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Dipolar Bose gas in a weak isotropic speckle disorder

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We investigate the properties of a homogeneous dipolar Bose gas in a weak three-dimensional isotropic speckle disorder at finite temperatures. By using the Bogoliubov theory (beyond the mean field), we calculate the condensate and the superfluid fractions as a function of density and strengths of disorder and interaction. The disorder impact on the anomalous density, the chemical potential, and the ground-state energy is also analyzed. We show that the peculiar interplay of the dipole-dipole interaction and weak disorder makes the superfluid fraction and sound velocity anisotropic.

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I. INTRODUCTION

Disordered Bose gas in a weak random external potential (dirty boson) represents an interesting model for studying the relation between Bose-Einstein condensation (BEC) and superfluidity and has been the subject of many theoretical investigations in the past two decades [1–10]. Experimentally, the dirty boson problem was first studied with superfluid helium in aerosol glasses (Vycor) [11–13]. Recently, several groups [14–20] have loaded ultracold atoms into optical potentials and studied BECs in the presence of disorder.

What happens to a homogeneous BEC if a weak random external potential is switched on? Indeed, the presence of a disordered potential may lead to decrease both BEC and superfluidity. Furthermore, one of the intriguing features of disordered Bose gas is the appearance of the so-called Anderson localization [21,22] in the noninteracting case. This phenomenon, which can be understood as the effect of multiple reflections of a plane wave by random scatterers or random potential barriers, has recently attracted a great deal of interest [16–18]. Experimentally, the random potential can be created using different techniques, one of which is the static laser speckle, whereas the potential felt by atoms is proportional to the speckle intensity with the sign of the detuning from the atomic transition [23]. Laser speckles, produced by passing an expanded laser beam through diffusive plates, are special in that they have: (i) exponential, i.e., strongly non-Gaussian intensity distribution and (ii) finite support of their power spectrum [24]. Recent progress in different experimental realizations of laser speckle disorder is reported in Refs. [23,25].

In their recent work Abdulaev and Pelster [26] have shown that a Gaussian approximation of the autocorrelation function of laser speckles, used in some recent papers, is inconsistent with the general background of laser speckle theory. They also pointed out that the concept of a quasi-three-dimensional (3D) speckle, which appears due to an extension of the autocorrelation function in the longitudinal direction of a transverse two-dimensional speckle, is not applicable for the true 3D speckle, since it requires an additional space dimension. In this context, they derived an appropriate autocorrelation function for an isotropic 3D laser speckle potential which has the Fourier transform given in Eq. (16) (see below).

Recent progress in the physics of ultracold gases have led to the creation of BECs with dipole-dipole interaction (DDI) and stimulated a tremendous boost in theoretical and experimental studies of weakly interacting Bose gases [27–29]. What is important in such systems is that the atoms interact via a DDI that is both long ranged and anisotropic. By virtue of this interaction, these systems are expected to open fascinating prospects for the observation of novel quantum phases in ultracold atomic gases. On the other hand, dipolar BECs confined in random media remain largely unexplored. One can quote, for example, a uniform dipolar Bose gas with a Gaussian disorder correlation function, a Lorentzian, and a δ-correlated disorder that have been explored recently by Krumnow and Pelster [30], Nikolic *et al.* [31], and Ghabour and Pelster [32].

In the present paper, we study the impact of a weak disorder potential with a 3D isotropic laser speckle autocorrelation function of Ref. [26] on the properties of a homogeneous dipolar Bose gas at finite temperatures. To this end, we use the Bogoliubov theory (beyond the mean field), and we calculate in particular the condensed depletion and the anomalous fraction. This latter quantity, which grows with increasing interactions and vanishes in noninteracting systems [33–36], is important to fully understand the interplay of disorder and interactions. We show, in addition, how the anisotropy of the DDI enhance quantum, thermal, and disorder fluctuations as well as the superfluid fraction.

The rest of the paper is organized as follows. In Sec. II, we describe our model of the dipolar dilute Bose gas in a general random potential. In Sec. III, we derive analytical expressions for the condensate fluctuations and some thermodynamic quantities for 3D isotropic laser speckle disorder potential at finite temperatures. We show that the competition between both contact interaction-disorder and DDI-disorder leads to enhance the condensate depletion, the anomalous density, disorder fluctuation, ground-state energy, equation of state, and the sound velocity. In Sec. IV, the superfluid fraction is obtained, and its characteristics are discussed. Finally, our conclusions and outlook remain in Sec. V.

II. THE MODEL

We consider the effects of an external random field on a dilute 3D dipolar Bose gas with dipoles oriented

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perpendicularly to the plane. The Hamiltonian of the system is written as

$$\hat{H} = \int d^3r\, \hat{\psi}^\dagger(\mathbf{r})\left(\frac{-\hbar^2}{2m}\Delta + U(\mathbf{r})\right)\hat{\psi}(\mathbf{r}) + \frac{1}{2}\int d^3r \int d^3r' \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}), \quad (1)$$

where $\psi^\dagger$ and ψ denote, respectively, the usual creation and annihilation field operators, the interaction potential $V(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}') + V_{dd}(\mathbf{r}-\mathbf{r}')$, and $g = 4\pi\hbar^2 a/m$ corresponds to the short-range part of the interaction with a being the scattering length. In what follows we suppose that the contact interactions are repulsive, i.e., $a > 0$. On the other hand, the dipole-dipole component reads

$$V_d(\vec{r}) = \frac{C_{dd}}{4\pi}\frac{1-3\cos^2\theta}{r^3}, \quad (2)$$

where the coupling constant C_{dd} is $\mu_0\mu^2$ for particles having a permanent magnetic dipole moment μ (μ_0 is the magnetic permeability in vacuum) and d^2/ϵ_0 for particles having a permanent electric dipole d (ϵ_0 is the permittivity of vacuum), m is the particle mass, and θ is the angle between the relative position of the particles $\mathbf{r}$ and the direction of the dipole. The characteristic dipole-dipole distance can be defined as $r_* = mC_{dd}/4\pi\hbar^2$. For most polar molecules r_* ranges from 10 to 10^4 Å. The disorder potential is described by vanishing ensemble averages $\langle U(\mathbf{r})\rangle = 0$ and a finite correlation of the form $\langle U(\mathbf{r})U(\mathbf{r}')\rangle = R(\mathbf{r},\mathbf{r}')$.

Passing to the Fourier transform and working in the momentum space, the Hamiltonian (1) takes the form

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar^2k^2}{2m}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k},\mathbf{p}} U_{\mathbf{k}-\mathbf{p}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{p}} + \frac{1}{2V}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}} f(\mathbf{p})\hat{a}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{a}^\dagger_{\mathbf{k}-\mathbf{q}}\hat{a}_{\mathbf{k}+\mathbf{p}}\hat{a}_{\mathbf{k}-\mathbf{p}}, \quad (3)$$

where V is a quantization volume and the interaction potential in momentum space is given by [36]

$$f(\mathbf{k}) = g[1 + \epsilon_{dd}(3\cos^2\theta_k - 1)], \quad (4)$$

here $\epsilon_{dd} = C_{dd}/3g$ is the dimensionless relative strength which describes the interplay between the DDI and the short-range interactions.

Assuming the weakly interacting regime where $r_* \ll \xi$ with $\xi = \hbar/\sqrt{mgn}$ being the healing length and n is the total density, we may use the Bogoliubov approach. Applying the inhomogeneous Bogoliubov transformations [2],

$$\hat{a}_{\mathbf{k}} = u_k\hat{b}_{\mathbf{k}} - v_k\hat{b}^\dagger_{-\mathbf{k}} - \beta_{\mathbf{k}}, \quad \hat{a}^\dagger_{\mathbf{k}} = u_k\hat{b}^\dagger_{\mathbf{k}} - v_k\hat{b}_{-\mathbf{k}} - \beta^*_{\mathbf{k}}, \quad (5)$$

where $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are operators of elementary excitations. The Bogoliubov functions u_k,v_k are expressed in a standard way: $u_k,v_k = (\sqrt{\varepsilon_k/E_k} \pm \sqrt{E_k/\varepsilon_k})/2$ where $E_k = \hbar^2k^2/2m$ is the energy of a free particle and

$$\beta_{\mathbf{k}} = \sqrt{\frac{n}{V}}\frac{E_k}{\varepsilon_k^2}U_k. \quad (6)$$

The Bogoliubov excitations energy is given by

$$\varepsilon_k = \sqrt{E_k^2 + 2\mu_{0d}(\theta)E_k}, \quad (7)$$

where $\mu_{0d} = n\lim_{k\to 0} f(\mathbf{k})$ is the zeroth-order chemical potential.

Importantly, the spectrum (7) is independent of the random potential. This independence holds in fact only in zeroth order in perturbation theory; conversely, higher-order calculations render the spectrum dependent on the random potential due to the contribution of the anomalous terms (see below). For $k \to 0$, the excitations are sound waves $\varepsilon_k = \hbar c_{sd}(\theta)k$, where $c_{sd}(\theta) = c_s\sqrt{1+\epsilon_{dd}(3\cos^2\theta - 1)}$ with $c_s = \sqrt{gn/m}$ is the sound velocity without DDI. Due to the anisotropy of the dipolar interaction, the sound velocity acquires a dependence on the propagation direction, which is fixed by the angle θ between the propagation direction and the dipolar orientation. This angular dependence of the sound velocity has been confirmed experimentally [37].

Therefore, the diagonal form of the Hamiltonian of the dirty dipolar Bose gas (3) can be written as

$$\hat{H} = E + \sum_{\vec{k}} \varepsilon_k\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}, \quad (8)$$

where $E = E_{0d} + \delta E + E_R$, $E_{0d}(\theta) = \mu_{0d}(\theta)N/2$ with N being the total number of particles,

$$\delta E = \frac{1}{2}\sum_{\mathbf{k}}[\varepsilon_k - E_k - nf(\mathbf{k})] \quad (9)$$

is the ground-state energy correction due to quantum fluctuations,

$$E_R = -\sum_{\mathbf{k}} n\langle|U_k|^2\rangle\frac{E_k}{\varepsilon_k^2} = -\sum_{\mathbf{k}} nR_k\frac{E_k}{\varepsilon_k^2} \quad (10)$$

gives the correction to the ground-state energy due to the external random potential.

The noncondensed and the anomalous densities are defined as $\tilde{n} = \sum_{\mathbf{k}}\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\rangle$ and $\tilde{m} = \sum_{\mathbf{k}}\langle\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\rangle$, respectively. Then invoking for the operators $\hat{a}_{\mathbf{k}}$ the transformation (5), setting $\langle\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\rangle = \delta_{\mathbf{k}'\mathbf{k}}N_k$ and putting the rest of the expectation values equal to zero, where $N_k = [\exp(\varepsilon_k/T) - 1]^{-1}$ are occupation numbers for the excitations. As we work in the thermodynamic limit, the sum over k can be replaced by the integral $\sum_{\mathbf{k}} = V\int d^3k/(2\pi)^3$, and using the fact that $2N(x) + 1 = \coth(x/2)$, we obtain

$$\tilde{n} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{E_k + f(\mathbf{k})n}{\varepsilon_k}\left[\coth\left(\frac{\varepsilon_k}{2T}\right) - 1\right] + n_R, \quad (11)$$

and

$$\tilde{m} = -\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{f(\mathbf{k})n}{\varepsilon_k}\coth\left(\frac{\varepsilon_k}{2T}\right) + n_R. \quad (12)$$

The contribution of the random potential comes through the last terms in Eqs. (11) and (12). These terms are defined as

$$n_R = \frac{1}{V}\sum_{\mathbf{k}}\langle|\beta_{\mathbf{k}}|^2\rangle = n\int\frac{d^3k}{(2\pi)^3}\frac{E_k^2}{\varepsilon_k^4}R_k. \quad (13)$$

Expressions (11) and (12) must satisfy the equality,

$$\tilde{n}_k(\tilde{n}_k+1)-|\tilde{m}_k|^2=\frac{1}{4\ \sinh^2(\varepsilon_k/2T)}+n_R\left(\frac{E_k+2f(\mathbf{k})n}{\varepsilon_k}\right)\times\coth\left(\frac{\varepsilon_k}{2T}\right). \tag{14}$$

Equation (14) clearly shows that $\tilde{m}$ is larger than $\tilde{n}$ at low temperatures irrespective of the presence of an external random potential or not. So the omission of the anomalous density in this situation is principally an unjustified approximation and wrong from the mathematical point of view [34–36].

III. BEC FLUCTUATIONS AND THERMODYNAMIC QUANTITIES

To proceed further in practical calculations, we must define the laser speckle potential: $U(\mathbf{r})=U_0+\Delta U(\mathbf{r})$, where U_0 is defined by the light far-field intensity as $U_0=\langle I\rangle$ and $\langle\Delta U(\mathbf{r})\rangle=0$. At the derivation of $U(\mathbf{r})$, it was assumed that the incident laser wave does not induce an atomic electron interlevel transition but merely deforms the atomic ground state. It is useful now to specify the relationship among the far-field intensity autocorrelation function $|C_I(\mathbf{r})|^2$, the laser speckle autocorrelation function $|C_A(\mathbf{r})|^2$, and the disorder potential correlation function. One can write then: $|C_I(\mathbf{r})|^2=\langle U(\mathbf{r}')U(\mathbf{r}'+\mathbf{r})\rangle$ and $|C_A(\mathbf{r})|^2=\langle\Delta U(\mathbf{r}')\Delta U(\mathbf{r}'+\mathbf{r})\rangle/U_0^2$. Therefore, using the Fourier transform, we get

$$|C_I(\mathbf{k})|^2=U_0^2[\delta(\mathbf{k})+|C_A(\mathbf{k})|^2], \tag{15}$$

where the autocorrelation function of the laser speckle is given by [26]

$$|C_A(\mathbf{k})|^2=\frac{3}{4\pi}(2\sigma)^3[(2\sigma k)^3-12(2\sigma k)+16], \tag{16}$$

where σ characterizes the correlation length of the disorder (for further computational details, see Ref. [26]). Interestingly, we see from the formula of $|C_A(\mathbf{k})|^2$ that its value becomes zero for $k=1/\sigma$. Hence, the momentum in (16) only varies in a finite interval from zero, in contrast to the case for a Gaussian function [30]. Accordingly, Eq. (16) makes our results completely different from those of Ref. [32] in particular at zero temperature. At finite temperatures one can expect that our results coincide with the above reference since we assume that disorder, quantum, and thermal fluctuations are too small that the Hamiltonian can be expanded in both leading order of disorder plus thermal and quantum fluctuations. Therefore, disorder and thermal fluctuations are additive and, thus, independent from each other.

Putting $R(\mathbf{k})=R|C_A(\mathbf{k})|^2$ [26], where $R=U_0^2$ stands for the disorder strength. Substituting the function (16) in Eq. (13) and performing the integration over the momentum form 0 to $1/\sigma$, we get the expression for the condensate fluctuation due to the external random potential,

$$n_R=\frac{m^2R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n}{a}}h(\epsilon_{dd},\alpha), \tag{17}$$

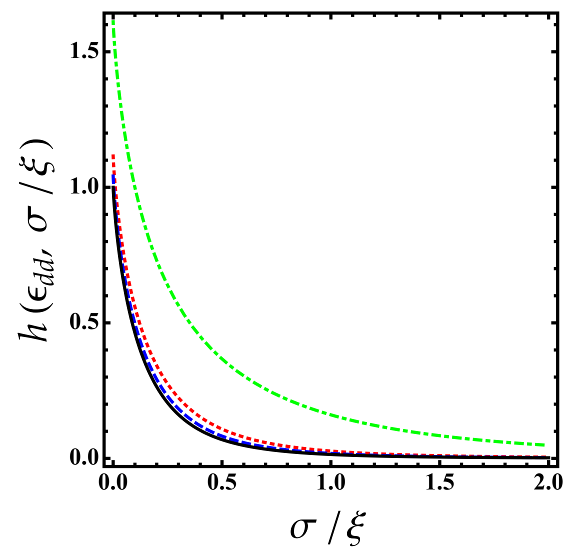

FIG. 1. (Color online) Behavior of the disorder function $h(\epsilon_{dd},\sigma/\xi)$ from Eq. (18) as a function of σ/ξ. Black line: $\epsilon_{dd}=0$ (pure contact interaction). Blue dashed line: $\epsilon_{dd}=0.38$ (Er atoms). Red dotted line: $\epsilon_{dd}=0.6$. Green dot-dashed line: $\epsilon_{dd}=0.95$. Here, the interaction can be adjusted by means of the Feshbach resonance.

where

$$h(\epsilon_{dd},\alpha)=\int_0^\pi d\theta\frac{\sin\theta\, S(\alpha)}{\sqrt{1+\epsilon_{dd}(3\cos^2\theta-1)}} \tag{18}$$

is depicted in Fig. 1, and the function,

$$S(\alpha)=\frac{1}{2\pi}\sqrt{\frac{\alpha}{2}}\left[4-(8\alpha+6)\ln\left(1+\frac{1}{2\alpha}\right)+2\sqrt{\frac{2}{\alpha}}\arctan\left(\frac{1}{\sqrt{2\alpha}}\right)\right],$$

with $\alpha=\sigma^2[1+\epsilon_{dd}(3\cos^2\theta-1)]/\xi^2$.

In the absence of the DDI ($\epsilon_{dd}=0$), we recover the result for the 3D BEC with short-range interparticle interaction of Ref. [26]. For $\sigma/\xi\to0$ and $\epsilon_{dd}=0$, we read off from Eq. (18) that one obtains $h(\epsilon_{dd},\alpha)\to1$ (see also Fig. 1). Therefore, we should reproduce the Huang and Meng result [2] for the condensate depletion in this limit. For $\sigma/\xi\to0$, we get from Eq. (18) that $h(\epsilon_{dd},0)=\mathcal{Q}_{-1}(\epsilon_{dd})$. Thus, the disorder fluctuation (17) becomes identical to that obtained in 3D dipolar BEC with δ-correlated disorder $n_R=(m^2R/8\pi^{3/2}\hbar^4)\sqrt{n/a}\,\mathcal{Q}_{-1}(\epsilon_{dd})$ [32] where the contribution of the DDI is expressed by the functions $\mathcal{Q}_j(\epsilon_{dd})=(1-\epsilon_{dd})^{j/2}\,{}_2F_1(-\frac{j}{2},\frac{1}{2};\frac{3}{2};\frac{3\epsilon_{dd}}{\epsilon_{dd}-1})$, where ${}_2F_1$ is the hypergeometric function. Note that functions $\mathcal{Q}_j(\epsilon_{dd})$ attain their maximal values for $\epsilon_{dd}\approx1$ and become imaginary for $\epsilon_{dd}>1$ [36,38].

On the other hand, the disorder function (18) decreases with increasing disorder correlation length while it rises for increasing ϵ_{dd} (see Fig. 1) and diverges in the limit $\epsilon_{dd}>1$. Another important consequence is that when a vanishes, n_R becomes infinite. This means that the system would collapse if there were no repulsive interactions between particles.

Upon calculating the integral in Eq. (11), we get for the condensate depletion,

$$\frac{\tilde{n}}{n} = \frac{8}{3}\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_3(\epsilon_{dd}) + \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{\pi T}{gn}\right)^2\mathcal{Q}_{-1}(\epsilon_{dd}) + 2\pi R'\sqrt{\frac{na^3}{\pi}}h(\epsilon_{dd},\alpha), \tag{19}$$

where $R' = R/g^2 n$ is a dimensionless disorder strength.

The integral in Eq. (12) is ultraviolet divergent. This divergence is well known to be unphysical since it is caused by the usage of the contact interaction potential. A general way of treating such integrals is as follows. First, one restricts to asymptotically weak coupling and introduces the Beliaev-type second-order coupling constant [36],

$$f_R(\mathbf{k}) = f(\mathbf{k}) - \frac{m}{\hbar^2}\int\frac{d^3q}{(2\pi)^3}\frac{f(-\mathbf{q})f(\mathbf{q})}{2E_q}. \tag{20}$$

After the subtraction of the ultraviolet divergent part, the anomalous fraction turns out to be given

$$\frac{\tilde{m}}{n} = 8\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_3(\epsilon_{dd}) - \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{\pi T}{gn}\right)^2\mathcal{Q}_{-1}(\epsilon_{dd}) + 2\pi R'\sqrt{\frac{na^3}{\pi}}h(\epsilon_{dd},\alpha). \tag{21}$$

The leading terms in Eqs. (19) and (21) represent the quantum fluctuation [36]. The subleading terms, which represent the thermal fluctuation [36], are calculated at temperatures $T \ll gn$ where the main contribution to integrals (11) and (12) comes from the region of small momenta ($\varepsilon_k = \hbar c_{sd} k$). The situation is quite different at higher temperatures, i.e., $T \gg gn$ where the main contribution to integrals (11) and (12) comes from the single-particle excitations. Hence, the thermal contribution of $\tilde{n}$ becomes identical to the density of noncondensed atoms in an ideal Bose gas [36], whereas the thermal contribution of $\tilde{m}$ tends to zero since the gas is completely thermalized in this range of temperature [33,34,36]. The last terms in (19) and (21) account for the effect of disorder on the noncondensed and the anomalous densities.

Equation (21) clearly shows that at zero temperature, the anomalous density is larger than the noncondensed density for any range of the dipolar interaction as well as for any value of the strength and the correlation length of the disorder as has been anticipated above. Moreover, $\tilde{m}$ changes its sign with increasing temperature in agreement with the uniform Bose gas with a pure contact interaction [36]. Likewise, the anomalous density obtained in (21) permits us to determine in a straightforward manner the equation of state and thus, leads to a finite compressibility (see below). Remarkably, Eqs. (19) and (21) reproduce the short-range interaction results since $\mathcal{Q}_j(\epsilon_{dd} = 0) = 1$. Furthermore, the DDI enhances quantum, thermal, and disorder fluctuations of the condensate for increasing ϵ_{dd} as is shown in Fig. 1.

The Bogoliubov approach assumes that fluctuations should be small. We thus conclude from Eqs. (19) and (21) that at $T = 0$, the validity of the Bogoliubov theory requires inequalities $\sqrt{na^3}\mathcal{Q}_3(\epsilon_{dd}) \ll 1$ and $R'\sqrt{na^3}h(\epsilon_{dd},\alpha) \ll 1$. For $R' = 0$, this parameter differs only by the factor $\mathcal{Q}_3(\epsilon_{dd})$ from the universal small parameter of the theory $\sqrt{na^3} \ll 1$ in the absence of DDI. At $T \ll gn$, the Bogoliubov theory requires the condition $(T/gn)\sqrt{na^3}\mathcal{Q}_{-1}(\epsilon_{dd}) \ll 1$. The appearance of the extra factor (T/gn) originates from the thermal fluctuation corrections.

The presence of quantum and disorder fluctuations leads also to corrections of the chemical potential which are given by $\delta\mu_d = \sum_{\mathbf{k}} f(\mathbf{k})[v_k(v_k - u_k)] = \sum_{\mathbf{k}} f(\mathbf{k})(\tilde{n} + \tilde{m})$ [35,36,39]. Inserting the definitions (11) and (12) into the expression of $\delta\mu_d$, we find after integration,

$$\frac{\delta\mu_d}{\mu_\delta} = \frac{32}{3}\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_5(\epsilon_{dd}) + 4\pi R'\sqrt{\frac{na^3}{\pi}}h_1(\epsilon_{dd},\alpha), \tag{22}$$

where $h_1(\epsilon_{dd},\alpha) = \int_0^\pi d\theta\, \sin\theta\sqrt{1 + \epsilon_{dd}(3\cos^2\theta - 1)}S(\alpha)$ and $\mu_\delta = gn$.

In the absence of the disordered potential ($R' = 0$), Eq. (22) coincides with that derived recently in Refs. [36,38]. For a condensate with a pure contact interaction [$\mathcal{Q}_5(\epsilon_{dd} = 0) = 1$] and for $R' = 0$, the obtained correction to the chemical potential (22) excellently agrees with the seminal Lee-Huang-Yang quantum corrected equation of state [40].

The energy shift due to the interaction and the quantum fluctuations (9) is ultraviolet divergent. The difficulty is overcome if one takes into account the second-order correction to the coupling constant (20). A simple calculation yields [36,38]

$$\delta E = \frac{64}{15}Vgn^2\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_5(\epsilon_{dd}). \tag{23}$$

However, the energy shift due to the external random potential (15) is evaluated as

$$\frac{E_R}{E_\delta} = 16\pi R'\sqrt{\frac{na^3}{\pi}}h_1(\epsilon_{dd},\alpha), \tag{24}$$

where $E_\delta = Ngn/2$.

When $\sigma \ll \xi$, the energy shift due to the external random potential (10) is ultraviolet divergent. Again, by introducing the renormalized coupling constant (20) one gets: $E_R/E_\delta = 16\pi R'\sqrt{na^3/\pi}\mathcal{Q}_1(\epsilon_{dd})$ which well coincides with the result obtained with the δ-correlated disorder of Ref. [32].

IV. SUPERFLUID FRACTION

The superfluid fraction n_s/n can be found from the normal fraction n_n/n, which is determined by the transverse current-current correlator $n_s/n = 1 - n_n/n$. We apply a Galilean boost with the total momentum of the moving system $\mathbf{P} = mv(n\mathbf{v_s} + n_n\mathbf{v_n})$, where $\mathbf{v_s}$ denotes the superfluid velocity and $\mathbf{v_n} = \mathbf{u} - \mathbf{v_s}$ is the normal fluid velocity with $\mathbf{u}$ being a boost velocity [31]. The superfluid fraction is then written

$$\frac{n_s^{ij}}{n} = \delta_{ij} - 4\int\frac{d^3k}{(2\pi)^3}\frac{\hbar^2}{2m}\frac{nR_k k_i k_j}{E_k[E_k - 2nf(\mathbf{k})]^2} - \frac{2}{Tn}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\hbar^2}{2m}\frac{k_i k_j}{4\sinh^2(\varepsilon_k/2T)}\right]. \tag{25}$$

We remark that if $\tilde{m}$ were omitted from expression (25), then the related integral would be divergent leading to the meaningless value $n_s \to \infty$. This indicates that the presence of the anomalous density is crucial for the occurrence of the superfluidity in Bose gases [8,36], which is in fact natural since both quantities are caused by atomic correlations.

Equation (25) yields a superfluid density that depends on the direction of the superfluid motion with respect to the orientation of the dipoles. In the parallel direction, the superfluid fraction reads

$$\frac{n_s^{\parallel}}{n} = 1 - 4\pi R'\sqrt{\frac{na^3}{\pi}}h^{\parallel}(\epsilon_{dd},\alpha) - \frac{2\pi^2\hbar}{45mnc_s}\left(\frac{T}{\hbar c_s}\right)^4 \mathcal{Q}_{-5}^{\parallel}(\epsilon_{dd}), \tag{26}$$

where the function,

$$h^{\parallel}(\epsilon_{dd},\alpha) = \int_0^{\pi} d\theta \frac{\sin\theta\,\cos^2\theta\, S(\alpha)}{\sqrt{1+\epsilon_{dd}(3\cos^2\theta - 1)}} \tag{27}$$

is decreasing with increasing ϵ_{dd} for fixed σ/ξ as is depicted in Fig. 2(a). And the functions $\mathcal{Q}_j^{\parallel}(\epsilon_{dd}) = \frac{1}{3}(1-\epsilon_{dd})^{j/2}\,{}_2F_1(-\frac{j}{2},\frac{5}{2};\frac{3}{2};\frac{3\epsilon_{dd}}{\epsilon_{dd}-1})$ have the following properties: $\mathcal{Q}_j^{\parallel}(\epsilon_{dd}=0) = 1/3$ and imaginary for $\epsilon_{dd} > 1$ [32].

In the perpendicular direction, the superfluid fraction (25) takes the form

$$\frac{n_s^{\perp}}{n} = 1 - 2\pi R'\sqrt{\frac{na^3}{\pi}}h^{\perp}(\epsilon_{dd},\alpha) - \frac{\pi^2\hbar}{45mnc_s}\left(\frac{T}{\hbar c_s}\right)^4 \mathcal{Q}_{-5}^{\perp}(\epsilon_{dd}), \tag{28}$$

where the function,

$$h^{\perp}(\epsilon_{dd},\alpha) = \int_0^{\pi} d\theta \frac{\sin\theta(1-\cos^2\theta)S(\alpha)}{\sqrt{1+\epsilon_{dd}(3\cos^2\theta - 1)}} = h(\epsilon_{dd},\alpha) - h^{\parallel}(\epsilon_{dd},\alpha) \tag{29}$$

is increasing with ϵ_{dd} for fixed σ/ξ as is displayed in Fig. 2(b). And $\mathcal{Q}_j^{\perp}(\epsilon_{dd}) = \mathcal{Q}_j(\epsilon_{dd}) - \mathcal{Q}_j^{\parallel}(\epsilon_{dd})$.

The third terms in (26) and (28), which represent the thermal contribution of $n_s^{\perp}$ and $n_s^{\parallel}$, are similar to those obtained in Ref. [32] for δ-correlated disorder as we have anticipated above. These thermal terms are calculated at low temperatures $T \ll ng$. Whereas, at $T \gg ng$, there is copious evidence that both thermal terms of n_s coincide with the noncondensed density of an ideal Bose gas. Furthermore, we read off from Eqs. (26) and (28) that for $\epsilon_{dd} \leqslant 0.5$, the thermal contribution of $n_s^{\perp}$ is smaller than that of $n_s^{\parallel}$, whereas the situation is inverted for $\epsilon_{dd} > 0.5$.

For $\sigma/\xi \to 0$ and $\epsilon_{dd} = 0$, both components of the superfluid fraction reduce to $n_s/n = 1 - 4n_R/3n$, which well recovers the earlier results of Refs. [2,4,5] for the isotropic contact interaction. For $\sigma/\xi \to 0$, we have $h^{\parallel}(\epsilon_{dd},0) = \mathcal{Q}_{-1}^{\parallel}(\epsilon_{dd})$ and $h^{\perp}(\epsilon_{dd},0) = \mathcal{Q}_{-1}^{\perp}(\epsilon_{dd})$. As a result, the disorder correction to superfluid fraction (17) becomes identical to that obtained in the 3D dipolar BEC with δ-correlated disorder [32].

Figure 2 shows also that for increasing ε_{dd}, $h^{\parallel}(\epsilon_{dd},\alpha)$ decreases, whereas $h(\epsilon_{dd},\alpha)$ increases for fixed σ/ξ. Therefore, this reveals that there exists a critical value of interaction ϵ_{dd}^c beyond which the system has the surprising property that the disorder-induced depletion of the parallel superfluid density is smaller than the condensate depletion even at $T = 0$. This

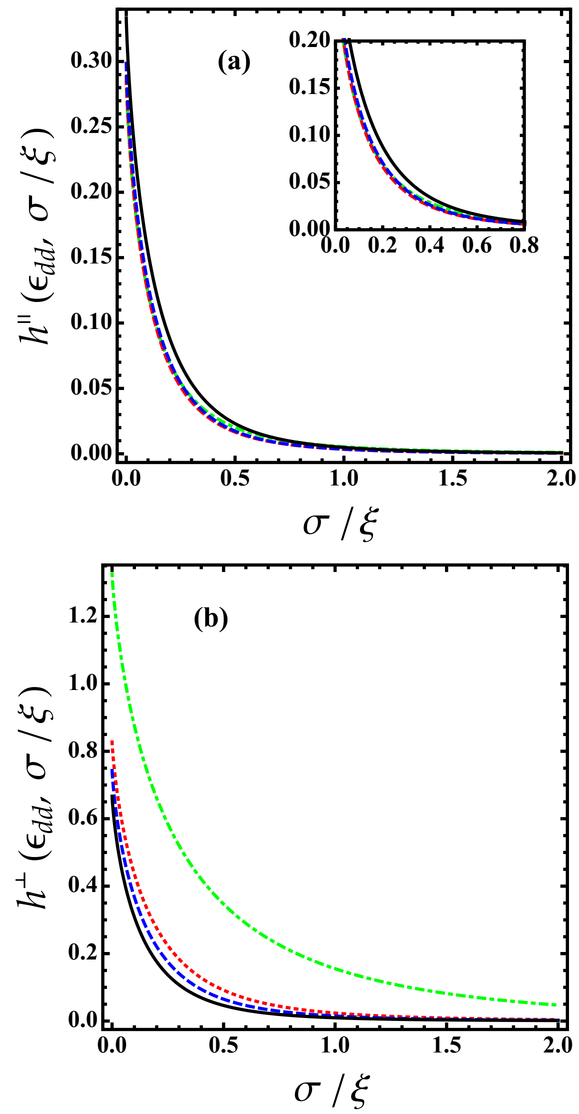

FIG. 2. (Color online) Behavior of the disorder functions (a) $h^{\parallel}(\epsilon_{dd},\sigma/\xi)$ and (b) $h^{\perp}(\epsilon_{dd},\sigma/\xi)$ as a function of σ/ξ. Black line: $\epsilon_{dd} = 0$. Blue dashed line: $\epsilon_{dd} = 0.38$. Red dotted line: $\epsilon_{dd} = 0.6$. Green dot-dashed line: $\epsilon_{dd} = 0.95$.

can be attributed to the fact that the localized particles cannot contribute to superfluidity and, hence, form obstacles for the superfluid flow. For large disorder correlation length, i.e., $\sigma \gg \xi$, ϵ_{dd}^c reduces indicating that the localized particles are localized in the respective minima for the disorder potential only for a finite localization time [41]. This localization time remains to be analyzed in more detail in a future paper. In addition, the superfluid fraction can be either larger or smaller than the condensate fraction $n_c/n = 1 - \tilde{n}/n$, depending on temperature, on interaction, and on the strength of disorder. Increasing R' leads to the simultaneous disappearance of the superfluid and condensate fractions.

Note that the sound velocity of a dipolar BEC in a weak external disorder potential can be calculated within the hydrodynamic approach as $c_s^2(\mathbf{q}) = (\partial\mu/m\,\partial n)\mathbf{q}^T\hat{n}_s\mathbf{q}$ [30,31] where the tensorial property of the superfluid density has been taken into account. From Eqs. (26) and (28) it follows that the sound velocity can also be separated into a parallel and a perpendicular component. Both components change via effects of the interaction strength ϵ_{dd}, disorder strength R', and the ratio σ/ξ. One can easily show also that the sound velocity is consistent with the inverse compressibility $\kappa^{-1} = n^2\partial\mu/\partial n$ [42] where the increase in κ^{-1} tends to increase the sound velocity and vice versa.

V. CONCLUSION

In this paper, we have studied the properties of a homogeneous dipolar Bose gas in the presence of a weak disorder with an autocorrelation function for an isotropic 3D laser speckle potential at finite temperatures. Using the Bogoliubov approach, we have calculated the condensate fluctuation due to disorder as well as the corresponding corrections to the condensed depletion, the anomalous fraction, the chemical potential, and the ground-state energy. We have pointed out that the interplay between the anisotropy of the DDI and the external random potential leads to modify both the BEC and the superfluidity characteristics. Furthermore, we have reproduced the expression of the condensate fluctuations and thermodynamics quantities obtained in the literature in the absence of both the DDI and the disordered potential. We discuss the validity criterion of the Bogoliubov approach in a dirty dipolar BEC.

Finally, an interesting question that begs to be asked is how the interplay of disorder and DDI can affect Anderson localization or the quantum phases that arise due to disorder in the regime of strong correlations.

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