/\mathcal{F}(g) \approx 1$ . Indeed  $\mathcal{F}_m(g)$  is larger than  $F_{p_f}(g)$  as  $\theta$  increases from zero up to the value  $(\pi/2 - g\Delta)$ , when their contributions to the Fisher information coincide, if one neglects contributions of  $O(g^2)$ .

## 2. Analytical results for the postselection state $|\psi_f\rangle = |\psi_i\rangle$

In this case, one should take  $\theta_i = \theta_f = \theta$  and  $\phi = 0$ . From Eqs. (F2) and (F3) one gets

$$\frac{\mathcal{F}_m(g)}{\mathcal{F}} = \frac{1}{2}(1 + \cos^2 \theta) - \frac{1}{2}(1 - 4g^2\Delta^2)e^{-2g^2\Delta^2} \sin^2 \theta - \frac{2g^2\Delta^2 e^{-4g^2\Delta^2} \sin^4 \theta}{1 + \cos^2 \theta + e^{-2g^2\Delta^2} \sin^2 \theta}, \quad (\text{F8})$$

$$\frac{F_{p_f}(g)}{\mathcal{F}} = \frac{4g^2\Delta^2 e^{-4g^2\Delta^2} \sin^2 \theta}{(1 + \cos^2 \theta + e^{-2g^2\Delta^2} \sin^2 \theta)(1 - e^{-2g^2\Delta^2})}. \quad (\text{F9})$$

Expansion of these expressions for  $g\Delta \ll 1$  leads to

$$\frac{\mathcal{F}_m(g)}{\mathcal{F}} \approx \cos^2 \theta - g^2\Delta^2(\sin^4 \theta - 3\sin^2 \theta), \quad (\text{F10})$$

$$\frac{F_{p_f}(g)}{\mathcal{F}} \approx \sin^2 \theta + g^2\Delta^2(\sin^4 \theta - 3\sin^2 \theta). \quad (\text{F11})$$

Therefore, in this case the Fisher information corresponding to the postselection statistics must be taken into account over a broader range of initial states, as compared to the postselection strategy considered before.

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