Geometry, robustness, and emerging unitarity in dissipation-projected dynamics

Paolo Zanardi and Lorenzo Campos Venuti
Department of Physics and Astronomy and Center for Quantum Information Science and Technology, University of Southern California, Los Angeles, California 90089-0484, USA

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Quantum information can be encoded in the set of steady states (SSS) of a driven-dissipative system. Nonsteady states are separated by a large dissipative gap that adiabatically decouples them while the dynamics inside the SSS is governed by an effective, dissipation-projected, Hamiltonian. The latter results from the interplay between a weak driving and the fast relaxation process that continuously projects the system back to the SSS. This amounts to a different type of environment-induced quantum Zeno effect. We prove that the dissipation-projected dynamics is of geometric nature and that it is robust against different types of Hamiltonian and dissipative perturbations. Remarkably, in some cases an effective unitary dynamics can emerge out of purely dissipative interactions.

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I. INTRODUCTION

Since the earliest days of quantum information processing (QIP) weak coupling to the environmental degrees of freedom has been regarded as one of the essential prerequisites. In fact, decoherence and dissipation generally spoil the unitary character of the quantum dynamics and induce errors into the computational process. In order to overcome such an obstacle, a variety of techniques have been devised including quantum error correction [1], decoherence-free subspaces (DFSs) [2–4], noiseless subsystems (NS) [5–8], and geometric or holonomic quantum computation [9–11].

However, it has been recently realized that dissipation and decoherence may even play a positive role to the aim of coherent quantum manipulations. Indeed, it has been shown that properly engineered, dissipative dynamics can in principle be used to enact QIP primitives (see Ref. [12] for an early important contribution in this direction) such as quantum state preparation [13–15], quantum simulation [16], and computation [17]. Exotic physical properties such as topological order [18] and non-Abelian synthetic gauge fields [19] can also be achieved by engineered dissipation.

In a nutshell, the idea is that one can design *driven-dissipative* systems such that their *steady states* enjoy some computationally desirable property. For example, in Ref. [14] the unique steady state is maximally entangled, while in Ref. [17] the steady states encode for an arbitrary quantum computation! Moreover, the irreversible and attractive nature of dissipative dynamics endows these techniques with a degree of robustness against imperfections in preparation and control. All this leads to a dramatic paradigm shift in QIP: *noise and dissipation should not be viewed as detrimental but may in fact be considered as a resource*.

In this paper we build upon our recent discovery on how to enact coherent dynamics over the set of steady states (SSS) of a strongly dissipative system [20]. Quantum information is encoded in sectors of the SSS while nonsteady states are separated by the large dissipative gap that adiabatically decouples them away. A weak Hamiltonian control gives rise to an effective dynamics *inside* the SSS that is ruled by a dissipation-projected Hamiltonian. The latter results from a nontrivial interplay between the control and the fast relaxation process that continuously projects the system back onto the SSS. This amounts to a different type of environment-induced quantum Zeno effect [21,22].

In this paper we show that the dissipation-projected dynamics is geometric in nature. This means that this approach can be regarded as a dissipative extension of the fault-tolerant techniques of geometric and holonomic quantum computation [9–11]. We also prove that the dissipation-projected Hamiltonians are protected against several types of perturbations (unitary and dissipative) and may allow for robust QIP. Finally we show how an effective unitary evolution may emerge out of suitable dissipative perturbations of a purely dissipative dynamics. This "emerging unitarity" phenomenon is perhaps the single most surprising one of our results.

II. THE DISSIPATION-PROJECTION THEOREM

We consider quantum open systems whose dynamics is described by the equation

$$\frac{d\rho(t)}{dt} = \mathcal{L}\,\rho(t). \tag{1}$$

The superoperator $\mathcal{L}$ is referred as to the Liouvillian. An open quantum system generically admits a unique *steady state* ρ_∞ that is approached by the time-evolving density matrix $\rho(t)$ as the time goes to infinity. Asymptotically the information-theoretic distance $D(\rho(t),\rho_\infty) := \frac{1}{2}\|\rho(t)-\rho_\infty\|_1$ decays exponentially with time where the time scale τ_R is referred to as the *relaxation time*. For $t \gg \tau_R$ the time-evolved state becomes indistinguishable from the steady state. According to Eq. (1) the steady state satisfies $\mathcal{L}(\rho_\infty) = 0$, i.e., it lies lies in the kernel of the Liouvillian. Uniqueness of the steady state translates into a one-dimensional kernel. In this paper we focus on the case in which the Liouvillian can be decomposed as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ in such a way that

(i) The relaxation time of $\mathcal{L}_0$ is the shortest time scale of the problem. Equivalently, the dissipative gap of $\mathcal{L}_0$, τ_R^{-1}, is the largest energy scale.

(ii) The kernel of $\mathcal{L}_0$ is *high dimensional* and attractive (the nonzero eigenvalues of $\mathcal{L}_0$ have a negative real part).

We denote by $\mathcal{P}_0$ ($\mathcal{Q}_0 = 1 - \mathcal{P}_0$) the spectral projection of $\mathcal{L}_0$ with eigenvalue zero (one). The steady-state set (SSS) is given by those states ρ such that $\mathcal{P}_0(\rho) = \rho$. The critical assumption is that the SSS is high dimensional. A prototypical instance of this *nongeneric* situation is the following:

Example 0. Suppose a system S is joined to a system B and that the dissipation acts only on the latter. Let ρ_B denote the (generically) unique steady state of B and by ρ any state of S. It is then obvious that any bipartite state of the form $\rho\otimes\rho_B$ is a steady state of the full dynamical system when the S and B are decoupled. Clearly, any transformation over S is a symmetry of the dynamics. For the sake of concreteness, one may think of a two-level atom S weakly coupled to a leaky cavity mode B. To a good approximation dissipation acts directly just on B. Formally, the Hilbert space is $\mathcal{H}=\mathcal{H}_S\otimes\mathcal{H}_B$ and $\mathcal{L}_0=\mathbf{1}_S\otimes\mathcal{L}_B$ where the Liouvillian $\mathcal{L}_B$ admits a *unique* steady state ρ_B. In this case $\mathcal{P}_0(X)=\mathrm{Tr}_B(X)\otimes\rho_B$, the kernel of $\mathcal{L}_0$ has dimension $(\dim\mathcal{H}_S)^2$, and the SSS can be identified with the state space of S. This apparently trivial example is later considerably generalized by resorting to the theory of NSs [5].

The fundamental technical result we build upon is the following fact proved in Ref. [20] (see also Sec. A):

Projection theorem. Suppose $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1$ with $\|\mathcal{L}_1\|=O(1/T)$, then

$$\sup_{t\in[0,T]}\|(\mathcal{E}_t-e^{t\mathcal{L}_{\rm eff}})\mathcal{P}_0\|=O(1/T),\qquad(2)$$

where $\mathcal{L}_{\rm eff}:=\mathcal{P}_0\,\mathcal{L}\,\mathcal{P}_0=\mathcal{P}_0\,\mathcal{L}_1\,\mathcal{P}_0$ and $\mathcal{E}_t=e^{t\mathcal{L}}$.

In words, if the system is prepared at time $t=0$ inside the SSS, then, in the large T limit, the time evolution leaves the SSS invariant and it is governed by the effective generator $\mathcal{L}_{\rm eff}$.

In several of the applications we discuss below the perturbation is of Hamiltonian type, i.e., $\mathcal{L}_1=-i[K,\bullet]$, $(K=K^\dagger)$; in that case it is denoted by $\mathcal{K}$. The key point is that $\mathcal{K}_{\rm eff}=\mathcal{P}_0\,\mathcal{K}\,\mathcal{P}_0$ turns out to be an Hamiltonian; it is referred to as the *dissipation-projected* Hamiltonian. Physically, this means that strong dissipation, while dressing the Hamiltonian by a continuous projection onto the SSS, does not alter its unitary character. Nonsteady states are adiabatically decoupled away. The SSS and unitarity are protected by the large dissipative gap of $\mathcal{L}_0$.

For example, in *Example 0* discussed above, where the Liouvillian $\mathcal{L}_B$ has a *unique* steady state ρ_B, one finds $\mathcal{K}_{\rm eff}=-i[K_{\rm eff},\bullet]$, where $K_{\rm eff}=\mathrm{Tr}_B(K\rho_B)\otimes\mathbf{1}_B$. We see that in fact $\mathcal{K}_{\rm eff}$ is Hamiltonian.

In Sec. A we prove Eq. (2) and we give a rigorous estimate for the coefficient in its right-hand side (RHS). It turns out [see Eq. (A14)] that the numerical factor is $c\tau_R$, where τ_R is the relaxation time of the unperturbed dynamics and c is a $O(1)$ constant. This fact is important as it implies that the error can be made small, either by making T larger (which also makes the waiting time $O(T)$ longer) or by making dissipation faster (i.e., τ_R smaller). Indeed, measuring times in unit of τ_R one realizes that the expansion parameter in Eq. (2) is really τ_R/T. In other terms, the "long T limit" just means that the Hamiltonian norm has to be much smaller than the dissipative gap $[=O(\tau_R^{-1})]$. The latter represents the physical quantity that in real applications has to be engineered in order to make it as large as possible. Equivalently, one wants to make the relaxation time τ_R as short as possible. We have to operate in the deep dissipative regime.

III. DISSIPATIVE HOLONOMIES

Let us now discuss the intimate relation between our basic result (2) and geometric and holonomic quantum computation [9,10]. We show that the effective evolution (2) is in fact *geometric* and is given by a superoperator holonomy.

The possibility of merging dissipation dynamics and holonomic quantum computation [10,11,23] by reservoir engineering was first suggested in Refs. [24,25]. More specifically, in Ref. [25] a time-dependent Lindbladian dynamics admitting a DFS was considered, and it was shown that under a suitable adiabatic condition, a state initially in a DFS remains inside the subspace and, hence, is rigidly transported around the Hilbert space together with the DFS. The evolution is, in fact, coherent, although entirely produced by an incoherent phenomenon. Moreover, when the DFS eventually returns to its initial configuration, the net effect is a holonomic transformation on the states in the subspace. Counterintuitively, the effect of the dissipation on the (time-dependent) DFS can be made smaller by making the dissipation rate larger. The authors qualitatively explain this phenomenon in terms of some sort of environment-induced quantum Zeno effect where the action of a strong environment can be regarded as a measuring apparatus continuously monitoring the slowly moving DFS.

In order to establish a connection between these findings and the results we have discussed so far it suffices to move to a rotated reference frame by defining $\tilde\rho(t):=\mathcal{U}_t^\dagger\rho(t)$ where $\mathcal{U}_t(X):=e^{t\mathcal{K}}(X)=e^{-itK}Xe^{itK}$. In this rotated frame $\tilde\rho(t)$ evolves in a *time-dependent* bath

$$\frac{d\tilde\rho(t)}{dt}=\mathcal{L}_t\,\tilde\rho(t),\quad \mathcal{L}_t:=\mathcal{U}_t^\dagger\mathcal{L}_0\mathcal{U}_t.\qquad(3)$$

In the rotated frame, the dynamical semigroup is given by $\tilde{\mathcal{E}}_t=\mathcal{U}_t^\dagger\mathcal{E}_t$ and a state $\tilde\rho_t$ is an *instantaneous* steady state of $\mathcal{L}_t$ if and only if (iff) $\tilde\rho_t=\mathcal{U}_t^\dagger\rho_0$, where ρ_0 is a steady state of $\mathcal{L}_0$. It follows that the projector onto the kernel of $\mathcal{L}_t$ is given by $\mathcal{P}_t=\mathcal{U}_t^\dagger\mathcal{P}_0\mathcal{U}_t=e^{-t\mathcal{K}}\mathcal{P}_0e^{t\mathcal{K}}$. Moreover, in the rotated frame the dissipation-projected dynamics is *geometric*.

Proposition 1. (a) The projection theorem (2) can be reformulated in the form

$$\|\tilde{\mathcal{E}}_t\,\mathcal{P}_0-\mathbf{T}\exp\left(\int_0^t d\tau\,[\dot{\mathcal{P}}_\tau,\mathcal{P}_\tau]\right)\mathcal{P}_0\|=O(1/T),\qquad(4)$$

where $\mathbf{T}$ denotes the chronological ordering symbol.

(b) The $\mathbf{T}$-ordered geometric superoperator in Eq. (4) can be rewritten as

$$X(t)=\lim_{N\to\infty}\mathbf{T}\prod_{j=1}^N\mathcal{P}_{t_j}=e^{-t\mathcal{K}}\lim_{N\to\infty}(e^{\frac{t}{N}\mathcal{K}}\mathcal{P}_0)^N,\qquad(5)$$

where $t_j=jt/N$, $j=0,\ldots,N$. Namely, the evolution corresponds to an infinite, time-ordered, succession of projections onto the instantaneous SSS, equivalently, to a succession of $\mathcal{P}_0$ *interleaved* with infinitesimal unitaries evolutions $e^{\frac{t}{N}\mathcal{K}}$.

Proof. (a) From unitarity of $\mathcal{U}_t=e^{t\mathcal{K}}$ and Eq. (2) one has

$$\|\mathcal{E}_t\mathcal{P}_0-e^{t\mathcal{P}_0\mathcal{K}\mathcal{P}_0}\mathcal{P}_0\|=\|\tilde{\mathcal{E}}_t\mathcal{P}_0-X(t)\|=O(1/T),\qquad(6)$$

where $X(t) := e^{-t\mathcal{K}} e^{t\mathcal{P}_0\mathcal{K}\mathcal{P}_0}\mathcal{P}_0$. By differentiation

$$\begin{aligned}\dot{X}(t) &= -\mathcal{K}e^{-t\mathcal{K}}e^{t\mathcal{P}_0\mathcal{K}\mathcal{P}_0}\mathcal{P}_0 + e^{-t\mathcal{K}}\mathcal{P}_0\mathcal{K}\mathcal{P}_0 e^{t\mathcal{P}_0\mathcal{K}\mathcal{P}_0}\mathcal{P}_0\\ &= -\mathcal{K}\mathcal{P}_t X(t) + \mathcal{P}_t\mathcal{K}X(t) = [-\mathcal{K},\mathcal{P}_t]X(t)\\ &= \dot{\mathcal{P}}_t\, X(t).\end{aligned} \tag{7}$$

Notice also that $\mathcal{P}_t X(t) = X(t)$ and $\mathcal{P}_t\dot{\mathcal{P}}_t\mathcal{P}_t = 0$, whence $\dot{X}(t) = \dot{\mathcal{P}}_t\, X(t) = \dot{\mathcal{P}}_t\mathcal{P}_t\, X(t) = (\dot{\mathcal{P}}_t\mathcal{P}_t - \mathcal{P}_t\dot{\mathcal{P}}_t)\,\mathcal{P}_t X(t)$, namely

$$\frac{dX(t)}{dt} = [\dot{\mathcal{P}}_t,\mathcal{P}_t]\, X(t) \Rightarrow X(t) = \mathbf{T}\, e^{\int_0^t d\tau\, [\dot{\mathcal{P}}_\tau,\mathcal{P}_\tau]} X(0). \tag{8}$$

Equation (4) is now obtained by using Eq. (6) and $X(0) = \mathcal{P}_0$.

(b) Proceeding formally, if $\tilde{X}(t) = \prod_{\tau\in[0,t]}\mathcal{P}_\tau$ then

$$\begin{aligned}\tilde{X}(t+dt) - \tilde{X}(t) &= \mathcal{P}_{t+dt}\tilde{X}(t) - \tilde{X}(t) = (\mathcal{P}_{t+dt} - \mathcal{P}_t)\tilde{X}(t)\\ &= \dot{\mathcal{P}}_t\tilde{X}(t)dt + O(dt^2),\end{aligned} \tag{9}$$

whence $\dot{\tilde{X}}(t) = \dot{\mathcal{P}}_t\tilde{X}(t)$. Since $X(t)$ and $\tilde{X}(t)$ fulfill the same ordinary differential equation and the same initial condition $X(0) = \tilde{X}(0) = \mathcal{P}_0$ they have to be the same function. This proves the first equality in Eq. (5) while the second can be verified by direct inspection using the definition of the $\mathcal{P}_t$'s. ■

The integral in Eq. (4) is clearly invariant under time reparametrizations $\tau \to \tau' = \tau'(\tau)$ and it is therefore of geometric nature; i.e., it depends only on the path $t \to \mathcal{P}_t$ in the space of (super) projections. We also see that the superoperator holonomy is the line integral of the "tautological" connection $\mathcal{A} = [\dot{\mathcal{P}}(\tau),\mathcal{P}(\tau)]$ [26].

If one replaces in Eq. (5) the projection $\mathcal{P}_0$ with a more general CP map, e.g., generalized measurement, basically *all* the quantum-Zeno-like QIP protocols recently discussed in the literature are recovered [24,27–29]. In all these works the geometric and holonomic nature of the resulting dynamics have been discussed on the basis of the particular case at hand, and a general comprehensive theoretical understanding seems to be lacking. The formalism discussed in this paper may be able to provide such an underlying conceptual framework.

IV. SSS AND INTERACTION ALGEBRAS

In this section we discuss an important class of dissipative systems whose SSS can be fully characterized on general algebraic grounds and at the same time describes physically relevant cases.

Let us consider the most general dissipative generator $\mathcal{L}_0$ of a Markovian quantum dynamical semigroup $\mathcal{E}_t := e^{t\,\mathcal{L}_0}$. Thanks to the Lindblad theorem [30] the Liouvillian can be written as

$$\mathcal{L}_0(\rho) = \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha,\rho\}\right). \tag{10}$$

The L_α's are the so-called Lindblad operators. Let us now define two operator algebras associated with Eq. (10):

$$\mathcal{A} := \mathrm{Alg}[\{L_\alpha, L_\alpha^\dagger\}_\alpha],\quad \mathcal{A}' := \{X\in\mathbf{L}(\mathcal{H})\,/\,[X,\mathcal{A}] = 0\}. \tag{11}$$

The algebra $\mathcal{A}$ is the associative algebra (with unit) generated by the Lindblad operators L_α and their Hermitian conjugates, and it is referred to as the *interaction algebra* [5] and $\mathcal{A}'$ as its *commutant*. These algebras play a fundamental role in unifying all the quantum information stabilization techniques developed so far [6,7]. Both $\mathcal{A}$ and $\mathcal{A}'$ are closed under Hermitian conjugation and can be regarded as (finite-dimensional) C^* algebras. Standard structure theorems then imply that the state space breaks down into d_J-dimensional irreducible representations of $\mathcal{A}$ (labeled by J), each of them appearing with multiplicity n_J:

$$\mathcal{H} \cong \bigoplus_J \mathbf{C}^{n_J}\otimes\mathbf{C}^{d_J}. \tag{12}$$

From this it follows that at the algebra level one has

$$\mathcal{A} \cong \bigoplus_J \mathbb{1}_{n_J}\otimes\mathrm{L}(\mathbf{C}^{d_J}),\quad \mathcal{A}' \cong \bigoplus_J \mathrm{L}(\mathbf{C}^{n_J})\otimes\mathbb{1}_{d_J}. \tag{13}$$

From the first equation in Eqs. (13) it follows that the Liouvillian (10) preserves the direct-sum structure of the Hilbert space, i.e., $\mathcal{L}$ is J block diagonal, and that it has a trivial action on the $\mathbf{C}^{n_J}$ factors. For this reason the latter are termed "noiseless subsystems" and is where quantum information can be stored safely from the influence of the environment described by of $\mathcal{L}_0$ (more on this in the next section) [5]. In each of the $\mathcal{L}_0$-invariant J blocks the situation coincides with the one of Example 0. In other terms Eq. (10) corresponds to a direct sum of bipartite systems in which the noise acts just on one of the two (virtual) subsystems, i.e., $\mathbf{C}^{d_J}$. In this sense this class of models can be regarded as a far-reaching generalization of Example 0 [20].

We now assume that $\sum_\alpha [L_\alpha, L_\alpha^\dagger] = 0$. Under these assumptions the dynamical semigroup $\{e^{t\mathcal{L}_0}\}_{t\geqslant 0}$ leaves the identity fixed as $\mathcal{L}_0(\mathbb{1}) = 0$ and $\mathrm{Ker}\,\mathcal{L}_0 = \mathcal{A}'$ [31]. Such a $\mathcal{L}_0$ is referred to as *unital*. From the second equation in Eqs. (13), we see that the SSS is given by the convex hull of states of the form $\omega_J\otimes\mathbb{1}_{d_J}/d_J$, where ω_J is a state over the factor $\mathbf{C}^{n_J}$. Since $\mathrm{Ker}\,\mathcal{L}_0 = \mathcal{A}'$ it follows that $\mathcal{P}_0$ is the projection onto the commutant algebra $\mathcal{A}'$, namely [20]

$$\begin{aligned}\mathcal{P}_0(X) &= \int dU\, UXU^\dagger\\ &= \sum_J \mathrm{Tr}_{d_J}(\Pi_J\, X\, \Pi_J)\otimes\mathbb{1}_{d_J}/d_J \in \mathcal{A}',\end{aligned} \tag{14}$$

where the Haar-measure integral is performed over the unitary group of the algebra $\mathcal{A}$ and $\Pi_J := \mathbb{1}_{n_J}\otimes\mathbb{1}_{d_J}$ are the projectors on the $\mathbf{C}^{n_J}\otimes\mathbf{C}^{d_J}$ sectors of $\mathcal{H}$. In Ref. [20] we have shown that

$$\mathcal{K}_{\mathrm{eff}}|_{\mathrm{Ker}\,\mathcal{L}_0} = -i[K_{\mathrm{eff}},\bullet],\quad K_{\mathrm{eff}} := \mathcal{P}_0(K)\in\mathcal{A}'. \tag{15}$$

The effective Hamiltonian $\mathcal{P}_0(K)$ clearly commutes with the whole unitary group of the interaction algebra. In this sense K_{eff} is a dissipation-projection symmetrized [32] version of K. As a consequence, its action is trivial on the "noise-full" $\mathbf{C}^{d_J}$ factors in Eq. (12). In other terms dissipation can also be regarded as a resource to the end of *dynamical decoupling* [32–35].

V. ROBUSTNESS

One of the main motivations behind the type of dissipation-assisted manipulations we are considering is that it features built-in resilience against *certain* types of perturbations. This means that dissipation, besides providing assistance for QIP, may provide *protection*. This stems from the simple observation that the projection theorem (2) clearly indicates that any extra term $\mathcal{V}$ in the Liouvillian, either Hamiltonian or dissipative, such that $\|\mathcal{V}\| = O(1/T) =$ and

$$\mathcal{P}_0\,\mathcal{V}\,\mathcal{P}_0 = 0 \tag{16}$$

will not contribute to the effective dynamics (2). For instance, in the context of Example 0 any pair of Hamiltonians K_1 and K_2 such that $\mathrm{Tr}_B[\rho_B(K_1 - K_2)] = \lambda \mathbb{1}_S$, ($\lambda \in \mathbf{R}$) generate the same projected dynamics.

A. Hamiltonian perturbations: Unital case

In the interaction algebra case associated with the unital Liouvillian in Eq. (10), one can prove the following result which is reminiscent of the correctability condition in operator error correction [36] [see, e.g., Eq. (4) therein].

Proposition 2. Equation (16) is satisfied by an Hamiltonian perturbation V iff

$$\mathcal{P}_0(V) \in \mathcal{A} \cap \mathcal{A}' =: \mathcal{Z}(\mathcal{A}). \tag{17}$$

The solution space of the Hamiltonian robustness, Eq. (17), is a linear subspace of the full operator algebra $\mathrm{L}(\mathcal{H})$ with codimension $\sum_J (n_J^2 - 1)$. This subspace, in particular, contains the kernel of $\mathcal{P}_0$ and the interaction algebra $\mathcal{A}$.

Proof. From Eq. (15) we see that the condition (16), for $\mathcal{V} = -i[V,\bullet]$, means $[\mathcal{P}_0(V),\rho] = 0, \forall \rho \in \mathcal{A}'$, namely the projected dynamics does not change by perturbing K with any term V such that $\|V\| = O(1/T)$ and $\mathcal{P}_0(V) \in \mathcal{A}'' = \mathcal{A}$. Since, by construction $\mathcal{P}_0(V) \in \mathcal{A}'$ as well, one finds that Eq. (16) is satisfied by an Hamiltonian perturbation V iff Eq. (17) is satisfied. Moreover, if $V \in \mathcal{Z}(\mathcal{A}) \Rightarrow \mathcal{P}_0(V) = V$ implies that the solution space of Eq. (17) is the linear space $\mathrm{Ker}\,\mathcal{P}_0 + \mathcal{Z}(\mathcal{A})$. More concretely, the Hamiltonian perturbations V fulfilling the robustness condition Eq. (17) have the form

$$V = V^{\mathrm{off}} + \sum_{J\beta} X_J^\beta \otimes Y_J^\beta + \sum_J \lambda_J \mathbb{1}_{n_J} \otimes \mathbb{1}_{d_J}, \tag{18}$$

where $\mathrm{Tr}(Y_J^\beta) = 0, (\forall J,\beta)$, and V^{off} is off diagonal in the decomposition (13). The first two terms in Eq. (18) represent $\mathrm{Ker}\,\mathcal{P}_0$ whose dimension is then $\sum_{J\neq J'}(n_J d_J)(n'_J d'_J) + \sum_J n_J^2 (d_J^2 - 1) = (\sum_J n_J d_J)^2 - \sum_J n_J^2 = \dim \mathrm{L}(\mathcal{H}) - \dim \mathcal{A}'$. The third term in Eq. (18) represents the center $\mathcal{Z}(\mathcal{A})$ of the interaction algebra whose dimension is $\sum_J 1$. Overall we see that the solution space of Eq. (17) has dimension $\dim \mathrm{L}(\mathcal{H}) - \sum_J n_J^2 + \sum_J 1 = \dim \mathrm{L}(\mathcal{H}) - \sum_J (n_J^2 - 1)$; i.e., it has codimension $\sum_J (n_J^2 - 1)$. ■

For example, in the collective decoherence case the interaction algebra $\mathcal{A}$ is the algebra of permutation-invariant operators acting on the N-qubit space [2,3]. Since

$$\mathcal{P}_0(\sigma_j^\alpha) = \int_{SU(2)} dU\, U^{\otimes N} \sigma_j^\alpha U^{\dagger \otimes N} = \int_{SU(2)} dU\, U \sigma_j^\alpha U^\dagger$$
$$= \mathrm{Tr}(\sigma_j^\alpha)\,\mathbb{1} = 0 \quad (\alpha = x,y,z;\ j = 1,\ldots,N),$$

it follows that all symmetry-breaking V's of the form $V = \sum_{j=1.\alpha=x,y,z}^N \delta_\alpha^j \sigma_j^\alpha$ with $|\delta_\alpha^j| = O(1/T)$ can be tolerated.

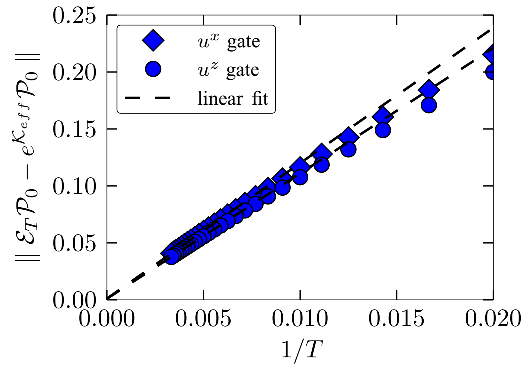

FIG. 1. (Color online) Robustness against dissipative errors. Distance from the exact evolution and the effective one as a function of $1/T$. The unperturbed Lindbladian is of the form $\mathcal{L}_0 = \sum_\alpha \gamma_\alpha \mathcal{L}^\alpha$, where $\mathcal{L}^\alpha$ is generated by a collective Lindblad operator $S^\alpha = \sum_{j=1}^N \sigma_j^\alpha$ with $N = 4$. The control Hamiltonians are $\mathcal{K}^\alpha = -i[H^\alpha,\bullet]$ with $H^x = (3/2)(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z) + \mathbb{1}$ and $H^z = -(\sqrt{3}/2)(\sigma_1^z\sigma_2^z - \sigma_2^z\sigma_3^z) + \sigma_1^z$. The effective dynamics generate the unitary gates $u^\alpha = \exp(-i\theta\sigma^\alpha)$ with an arbitrary angle ϑ up to an error $O(T^{-1})$ (see also Ref. [20]). The same dynamics is obtained (up to an error $O(T^{-1})$ with possibly a different prefactor) with an error on $\mathcal{L}_0$, which replaces $S^\alpha \to S^\alpha + T^{-1}X^\alpha$. For the numerical simulation we used $X^\alpha = g\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 S^z$. The plot is obtained by fixing $\gamma^\alpha = g = \vartheta = 1$. The norm used is the maximum singular value of the maps realized as matrices over $\mathcal{H}^{\otimes 2}$. The linear fit is obtained using the four most significant points.

B. Perturbation of the Lindblad operators

Besides unitary perturbations $K \to K + V$ in practical applications one has also to consider dissipative ones $\mathcal{L}_0 \to \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_1$ denotes a dissipative Liouvillian with $\|\mathcal{L}_1\| = O(1/T)$. It is important to stress that the resilience of the projected dynamics extends to nonunitary perturbations, e.g., extra noise sources.

To begin with, we observe that in the unit-preserving case all Lindbladian perturbations $\mathcal{L}_1$ of the form of Eq. (10) whose Lindblad operators are in the interaction algebra $\mathcal{A}$ [see Eq. (11)], satisfy $\mathcal{P}_0\mathcal{L}_1\mathcal{P}_0 = 0$. Let us then consider perturbations that take the Lindblad operators outside of $\mathcal{A}$. More precisely, we consider Eq. (10) with Lindblad operators given by collective spin operators $S^\mu = \sum_{j=1}^N \sigma_j^\mu$ ($\mu = x,y,z$) and then we perturb them by permutational symmetry-breaking terms $S^\mu \mapsto S^\mu + T^{-1}X^\mu$, where $\|X^\mu\| = O(1)$. This leads to a perturbed Liouvillian $\tilde{\mathcal{L}}_0 = \mathcal{L}_0 + T^{-1}\mathcal{L}_1 + T^{-2}\mathcal{L}_2$, where

$$\mathcal{L}_1(\rho) = \sum_\mu \left(X^\mu \rho S^\mu + S^\mu \rho X^\mu - \frac{1}{2}\{\{X^\mu, S^\mu\},\rho\} \right) \tag{19}$$

and $\mathcal{L}_2$ is a quadratic expression in the X^μ's.

Proposition 3. If $\mathcal{L}_1$ is given by Eq. (19) then $\mathcal{P}_0\,\mathcal{L}_1\,\mathcal{P}_0 = 0$.

Proof. To see that is enough to notice that $\rho \in \mathcal{A}' \Rightarrow \rho = \oplus_J \rho_J \otimes \mathbb{1}_{d_J}$, and $S^\mu = \oplus_J \mathbb{1}_{n_J} \otimes S_J^\mu$: Since $\mathcal{P}_0$ annihilates any off-diagonal contribution in Eq. (19) one can assume a J block-diagonal structure for the perturbation $X^\mu = \oplus_J X_J^\mu \otimes Y_J^\mu$. Therefore by considering, for example, the term

$\mathcal{P}_0(X^\mu \rho S^\mu) = \mathcal{P}_0(X^\mu S^\mu \rho) = \mathcal{P}_0(X^\alpha S^\mu)\rho$ one obtains

$$\mathcal{P}_0(X^\mu S^\mu) = \mathcal{P}_0\left(\oplus X_J^\mu \otimes Y_J S_J^\mu\right) = \oplus_J \mathrm{Tr}_{d_J}\left(Y_J S_J^\mu\right) X_J^\alpha \otimes \mathbb{1}/d_J$$
$$= \oplus_J \mathrm{Tr}_{d_J}\left(S_J^\mu Y_J\right) X_J^\mu \otimes \mathbb{1}/d_J = \mathcal{P}_0(S^\mu X^\mu). \quad (20)$$

This shows that any term in $\mathcal{P}_0\,\mathcal{L}_1\,\mathcal{P}_0$ arising, e.g., from the first term in Eq. (19) is canceled by an identical one arising from the anticommutator side as $\mathcal{P}_0(X^\mu S^\mu \rho + S^\mu X^\mu \rho) = 2\mathcal{P}_0(S^\mu X^\mu)\rho$. Notice that in particular for $X^\mu \in \mathcal{A}'$ one has the stronger property $\mathcal{L}_1\mathcal{P}_0 = 0$. ■

In Fig. 1 we report a numerical simulation for the four-qubits system discussed in the former section. The simulation confirms that for small $1/T$ the Liouvillians $\mathcal{L}_0$ and $\tilde{\mathcal{L}}_0$ generate the same projected dynamics [20]. In words: one can exploit (symmetric) noise to wash out other noise.

VI. EMERGING UNITARITY

A unitary dynamics gives rise to a nonunitary one as soon as some unobserved degrees of freedom are traced out. This is an ubiquitous situation in physics. The converse process, to obtain a unitary evolution from an underlying dissipative one, appears a much more difficult task. Here we show how this phenomenon of *emerging unitarity* manifests itself in the context we have discussed in this paper.

We begin by slightly generalizing our setup, i.e., going beyond the unital case where the kernel of $\mathcal{L}_0$ is the commutant of the interaction algebra. According to Ref. [38] any generator of the Lindblad type (10) is such that its SSS is given by states ρ of the form $\rho = \sum_J^\oplus \lambda_J\, \omega_J \otimes \rho_{0,J}$, where the state space block structure is still of the form (12). Here the ω_J are arbitrary states in $\mathbf{C}^{n_J}$, the $\rho_{0,J}$'s are uniquely defined states in $\mathbf{C}^{n_J}$, and the λ_J's are non-negative scalars. In the unital case we mostly considered so far $\rho_{0,J} = \mathbb{1}_{d_J}/d_J$. The robustness calculation of the former section can be now generalized to this nonunital Liouvillian case. As before, we perturb the (not necessarily Hermitean) Lindblad operators $L_\alpha \mapsto L_\alpha + \delta L_\alpha$ and consider as perturbation the first-order variation of $\mathcal{L}_0$:

$$\mathcal{L}_1(\rho) = \sum_\alpha \left(\delta L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}(\delta L_\alpha^\dagger L_\alpha + L_\alpha^\dagger \delta L_\alpha)\,\rho + \mathrm{H.c.}\right). \quad (21)$$

If $\|\delta L_\alpha\| = O(1/T)$ then Eq. (2) holds with $\tilde{\mathcal{K}}$ replaced by $\mathcal{L}_1$.

Proposition 4. Let us add, to a Liouvillian generator of the type (10), a perturbation of the form (21). Then, for ρ in the SSS, one has that $\mathcal{P}_0\mathcal{L}_1\mathcal{P}_0(\rho) = -i[A,\rho]$, where

$$A = \mathrm{Im} \sum_\alpha \sum_J^\oplus \mathrm{Tr}_{d_J}\left(\delta L_\alpha^\dagger L_\alpha\left(\mathbb{1}_{n_J} \otimes \rho_{0,J}\right)\right) \otimes \mathbb{1}_{d_J} = A^\dagger, \quad (22)$$

in which $\mathrm{Im}\,X := \frac{1}{2i}(X - X^\dagger)$. In particular, for the unital case $\mathcal{L}_0(\mathbb{1}) = 0$, one can write

$$A = \mathrm{Im}\,\mathcal{P}_0\left(\sum_\alpha \delta L_\alpha^\dagger L_\alpha\right). \quad (23)$$

Proof. Let us consider the Lindbladian $\mathcal{L}_0(\rho) = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$. Perturbing the Lindblad operators $L \mapsto L + \delta L$ one finds the variation $\mathcal{L}_0 \mapsto \mathcal{L}_0 + \delta\mathcal{L}$, where

$$\delta\mathcal{L}(\rho) = \delta L \rho L^\dagger - \tfrac{1}{2}(L^\dagger \delta L + \delta L^\dagger L)\,\rho + \mathrm{H.c.}, \quad (24)$$

Without loss of generality we can consider δL as block diagonal in the decomposition (13) and work on a fixed J sector $\mathbf{C}^{n_J} \otimes \mathbf{C}^{d_J}$. In that sector we write $\delta L = X \otimes Y$, $L = \mathbb{1} \otimes \tilde{L}$ and $\rho = \omega \otimes \rho_0$ in the SSS. Here ρ_0 denotes the unique steady state in the $\mathbf{C}^{d_J}$ factor of the J block, i.e., $\mathbb{1}_{d_J}/d_J$ in the unital case. The first three terms in Eq. (24) give rise to the following three contributions respectively:

$$X\omega \otimes Y\rho_0\tilde{L}^\dagger, \quad -\tfrac{1}{2}X\omega \otimes \tilde{L}^\dagger Y \rho_0, \quad -\tfrac{1}{2}X^\dagger\omega \otimes Y^\dagger \tilde{L}\,\rho_0.$$

Applying $\mathcal{P}_0 : A \otimes B \mapsto \mathrm{Tr}(B)\, A \otimes \rho_0$ and adding the h.c., terms one finds

$$\mathcal{P}_0\delta\mathcal{L}\mathcal{P}_0(\rho) = \left(\frac{\alpha}{2}X\omega - \frac{\bar{\alpha}}{2}X^\dagger\omega\right) \otimes \rho_0 + \mathrm{h.c.}$$
$$= -i\left[\frac{\bar{\alpha}X^\dagger - \alpha X}{2i}, \omega\right] \otimes \rho_0, \quad \alpha := \mathrm{Tr}(\tilde{L}^\dagger Y \rho_0). \quad (25)$$

On the other hand $\delta L^\dagger L = (X^\dagger \otimes Y^\dagger)(\mathbb{1} \otimes \tilde{L}) = X^\dagger \otimes Y^\dagger \tilde{L}$ from which we see that Eq. (25) can be written as $-i[\tilde{A}, \omega \otimes \rho_0]$, where $\tilde{A} = \mathrm{Im}\,\mathrm{Tr}_{d_J}(\delta L^\dagger L\,(\mathbb{1} \otimes \rho_0)) \otimes \mathbb{1}_{d_J}$. Here the index d_J denotes the second factor, i.e., $\mathbf{C}^{d_J}$ in the bipartition of the given J block. Let us now consider a Liouvillian $\mathcal{L}_0$ with more than one Lindblad operator $\{L_\alpha\}_\alpha$. Putting together all the different J blocks, we obtain $\mathcal{P}_0\mathcal{L}_1\mathcal{P}_0(\rho) = -i[A,\rho]$, where A is given by Eq. (22) with $\rho = \sum_J^\oplus \omega_J \otimes \rho_{0,J}$, $(\mathcal{P}_0(\rho) = \rho)$. In the unital case one has $\rho_{0,,J} = \mathbb{1}_{d_J}/d_J$ and Eq. (23) follows from Eqs. (22) and (14). ■

Remark. (i) If all the L_α's and perturbations δL_α's are Hermitean $A = 0$; (ii) if $\delta L_\alpha \in \mathcal{A}'(\Rightarrow (X_\alpha)_J \sim \mathbb{1}_{n_J})$ then $A_J \sim \mathbb{1}_{n_J} \Rightarrow \mathcal{P}_0\delta\mathcal{L}\mathcal{P}_0 = 0$.

While mathematically simple, Proposition 4 is, on physical grounds, a remarkable and surprising result. Combined with the dissipation-projection theorem Eq. (2), it indeed implies that a small, generic, Lindblad perturbation induces an effective unitary dynamics over the SSS generated by the Hamiltonian (22), *even in absence of any Hamiltonian term in both the unperturbed and unperturbed Liouvillian*. In principle, by tailoring the dissipative terms δL_α's, one can obtain a desired effective unitary generator A.

A. Examples

To illustrate this mechanism we first consider a simple two-qubit example. We set $L = \mathbb{1} \otimes \sigma^z$ and $\delta L = \sigma^- \otimes \sigma^z$. In this case $\mathcal{P}_0(X) = \mathrm{Tr}_B(X(\mathbb{1} \otimes |0\rangle\langle 0|)) \otimes |0\rangle\langle 0| + \mathrm{Tr}_B(X(\mathbb{1} \otimes |1\rangle\langle 1|)) \otimes |1\rangle\langle 1|$, where Tr_B denotes the partial trace over the second qubit. Using Eqs. (25) and (22) with $J = 0,1$; $d_0 = d_1 = 1$; $\rho_{0,0} = |0\rangle\langle 0|$, $\rho_{0,1} = |1\rangle\langle 1|$; $X = \sigma^-$ and $Y = \sigma^z$ one finds

$$A = \mathrm{Im}\,\mathcal{P}_0(\delta L^\dagger L) = \sigma^y \otimes |0\rangle\langle 0| + \sigma^y \otimes |1\rangle\langle 1| = \sigma^y \otimes \mathbb{1}. \quad (26)$$

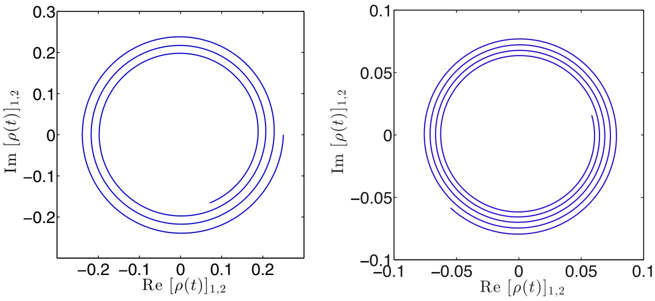

FIG. 2. (Color online) Unitarity from dissipation: a unitary dynamics is generated out of a purely dissipative one. To highlight the unitary character we use the method described in Ref. [37]. For unitary dynamics one has $\rho(t) = e^{-itH_{\rm eff}}\rho_0 e^{itH_{\rm eff}} = \sum_{n,m} e^{-it(E_n-E_m)}[\rho_0]_{n,m}|n\rangle\langle m|$, with $H_{\rm eff} = \sum_n E_n|n\rangle\langle n|$. In the figure we plot the real and imaginary part of $\langle 1|\rho(t)|2\rangle$. Left panel: two-qubit example generating the effective Hamiltonian $H_{\rm eff} = \sigma^y \otimes \mathbb{1}$. The time evolution window is $t \in [T/100, 10T]$ and $T = 100$ in arbitrary units. Right panel: four-qubit example with perturbed collective noise. The effective Hamiltonian is given in Eq. (28). The time-evolution window is $[T/2, 4T]$ with $T = 100$.

A numerical simulation of this dissipation-generated gate is shown in the left panels of Figs. 2 and 3. To highlight the unitary character of the dynamics we use the fact that matrix elements of the density matrix $\rho(t)$ evolve as phases in the Hamiltonian eigenbasis and therefore result in circles on the complex plane (Fig. 2). Similarly as $\mathcal{E}_T$ approaches a unitary evolution within the SSS, the eigenvalues of $\mathcal{E}_T\mathcal{P}_0$ converge to the unit circle increasing T as depicted in Fig. 3.

Let us now consider a four-qubit system subject to general collective decoherence [2,3]. In this case $\mathcal{L}_0$ has the form (10) with the Lindblad operators given by collective spin operators, i.e., $L_\alpha = \sum_{j=1}^4 \sigma_j^\alpha$, $(\alpha = x,y,z)$. In this unital case the interaction algebra $\mathcal{A}$ coincides with algebra of totally symmetric operators and the commutant $\mathcal{A}'$ is 14-dimensional

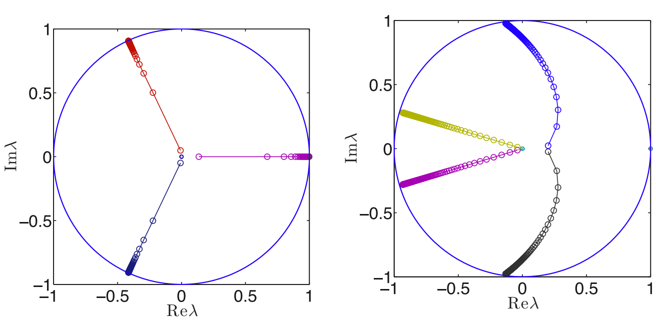

FIG. 3. (Color online) Unitarity from dissipation: a unitary dynamics is generated out of a purely dissipative one. Here we plot real and imaginary part of the eigenvalues of the exact map $\mathcal{E}_T\mathcal{P}_0 = \exp[T(\mathcal{L}_0 + \frac{1}{T}\mathcal{L}_1 + \frac{1}{T^2}\mathcal{L}_2)]\mathcal{P}_0$ for different T. The examples are the same as in Fig. 2, i.e., two- (four-) qubit example on the left (right) panel. By increasing T the eigenvalues tend to go on the unit circle. Left panel: $T \in [1,1000]$, the eigenvalues approach $e^{\pm 2i}, 1$, consistent with Eq. (26). Right panel: $T \in [5,600]$. The eigenvalues of the effective Hamiltonians (28) are $\pm 8, 0$. Consequently the eigenvalues of $\mathcal{E}_T P_0$ approach $e^{\pm 16i}, e^{\pm 8i}, 1$.

[20] and generated by qubit permutation operators in the group $\mathcal{S}_4$. We consider perturbations of the form $\delta L_\alpha = U\, L_\alpha = L_\alpha\, U$, where $U \in \mathcal{S}_4$. Then $\sum_\alpha \delta L_\alpha^\dagger L_\alpha = 4U^\dagger \boldsymbol{S}^2$, where $\boldsymbol{S}$ is the total spin operator. We also have $\mathcal{P}_0(U^\dagger \boldsymbol{S}^2) = U^\dagger \boldsymbol{S}^2$ and $U^\dagger \boldsymbol{S}^2 - U\boldsymbol{S}^2 = 2(U^\dagger - U)$. We further fix U to perform the right-shift permutation $(1,2,3,4) \to (4,1,2,3)$. One obtains

$$A = 4\,\mathrm{Im}\,\mathcal{P}_0(U^\dagger \boldsymbol{S}^2) = 8\,\mathrm{Im}(U^\dagger) \tag{27}$$

$$= [(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \times \boldsymbol{\sigma}_3] \cdot \boldsymbol{\sigma}_4 + [\boldsymbol{\sigma}_2 \times (\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_4)] \cdot \boldsymbol{\sigma}_1\,, \tag{28}$$

where in the last equation we used the fact that $U = S_{2,3}S_{3,4}S_{1,4}$ with $S_{i,j} = (\mathbb{1} + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)/2$ the operator swapping site i with j. A numerical simulation confirming this unitary behavior emerging from a dissipative dynamics is shown in the right panels of Figs. 2 and 3.

VII. CONCLUSIONS

The traditional avenue to quantum information processing (QIP) primitives, such as quantum gates, requires the dissipation due to the environment to be as small a possible compared to the control Hamiltonian. A number of powerful techniques have been develop to combat the detrimental effects of dissipation [1]. However, over the past few years there has been a growing amount of evidence that dissipation may, on the contrary, provide a *resource* for QIP; see, e.g., Refs. [13,14,16–19].

In this spirit in Ref. [20] we have shown how it is possible to generate coherent quantum manipulations also in the opposite regime in which the dissipation is much stronger than the control Hamiltonian. The only requirement is essentially that the dissipation must provide a degenerate set of steady states (SSS). The coherent control drives the system away from the SSS but the strong dissipation effectively projects the dynamics back onto the SSS. As a consequence, a quantum evolution governed by an effective Hamiltonian coherently unfolds within the SSS [20].

In this paper we further investigated the consequences of this approach. The following are the main findings of this paper: (i) We provided further details on the rigorous estimate of the error between the exact evolution and the effective projected dynamics. (ii) Moving to a suitable rotated frame, we have shown that the effective dynamics in the SSS is of geometric origin, i.e., it is the holonomy associated with a superoperator-valued connection. (iii) The effective dynamics is protected against a large class of Hamiltonian and dissipative perturbations. (iv) As a corollary of this result, we have shown that certain dissipative perturbations of purely, dissipative, Lindbladian nature (i.e., one for which the eigenvalues are real negative) generate an effective unitary dynamics.

Dissipative dynamics is easily obtained from a unitary one as soon as some degrees of freedom are traced out. On the contrary, the emergent unitarity phenomenon we have discussed is a quite surprising example of a unitary dynamics obtained from a purely dissipative one. Understanding its fundamental origin and potential application in QIP is a worthwhile topic for future investigations.

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APPENDIX: ERROR ESTIMATE

Here we estimate the error term appearing in the right-hand side (RHS) of Eq. (2). In this section we use the following notation for the total Liouvillian: $\mathcal{L}(x) = \mathcal{L}_0 + x\mathcal{L}_1$, where $\mathcal{L}_0$ is the dominant, dissipative term, $\mathcal{L}_1$ is the perturbation (which often will be taken as a unitary generator, i.e., $\mathcal{L}_1 = \mathcal{K}$), and x is a small dimensionless parameter. The relation between x and the T in the main text is $T^{-1} = x\tau_0^{-1}$, where τ_0 is some time scale which will become more explicit below. We show that the expression inside the norm in the left-hand side (LHS) of Eq. (2) is analytic in x around zero starting with a linear term and we estimate its coefficient. Let us assume that Ker($\mathcal{L}_0$) is d dimensional, i.e., d eigenvalues of $\mathcal{L}_0$ are zero. If we turn on the perturbation $x\mathcal{L}_1$, some of these eigenvalues will move a little bit. The collection of all these d eigenvalues forms the so-called λ group [39] and identifies an invariant subspace of $\mathcal{L}(x)$. The projection $\mathcal{P}(x)$ onto such subspace turns out to be an analytic function of x [39]. As shown in Ref. [39], the restriction of $\mathcal{L}(x)$ to the λ group, $\mathcal{R}(x) := \mathcal{P}(x)\mathcal{L}(x) = \mathcal{L}(x)\mathcal{P}(x) = \mathcal{P}(x)\mathcal{L}(x)\mathcal{P}(x)$, is also an analytic function of x. The λ-group projection $\mathcal{P}(x)$ is a standard tool in the spectral theory of linear operators (see, e.g., the classic Ref. [39]). It is basically the sum of all the (not necessarily orthogonal) spectral projections associated with λ group itself and is analytic even at the exceptional points where the degeneracy changes. It can be obtained by integrating the resolvent $(z - \mathcal{L}(x))^{-1}$ along a contour enclosing all (and only) the eigenvalues belonging to the λ group.

One has the following expansions:

$$\mathcal{P}(x) = \mathcal{P}_0 + x\mathcal{P}_1 + O(x^2), \tag{A1}$$

$$\mathcal{R}(x) = x\mathcal{R}_1 + x^2\mathcal{R}_2 + x^3\mathcal{R}_3 + O(x^4). \tag{A2}$$

The Liouvillian character of $\mathcal{L}_0$ (i.e., that fact that $\mathcal{P}_0 = \lim_{t\to\infty} e^{t\mathcal{L}}$) assures that zero is a semisimple eigenvalue of $\mathcal{L}_0$; i.e., there is no Jordan block associated with the zero eigenvalue of $\mathcal{L}_0$. Although this is not strictly required, it does simplify the following formulae. In the semisimple case one has, for instance, [39]

$$\mathcal{P}_1 = -(\mathcal{P}_0\mathcal{L}_1\mathcal{S} + \mathcal{S}\mathcal{L}_1\mathcal{P}_0), \tag{A3}$$

$$\mathcal{R}_1 = \mathcal{P}_0\mathcal{L}_1\mathcal{P}_0, \tag{A4}$$

$$\mathcal{R}_2 = -(\mathcal{P}_0\mathcal{L}_1\mathcal{P}_0\mathcal{L}_1\mathcal{S} + \mathcal{P}_0\mathcal{L}_1\mathcal{S}\mathcal{L}_1\mathcal{P}_0 + \mathcal{S}\mathcal{L}_1\mathcal{P}_0\mathcal{L}_1\mathcal{P}_0). \tag{A5}$$

In the above formulas, $\mathcal{S}$ is the projected resolvent of $\mathcal{L}_0$ at zero satisfying $\mathcal{L}_0\mathcal{S} = \mathcal{S}\mathcal{L}_0 = \mathbb{1} - \mathcal{P}_0$ and is given by

$$\mathcal{S} = -\lim_{z\to 0}(z - \mathcal{L}_0)^{-1}(1 - \mathcal{P}_0) \tag{A6}$$

$$= -\sum_{k\neq 0}\left[(-\lambda_k)^{-1}\mathcal{P}^{(k)} + \sum_{n=1}^{m_k-1}(-\lambda_k)^{-n-1}(\mathcal{D}^{(k)})^n\right]. \tag{A7}$$

In the last equation we assumed that $\mathcal{L}_0$ has the following Jordan decomposition:

$$\mathcal{L}_0 = \sum_k(\lambda_k\mathcal{P}^{(k)} + \mathcal{D}^{(k)}) \tag{A8}$$

with λ_k eigenvalues with (algebraic) multiplicity m_k, projectors $\mathcal{P}^{(k)}$, and nilpotent blocks $\mathcal{D}^{(k)}$ (note that $\lambda_0 = 0$ and $\mathcal{P}^{(0)} \equiv \mathcal{P}_0$). Note that all the $\mathcal{L}_j, \mathcal{R}'_j s$ have the dimension of inverse of time, the $\mathcal{P}_j$'s are dimensionless, and $\mathcal{S}$ has units of time. In particular $x\mathcal{R}_1$ is precisely $\mathcal{L}_{\text{eff}}$ in our applications. We denote it $\mathcal{R}_{\text{eff}} := tx\mathcal{R}_1$ here for notational consistency.

Clearly $\mathcal{P}(x)$ commutes with $\mathcal{L}(x)$ and so one has the identity

$$e^{t\mathcal{L}(x)}\mathcal{P}(x) = e^{t\mathcal{R}(x)}\mathcal{P}(x). \tag{A9}$$

We now choose times t such that tx is bounded by a given finite time in some unit, i.e., $tx = O(1)\tau_0$. In the following we adopt the so-called $1-1$ norm for maps $\mathcal{E}$, i.e., $\|\mathcal{E}\| := \sup_{\|X\|_1=1}\|\mathcal{E}(X)\|_1$. This is a submultiplicative and automorphism-invariant norm for superoperators such that $\|e^{t\mathcal{L}(x)}\| \leqslant 1$.

Note that $\|e^{t\mathcal{L}(x)}\| \leqslant 1$ because the evolution maps states to states (i.e., because of complete positivity). Instead from $\|e^{t\mathcal{R}(x)}\| \leqslant \exp\|t\mathcal{R}(x)\|$ we get $\|e^{t\mathcal{R}(x)}\| \leqslant O(1)$. Hence we obtain

$$(e^{t\mathcal{L}(x)} - e^{t\mathcal{R}(x)})\mathcal{P}_0 = -(e^{t\mathcal{L}(x)} - e^{t\mathcal{R}(x)})x\mathcal{P}_1 + O(x^2). \tag{A10}$$

Now define $\Delta := e^{t\mathcal{R}(x)} - e^{tx\mathcal{R}_1}$. Clearly $\Delta = O(x)$ (we later determine the coefficient), so we finally get

$$(e^{t\mathcal{L}(x)} - e^{tx\mathcal{R}_1})\mathcal{P}_0 = +\Delta\mathcal{P}_0 - (e^{t\mathcal{L}(x)} - e^{tx\mathcal{R}_1})x\mathcal{P}_1 + O(x^2). \tag{A11}$$

Using Dyson expansion one can easily estimate Δ:

$$\Delta = tx^2\int_0^1 e^{(1-s)\mathcal{R}_{\text{eff}}}\mathcal{R}_2 e^{s\mathcal{R}_{\text{eff}}}ds + O(x^2). \tag{A12}$$

We now take the norm of Eq. (A11), use triangle inequality, and bound all the resulting terms. Defining $C = \sup_{s\in[0,1]}\|e^{s\mathcal{R}_{\text{eff}}}\|$, we get $\|\Delta\| \leqslant tx^2C^2\|\mathcal{R}_2\| + O(x^2)$, $\|\mathcal{R}_2\| \leqslant 3\|\mathcal{P}_0\|^2\|\mathcal{L}_1\|^2\|\mathcal{S}\|$, and $\|\mathcal{P}_1\| \leqslant 2\|\mathcal{P}_0\|\|\mathcal{L}_1\|\|\mathcal{S}\|$. Putting things together we finally obtain

$$\begin{aligned}\|(e^{t\mathcal{L}(x)} - e^{\mathcal{R}_{\text{eff}}})\mathcal{P}_0\| \leqslant{}& x\|\mathcal{S}\|\|\mathcal{L}_1\|\|\mathcal{P}_0\| \\ &\times(3txC^2\|\mathcal{P}_0\|^2\|\mathcal{L}_1\| \\ &+2(C+1)) + O(x^2).\end{aligned} \tag{A13}$$

In order to make more apparent the connection with physical constants, we define dimensionless (tilded) operators via $\mathcal{L} = \gamma_0\tilde{\mathcal{L}}_0 + \gamma_1\tilde{\mathcal{L}}_1$ such that $\gamma_0^{-1} = \tau_R$ is the (short) relaxation time of the unperturbed dissipation and $\gamma_1^{-1} = T$ is the time scale of the control term. By measuring times in units of τ_R the evolution becomes $\exp[(t/\tau_R)(\tilde{\mathcal{L}}_0 + x\tilde{\mathcal{L}}_1)]$ and we see that $x = \gamma_1/\gamma_0 = \tau_R/T$ is the small parameter. The requirement that the effective generator $\tilde{\mathcal{R}}_{\text{eff}} = t\gamma_1\tilde{\mathcal{P}}_0\tilde{\mathcal{L}}_1\tilde{\mathcal{P}}_0$ is finite and nonzero implies $t\gamma_0 x = t\gamma_1 = O(1)$. This means that the waiting time is given by $t = O(T)$. The bound Eq. (A13) then translates into (all the tilded operators are dimensionless)

$$\|(e^{t\mathcal{L}} - e^{\tilde{\mathcal{R}}_{\rm eff}})\mathcal{P}_0\| \leqslant \frac{\tau_R}{T}\|\tilde{\mathcal{S}}\|\|\tilde{\mathcal{L}}_1\|\|\tilde{\mathcal{P}}_0\|\left(3\frac{t}{T}C^2\|\tilde{\mathcal{P}}_0\|^2\|\tilde{\mathcal{L}}_1\| + 2(C+1)\right) + O(x^2). \tag{A14}$$

The relation above can often be further simplified. For example, if $\mathcal{L}_0$ generate a positive map $\|\tilde{\mathcal{P}}_0\| = 1$, whereas if $\mathcal{R}_{\rm eff}$ generates a unitary one has $C = 1$.

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